Available online at www.sciencedirect.com

ScienceDirect

Ann. I. H. Poincaré – AN 36 (2019) 1401–1430

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Multiple solutions for the non-Abelian Chern–Simons–Higgs vortex equations ☆

Xiaosen Han [a], Gabriella Tarantello [b,*]

[a] *Institute of Contemporary Mathematics, School of Mathematics and Statistics, Henan University, Kaifeng 475004, PR China*
[b] *Dipartimento di Matematica, Università degli Studi di Roma "Tor Vergata", Via della Ricerca Scientifica, 00133 Rome, Italy*

Received 3 June 2018; received in revised form 18 January 2019; accepted 22 January 2019
Available online 25 January 2019

Abstract

In this paper we study the existence of multiple solutions for the non-Abelian Chern–Simons–Higgs $(N \times N)$-system:

$$\Delta u_i = \lambda \left(\sum_{j=1}^{N} \sum_{k=1}^{N} K_{kj} K_{ji} \mathrm{e}^{u_j} \mathrm{e}^{u_k} - \sum_{j=1}^{N} K_{ji} \mathrm{e}^{u_j} \right) + 4\pi \sum_{j=1}^{n_i} \delta_{p_{ij}}, \quad i = 1, \ldots, N;$$

over a doubly periodic domain Ω, with coupling matrix K given by the Cartan matrix of $SU(N+1)$, (see (1.2) below). Here, $\lambda > 0$ is the coupling parameter, δ_p is the Dirac measure with pole at p and $n_i \in \mathbb{N}$, for $i = 1, \ldots, N$. When $N = 1, 2$ many results are now available for the periodic solvability of such system and provide the existence of different classes of solutions known as: topological, non-topological, mixed and blow-up type. On the contrary for $N \geq 3$, only recently in [27] the authors managed to obtain the existence of one doubly periodic solution via a minimization procedure, in the spirit of [46]. Our main contribution in this paper is to show (as in [46]) that actually the given system admits a second doubly periodic solutions of "Mountain-pass" type, provided that $3 \leq N \leq 5$. Note that the existence of multiple solutions is relevant from the physical point of view. Indeed, it implies the co-existence of different non-Abelian Chern–Simons condensates sharing the same set (assigned component-wise) of vortex points, energy and fluxes. The main difficulty to overcome is to attain a "compactness" property encompassed by the so-called Palais–Smale condition for the corresponding "action" functional, whose validity remains still open for $N \geq 6$.

Keywords: Chern–Simons–Higgs equations; Doubly periodic solutions; Mountain-pass solution

☆ This work is supported by PRIN12: "Variational and Perturbative Aspects in Nonlinear Differential Problems", PRIN15: "Variational methods, with applications to problems in mathematical physics and geometry", FIRB project: "Analysis and Beyond", MIUR Excellence Project: Department of Mathematics, University of Rome Tor Vergata, CUP E83C18000100006, National Natural Science Foundation of China under Grants 11671120 and 11471100, and HASTIT (18HASTIT028).

* Corresponding author.
E-mail address: tarantel@mat.uniroma2.it (G. Tarantello).

https://doi.org/10.1016/j.anihpc.2019.01.002

1. Introduction

In recent years the Chern–Simons forms proposed by Chern and Simons [12,13] concerning secondary characteristic classes have played a very important role both in theoretical and applied sciences. In this respect we mention, knot invariants [19], Jones polynomial [59], quantum field theory [4], string theory [42,60], high-temperature superconductivity [37,43,56,58], optics [7], and condensed matter physics [32,49,50].

In superconductivity, Hong–Kim–Pac [28] and Jackiw–Weinberg [34] introduced the Chern–Simons terms into the Abelian Higgs model to describe particles carrying both magnetic and electric charges. In addition, in [28] and [34] the authors showed that, by neglecting the Maxwell term, one could attain a self-dual BPS-regime (Bogomol'nyi [8] and Prasad–Sommerfield [48]) with a 6th-order potential. Since then, many other physical Chern–Simons models have been introduced with analogous features [6,9,23,29,33,38]. Starting with the work in [10,45,52–54,57], a rather complete description of (electro-magnetic) abelian Chern–Simons vortices is now available in literature, see [55,62] for a detailed account.

However, more recently there has been a growing interest towards non-Abelian vortices concerning particle interactions other than electro-magnetic ones (e.g. weak, strong, electro-weak, etc). Indeed within the general framework of Supersymmetry, it has been noted that non-Abelian vortices assume a relevant role towards the delicate issue of "confinement". With this point of view, and after the "pure" non-Abelian Chern–Simons–Higgs model of Dunne [20–22], several other models have been discussed in [18,35,36,39,40,44,47], which have introduced also genuinely new non-Abelian ansatz in order to attain self-duality. In this way, one can reduce the equations of motion governing non-Abelian Chern–Simons–Higgs vortices in the (self-dual) BPS-regime into the following nonlinear elliptic system of PDEs:

$$\Delta u_i + \lambda \left(\sum_{j=1}^{n} K_{ji} \mathrm{e}^{u_j} - \sum_{j=1}^{n} \sum_{k=1}^{n} K_{kj} K_{ji} \mathrm{e}^{u_j} \mathrm{e}^{u_k} \right) = 4\pi \sum_{j=1}^{n_i} \delta_{p_{ij}}, \quad i = 1, \cdots, n, \tag{1.1}$$

with a suitable coupling matrix $K = (K_{ij})$ determined by the physical model under consideration. In (1.1), we have $\lambda > 0$ a coupling parameter and $n_i \in \mathbb{N}$ is the number of assigned (vortex) points $p_{i1}, \dots, p_{in_i}$ (counted with multiplicity) for the i-th component, $i = 1, \dots, n$.

For the "pure" Chern–Simons–Higgs model in [20], the matrix $K = (K_{ij})$ coincides with the Cartan matrix corresponding to the (non-Abelian) gauge group G describing the internal symmetries of the model. Typically G admits a finite-dimensional semi-simple Lie algebra L and $n = \operatorname{rank} L$.

For example, to the gauge group $G = SU(N+1)$ of rank N corresponds the following $N \times N$ Cartan matrix K:

$$K \equiv \begin{pmatrix} 2 & -1 & 0 & \dots & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & -1 & 2 & -1 \\ 0 & \dots & & 0 & -1 & 2 \end{pmatrix}. \tag{1.2}$$

The first rigorous existence result about the system (1.1) in $\mathbb{R}^2$ is due to Yang [61], who uses a direct minimization approach to establishes a planar (topological) solution for a general class of coupling matrices K, which include all possible choices of Cartan matrices. The existence of non-topological planar solutions was pursued by a perturbation approach (in the spirit of [11]) for the Lie-Algebras of rank 2 given respectively by A_2, B_2 and G_2 in [2,3], see also [14] and [17]. While, the existence of mixed-type planar $SU(3)$-vortices can be found in [15,16]. See also [30,31] for results in the skew-symmetric case.

The periodic case was first dealt in [46], where the authors proved the existence of multiple doubly periodic $SU(3)$-vortices, solution of (1.1)–(1.2) with $N = 2$. This result was extended in [26], where (1.1) is considered with a general 2×2 nonsingular coupling matrix K, including all Cartan's type. See also [41] for the construction of bubbling solutions.

However, when the system (1.1) involves three or more components over a doubly periodic domain, then the results available are less satisfactory. In fact, only recently Han–Yang [27] were able to extend the constraint-minimization approach of [46] and established the existence of a doubly periodic solution for the system (1.1)–(1.2) with $N \geq 3$. A possible extension of [27] to the system (1.1) with a more general $n \times n$ nonsingular coupling matrix K of Cartan-type, was claimed in Han–Lin–Yang [25].

