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Number theory

Period relations for automorphic induction and applications, I

Relations de périodes pour l'induction automorphe et applications, I

Jie Lin

Institut de mathématiques de Jussieu, 4, place Jussieu, 75005 Paris, France

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ABSTRACT

Let K be a quadratic imaginary field. Let Π (resp. Π') be a regular algebraic cuspidal representation of $GL_n(\mathbb{A}_K)$ (resp. $GL_{n-1}(\mathbb{A}_K)$), which is moreover cohomological and conjugate self-dual. When Π is a cyclic automorphic induction of a Hecke character χ over a CM field, we show relations between automorphic periods of Π defined by Harris and those of χ. Consequently, we refine a formula given by Grobner and Harris for critical values of the Rankin–Selberg L-function $L(s, \Pi \times \Pi')$. This completes the proof of an automorphic version of Deligne's conjecture in certain cases.

RÉSUMÉ

Soit K un corps quadratique imaginaire. Soit Π (resp. Π') une représentation cuspidale régulière algébrique de $GL_n(\mathbb{A}_K)$ (resp. $GL_{n-1}(\mathbb{A}_K)$), qui est, de plus, cohomologique et auto-duale. Si Π est une induction automorphe cyclique d'un caractère de Hecke χ sur un corps CM, on montre les relations entre les périodes automorphes de Π définies par Harris et celles de χ. Par conséquent, on affine une formule de Grobner et Harris pour les valeurs critiques de $L(s, \Pi \times \Pi')$, L étant la fonction de Rankin–Selberg. Cela complète la démonstration d'une version automorphe de la conjecture de Deligne dans certains cas.

0. Introduction

In [7], M. Harris has defined complex invariants, called automorphic periods, for certain automorphic representations over a quadratic imaginary field. We believe that these periods are functorial. In this note, we treat the case when the representation is a cyclic automorphic induction of a Hecke character over a CM field. More precisely, let K be a quadratic imaginary field and $F \supset K$ be a CM field that is cyclic over K. Let χ be certain Hecke character of F and $\Pi(\chi)$ be the automorphic induction of χ with respect to F/K. We show the relations between automorphic periods of $\Pi(\chi)$ and CM periods of χ. Our main result is Theorem 3.2 below.

These relations allow us to simplify a formula obtained by Grobner and Harris on the critical values for the Rankin–Selberg L-function of $\Pi \times \Pi'$ where Π and Π' are certain automorphic representations of $GL_n(\mathbb{A}_K)$ and $GL_{n-1}(\mathbb{A}_K)$ (cf. [5]).

E-mail address: linjie_emma@hotmail.com.

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We first refine the formula in the case when Π and Π' are both induced from characters and then to more general cases. We see finally that our result is compatible with Deligne's conjecture.

1. Notation and conventions

Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

Let $K \subset \overline{\mathbb{Q}}$ be a quadratic imaginary field and n be an integer of at least 2. Let ε_K be the Artin character of $\mathbb{A}_{\mathbb{Q}}$ associated with the extension $K/\mathbb{Q}$. We fix ψ an algebraic Hecke character of K with infinity type $z^1\bar{z}^0$ such that $\psi\psi^c = \|\cdot\|_{\mathbb{A}_K}$. The existence follows from Lemma 4.1.4 in [3].

Let F^+ (resp. F'^+) be a totally real field of degree n (resp. $n-1$) over $\mathbb{Q}$. We set $F = KF^+$ (resp. $F' = KF'^+$) a CM field. We put $L = F \otimes_K F'$. It is easy to see that L is a CM field of degree $n(n-1)$ over K.

Let $\iota \in G_{\mathbb{Q}} := Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ be the complex conjugation. We may consider it as an element of $Gal(F/F^+)$ or $Gal(F'/F'^+)$. For $z \in \mathbb{C}$, we write $\bar{z}$ for its complex conjugation. For any number field E, let Σ_E be the set of complex embeddings of E. For $\sigma \in \Sigma_F$, we define $\bar{\sigma} := \iota \circ \sigma$ the complex conjugation of σ.

