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Complex analysis

Faber polynomial coefficient estimates for certain subclasses of meromorphic bi-univalent functions

Estimations à l'aide des polynômes de Faber des coefficients de certaines fonctions méromorphes bi-univalentes

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ABSTRACT

Making use of the Faber polynomial coefficient expansions to a class of meromorphic bi-univalent functions, we obtain the general coefficient estimates for such functions and study their initial coefficient bounds. The coefficient bounds presented here are new in their own kind.

RÉSUMÉ

Utilisant les développements des coefficients en termes de polynômes de Faber, nous obtenons des estimations du coefficient général des éléments d'une classe de fonctions méromorphes bi-univalentes. Nous étudions aussi les bornes pour leurs coefficients initiaux. Les bornes présentées ici sont nouvelles dans leur genre.

1. Introduction

Let Σ' be the family of meromorphic functions g of the form

$$g(z) = z + b_0 + \sum_{n=1}^{\infty} b_n \frac{1}{z^n}, \tag{1.1}$$

that are univalent in $\Delta := \{z \in \mathbb{C} : 1 < |z| < \infty\}$. The coefficients of $h = g^{-1}$, the inverse map of g, are given by the Faber polynomial [9]:

$$h(w) = g^{-1}(w) = w + \sum_{n=0}^{\infty} \frac{B_n}{w^n} = w - b_0 - \sum_{n\geq 1} \frac{1}{n} K_{n+1}^n \frac{1}{w^n}, \quad w \in \Delta, \tag{1.2}$$

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where

$$K_{n+1}^{n} = nb_0^{n-1}b_1 + n(n-1)b_0^{n-2}b_2 + \frac{1}{2}n(n-1)(n-2)b_0^{n-3}\left(b_3 + b_1^2\right) + \frac{n(n-1)(n-2)(n-3)}{3!}b_0^{n-4}(b_4 + 3b_1b_2) + \sum_{j\geq 5} b_0^{n-j}V_j \tag{1.3}$$

and V_j with $5 \leq j \leq n$ is a homogeneous polynomial of degree j in the variables $b_1, b_2, \ldots, b_n$.

Gong [11] demonstrated the significance of the Faber polynomials [9] in mathematical sciences, particularly in geometric function theory. The recent works of [1,2,12,13] and [20] dealing with the Taylor expansion of the inverse function $h = g^{-1}$ are estimable and fit our study for the bi-univalent functions.

In 1923, Lowner [15] proved that the inverse of the Koebe function $k(z) = \frac{z}{(1-z)^2}$ endows with the best upper bounds for the coefficients of the inverses of analytic univalent functions. Even though estimates for the coefficients of the inverse of analytic univalent functions have been obtained in an astonishingly uncomplicated approach (see [8, p. 104]), the case turns out to be a challenge when the bi-univalency condition is imposed on these functions. Further, Lewin [14] examined the bi-univalent function class Σ and proved that

$$|a_2| < 1.51.$$

On the other hand, Brannan and Taha [4] and Netanyahu [16] made an effort to introduce various subclasses of the bi-univalent function class Σ and found non-sharp coefficient estimates on the initial coefficients $|a_2|$ and $|a_3|$ of

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disc

$$\mathbb{U} = \left\{z : z \in \mathbb{C} \text{ and } |z| < 1\right\}.$$

But the coefficient problem for each one of the following Taylor–Maclaurin coefficients

$$|a_n|, \quad n \in \mathbb{N} \setminus \{1, 2\};\ \mathbb{N} := \{1, 2, 3, \ldots\}$$

is still an open problem.

Following Brannan and Taha [4], in recent times numerous researchers (see [3,5,7,10,12,13,17–19]) introduced and examined some interesting subclasses of the bi-univalent functions and they have established non-sharp estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$.

Motivated by the works [6,12,13], we define the following subclass $\mathcal{R}_{\Sigma'}(\alpha, \lambda, \delta)$ of the function class Σ'.