The aim of this paper is to show that actually (1.1)–(1.2) with $N \geq 3$ admits a second doubly periodic solution (other than the topological one in [27]), which we obtain via a min-max procedure of "mountain-pass" type [1]. As already mentioned, the multiple solvability of the system (1.1) is relevant from the physical point of view. Indeed, it indicates that (asymptotically) each "vacua" states of the system may support a vortex configuration with the same set of vortex points (assigned component-wise) at the same (quantized) energy level. The main difficulty to apply a variational approach to the "action" functional corresponding to (1.1) (see (2.17) below) is to show that it satisfies a "compactness" property, expressed by the so-called Palais–Smale (PS)-condition. Such condition becomes rather involved when we deal with three or more components, which might allow enough room for a compactness loss. We manage to resolve such a compactness issue for (1.1)–(1.2), when $N = 3, 4, 5$, and prove the following:

Theorem 1.1. *Consider the non-Abelian Chern–Simons–Higgs system* (1.1) *over a doubly periodic domain* Ω *and with the matrix* K *in* (1.2) *(i.e. the Cartan matrix of* $SU(N+1)$*). For* $N = 3, 4, 5$ *and any given set of points* $p_{j1}, \ldots, p_{jn_j}$ $(j = 1, \ldots, N)$ *on* Ω *repeated with multiplicity, there exists a large constant* $\lambda_1 > 0$ *such that when* $\lambda > \lambda_1$ *the system* (1.1) *admits at least two distinct solutions.*

Remark 1.1. The constant λ_1 in our statement satisfies the following lower bound:

$$\lambda_1 > \lambda_0 \equiv \frac{16\pi}{|\Omega|} \frac{\sum_{i=1}^{N} \sum_{j=1}^{N} (K^{-1})_{ij} n_j}{\sum_{i=1}^{N} \sum_{j=1}^{N} (K^{-1})_{ij}}. \tag{1.3}$$

In fact the condition $\lambda > \lambda_0$ is necessary for the existence of a doubly periodic solution of (1.1)–(1.2), as shown in [27].

As a final remark we mention that, on the basis of the physical motivation, we have focused on the multiple solvability of (1.1) over a doubly periodic domain, or equivalently we have considered solutions of (1.1) defined on a flat bi-dimensional torus. However, our (variational) approach allows one to obtain (without any additional effort) a similar multiplicity result for solutions of (1.1) over a closed Riemann surface.

The rest of our paper is organized as follows. In Section 2 we present the variational formulation of the problem and furnish a new approach (different from [27]) to solve the associated constraint equations for the system (1.1)–(1.2). In Section 3 we prove our main theorem by showing first that the solution obtained in [27] corresponds to a local minimum for the "action" functional I in (2.17) below, which we show then to admit a mountain-pass structure [1]. Section 4 is devoted to the proof of the Palais–Smale-condition. The last section is a linear algebra Appendix which contains useful facts needed in Section 2.

2. Variational formulation and resolution of the constraints

In this section, we carry out a variational formulation for (1.1) and solve the associated constrained problem when K is the Cartan matrix (1.2) of $SU(N+1)$, with $N \geq 3$. It is well known that K in (1.2) is non-degenerate and positive definite.

Moreover by setting:

$$A \equiv K^{-1} = (a_{ij})_{N\times N}, \tag{2.1}$$

we easily check that,

$$a_{jk} = a_{kj} = \frac{1}{N+1}\Big[\min\{j,k\}(N+1-\max\{j,k\})\Big], \quad j,k = 1, \ldots N. \tag{2.2}$$

Let

$$r_j \equiv \sum_{k=1}^{N} a_{jk} = \frac{1}{2} j(N+1-j), \quad j=1,\dots,N \tag{2.3}$$

and note that,

$$\sum_{j=1}^{N} r_j = \frac{N(N+1)(N+2)}{12}.$$

Furthermore, consistently with (2.3), it is convenient to set

$$r_j = 0, \text{ for } j \le 0 \text{ or } j \ge N+1. \tag{2.4}$$

Define,

$$R \equiv \operatorname{diag}\{r_1,\dots,r_N\}, \tag{2.5}$$

and let

$$M \equiv RKR = \begin{pmatrix} 2r_1^2 & -r_1r_2 & 0 & \dots & \dots & 0 \\ -r_1r_2 & 2r_2^2 & -r_2r_3 & 0 & \dots & 0 \\ 0 & -r_2r_3 & 2r_3^2 & -r_3r_4 & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & -r_{N-2}r_{N-1} & 2r_{N-1}^2 & -r_{N-1}r_N \\ 0 & \dots & & 0 & -r_{N-1}r_N & 2r_N^2 \end{pmatrix}. \tag{2.6}$$

In what follows we replace the given unknown u_i by its translation $u_i \to u_i + \ln r_i$, which (by an abuse of notation) we still denote by u_i, namely:

$$u_i \to u_i + \ln r_i, \quad i=1,\dots,N, \tag{2.7}$$

with r_i given by (2.3).

Furthermore we use the following notations:

$$\mathbf{u} \equiv (u_1,\dots,u_N)^\tau,\ \mathrm{U} \equiv \operatorname{diag}\{\mathrm{e}^{u_1},\dots,\mathrm{e}^{u_N}\},\ \mathbf{U} \equiv (\mathrm{e}^{u_1},\dots,\mathrm{e}^{u_N})^\tau, \tag{2.8}$$

$$\mathbf{1} \equiv (1,\dots,1)^\tau, \quad \mathbf{s} \equiv \left(\sum_{s=1}^{n_1} \delta_{p_{1s}},\dots,\sum_{s=1}^{n_N} \delta_{p_{Ns}}\right)^\tau, \tag{2.9}$$

which help us to write (1.1) as follows:

$$\Delta \mathbf{u} = \lambda K \mathrm{U} M(\mathbf{U} - \mathbf{1}) + 4\pi \mathbf{s}, \tag{2.10}$$

once we take into account that,

$$M\mathbf{1} = R\mathbf{1}. \tag{2.11}$$

To find a doubly periodic solution of (1.1), we define the following background functions [5],

$$\Delta u_i^0 = 4\pi \sum_{s=1}^{n_i} \delta_{p_{is}} - \frac{4\pi n_i}{|\Omega|}, \quad \int_\Omega u_i^0 \mathrm{d}x = 0, \tag{2.12}$$

and observe that $\mathrm{e}^{u_i^0} \in L^\infty(\Omega)$, $\forall i=1,\dots,N$. We set $u_i = u_i^0 + v_i$, $i=1,\dots,N$, and we will use the following N-vector notation:

$$\mathbf{v} \equiv (v_1,\dots,v_N)^\tau, \quad \mathbf{n} \equiv (n_1,\dots,n_N)^\tau, \quad \mathbf{0} \equiv (0,\dots,0)^\tau. \tag{2.13}$$

In this way, the system (2.10) can be rewritten component-wise as follows:

$$\Delta v_i = \lambda\left(\sum_{j=1}^{N}\sum_{k=1}^{N} K_{jk}K_{ij}r_jr_k e^{u_j^0+v_j}e^{u_k^0+v_k} - \sum_{j=1}^{N} K_{ij}r_j e^{u_j^0+v_j}\right) + \frac{4\pi n_i}{|\Omega|}, \quad i=1,\dots,N. \tag{2.14}$$