Let Φ be a subset of Σ_F. We say that Φ is a **CM type** of F if $\Phi \cup \iota\Phi = \Sigma_F$ and $\Phi \cap \iota\Phi = \varnothing$. Let $\{\sigma_1, \sigma_2, \cdots, \sigma_n\}$ be the elements of Σ_F which are the identity on K. We know that $\{\sigma_1, \sigma_2, \cdots, \sigma_n\}$ is a CM type of F.

Let χ be a Hecke character of F with infinity-type $\chi_\infty(z) = \prod_{i=1}^n \sigma_i(z)^{a_i}\bar{\sigma}_i(z)^{b_i}$. We suppose that χ is **algebraic**, i.e. $a_i, b_i \in \mathbb{Z}$, which implies that $a_i + b_i = -w(\chi)$ an integer independent of i, and **critical**, i.e. $a_i \neq b_i$ for all i. We can then define Φ_χ, a unique CM type associated with χ, as follows: for each i, $\sigma_i \in \Phi_\chi$ if $a_i < b_i$, otherwise $\bar{\sigma}_i \in \Phi_\chi$. In this case, we say that χ is **compatible** with Φ_χ.

For such χ, one can define $E(\chi_\infty) \subset \mathbb{C}$, a number field, as in (1.1.2) of [6]. It is the field of definition of $\sum(a_i\sigma_i + b_i\overline{\sigma_i}) \in \mathbb{Z}^{\Sigma_F}$. We denote by $E(\chi)$ the field generated by the values of χ on $\mathbb{A}_{F,f}$ over $E(\chi_\infty)$. It is still a number field. We assume that $E(\chi)$ contains F for simplicity of notation.

With any $\Psi \subset \Sigma_F$ such that $\Psi \cap \iota\Psi = \varnothing$, one can associate a non-zero complex number $p_F(\chi, \Psi)$ that is well defined modulo $E(\chi)^\times$ (cf. the appendix of [9]). We call it a **CM period**. Sometimes we write $p(\chi, \Psi)$ instead of $p_F(\chi, \Psi)$ if there is no ambiguity concerning the base field F.

The special values of an L-function for a Hecke character over a CM field can be interpreted in terms of CM periods. The following theorem is proved by Blasius. We state it as in Proposition 1.8.1 in [6] where ω should be replaced by $\check{\omega} := \omega^{-1,c}$ (for this erratum, see the notation and conventions part on page 82 in [7]).

Theorem 1.1. *Let χ be as before. We denote D_{F^+} the absolute discriminant of F^+. For m a critical value of χ in the sense of Deligne, we have*

$$\left(L\left(\chi^\sigma, m\right)\right)_{\sigma\in\Sigma_{E(\chi)}} \sim_{E(\chi)} D_{F^+}^{1/2}(2\pi i)^{mn}\left(p\left(\check{\chi}^\sigma, \Phi_{\chi^\sigma}\right)\right)_{\sigma\in\Sigma_{E(\chi)}}.$$

We now introduce the notation $\sim_{E(\chi)}$ in the previous theorem. Let E be a finite extension of K. We identify $\mathbb{C}^{\Sigma_E}$ with $E \otimes \mathbb{C}$ by the inverse of the map that sends $t \otimes z$ to $(\sigma(t)z)_{\sigma\in\Sigma_E}$ for all $t \in E$ and $z \in \mathbb{C}$. This is a morphism of algebras where the multiplication on the former is the usual multiplication through each coordinates. Similarly, let $\Sigma_{E;K}$ be the subset of Σ_E containing embeddings of E into $\mathbb{C}$ that are the identity on K. We may identify $\mathbb{C}^{\Sigma_{E;K}}$ with $E \otimes_K \mathbb{C}$.

Definition 1.1. Let A, B be two elements in $E \otimes \mathbb{C}$ (resp. $E \otimes_K \mathbb{C}$). We say that $A \sim_E B$ (resp. $A \sim_{E;K} B$) if one of the following conditions is satisfied: (i) $A = 0$, (ii) $B = 0$ or (iii) $A, B \in (E \otimes \mathbb{C})^\times$ (resp. $(E \otimes_K \mathbb{C})^\times$) with $AB^{-1} \in E^\times \subset (E \otimes \mathbb{C})^\times$ (resp. $(E \otimes_K \mathbb{C})^\times$).