For $0 \leq \alpha < 1$, $\lambda \geq 1$ and $\delta \geq 0$, let $\mathcal{R}_{\Sigma'}(\alpha, \lambda, \delta)$ be the class of meromorphic bi-univalent functions $g \in \Sigma'$ so that:

$$\Re\left((1 - \lambda + 2\delta)\frac{g(z)}{z} + (\lambda - 2\delta)g'(z) + \delta z g''(z)\right) > \alpha, \quad z \in \Delta \tag{1.4}$$

and

$$\Re\left((1 - \lambda + 2\delta)\frac{h(w)}{w} + (\lambda - 2\delta)h'(w) + \delta w h''(w)\right) > \alpha, \quad w \in \Delta. \tag{1.5}$$

It is interesting to note that, for $\delta = 0$ the class $\mathcal{R}_{\Sigma'}(\alpha, \lambda, \delta)$ reduces to the class $\mathcal{B}\Sigma(\alpha; \lambda)$ introduced and studied by Hamidi et al. [12].

In this paper, we use the Faber polynomial expansions to obtain bounds for the general coefficients $|a_n|$ of meromorphic bi-univalent functions in $\mathcal{R}_{\Sigma'}(\alpha, \lambda, \delta)$ as well as providing estimates for the initial coefficients of these functions. As an outcome, we are able to prove the following Theorems 2.1 and 2.2 by employing the techniques of Hamidi et al. [12].

2. Coefficient bounds

We begin this section by obtaining the general coefficient:

Theorem 2.1. *Let g be given by (1.1). For $0 \leq \alpha < 1$, $\lambda \geq 1$ and $\delta \geq 0$ if $g \in \mathcal{R}_{\Sigma'}(\alpha, \lambda, \delta)$ and $b_k = 0$; $0 \leq k \leq n-1$, then*

$$|b_n| \leq \frac{2(1-\alpha)}{|1 + (n+1)(\delta - \lambda) + (n+1)^2\delta|}, \quad n \geq 1. \tag{2.1}$$

Proof. For meromorphic functions g of the form (1.1), we have

$$(1-\lambda+2\delta)\frac{g(z)}{z}+(\lambda-2\delta)g'(z)+\delta z g''(z)=1+\sum_{n=0}^{\infty}\left[1+(n+1)(\delta-\lambda)+(n+1)^2\delta\right]\frac{b_n}{z^{n+1}} \tag{2.2}$$

and

$$\begin{aligned}(1-\lambda+2\delta)&\frac{h(w)}{w}+(\lambda-2\delta)h'(w)+\delta w h''(w)\\ &=1+\sum_{n=0}^{\infty}\left[1+(n+1)(\delta-\lambda)+(n+1)^2\delta\right]\frac{B_n}{w^{n+1}}\\ &=1-(1-\lambda+2\delta)\frac{b_0}{w}-\sum_{n=1}^{\infty}\left[1+(n+1)(\delta-\lambda)+(n+1)^2\delta\right]\frac{1}{n}\frac{K_{n+1}^{n}(b_0,b_1,\dots,b_n)}{w^{n+1}}.\end{aligned} \tag{2.3}$$

Following [12], since $g\in\mathcal{R}_{\Sigma'}(\alpha,\lambda,\delta)$, by definition, there exist two positive real-part functions $p(z)=1+\sum_{n=1}^{\infty}c_n z^{-n}$ and $q(w)=1+\sum_{n=1}^{\infty}d_n w^{-n}$, where $\Re(p(z))>0$ and $\Re(q(w))>0$ in Δ so that

$$(1-\lambda+2\delta)\frac{g(z)}{z}+(\lambda-2\delta)g'(z)+\delta z g''(z)=1+(1-\alpha)\sum_{n=1}^{\infty}K_n^1(c_1,c_2,\dots,c_n)\frac{1}{z^n} \tag{2.4}$$

and

$$(1-\lambda+2\delta)\frac{h(w)}{w}+(\lambda-2\delta)h'(w)+\delta w h''(w)=1+(1-\alpha)\sum_{n=1}^{\infty}K_n^1(d_1,d_2,\dots,d_n)\frac{1}{w^n}. \tag{2.5}$$