To formulate the system (2.14) in a variational form, as in [25], we rewrite (2.14) equivalently as follows:

$$\Delta A\mathbf{v} = \lambda \mathrm{U}M(\mathbf{U}-\mathbf{1}) + \frac{\mathbf{b}}{|\Omega|}, \tag{2.15}$$

where, the matrices A and M are defined in (2.1) and (2.6) respectively, and we have set:

$$\mathbf{b} \equiv (b_1,\dots,b_N)^\tau \quad \text{with} \quad b_j = 4\pi\sum_{k=1}^{N} a_{jk}n_k > 0,\ j=1,\dots,N. \tag{2.16}$$

Since the matrices A and M defined in (2.1) are symmetric, we obtain a variational formulation for the system (2.15), by considering the following (action) functional:

$$I(\mathbf{v}) = \frac{1}{2}\sum_{i=1}^{2}\int_\Omega \partial_i\mathbf{v}^\tau A\partial_i\mathbf{v}\mathrm{d}x + \frac{\lambda}{2}\int_\Omega (\mathbf{U}-\mathbf{1})^\tau M(\mathbf{U}-\mathbf{1})\mathrm{d}x + \frac{1}{|\Omega|}\int_\Omega \mathbf{b}^\tau\mathbf{v}\mathrm{d}x. \tag{2.17}$$

Indeed, the functional (2.17) is well-defined and of class C^1 on the Hilbert (product) space $(W^{1,2}(\Omega))^N$ considered with the usual norm:

$$\|\mathbf{w}\|^2 = \|\mathbf{w}\|_2^2 + \|\nabla\mathbf{w}\|_2^2 = \sum_{i=1}^{N}\int_\Omega (w_i^2 + |\nabla w_i|^2)\mathrm{d}x,$$

for any $\mathbf{w} = (w_1,\dots,w_N)^\tau$, $w_i \in W^{1,2}(\Omega)$, $i=1,\dots,N$.

It is easy to check that every critical point of I in $(W^{1,2}(\Omega))^N$ defines a (weak) solution for (2.15). Although I is not bounded from below, we show that it admits a local minimum.

To this purpose, it is useful to consider a constrained minimization problem, firstly introduced in [10] for the abelian Chern–Simons–Higgs equation, and subsequently refined in [27,46] for the non-Abelian Chern–Simons–Higgs system (1.1). The main difficulty to pursue such a constraint approach is to show that the given "natural" constraints are actually uniquely solvable with respect to the mean value of each component.

To be more precise, we use the decomposition: $W^{1,2}(\Omega) = \mathbb{R}\oplus \dot{W}^{1,2}(\Omega)$, where,

$$\dot{W}^{1,2}(\Omega) \equiv \left\{ w\in W^{1,2}(\Omega) \middle| \int_\Omega w\mathrm{d}x = 0\right\}$$

is a closed subspace of $W^{1,2}(\Omega)$. Therefore, for any $v_i \in W^{1,2}(\Omega)$, we set $v_i = c_i + w_i$, with $w_i \in \dot{W}^{1,2}(\Omega)$, and $c_i = \frac{1}{|\Omega|}\int_\Omega v_i\mathrm{d}x$, $i=1,\dots,N$. Consequently, the integration of (2.14) over Ω gives the following natural constraints:

$$2r_j e^{2c_j}\int_\Omega e^{2(u_j^0+w_j)}\mathrm{d}x - e^{c_j}\int_\Omega e^{u_j^0+w_j}\left(1 + r_{j-1}e^{c_{j-1}}e^{u_{j-1}^0+w_{j-1}} + r_{j+1}e^{c_{j+1}}e^{u_{j+1}^0+w_{j+1}}\right)\mathrm{d}x + \frac{b_j}{\lambda r_j} = 0,$$
$$j=1,\dots,N. \tag{2.18}$$

Clearly, for any $\mathbf{w} = (w_1,\dots,w_N)^\tau$ with $w_i \in \dot{W}^{1,2}(\Omega)$ $i=1,\dots,N$, the equations (2.18) are solvable with respect to e^{c_j} only if,

$$\left(\int_\Omega e^{u_j^0+w_j}\left(1 + r_{j-1}e^{c_{j-1}}e^{u_{j-1}^0+w_{j-1}} + r_{j+1}e^{c_{j+1}}e^{u_{j+1}^0+w_{j+1}}\right)\mathrm{d}x\right)^2 \geq \frac{8b_j}{\lambda}\int_\Omega e^{2(u_j^0+w_j)}\mathrm{d}x,$$
$$j=1,\dots,N. \tag{2.19}$$

On the other hand, (2.19) can be ensured by requiring that,

$$\left(\int_\Omega e^{u_j^0+w_j}\mathrm{d}x\right)^2 \geq \frac{8b_j\int_\Omega e^{2(u_j^0+w_j)}\mathrm{d}x}{\lambda}, \quad j=1,\dots,N. \tag{2.20}$$

Thus, we define the admissible set:

$$\mathscr{A} \equiv \left\{(w_1,\dots,w_N)^\tau \,\middle|\, w_j \in \dot{W}^{1,2}(\Omega) \text{ satisfies (2.20)}, \ \forall j=1,\dots,N\right\}. \tag{2.21}$$

Therefore, for any $\mathbf{w}\in\mathscr{A}$, to get a solution of (2.18), it is equivalent to show that,

$$e^{c_j} = \frac{1}{4r_j\int_\Omega e^{2(u_j^0+w_j)}\mathrm{d}x}\left\{Q_j-(-1)^{\varepsilon_j}\sqrt{Q_j^2-\frac{8b_j\int_\Omega e^{2(u_j^0+w_j)}\mathrm{d}x}{\lambda}}\right\},\ j=1,\dots,N, \tag{2.22}$$

admits a (unique) solution, with fixed $\varepsilon_j\in\{0,1\}$ and,

$$Q_j \equiv \int_\Omega e^{u_j^0+w_j}\left(1+r_{j-1}e^{c_{j-1}}e^{u_{j-1}^0+w_{j-1}}+r_{j+1}e^{c_{j+1}}e^{u_{j+1}^0+w_{j+1}}\right)\mathrm{d}x. \tag{2.23}$$

In [25] the above equations (2.22) are shown to be uniquely solvable when one takes $\varepsilon_j=1, \forall j=1,\dots,N$. In what follows, we shall handle such a uniqueness solvability issue of (2.22), for any choice of $\varepsilon_j\in\{0,1\}$.

To this purpose we set $t_j=e^{c_j}>0$, $j=1,\dots N$ and we show that, for any assigned $\varepsilon_j\in\{0,1\}$, the N-system of equations:

$$t_j-\frac{1}{4r_j\int_\Omega e^{2(u_j^0+w_j)}\mathrm{d}x}\left[\hat{Q}_j-(-1)^{\varepsilon_j}\sqrt{\hat{Q}_j^2-\frac{8b_j}{\lambda}\int_\Omega e^{2(u_j^0+w_j)}\mathrm{d}x}\right]=0,\ j=1,\dots,N, \tag{2.24}$$

admits a unique non-degenerate solution, smoothly depending on $(w_1,\dots,w_N)$, where

$$\hat{Q}_j \equiv \int_\Omega e^{u_j^0+w_j}\left(1+r_{j-1}t_{j-1}e^{u_{j-1}^0+w_{j-1}}+r_{j+1}t_{j+1}e^{u_{j+1}^0+w_{j+1}}\right)\mathrm{d}x. \tag{2.25}$$