Note that this relation is symmetric but not transitive unless we know that everything is non-zero.

Let $(a(\sigma))_{\sigma\in G_K}$ be some complex numbers such that $a(\sigma) = a(\sigma')$ if $\sigma|_E = \sigma'|_E$ for any $\sigma, \sigma' \in G_K$. For example, for $E = E(\chi)$ and s a complex number, the values $(L(s, \chi^\sigma))_{\sigma\in G_K}$ satisfy the above condition. We can define $a(\sigma)$ for $\sigma \in \Sigma_{E;K}$ by taking $\tilde{\sigma}$, any lift of σ in G_K, and defining $a(\sigma)$ to be $a(\tilde{\sigma})$. We consider $(a(\sigma))_{\sigma\in\Sigma_{E;K}}$ as an elements in $\mathbb{C}^{\Sigma_{E;K}}$.

Definition–Lemma 1.1. *Let $b(\sigma)_{\sigma\in G_K}$ be some complex numbers with the same property as $a(\sigma)_{\sigma\in G_K}$. We assume $b(\sigma) \neq 0$ for all $\sigma \in G_K$. We fix $\sigma_0 \in \Sigma_{E;K}$. We then have $(a(\sigma))_{\sigma\in\Sigma_{E;K}} \sim_{E;K} (b(\sigma))_{\sigma\in\Sigma_{E;K}}$ if and only if $\frac{a(\sigma_0)}{b(\sigma_0)} \in \overline{\mathbb{Q}}$ and $\tau(\frac{a(\sigma_0)}{b(\sigma_0)}) = \frac{a(\tau\sigma_0)}{b(\tau\sigma_0)}$ for all $\tau \in G_K$.*

In this case, we say $a \sim_E b$ ***equivariant under action of*** *G_K. In particular, $\frac{a(\sigma)}{b(\sigma)} \in E$ for all $\sigma \in G_K$.*

At last, we introduce certain notation concerning Hecke characters of K.

Definition 1.2. For η an algebraic Hecke character of K with infinity type $z^{a(\eta)}\bar{z}^{b(\eta)}$, we define:

- $\check{\eta} = \eta^{-1,c}$ a Hecke character of K,

- $\widetilde{\eta}(z) = \eta(z)/\eta(\bar{z})$ a Hecke character of K,
- η_0 the Hecke character of $\mathbb{Q}$ such that $\eta\eta^c = (\eta_0 \circ N_{\mathbb{A}_K/\mathbb{A}_\mathbb{Q}})\|\cdot\|^{a(\eta)+b(\eta)}$,
- $\eta^{(2)} = \eta^2/\eta_0 \circ N_{\mathbb{A}_K/\mathbb{A}_\mathbb{Q}}$.

2. Unitary similitude group and base change

In this section, we recall a result on the base change of representations for similitude unitary groups. Let G be a connected quasi-split reductive group over $\mathbb{Q}$ and $G' = Res_{K/\mathbb{Q}} G_K$. Roughly speaking, the base change is a correspondence from certain automorphic representations of $G(\mathbb{A}_\mathbb{Q})$ to certain automorphic representations of $G'(\mathbb{A}_\mathbb{Q}) = G(\mathbb{A}_K)$. We refer to Section 26 of [1] for more details.

Over a local field, this correspondence can be defined concretely for unramified representations (cf. [12]) and is in fact a map from the set of unramified representations of G to that of G'. This allows us to give a precise definition for global base change. For π an admissible irreducible representation of $G(\mathbb{A}_\mathbb{Q})$, we say Π, a representation of $G(\mathbb{A}_K)$, is a **(weak) base change** of π if for all v, a finite place of $\mathbb{Q}$ at which π is unramified and G is quasi-split, and for all w, a place of K over v, Π_w is the base change of π_v. In this case, we say that Π **descends to** π by base change.

For example, if v is a place of $\mathbb{Q}$ split in K, let w be a place of K above v. We know $\mathbb{Q}_v \cong K_w$ and hence $G(\mathbb{Q}_v) = G(K_w)$. The local base change map is the identity.

