Mathematical analysis

Non-universality of the Nazarov–Sodin constant

Non-universalité de la constante de Nazarov–Sodin

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A B S T R A C T

We prove that the *Nazarov–Sodin constant*, which up to a natural scaling gives the leading order growth for the expected number of nodal components of a random Gaussian field, genuinely depends on the field. We then infer the same for "arithmetic random waves", i.e. random toral Laplace eigenfunctions.

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R É S U M É

On démontre que la constante de Nazarov–Sodin, qui, à un changement d'échelle près, donne le terme principal de l'ordre de croissance du nombre de composantes nodales d'un champ aléatoire gaussien, dépend effectivement du champ. On en déduit que le résultat reste vrai pour les «ondes aléatoires arithmétiques», c'est-à-dire pour les fonctions propres du laplacien aléatoire sur un tore.

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1. The Nazarov–Sodin constant

Let $m \geq 2$, and

$$f : \mathbb{R}^m \to \mathbb{R}$$

be a *stationary* centred Gaussian random field, and $r_f : \mathbb{R}^m \to \mathbb{R}$ its covariance function defined as

$$r_f(x) = \mathbb{E}\big[f(y)f(y+x)\big].$$

Given such an f, let $\rho = \rho_f$ denote its spectral measure, i.e. the Fourier transform of r_f (assumed to be a probability measure); note that prescribing ρ defines f uniquely. We further assume that a.s. f is sufficiently smooth, and that the distribution of $\nabla f(x)$ is non-degenerate.

Let $N(f; R)$ be the number of connected components of $f^{-1}(0)$ in $B_0(R)$ (the radius-R ball centred at 0), usually referred to as the *nodal components* of f; $N(f; R)$ is a random variable. Nazarov and Sodin [9, Theorem 1] proved that under the above conditions the expected number of nodal components of f is

$$\mathbb{E}\big[N(f; R)\big] = c_{NS}(\rho_f)R^m + o\big(R^m\big), \tag{1}$$

where $c_{\text{NS}}(\rho_f) \geq 0$ is referred to as the Nazarov–Sodin constant of f (we will consider c_{NS} as a function of the spectral density of ρ_f rather than of f).

For $m = 2$, $\rho = \rho_{\mathcal{S}^1}$ the uniform measure on the unit circle $\mathcal{S}^1 \subseteq \mathbb{R}^2$ (i.e. $d\rho = \frac{d\theta}{2\pi}$ on $\mathcal{S}^1$ vanishing outside the circle), the corresponding random field f_{RWM} is known as *random monochromatic waves*; Berry [2] suggested that f_{RWM} may serve as a *universal model* to Laplace eigenfunctions on generic surfaces in the high-energy limit—the *Random Wave Model*. The corresponding *universal* Nazarov–Sodin constant c_{RWM} is known to be strictly positive [7], and in [3] its value was predicted using a certain percolation model. However, recent numerics by Nastacescu, as well as by Konrad, show a small deviation from these predictions.

Let $(\mathcal{M}^m, g)$ be a smooth manifold. Here the restriction of a fixed random field $f : \mathcal{M} \to \mathbb{R}$ to growing domains, as was considered on the Euclidean space, makes no sense. Instead we consider a sequence of random fields $\{f_L\}_{L \in \mathcal{L}}$ (for L lying in some discrete subset $\mathcal{L} \subseteq \mathbb{R}$), and the total number $N(f_L)$ of nodal components of f_L on M. Here we may define a scaled covariance function of f_L around a fixed point $x \in \mathcal{M}$ on its tangent space $T_x(\mathcal{M}) \cong \mathbb{R}^m$ via the exponential map at x, and assume that for a.e. $x \in \mathcal{M}$ the scaled covariance converges, locally uniformly, to a covariance function of a limiting stationary Gaussian field around x.

For the setup as above Nazarov–Sodin proved ([9], Theorem 4) that

$$N(f_L) = \overline{c_{\text{NS}}} \cdot L^m + o(L^m),$$

for some $\overline{c_{\text{NS}}} \geq 0$ depending on the limiting fields only, namely their Nazarov–Sodin constants. This result applies in particular to random band-limited functions on a generic Riemannian manifold, considered in [8], with the constant $\overline{c_{\text{NS}}} > 0$ strictly positive.