For fixed $\varepsilon_j\in\{0,1\}$, $j=1,\dots,N$ we set:

$$\varepsilon=(\varepsilon_1,\dots,\varepsilon_N), \tag{2.26}$$

and for $s\in[0,1]$ we consider the following one-parameter family of functions

$$\begin{aligned}
&\phi_{j,s}(t_{j-1},t_j,t_{j+1},\varepsilon_j)\\
&\equiv t_j-\frac{1}{4r_j\int_\Omega e^{2(u_j^0+w_j)}\mathrm{d}x}\left[\tilde{Q}_j(s)-(-1)^{\varepsilon_j}\sqrt{\tilde{Q}_j^2(s)-\frac{8b_j}{\lambda}\int_\Omega e^{2(u_j^0+w_j)}\mathrm{d}x}\right]\\
&\equiv t_j-\varphi_{j,s}(t_{j-1},t_{j+1},\varepsilon_j), \quad j=1,\dots,N,
\end{aligned} \tag{2.27}$$

where

$$\tilde{Q}_j(s)\equiv\int_\Omega e^{u_j^0+w_j}(1+sr_{j-1}t_{j-1}e^{u_{j-1}^0+w_{j-1}}+sr_{j+1}t_{j+1}e^{u_{j+1}^0+w_{j+1}})\mathrm{d}x. \tag{2.28}$$

We set,

$$\Phi_{s,\varepsilon}(t_1,\dots,t_N)=\left(\phi_{1,s}(t_1,t_2,\varepsilon_1),\phi_{2,s}(t_1,t_2,t_3,\varepsilon_2),\dots,\phi_{N,s}(t_{N-1},t_N,\varepsilon_N)\right). \tag{2.29}$$

In what follows, we always use C to denote a universal positive constant whose value may change from line to line.

Lemma 2.1. *There exists a constant $C > 1$, such that for given $\varepsilon = (\varepsilon_1, \dots, \varepsilon_N)$ with $\varepsilon_j \in \{0, 1\}$, $s \in [0, 1]$, and $(t_1, \dots, t_N)$ satisfying: $\Phi_{s,\varepsilon}(t_1, \dots, t_N) = 0$, we have:*

$$\frac{1}{C}\frac{1}{\left(\int_\Omega e^{2(u_j^0+w_j)}dx\right)^{1/2}} \le t_j \le \frac{C}{\left(\int_\Omega e^{2(u_j^0+w_j)}dx\right)^{1/2}},\ j = 1, \dots, N. \tag{2.30}$$

Proof. To establish (2.30) we observe that, $t_j > 0$, $\forall\, j = 1, \dots, N$, and by setting:

$$u_j = u_j^0 + w_j + \ln t_j, \quad j = 1, \dots, N,$$

we see that u_j satisfies

$$2r_j^2 \int_\Omega e^{2u_j}dx - sr_{j-1}r_j \int_\Omega e^{u_{j-1}+u_j}dx - sr_jr_{j+1}\int_\Omega e^{u_j+u_{j+1}}dx - r_j\int_\Omega e^{u_j}dx \le 0,$$
$$\forall j = 1, \dots, N \text{ and } s \in [0, 1].$$

Since M in (2.6) is positive definite, with the help of Hölder's inequality and in view of the notation (2.8), we find constants $\alpha_0 > 0$, $\beta_0 > 0$ such that,

$$\alpha_0 \sum_{j=1}^N \int_\Omega e^{2u_j}dx \le \int_\Omega \mathbf{U}^\tau M\mathbf{U}dx \le \sum_{j=1}^N r_j \int_\Omega e^{u_j}dx$$
$$\le |\Omega|^{\frac12}\sum_{j=1}^N r_j \left(\int_\Omega e^{2u_j}dx\right)^{\frac12} \le \beta_0\left(\sum_{j=1}^N\int_\Omega e^{2u_j}dx\right)^{\frac12}. \tag{2.31}$$

Hence (2.31) implies that,

$$\int_\Omega e^{2u_j}dx \le C, \quad j = 1, \dots, N \tag{2.32}$$

from which we readily get,

$$t_j \le \frac{C}{\left(\int_\Omega e^{2(u_j^0+w_j)}dx\right)^{1/2}}, \quad j = 1, \dots, N.$$

To obtain the reverse inequality, in view of (2.32), we can estimate

$$\hat{Q}_j = \int_\Omega e^{u_j^0+w_j}(1 + sr_{j-1}t_{j-1}e^{u_{j-1}^0+w_{j-1}} + sr_{j+1}t_{j+1}e^{u_{j+1}^0+w_{j+1}})dx$$
$$\le \left(\int_\Omega e^{2u_j^0+2w_j}dx\right)^{\frac12}\left(|\Omega|^{\frac12} + r_{j-1}\left(\int_\Omega e^{2u_{j-1}}dx\right)^{\frac12} + r_{j+1}\left(\int_\Omega e^{2u_{j+1}}dx\right)^{\frac12}\right)$$
$$\le C\left(\int_\Omega e^{2u_j^0+2w_j}dx\right)^{\frac12}, \tag{2.33}$$

for suitable $C > 0$ (depending only on r_j, $j = 1, \dots, N$).

In case $\varepsilon_j = 1$, then we can use (2.21) to derive:

$$t_j \ge \frac{\int_\Omega e^{u_j^0+w_j}dx}{4r_j\int_\Omega e^{2u_j^0+2w_j}dx} \ge \sqrt{\frac{8b_j}{\lambda}}\frac{1}{4r_j\left(\int_\Omega e^{2u_j^0+2w_j}dx\right)^{\frac12}}$$

and (2.30) is established in this case.

In case $\varepsilon_j = 0$, then we can use (2.24) to deduce:

$$t_j \geq \frac{b_j}{\lambda r_j} \frac{1}{\int_\Omega e^{u_j^0 + w_j}(1 + r_{j-1}e^{u_{j-1}} + r_{j+1}e^{u_{j+1}})dx}$$
$$\geq \frac{b_j}{\lambda r_j} \frac{1}{C} \frac{1}{\left(\int_\Omega e^{2u_j^0 + 2w_j} dx\right)^{\frac{1}{2}}}$$

and (2.30) follows in this case as well. □

As a consequence of Lemma 2.1, we can take $R \gg 1$ sufficiently large, such that the topological degree of $\Phi_{s,\varepsilon}$ on $\Omega_R = \{(t_1, \dots, t_N) : 0 < t_j < R,\ j = 1, \dots, N\}$ is well defined for every $s \in [0, 1]$ and for every $\varepsilon = (\varepsilon_1, \dots, \varepsilon_N)$ with $\varepsilon_j \in \{0, 1\}$, $j = 1, \dots, N$.

By the homotopy invariance of the topological degree we find

$$\deg(\Phi_{s=1,\varepsilon}, \Omega_R, 0) = \deg(\Phi_{s,\varepsilon}, \Omega_R, 0) = \deg(\Phi_{s=0,\varepsilon}, \Omega_R, 0).$$

On the other hand, for any given $(w_1, \dots, w_N) \in \mathscr{A}$, we have: $\Phi_{s=0,\varepsilon}(t_1, \dots, t_N) = (t_1 - a_1, \dots, t_N - a_N)$ with

$$a_j = \frac{1}{4r_j \int_\Omega e^{2(u_j^0 + w_j)} dx} \times$$
$$\times \left(\int\limits_\Omega e^{u_j^0 + w_j} dx + (-1)^{\varepsilon_j} \sqrt{\left(\int\limits_\Omega e^{u_j^0 + w_j} dx \right)^2 - \frac{8b_j}{\lambda} \int\limits_\Omega e^{2(u_j^0 + w_j)} dx} \right), \tag{2.34}$$

$j = 1, \dots, N$. Thus, we obtain that,

$$\deg(\Phi_{s=0,\varepsilon}, \Omega_R, 0) = 1.$$

As a consequence, $\deg(\Phi_{s=1,\varepsilon}, \Omega_R, 0) = 1$ and we conclude that, for any given $(w_1, \dots, w_N) \in \mathscr{A}$ the system (2.24) admits at least one solution.