Let $r, s \in \mathbb{N}$ such that $r + s = n$. Fix q_1, q_2 two places of $\mathbb{Q}$ that split in K. Take $D_{r,s}$ to be a division algebra of dimension n^2 with center K and endowed with $* : D_{r,s} \to D_{r,s}$ an involution of second kind. Moreover, we want $(D_{r,s}, *)$ to be quasi-split at all finite places that do not equal to q_1 or q_2, to be a division algebra at one or two places between q_1 and q_2, and to have infinity sign (r, s). The calculation of local invariants of unitary groups in Chapter 2 of [2] shows that such a division algebra exists.

We denote $U(r, s)$ the unitary group over $\mathbb{Q}$ associated with $(D_{r,s}, *)$ and write $GU(r, s)$ for the similitude group of $U(r, s)$. One can show that $GU(r, s)_K \cong U(r, s)_K \times \mathbb{G}_{m,K}$. In particular, $GU(\mathbb{A}_K) \cong GL_n(\mathbb{A}_K) \times \mathbb{A}_K^\times$. For Π a cuspidal representation of $GL_n(\mathbb{A}_K)$ and ξ a Hecke character of K, $\Pi \otimes \xi$ defines a cuspidal representation of $GU(\mathbb{A}_K)$. Conversely, by the tensor product theorem, every irreducible automorphic representation of $GU(\mathbb{A}_K)$ can be written in the form $\Pi \otimes \xi$. Moreover, Π and ξ are unique up to isomorphisms.

Let us consider now the base change for $G = GU(r, s)$. Theorem 2.1.2 and Theorem 3.1.2 of [10] tell us when $\Pi \otimes \xi$ descends to a representation of $G(\mathbb{A}_\mathbb{Q})$. In this note, we start with a representation of $GL_n(\mathbb{A}_K)$. The following lemma will be useful (cf. Lemma VI.2.10 of [11]):

Lemma 2.1. *Let Π be a conjugate self-dual cuspidal representation of $GL_n(\mathbb{A}_K)$. We assume that Π is cohomological and supercuspidal at places over q_1 and q_2. There always exists ξ, a Hecke character of K, such that $\Pi \otimes \xi$ descends to a representation of $G(\mathbb{A}_\mathbb{Q})$.*

3. Automorphic period

In this note, a **motive** M simply means a pure motive for absolute Hodge cycles in the sense of Deligne. We refer the reader to [4] for detailed definitions. We recall that an integer m is **critical** for M if neither $L_\infty(M, s)$ nor $L_\infty(\check{M}, 1 - s)$ has a pole at $s = m$ where $\check{M}$ is the dual of M. In this case, we say m is **critical** for M.

The **Hodge type** of M is defined by the set $T = T(M)$ consisting of pairs (p, q) such that $M^{p,q} \neq 0$. We assume that M is pure, namely there exists an integer w such that $p + q = w$ for all $(p, q) \in T(M)$. In [4], the author has determined the critical points in terms of the Hodge type of M.

Let $n \geq 1$ be an integer, K be a quadratic imaginary field and $\Pi = \Pi_f \otimes \Pi_\infty$ be a regular cohomological cuspidal representation of $GL_n(\mathbb{A}_K)$. We denote V the representation space for Π_f. For $\sigma \in Aut(\mathbb{C})$, we define another $GL_n(\mathbb{A}_{K,f})$-representation Π_f^σ to be $V \otimes_{\mathbb{C},\sigma} \mathbb{C}$. Let $\mathbb{Q}(\Pi)$ be the subfield of $\mathbb{C}$ fixed by $\{\sigma \in Aut(\mathbb{C}) \mid \Pi_f^\sigma \cong \Pi_f\}$. We call it the **rationality field** of Π. This is in fact a number field and Π_f has a rational structure on $\mathbb{Q}(\Pi)$. In other words, there exists V, a $GL_n(\mathbb{A}_{\mathbb{Q},f})$-module over $\mathbb{Q}(\Pi)$, such that $\Pi_f = V \otimes_{\mathbb{Q}(\Pi)} \mathbb{C}$ as $GL_n(\mathbb{A}_{\mathbb{Q},f})$-module.