Note that, according to the Caratheodory Lemma [8],

$$|c_n|\le 2 \quad\text{and}\quad |d_n|\le 2,\quad n\ge 1.$$

Comparing the corresponding coefficients of (2.2) and (2.4) yields:

$$\left[1+(n+1)(\delta-\lambda)+(n+1)^2\delta\right]b_n=(1-\alpha)K_{n+1}^1(c_1,c_2,\dots,c_{n+1}) \tag{2.6}$$

and, similarly from (2.3) and (2.5), we obtain:

$$\left[1+(n+1)(\delta-\lambda)+(n+1)^2\delta\right]B_n=(1-\alpha)K_{n+1}^1(d_1,d_2,\dots,d_{n+1}). \tag{2.7}$$

We observe that for $b_k=0$; $0\le k\le n-1$, we have $B_n=-b_n$ so we have,

$$\left[1+(n+1)(\delta-\lambda)+(n+1)^2\delta\right]b_n=(1-\alpha)c_{n+1} \tag{2.8}$$

and

$$\left[1+(n+1)(\delta-\lambda)+(n+1)^2\delta\right]B_n=(1-\alpha)d_{n+1}. \tag{2.9}$$

Now taking absolute values to (2.8), (2.9) and applying the Caratheodory Lemma, we obtain:

$$\begin{aligned}|b_n|&=\frac{(1-\alpha)|c_{n+1}|}{|1+(n+1)(\delta-\lambda)+(n+1)^2\delta|}=\frac{(1-\alpha)|d_{n+1}|}{|1+(n+1)(\delta-\lambda)+(n+1)^2\delta|}\\ &\le\frac{2(1-\alpha)}{|1+(n+1)(\delta-\lambda)+(n+1)^2\delta|}.\qquad\square\end{aligned}$$

Relaxing the coefficient restrictions imposed on Theorem 2.1, in the following theorem, we obtain estimates for the early coefficients of functions g in $\mathcal{R}_{\Sigma'}(\alpha,\lambda,\delta)$ as well as a bound for the coefficient body $(b_2+b_0b_1)$.

Theorem 2.2. *Let g be given by (1.1). For $0\le\alpha<1$, $\lambda>1$ and $\delta\ge 0$ if $g\in\mathcal{R}_{\Sigma'}(\alpha,\lambda,\delta)$, then*

$$|b_0|\le\begin{cases}\frac{2(1-\alpha)}{1-\lambda+2\delta}, & \lambda<1+2\delta,\\ \frac{2(1-\alpha)}{-1+\lambda-2\delta}, & \lambda>1+2\delta,\end{cases}$$

$$|b_1| \le \begin{cases} \frac{2(1-\alpha)}{1-2\lambda+6\delta}, & \lambda < \frac{1}{2}+3\delta, \\ \frac{2(1-\alpha)}{-1+2\lambda-6\delta}, & \lambda > \frac{1}{2}+3\delta, \end{cases}$$

$$|b_2| \le \begin{cases} \frac{2(1-\alpha)}{1-3\lambda+12\delta}, & \lambda < \frac{1}{3}+4\delta, \\ \frac{2(1-\alpha)}{-1+3\lambda-12\delta}, & \lambda > \frac{1}{3}+4\delta, \end{cases}$$

$$|b_2+b_0b_1| \le \begin{cases} \frac{2(1-\alpha)}{1-3\lambda+12\delta}, & \lambda < \frac{1}{3}+4\delta, \\ \frac{2(1-\alpha)}{-1+3\lambda-12\delta}, & \lambda > \frac{1}{3}+4\delta. \end{cases}$$

Proof. Comparing Eqs. (2.2) and (2.4) for $n = 0, 1, 2$, we obtain:

$$(1-\lambda+2\delta)b_0 = (1-\alpha)c_1$$
$$(1-2\lambda+6\delta)b_1 = (1-\alpha)c_2$$

and

$$(1-3\lambda+12\delta)b_2 = (1-\alpha)c_3.$$

On the other hand, from (2.3) and (2.5), for $n = 2$, we obtain:

$$-(1-3\lambda+12\delta)(b_2+b_0b_1) = (1-\alpha)d_3.$$

Solving the equations above for b_0, b_1, b_2 and $(b_2+b_0b_1)$, respectively, taking their absolute values and then applying the Caratheodory Lemma, we obtain the desired results. □

Remark 2.3. For $\delta = 0$, the bounds obtained in Theorems 2.1 and 2.2 coincide with the ones of [12, Theorem 1.1] and [12, Theorem 1.2], respectively.

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