To show that such a solution is actually unique and depends smoothly on $(w_1, \dots, w_N)$, we show that $\forall s \in [0, 1]$ every solution of the equation:

$$\Phi_{s,\varepsilon} = 0 \tag{2.35}$$

is actually non-degenerate. More precisely the following holds:

Theorem 2.1. *For every $(w_1, \dots, w_N) \in \mathscr{A}$, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_N)$, $\varepsilon_j \in \{0, 1\}$, $j = 1, \dots, N$ and $s \in [0, 1]$, every solution of the equation (2.35) is nondegenerate.*

Proof. By the above calculation for $\Phi_{s=0,\varepsilon}$, we see that the claim obviously holds for $s = 0$. So we are left to consider the case where $0 < s \leq 1$. To this purpose, we compute the Jacobian of $\Phi_{s,\varepsilon}$. According to (2.27) and (2.28), we easily find that $\frac{\partial \phi_{j,s}}{\partial t_j} = 1$ and

$$\frac{\partial \Phi_{s,\varepsilon}}{\partial t} = \begin{pmatrix} 1 & \frac{\partial \phi_{1,s}}{\partial t_2} & 0 & \cdots & \cdots & 0 \\ \frac{\partial \phi_{2,s}}{\partial t_1} & 1 & \frac{\partial \phi_{2,s}}{\partial t_3} & 0 & \cdots & 0 \\ 0 & \frac{\partial \phi_{3,s}}{\partial t_2} & 1 & \frac{\partial \phi_{3,s}}{\partial t_4} & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \cdots & & \ddots & \frac{\partial \phi_{N-1,s}}{\partial t_{N-2}} & 1 & \frac{\partial \phi_{N-1,s}}{\partial t_N} \\ 0 & \cdots & \cdots & 0 & \frac{\partial \phi_{N,s}}{\partial t_{N-1}} & 1 \end{pmatrix}. \tag{2.36}$$

As above, by setting $u_j = u_j^0 + w_j + \ln t_j$, $j = 1, \dots, N$, we find,

$$\frac{\partial \phi_{j,s}}{\partial t_{j+1}} = -\frac{s r_{j+1} \int_\Omega e^{u_j^0 + w_j + u_{j+1}^0 + w_{j+1}} dx}{4 r_j \int_\Omega e^{2(u_j^0 + w_j)} dx} \times \\ \times \left[1 - \frac{(-1)^{\varepsilon_j} \int_\Omega e^{u_j^0 + w_j}(1 + s r_{j-1} e^{u_{j-1}} + s r_{j+1} e^{u_{j+1}}) dx}{\sqrt{\left(\int_\Omega e^{u_j^0 + w_j}(1 + s r_{j-1} e^{u_{j-1}} + s r_{j+1} e^{u_{j+1}}) dx\right)^2 - \frac{8 b_j}{\lambda} \int_\Omega e^{2(u_j^0 + w_j)} dx}} \right]. \tag{2.37}$$

Therefore, if we use (2.27) we derive,

$$\frac{\partial \phi_{j,s}}{\partial t_{j+1}} = \frac{(-1)^{\varepsilon_j} s t_j r_{j+1} \int_\Omega e^{u_j^0 + w_j + u_{j+1}^0 + w_{j+1}} dx}{\sqrt{\left(\int_\Omega e^{u_j^0 + w_j}(1 + s r_{j-1} e^{u_{j-1}} + s r_{j+1} e^{u_{j+1}}) dx\right)^2 - \frac{8 b_j}{\lambda} \int_\Omega e^{2(u_j^0 + w_j)} dx}}, \tag{2.38}$$

$j = 1, \dots, N$. Similarly, we find:

$$\frac{\partial \phi_{j,s}}{\partial t_{j-1}} = \frac{(-1)^{\varepsilon_j} s t_j r_{j-1} \int_\Omega e^{u_j^0 + w_j + u_{j-1}^0 + w_{j-1}} dx}{\sqrt{\left(\int_\Omega e^{u_j^0 + w_j}(1 + s r_{j-1} e^{u_{j-1}} + s r_{j+1} e^{u_{j+1}}) dx\right)^2 - \frac{8 b_j}{\lambda} \int_\Omega e^{2(u_j^0 + w_j)} dx}}, \tag{2.39}$$

$j = 1, \dots, N$.

At this point, we are going to use the (linear algebra) results of the Appendix A, in order to show that det $\frac{\partial \Phi_{s,\varepsilon}}{\partial t} > 0$. To this purpose, from (2.36), (2.38) and (2.39), we see that the Jacobian of $\Phi_{s,\varepsilon}$ admits the same structure of the matrix $T_1^{(N)}$ defined in (A.2) of the Appendix, with

$$\beta_{j,1} = (-1)^{\varepsilon_j} \alpha_{j,1} \quad \text{and} \quad \beta_{j,2} = (-1)^{\varepsilon_j} \alpha_{j,2}, \qquad j = 1, \dots, N \tag{2.40}$$

and

$$\alpha_{j,1} = \frac{s t_{j-1} r_{j-1} \int_\Omega e^{u_j^0 + w_j + u_{j-1}^0 + w_{j-1}} dx}{\sqrt{\left(\int_\Omega e^{u_j^0 + w_j}(1 + s r_{j-1} e^{u_{j-1}} + s r_{j+1} e^{u_{j+1}}) dx\right)^2 - \frac{8 b_j}{\lambda} \int_\Omega e^{2(u_j^0 + w_j)} dx}}, \tag{2.41}$$

$$\alpha_{j,2} = \frac{s t_{j+1} r_{j+1} \int_\Omega e^{u_j^0 + w_j + u_{j+1}^0 + w_{j+1}} dx}{\sqrt{\left(\int_\Omega e^{u_j^0 + w_j}(1 + s r_{j-1} e^{u_{j-1}} + s r_{j+1} e^{u_{j+1}}) dx\right)^2 - \frac{8 b_j}{\lambda} \int_\Omega e^{2(u_j^0 + w_j)} dx}}. \tag{2.42}$$

Therefore, the assumptions (A.8) and (A.9) of the Appendix are satisfied. Furthermore, concerning the coefficients $\alpha_{j,1}$ and $\alpha_{j,2}$, defined in (2.41)–(2.42) $j = 1, \dots, N$, we observe that, since $(w_1, \dots, w_N) \in \mathscr{A}$, then for $s \in (0, 1]$ we can estimate:

$$\left(\int_\Omega e^{u_j^0 + w_j}(1 + s t_{j-1} r_{j-1} e^{u_{j-1}^0 + w_{j-1}} + s t_{j+1} r_{j+1} e^{u_{j+1}^0 + w_{j+1}}) dx \right)^2 \\ > \frac{8 b_j}{\lambda} \int_\Omega e^{2(u_j^0 + w_j)} dx + \left(s t_{j-1} r_{j-1} \int_\Omega e^{u_j^0 + w_j + u_{j-1}^0 + w_{j-1}} dx + s t_{j+1} r_{j+1} \int_\Omega e^{u_j^0 + w_j + u_{j+1}^0 + w_{j+1}} dx \right)^2.$$