Moreover, for all $\sigma \in Aut(\mathbb{C})$, Π_f^σ is the finite part of a cuspidal representation of $GL_n(\mathbb{A}_K)$ which is unique by the strong multiplicity one theorem, denoted by Π^σ. We know that Π^σ is determined by $\sigma|_{\mathbb{Q}(\Pi)} : \mathbb{Q}(\Pi) \hookrightarrow \mathbb{C}$. Therefore, we may define Π^σ for any $\sigma \in \Sigma_{Q(\Pi)}$ by lifting σ to an element in $Aut(\mathbb{C})$. In particular, we may define Π^σ for any $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ or $\sigma \in \Sigma_E$ where E is an extension of $\mathbb{Q}(\Pi)$.

When Π is a cohomological and conjugate self-dual, M. Harris has proved that there exists a motive associated with Π of rank n over K with coefficients in a number field. By restriction of scalars from K to $\mathbb{Q}$, we obtain (cf. [7]) that:

Theorem 3.1. *There exists E a finite extension of $\mathbb{Q}(\Pi)$ and M a regular pure motive of rank $2n$ over $\mathbb{Q}$ with coefficients in E such that $L(s, M, \sigma) = L(s + \frac{1-n}{2}, \Pi^\sigma)$ for all $\sigma : E \hookrightarrow \mathbb{C}$.*

Harris has also defined automorphic periods $P^{(s)}(\Pi)$ for certain integers $0 \leq s \leq n$, which is a complex number defined up to multiplication by an element in $E^\times$. If Π is supercuspidal at each places over q_1 and q_2, the automorphic period can be defined for every $0 \le s \le n$. More precisely, $P^{(s)}$ is defined when there exists ξ, a Hecke character of K, such that $\Pi \otimes \xi$ descends to a representation of $GU_{n-s,s}(\mathbb{A}_{\mathbb{Q}})$. With the supercuspidal condition, we know that this is true by Lemma 2.1. We assume this condition on Π throughout this note. Harris proved that special values of the automorphic L-function can be interpreted in terms of automorphic periods:

Theorem 3.2. *Let Π be as before with its infinity type $(z^{a_i}\bar{z}^{-a_i})_{1\le i\le n}$. Let η be an algebraic Hecke character of K with infinity type $\eta_\infty(z) = z^a\bar{z}^b$ such that for all $1 \le i \le n$, $b - a \ne 2a_i$.*

Write $\eta^c = \widetilde{\beta}\alpha$. Here α, β are Hecke characters of K with $\alpha_\infty(z) = z^\kappa$ and $\beta_\infty(z) = z^{-k}$, $\kappa, k \in \mathbb{Z}$. Define $s = s(\eta^c, \Pi^\vee) = \#\{i \mid a - b + 2a_i < 0\}$.

For $m \in \mathbb{Z}$ critical for $M(\Pi) \otimes M(\eta)$ and satisfies $m \ge \frac{n-\kappa}{2} = \frac{n-a-b}{2}$, we have:

$$L\big(m, M(\Pi) \otimes M(\eta)\big) \sim_{E(\Pi)E(\beta)E(\alpha)} (2\pi i)^{(m-\frac{n-1}{2})n}\mathcal{G}(\varepsilon_K)^{[\frac{n}{2}]}P^{(s)}(\Pi)\big[(2\pi i)^\kappa \mathcal{G}(\alpha_0)\big]^s\big[(2\pi i)^k p\big(\check{\beta}^{(2)}\check{\alpha}, 1\big)\big]^{n-2s}$$

equivariant under action of G_K. Here $\mathcal{G}(\alpha_0)$ refers to a Gauss sum of α_0.

Proposition 3.1. *Let Π be as in Theorem 3.2. For any fixed integer $0 \le s \le n$, there exists an algebraic Hecke character η and an integer m as in Theorem 3.2 such that $s(\eta^c, \Pi^\vee) = s$ and $L(m, M(\Pi) \otimes M(\eta)) \ne 0$.*

In [5], the authors gave an interpretation of special values of L-function for $GL_n \times GL_{n-1}$ over K. Let Π and Π' be two cuspidal representations of $GL_n(\mathbb{A}_K)$ and $GL_{n-1}(\mathbb{A}_K)$ that satisfy the conditions in Theorem 3.2 and some regular conditions (cf. *loc. cit.*). We have:

Theorem 3.3. *Let m be a non-negative integer. If $m + n - 1$ is critical for $M(\Pi) \otimes M(\Pi')$, then*

$$L\left(m + \frac{1}{2}, \Pi \times \Pi'\right) \sim_{E(\Pi)E(\Pi')} p\big(m, \Pi_\infty, \Pi'_\infty\big)Z(\Pi_\infty)Z\big(\Pi'_\infty\big)\prod_{j=1}^{n-1}P^{(j)}(\Pi)\prod_{k=1}^{n-2}P^{(k)}\big(\Pi'\big)$$

equivariant under action of G_K.

Here $p(m, \Pi_\infty, \Pi'_\infty)$ is a complex number depending only on m, Π_∞ and Π'_∞ (cf. Proposition 6.4 of loc. cit.); $Z(\Pi_\infty)$ (resp. $Z(\Pi'_\infty)$) is a complex number depending only on Π_∞ (resp. Π'_∞) (cf. Theorem 6.7 of loc. cit.).

4. Period relations for automorphic induction of Hecke characters

In this section, we consider the representation induced from Hecke characters. Let χ be a regular algebraic conjugate self-dual Hecke character of F. Here conjugate self-dual means $\chi^{-1} = \chi^c$.

We make the hypothesis that:

Hypothesis 4.1. For any v a place of K over q_1 and q_2, $\chi_v \ne \chi_v^\tau$ for all $\tau \in Gal(F_v/K_v)$ non trivial.

Under this hypothesis, $\Pi(\chi)$, the automorphic induction of χ from $GL_1(\mathbb{A}_F)$ to $GL_n(\mathbb{A}_K)$, is supercuspidal at all places over q_1 and q_2 (cf. Proposition 2.4 of [8]).

Definition–Lemma 4.1. *Let χ be as above. We define $\Pi_\chi := \Pi(\chi)$ if the degree of F over K is odd; $\Pi_\chi := \Pi(\chi) \otimes \|\cdot\|_{\mathbb{A}_K}^{-\frac{1}{2}}\psi$ otherwise where ψ is a Hecke character of K defined in Section 1.*

We have that Π_χ is a regular algebraic cuspidal which satisfies all the conditions in Theorem 3.2.

Up to finite extension, we may assume $E(\Pi_\chi) = E(\chi)$. We define $\Phi_{s,\chi}$, a CM type of F as follows: for each i such that a_i is one of the s smallest numbers in $\{a_i, 1 \le i \le n\}$, we have $\sigma_i \in \Phi_{s,\chi}$; otherwise $\bar{\sigma}_i \in \Phi_{s,\chi}$.

Theorem 4.1. *Let n be an integer. Let $F = F^+K$ with F^+ a totally real field of degree n over $\mathbb{Q}$ and K a quadratic imaginary field. Assume that F is cyclic over K. Let χ be a regular conjugate self-dual algebraic Hecke character of F satisfying Hypothesis 4.1. We have that the automorphic period of $\Pi = \Pi_\chi$ satisfies:*

$$P^{(s)}(\Pi) \sim_{E(\chi)} D_{F^+}^{1/2}\mathcal{G}(\varepsilon_K)^{-[\frac{n}{2}]}p(\check{\chi}, \Phi_{s,\chi}) \quad \text{if } n \text{ is odd}$$

$$P^{(s)}(\Pi) \sim_{E(\chi)E(\psi)} D_{F^+}^{1/2}(2\pi i)^{-\frac{n}{2}}\mathcal{G}(\varepsilon_K)^{-[\frac{n}{2}]}p(\check{\chi}, \Phi_{s,\chi})p(\psi)^s p\big(\psi^c\big)^{n-s} \quad \text{if } n \text{ is even}$$

equivariant under action of G_K.

This is the main result of this note. The idea is simple. We fix $0 \leq s \leq n$ an integer. We take η and m as in Proposition 3.1. When n is odd, we have $L(m, \Pi_\chi \otimes \eta) = L(m, \chi \otimes \eta \circ N_{\mathbb{A}_F/\mathbb{A}_K})$ by automorphic induction and with both sides non-zero. We may simplify the left-hand side of this equation by Theorem 3.2 and the right-hand side by Blasius' result. The CM periods of η that appeared in both sides unsurprisingly coincide, and we then deduce the above result.