2. Statement of results for arithmetic random waves

Let S be the set of all integers that admit a representation as a sum of two integer squares and $n \in S$. The toral Laplace eigenfunctions $f_n : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}$ of eigenvalue $-4\pi^2 n$ may be expressed as

$$f_n(x) = \sum_{\substack{\|\lambda\|^2 = n \\ \lambda \in \mathbb{Z}^2}} a_\lambda e^{2\pi i \langle x, \lambda \rangle} \tag{2}$$

with some coefficients a_λ satisfying $a_{-\lambda} = \overline{a_\lambda}$. We endow the space of eigenfunctions with a Gaussian probability measure by making the coefficient a_λ i.i.d. standard Gaussian (save for the relation $a_{-\lambda} = \overline{a_\lambda}$).

For this model, it is known [5] that various local properties of f_n, e.g., the total length of the nodal line $f_n^{-1}(0)$, depend on the limiting angular distribution of $\{\lambda \in \mathbb{Z}^2 : \|\lambda\|^2 = n\}$. More precisely, for $n \in S$ let

$$\mu_n = \frac{1}{r_2(n)} \sum_{\|\lambda\|^2 = n} \delta_{\lambda/\sqrt{n}},$$

where δ_x is the Dirac delta at x, be a probability measure on the unit circle $\mathcal{S}^1 \subseteq \mathbb{R}^2$. Then in order to exhibit an asymptotic law for the total length of $f_n^{-1}(0)$ such as its variance, or some other local properties of f_n, it is natural to pass to subsequences $\{n_j\} \subseteq S$ such that μ_{n_j} weakly converges to μ, a probability measure on $\mathcal{S}^1$. In this situation, we may identify μ as the spectral density of the limiting field around each point of the torus (when the unit circle is considered embedded, $\mathcal{S}^1 \subseteq \mathbb{R}^2$); such a limiting probability measure μ necessarily lies in the set $\mathcal{P}_{\text{Symm}}$ of probability measures on $\mathcal{S}^1$, invariant w.r.t. $\pi/2$-rotation and complex conjugation (i.e. $(x_1, x_2) \mapsto (x_1, -x_2)$). In fact, the family of weak-* partial limits of $\{\mu_n\}$ ("attainable" measures) is known [6] to be a proper subset of $\mathcal{P}_{\text{Symm}}$.

Let $N(f_n)$ as usual denote the total number of nodal components of f_n. An application of [9], Theorem 4 mentioned above implies that if, as above, $\mu_{n_j} \Rightarrow \mu$ with μ some probability measure on $\mathcal{S}^1$, we have

$$\mathbb{E}\big[N(f_n)\big] = c_{\text{NS}}(\mu)n + o(n), \tag{3}$$

with the same leading constant $c_{\text{NS}}(\mu)$ as for the scale-invariant model (1).

In order to state our results, first we will need the following notation: let

$$\nu_0 = \frac{1}{4} \sum_{k=0}^{3} \delta_{k \cdot \pi/2}$$

be the *Cilleruelo* measure [4], and

$$\nu_{\pi/4} = \frac{1}{4} \sum_{k=0}^{3} \delta_{\pi/4 + k \cdot \pi/2}$$

be the *tilted* Cilleruelo measure; these are the only measures in $\mathcal{P}_{\text{Symm}}$ supported on precisely four points. We prove the following concerning the range of possible constants $c_{\text{NS}}(\mu)$ appearing in (3).

Theorem 2.1. *For μ in the family of weak-* partial limits of $\{\mu_n\}$ the functional $c_{NS}(\mu)$ attains an interval of the form $I_{NS} = [0, d_{max}]$ with some $d_{max} > 0$. Equivalently,*

$$N(f_{n_j}) = c \cdot n_j + o(n_j)$$

for some $\{n_j\} \subseteq S$, if and only if $c \in I_{NS}$. Moreover, for $\mu \in \mathcal{P}_{Symm}$, $c_{NS}(\mu) = 0$ if and only if $\mu = \nu_0$ or $\mu = \nu_{\pi/4}$ (i.e., either the Cilleruelo or tilted Cilleruelo measures.)

Theorem 2.1 is a particular case of a more general result concerning arbitrary random fields on $\mathbb{R}^2$, presented in Section 3. Concerning the maximal Nazarov–Sodin constant $d_{max} > 0$, we believe that the following is true.