Consequently, by using the above estimate, for every s in $(0, 1]$ we have:

$$0 < \alpha_{j,2} < \frac{t_{j+1} r_{j+1} \int_\Omega e^{u_j^0 + w_j + u_{j+1}^0 + w_{j+1}} dx}{t_{j-1} r_{j-1} \int_\Omega e^{u_j^0 + w_j + u_{j-1}^0 + w_{j-1}} dx + t_{j+1} r_{j+1} \int_\Omega e^{u_j^0 + w_j + u_{j+1}^0 + w_{j+1}} dx} \equiv 1 - \tau_j \le 1$$

with

$$\tau_j = \frac{t_{j-1}r_{j-1}\int_\Omega e^{u_j^0+w_j+u_{j-1}^0+w_{j-1}}dx}{t_{j-1}r_{j-1}\int_\Omega e^{u_j^0+w_j+u_{j-1}^0+w_{j-1}}dx + t_{j+1}r_{j+1}\int_\Omega e^{u_j^0+w_j+u_{j+1}^0+w_{j+1}}dx} \in [0,1],$$

and in turn,

$$0 < \alpha_{j+1,1} < \frac{t_j r_j \int_\Omega e^{u_j^0+w_j+u_{j+1}^0+w_{j+1}}dx}{t_j r_j \int_\Omega e^{u_j^0+w_j+u_{j-1}^0+w_{j-1}}dx + t_{j+2}r_{j+2}\int_\Omega e^{u_j^0+w_j+u_{j+2}^0+w_{j+2}}dx} \equiv \tau_{j+1}.$$

In other words, for $j = 1, \dots, N$, the coefficients $\alpha_{j,1}$ and $\alpha_{j,2}$ in (2.41), (2.42) satisfy also the assumption (A.19) of Theorem A.1 of the Appendix, and therefore we may conclude that,

$$\det \frac{\partial \Phi_{s,\varepsilon}}{\partial t} = F_N^{(1)} > 0$$

and the proof is completed. □

Corollary 2.1. *For every* $(w_1, \dots, w_N)^\tau \in \mathscr{A}, \varepsilon = (\varepsilon_1, \dots, \varepsilon_N)$ *with* $\varepsilon_j \in \{0,1\}, j = 1, \dots, N$ *and* $s \in [0,1]$*, the equation:*

$$\Phi_{s,\varepsilon}(t_1, \dots, t_N) = 0,$$

admits a unique (non-degenerate) solution, smoothly depending on $(w_1, \dots, w_N)$*.*

Proof. It is clear that, for $s = 0$ the given statement follows form (2.34). Furthermore, by the Implicit Function Theorem, there exists $\delta > 0$ sufficiently small, such that for $s \in [0, \delta)$, problem (2.35) admits a unique (non-degenerate) solution. Let

$$s_0 \equiv \sup \left\{ \sigma \in [0,1] \middle| \text{ such that (2.35) admits a unique solution } \quad \forall s \in [0, \sigma] \right\}. \tag{2.43}$$

We claim that, $s_0 = 1$.

While it is clear that $s_0 > 0$, if by contradiction, we suppose that $s_0 < 1$, then there would exist $s_0 < s_n < 1$ and $t_n^{(1)} \neq t_n^{(2)} \in \mathbb{R}^N$ such that,

$$\Phi_{s_n,\varepsilon}(t_n^{(2)}) = \Phi_{s_n,\varepsilon}(t_n^{(1)}) = 0$$
$$s_n \searrow s_0 \quad \text{as} \quad n \to +\infty.$$

By virtue of Lemma 2.1, we can pass to a subsequence if necessary, to find that,

$$t_n^{(i)} \to t^{(i)} \quad \text{and} \quad \Phi_{s_0,\varepsilon}(t^{(i)}) = 0, \quad i = 1, 2.$$

By the non-degeneracy of $t^{(i)}$, $i = 1, 2$, (as given by Theorem 2.1), we can first rule out the possibility that, $t^{(1)} \neq t^{(2)}$. Indeed, if this was the case, then by the Implicit Function Theorem, for sufficiently small $\delta > 0$, and for $s \in (0,1)$ such that: $s_0 - \delta < s < s_0$ we would get that the equation (2.35) would admit at least two solutions, in contradiction with the definition of s_0 in (2.43). Thus, $t^{(1)} = t^{(2)} = \underline{t}$, and this would be again impossible, since the Implicit Function Theorem implies local uniqueness for solutions of (2.35) around $(s_0, \underline{t})$. □

3. Existence of multiple solutions

In this section we show that system (2.14) admits at least two distinct solutions provided that the parameter λ is sufficiently large and $N = 3, 4, 5$. For this purpose, by following [27,46], we consider the constrained functional

$$J(\mathbf{w}) \equiv I(\mathbf{w} + \mathbf{c}_+(\mathbf{w})), \quad \mathbf{w} \in \mathscr{A}, \tag{3.1}$$

where $\mathbf{c}_+(\mathbf{w})$ is the unique solution of the constraint equations (2.18) with all $\varepsilon_j = 1$, $j = 1, \dots, N$, (see Corollary 2.1). By minimizing the constrained functional $J(\mathbf{w})$ in $\mathscr{A}$, the authors of [27] establish the following:

Proposition 3.1 *([27]). There exists $\bar{\lambda} > 0$ such that for every $\lambda > \bar{\lambda}$ the functional J in (3.1) attains its minimum value at the point $\mathbf{w}_\lambda$ which belongs to the* **interior** *of $\mathscr{A}$. Namely,*

$$J(\mathbf{w}_\lambda) = \inf_{\mathscr{A}} J(\mathbf{w})$$

and $\mathbf{v}^*_\lambda = \mathbf{w}_\lambda + \mathbf{c}_+(\mathbf{w}_\lambda)$ defines a critical point for I in (2.17). □

By setting, $\mathbf{c}^*_\lambda = \mathbf{c}_+(\mathbf{w}_\lambda) = (c^*_{1,\lambda}, \dots, c^*_{N,\lambda})^\tau$ we may write,

$$\mathbf{v}^*_\lambda = (v^*_{1,\lambda}, \dots, v^*_{N,\lambda})^\tau = (c^*_{1,\lambda} + w_{1,\lambda}, \dots, c^*_{N,\lambda} + w_{N,\lambda})^\tau. \tag{3.2}$$

We will show that actually $\mathbf{v}^*_\lambda$ is a local minimum for I in (2.17).

Lemma 3.1. *The solution $\mathbf{v}^*_\lambda = (v^*_{1,\lambda}, \dots, v^*_{N,\lambda})^\tau$ of (2.14), as given by (3.2), defines a local minimum for functional I in (2.17).*

Proof. For fixed $\mathbf{w} \in \mathscr{A}$, we denote by $\mathbf{c}^*(\mathbf{w}) = (c^*_1(\mathbf{w}), \dots, c^*_N(\mathbf{w}))$ the unique solution of (2.22) with $\varepsilon_j = 1$, $\forall j = 1 \dots, N$, as given by Corollary 2.1. For $\mathbf{c} = (c_1, \dots, c_N)^\tau \in \mathbb{R}^N$ we easily check that,

$$\frac{\partial}{\partial c_j} I(\mathbf{w} + \mathbf{c})\Big|_{\mathbf{c}=\mathbf{c}^*(\mathbf{w})} = \frac{\partial}{\partial c_j} I(w_1 + c_1, \dots, w_N + c_N)\Big|_{\mathbf{c}=\mathbf{c}^*(\mathbf{w})} = 0, \quad j = 1, \dots, N. \tag{3.3}$$