5. Application: simplification of Archimedean local factors

We can now refine the Archimedean local factors in Theorem 3.3 first in the case where Π and Π' come from a Hecke character and then for general Π and Π'.

We take χ and χ' two algebraic regular conjugate self-dual Hecke characters of F and F' that satisfy Hypothesis 4.1 and some regular conditions. We may apply Theorem 3.3 to $\Pi_\chi \times \Pi'_\chi$. Our main result (Theorem 4.1) allows us to replace the automorphic periods by CM periods and we get:

$$p(m, \Pi_\infty, \Pi'_\infty) Z(\Pi_\infty) Z(\Pi'_\infty) \sim_{K E(\chi_\infty) E(\chi'_\infty)} (2\pi i)^{(m+\frac{1}{2})n(n-1)}$$

provided that $L(m + \frac{1}{2}, \Pi \times \Pi')$ does not vanish. This is always true when $m > 0$ since in this case, m is in the absolutely convergent range.

Note that the above result concerns only the infinity type. The following lemma allows us to generalize it.

Lemma 5.1. *If Π is an algebraic cuspidal representation of $GL_n(K)$, then there exists χ an algebraic Hecke character of F that satisfies Hypothesis 4.1 such that $\Pi_\infty \cong \Pi_{\chi,\infty}$. Furthermore, if Π is conjugate self-dual, we may have in addition that χ is conjugate self-dual.*

Note that an extra condition on the non-vanishing of the L-function will be needed when $m = 0$:

Hypothesis 5.1. For Π and Π' conjugate self-dual algebraic cuspidal representations of $GL_n(\mathbb{A}_K)$ and $GL_{n-1}(\mathbb{A}_K)$, there exists Hecke characters χ and χ' of F and F' such that χ and χ' are as in the previous lemma and $L(\frac{1}{2}, \Pi_\chi \times \Pi_{\chi'}) \neq 0$.

Theorem 5.1. *Let Π and Π' be cuspidal representations of $GL_n(\mathbb{A}_K)$ which are very regular, cohomological, conjugate self-dual, supercuspidal at places over at least two places of $\mathbb{Q}$ that split in K.*

Let $m \geq 0$ be an integer such that $m + n - 1$ is critical for $M(\Pi) \otimes M(\Pi')$. If $m = 0$, we assume moreover Hypothesis 5.1.

We then have the following equation equivariant under action of G_K:

$$p(m, \Pi_\infty, \Pi'_\infty) Z(\Pi_\infty) Z(\Pi'_\infty) \sim_{K E(\Pi_\infty) E(\Pi'_\infty)} (2\pi i)^{(m+\frac{1}{2})n(n-1)}.$$

Consequently, we have, equivariant under action of G_K,

$$L\left(m + \frac{1}{2}, \Pi \times \Pi'\right) \sim_{E(\Pi)E(\Pi')} (2\pi i)^{(m+\frac{1}{2})n(n-1)} \prod_{j=1}^{n-1} P^{(j)}(\Pi) \prod_{k=1}^{n-2} P^{(k)}(\Pi').$$

Remark 5.1. The above result is compatible with the Deligne conjecture and M. Harris' calculation on the Deligne period.

Recall that the Deligne conjecture predicts

$$L(n - 1 + m, M(\Pi) \otimes M(\Pi')) \sim c^+(M(\Pi) \otimes M(\Pi')(n - 1 + m))$$

where $c^+(\cdot)$ is Deligne's period defined in [4].

Eq. (4.12) of [5] gives

$$c^+(M(\Pi) \otimes M(\Pi')(n - 1 + m)) \sim (2\pi i)^{(m+\frac{1}{2})n(n-1)} \prod_{j=1}^{n-1} P_{\leq j}(\Pi) \prod_{k=1}^{n-2} P_{\leq k}(\Pi')$$

(see Chapter 4 of [5] for the notion). From the discussion after Theorem 4.27 in [5], we see that $P^{(s)} \sim P_{\leq s}$ in our case.

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