Conjecture 2.2. *For $\mu \in \mathcal{P}_{Symm}$, the maximal value d_{max} is uniquely attained by $c_{NS}(\mu_{\mathcal{S}^1})$, where $\mu_{\mathcal{S}^1}$ is the uniform measure on $\mathcal{S}^1 \subseteq \mathbb{R}^2$. In particular,*

$$d_{max} = c_{RWM}.$$

Question 2.3. What is the true asymptotic behaviour of $\mathbb{E}[f_{n_j}]$ for $\{n_j\}$ a Cilleruelo sequence, i.e. $\mu_{n_j} \Rightarrow \nu_0$? The latter might not admit an asymptotic law; in this case it would still be very interesting to know if the expected number of nodal components grows, in the sense that

$$\liminf_{j \to \infty} N(f_{n_j}) \to \infty.$$

In fact, we have reasons to believe that the stronger bound

$$N(f_{n_j}) \gg \sqrt{n_j}$$

holds.

Motivated by the fact that the nodal length variance only depends on the first non-trivial Fourier coefficient of the measure [5], and some other local computations, we raise the following question.

Question 2.4. Is it true that $c_{NS}(\mu)$ with $\mu \in \mathcal{P}_{Symm}$ supported on $\mathcal{S}^1$ only depends on finitely many Fourier coefficients, e.g. $\widehat{\mu}(4)$ or $(\widehat{\mu}(4), \widehat{\mu}(8))$?

3. Statement of results for random waves on $\mathbb{R}^2$

Let $\mathcal{P}_R$ be the collection of probability measures on $\mathbb{R}^2$ supported on the radius-R standard ball $B(R) \subseteq \mathbb{R}^2$; by the scale invariance we may assume that $R = 1$, and denote $\mathcal{P} := \mathcal{P}_1$.

Theorem 3.1. *The functional*

$$c_{NS} : \mathcal{P} \to \mathbb{R}_{\geq 0}$$

is continuous w.r.t. the weak-* topology on $\mathcal{P}$.

Some aspects of the proof of Theorem 3.1 can be found in Section 3.1.

Proposition 3.2. *Let ν_0 be the Cilleruelo measure on $\mathbb{R}^2$ as above. Then its Nazarov–Sodin constant vanishes, i.e.,*

$$c_{NS}(\nu_0) = 0.$$

Note that the result of Proposition 3.2 is in the same spirit as known constructions of (deterministic) eigenfunctions of arbitrarily high energy with few or bounded number of nodal components that arise in eigenspaces with a spectral measure given by the Cilleruelo measure (see the recent manuscript [1]). Proposition 3.2 can be proved by either considering an explicit construction of a random field f with the given spectral measure ν_0 and noting that for this model there are a.s. no compact nodal components, or, alternatively, by a local computation, e.g. of the number of "flips", i.e. points x with $f(x) = \frac{\partial}{\partial x_1} f(x) = 0$.

Combining Theorem 3.1, Proposition 3.2, and using the convexity of $\mathcal{P}$, we obtain the following corollary.

Corollary 3.3. *The Nazarov–Sodin constant $c_{NS}(\rho)$ for $\rho \in \mathcal{P}$ attains an interval of the form $[0, c_{max}]$ for some $0 < c_{max} < \infty$.*

As for the maximal value of the Nazarov–Sodin constant, we make the following conjecture.

Conjecture 3.4. *For $\rho \in \mathcal{P}$, the maximal value $c_{\max}$ is uniquely attained by $c_{NS}(\rho)$ for ρ the uniform measure on $\mathcal{S}^1 \subseteq \mathbb{R}^2$. In particular (cf. Conjecture 2.2),*

$$c_{\max} = d_{\max} = c_{RWM}.$$

3.1. On the proof of continuity

To prove Theorem 3.1, we follow the steps of Nazarov–Sodin [9] closely, controlling the various error terms encountered. One of the key aspects of our proof, different from Nazarov–Sodin's one, is proving a uniform version of (1) as below, perhaps of independent interest.

Proposition 3.5. *Let f_ρ be a random field with spectral density $\rho \in \mathcal{P}$. The limit*

$$c_{NS}(\rho) = \lim_{R \to \infty} \frac{\mathbb{E}[N(f_\rho; R)]}{R^2}$$

is uniform w.r.t. $\rho \in \mathcal{P}$. More precisely,

$$\mathbb{E}\big[N(f_\rho; R)\big] = c_{NS}(\rho)R^2 + O(R)$$

with constant involved in the "O"-notation universal.

Acknowledgements

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