Moreover, by a straightforward computation we find:

$$\begin{aligned}&\frac{\partial^2}{\partial c_j^2} I(w_1 + c_1, \dots, w_N + c_N)\\ &= \lambda \int_\Omega r_j e^{u_j^0 + c_j + w_j}\big(4 r_j e^{u_j^0 + c_j + w_j} - 1 - r_{j-1} e^{u_{j-1}^0 + c_{j-1} + w_{j-1}} - r_{j+1} e^{u_{j+1}^0 + c_{j+1} + w_{j+1}}\big) dx,\\ &\quad j = 1, \dots, N,\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}&\frac{\partial^2}{\partial c_j c_k} I(w_1 + c_1, \dots, w_N + c_N) = \frac{\partial^2}{\partial c_k c_j} I(w_1 + c_1, \dots, w_N + c_N),\\ &= -\lambda r_j r_k \int_\Omega e^{u_j^0 + c_j + w_j + u_k^0 + c_k + w_k} dx, \ \text{for } k \in \{j-1, j+1\} \text{ and } j = 1, \dots, N,\end{aligned} \tag{3.5}$$

while,

$$\begin{aligned}&\frac{\partial^2}{\partial c_j c_k} I(w_1 + c_1, \dots, w_N + c_N) = \frac{\partial^2}{\partial c_k c_j} I(w_1 + c_1, \dots, w_N + c_N) = 0,\\ &\text{for } k \notin \{j-1, j+1\} \text{ and } j = 1, \dots, N.\end{aligned} \tag{3.6}$$

By setting $\mathbf{v}^* = \mathbf{w} + \mathbf{c}^*(\mathbf{w}) = (v^*_1, \dots, v^*_N)^\tau$, then by the definition of $\mathbf{c}^*(\mathbf{w})$ and (2.22), we see that,

$$\begin{aligned}&\frac{\partial^2}{\partial c_j^2} I(v_1^*, \dots, v_N^*)\\ &= \lambda \left[\left(\int_\Omega r_j e^{u_j^0 + v_j^*}\big(1 + r_{j-1} e^{u_{j-1}^0 + v_{j-1}^*} + r_{j+1} e^{u_{j+1}^0 + v_{j+1}^*}\big) dx\right)^2 - \frac{8 b_j r_j^2 \int_\Omega e^{2(u_j^0 + v_j^*)} dx}{\lambda}\right]^{\frac{1}{2}},\\ &\quad \forall j = 1, \dots, N.\end{aligned} \tag{3.7}$$

In particular from (3.4)–(3.7) we conclude that,

$$\frac{\partial^2}{\partial c_j^2} I(v_1^*, \dots, v_N^*) + \frac{\partial^2}{\partial c_j c_{j-1}} I(v_1^*, \dots, v_N^*) + \frac{\partial^2}{\partial c_j c_{j+1}} I(v_1^*, \dots, v_N^*) > 0, \quad \forall\, j = 1, \dots, N. \tag{3.8}$$

Hence from (3.8) we infer that, for any given $\mathbf{w}$ in $\mathscr{A}$, the Hessian matrix of $I(\mathbf{w}+\mathbf{c})$, as a function of $\mathbf{c} = (c_1, \dots, c_N)$, is a strictly diagonally dominant tri-diagonal matrix at the point $\mathbf{c} = \mathbf{c}^*(\mathbf{w})$, and therefore it is strictly positive-definite. In particular this property holds for $\mathbf{w}_\lambda$ the minimum point of J in (3.1).

Therefore we conclude that, for $\delta > 0$ sufficiently small and for $(v_1, \dots, v_N)^\tau = (w_1 + c_1, \dots, w_N + c_N)^\tau$ satisfying:

$$\sum_{i=1}^{N} \|v_i - v_{i,\lambda}^*\| \le \delta, \tag{3.9}$$

we have that $(w_1, \dots, w_N)^\tau$ belongs to the interior of $\mathscr{A}$ and also (by the smooth dependence of $\mathbf{c}^*(\mathbf{w})$ with respect to $\mathbf{w}$ (see Corollary 2.1)) that the vector $\mathbf{c} = (c_1, \dots, c_N)$ is sufficiently close to $\mathbf{c}^*(\mathbf{w})$ to guarantee that,

$$I(v_1, \dots, v_N) = I(w_1 + c_1, \dots, w_N + c_N) \ge I(\mathbf{w} + \mathbf{c}^*(\mathbf{w})) = J(\mathbf{w})$$

As a consequence, for any v satisfying (3.9), we have:

$$I(\mathbf{v}) \ge \inf_{\mathbf{w} \in \mathscr{A}} J(\mathbf{w}) = I(\mathbf{v}_\lambda^*)$$

and the proof is completed. □

To proceed further, we need the following "compactness" property of I.

Proposition 3.2. *Let $3 \le N \le 5$ and $\{(v_{1,n}, \dots, v_{N,n})^\tau\} \in (W^{1,2}(\Omega))^N$ be such that,*

$$I(v_{1,n}, \dots, v_{N,n}) \to a_0 \quad as \quad n \to +\infty, \tag{3.10}$$

$$\|I'(v_{1,n}, \dots, v_{N,n})\|_* \to 0 \quad as \quad n \to +\infty, \tag{3.11}$$

where a_0 is a constant and $\|\cdot\|_*$ denotes the norm of the dual space of $(W^{1,2}(\Omega))^N$. Then $(v_{1,n}, \dots, v_{N,n})$ admits a strongly convergent subsequence in $(W^{1,2}(\Omega))^N$.

Using a standard terminology, Proposition 3.2 asserts that the functional I satisfies the Palais–Smale (PS)-condition. We suspect that such property should hold also when $N \ge 6$.

We provide the proof of Proposition 3.2 in the following section.

Based on Proposition 3.2, we can carry out the proof of Theorem 1.1 and obtain a second solution of (2.14) (other than $(v_{1,\lambda}^*, \dots, v_{N,\lambda}^*)^\tau$ in (3.2)) by a Mountain-pass construction.

To this purpose, we need to reduce to the case where we know that $\mathbf{v}_\lambda^*$ is a strict local minimum of I. Indeed, if on the contrary, for small $\delta > 0$, we have:

$$\inf_{\sum_{j=1}^{N} \|v_i - v_{i,\lambda}^*\| = \delta} I(v_1, \dots, v_N) = I(v_{1,\lambda}^*, \dots, v_{N,\lambda}^*),$$

then we conclude, from Corollary 1.6 of [24], that the functional I admits a one parameter family of degenerate local minimizers, and a second solution of (2.14) is certainly guaranteed in this case.

Thus, we can assume that $\mathbf{v}_\lambda^* = (v_{1,\lambda}^*, \dots, v_{N,\lambda}^*)^\tau$ is a strict local minimum for I. So that for sufficiently small $\delta > 0$, we have that,

$$I(v_{1,\lambda}^*, \dots, v_{N,\lambda}^*) < \inf_{\sum_{j=1}^{N} \|v_i - v_{i,\lambda}^*\| = \delta} I(v_1, \dots, v_N) \equiv \gamma_0. \tag{3.12}$$

On the other hand, we easily check that,

$$I(v_{1,\lambda}^{*}-\xi,\ldots,v_{N,\lambda}^{*}-\xi)\to-\infty \quad \text{as} \quad \xi\to+\infty.$$

Therefore, for a sufficiently large $\xi_0>1$, we let

$$\hat{v}_i\equiv v_{j,\lambda}^{*}-\xi_0, \quad j=1,\ldots,N,$$

and conclude that,

$$\sum_{j=1}^{N}\|\hat{v}_j-v_{j,\lambda}^{*}\|>\delta \tag{3.13}$$

and

$$I(\hat{v}_1,\ldots,\hat{v}_N)<I(v_{1,\lambda}^{*},\ldots,v_{N,\lambda}^{*})-1. \tag{3.14}$$

We introduce the set of paths,

$$\mathscr{P}\equiv\left\{\Gamma(t)\Big|\Gamma\in C\left([0,1],\,(W^{1,2}(\Omega))^N\right), \quad \Gamma(0)=(v_{1,\lambda}^{*},\ldots,v_{N,\lambda}^{*})^{\tau}, \quad \Gamma(1)=(\hat{v}_1,\ldots,\hat{v}_2)^{\tau}\right\}$$

and define:

$$a_0\equiv\inf_{\Gamma\in\mathscr{P}}\sup_{t\in[0,1]}I(\Gamma(t)).$$

Clearly,

$$a_0>I(v_{1,\lambda}^{*},\ldots,v_{N,\lambda}^{*}), \tag{3.15}$$

and in view of Proposition 3.2, we can use the "Mountain-pass" theorem of Ambrosetti–Rabinowitz [1] to obtain that a_0 defines a critical value of the functional I, to which it corresponds to a critical point different from $(v_{1,\lambda}^{*},\ldots,v_{N,\lambda}^{*})^{\tau}$. Thus the proof of Theorem 1.1 is completed. □

4. The (PS)-condition for $N=3,4,5$

We devote this section to establish the (PS)-condition.

Let $\{(v_{1,n},\ldots,v_{N,n})\}$ be a sequence in $(W^{1,2}(\Omega))^N$ satisfying (3.10)–(3.11) and denote by,

$$u_{j,n}=u_j^0+v_{j,n}, \quad j=1,\ldots,N,$$

where u_j^0 is given by (2.12). In what follows, we always use the decomposition:

$$v_{j,n}=c_{j,n}+w_{j,n}, \quad w_{j,n}\in\dot{W}^{1,2}(\Omega),\ c_{j,n}=\frac{1}{|\Omega|}\int_\Omega v_{j,n}\mathrm{d}x,\ j=1,\ldots,N,\forall n\in\mathbb{N}. \tag{4.1}$$

By recalling (2.3) and (2.4), we note that,

$$2r_j-r_{j-1}-r_{j+1}=1 \quad \forall j=1,\ldots,N, \tag{4.2}$$

and (4.2), allow us to obtain the following:

$$\begin{aligned}&I'(v_{1,n},\ldots,v_{N,n})(\phi_1,\ldots,\phi_N)\\&=\sum_{j,k=1}^{N}a_{jk}\int_\Omega\nabla w_{k,n}\cdot\nabla\phi_j\mathrm{d}x+\lambda\sum_{j=1}^{N}\int_\Omega r_j\mathrm{e}^{u_{j,n}}\Big(2r_j(\mathrm{e}^{u_{j,n}}-1)-r_{j-1}(\mathrm{e}^{u_{j-1,n}}-1)\\&\quad-r_{j+1}(\mathrm{e}^{u_{j+1,n}}-1)\Big)\phi_j\mathrm{d}x+\frac{1}{|\Omega|}\sum_{j=1}^{N}\int_\Omega b_j\phi_j\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j,k=1}^{N} a_{jk} \int_{\Omega} \nabla w_{k,n} \cdot \nabla \phi_j \mathrm{d}x + \lambda \sum_{j=1}^{N} \int_{\Omega} r_j \mathrm{e}^{u_{j,n}} (2 r_j \mathrm{e}^{u_{j,n}} - r_{j-1} \mathrm{e}^{u_{j-1,n}} - r_{j+1} \mathrm{e}^{u_{j+1,n}}) \phi_j \mathrm{d}x \\
&\quad - \lambda \sum_{j=1}^{N} r_j \int_{\Omega} \mathrm{e}^{u_{j,n}} \phi_j \mathrm{d}x + \frac{1}{|\Omega|} \sum_{j=1}^{N} \int_{\Omega} b_j \phi_j \mathrm{d}x \\
&= o(1) \|\boldsymbol{\phi}\|, \quad \forall \boldsymbol{\phi} = (\phi_1, \dots, \phi_N) \in (W^{1,2}(\Omega))^N.
\end{aligned} \tag{4.3}
$$

Thus, by taking $(\phi_1, \dots, \phi_N) = (1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$ separately in (4.3), we find:

$$
\begin{aligned}
&\lambda \int_{\Omega} \Big[2 r_j^2 \mathrm{e}^{2u_{j,n}} - r_j r_{j-1} \mathrm{e}^{u_{j,n} + u_{j-1,n}} - r_j r_{j+1} \mathrm{e}^{u_{j,n} + u_{j+1,n}} - r_j \mathrm{e}^{u_{j,n}} \Big] \mathrm{d}x + b_j \\
&= \lambda \int_{\Omega} \Big[2 r_j^2 \mathrm{e}^{u_{j,n}} (\mathrm{e}^{u_{j,n}} - 1) - r_j r_{j-1} \mathrm{e}^{u_{j,n}} (\mathrm{e}^{u_{j-1,n}} - 1) - r_j r_{j+1} \mathrm{e}^{u_{j,n}} (\mathrm{e}^{u_{j+1,n}} - 1) \Big] \mathrm{d}x + b_j \\
&= o(1), \ j = 1, \dots, N.
\end{aligned} \tag{4.4}
$$

Hence by using still (4.2), we can sum up the identities (4.4) over $j = 1, \dots, N$, and arrive at the following identity:

$$
\begin{aligned}
&\lambda \int_{\Omega} (\mathbf{U}_n - \mathbf{1})^{\tau} M (\mathbf{U}_n - \mathbf{1}) \mathrm{d}x + \lambda \sum_{j=1}^{N} r_j \int_{\Omega} \mathrm{e}^{u_{j,n}} \mathrm{d}x \\
&- \frac{N(N+1)(N+2)}{12} \lambda |\Omega| + \sum_{j=1}^{N} b_j = o(1),
\end{aligned} \tag{4.5}
$$

with

$$
\mathbf{U}_n = (\mathrm{e}^{u_{1,n}}, \dots, \mathrm{e}^{u_{N,n}})^{\tau}.
$$

Since the matrix M is positive definite, from (4.5) we see that, as $n \to +\infty$:

$$
\int_{\Omega} (\mathbf{U}_n - \mathbf{1})^{\tau} M (\mathbf{U}_n - \mathbf{1}) \mathrm{d}x \leq \frac{N(N+1)(N+2)}{12} |\Omega| + o(1), \tag{4.6}
$$

$$
\int_{\Omega} \mathrm{e}^{u_{j,n}} \mathrm{d}x \leq \frac{N(N+1)(N+2)}{12} |\Omega| + o(1), \quad j = 1, \dots, N. \tag{4.7}
$$

By Jensen's inequality, from (4.1), (4.6) and (4.7), we have:

$$
\mathrm{e}^{c_{j,n}} \leq \frac{N(N+1)(N+2)}{12} + o(1), \quad j = 1, \dots, N. \tag{4.8}
$$

In addition, from (4.6) and (4.7), we derive that

$$
\int_{\Omega} (\mathrm{e}^{u_j^0 + v_{j,n}} - 1)^2 \mathrm{d}x \leq C, \quad j = 1, \dots, N, \tag{4.9}
$$

$$
\int_{\Omega} \mathrm{e}^{2u_j^0 + 2v_{j,n}} \mathrm{d}x \leq C, \quad j = 1, \dots, N, \tag{4.10}
$$

for some suitable constant $C > 0$.

Therefore, if we take $(\phi_1, \dots, \phi_N) = (w_{1,n}, \dots, w_{N,n})$ in (4.3), in view of (4.8)–(4.10), we find positive constants $\beta_1 > 0$ and $\beta_2 > 0$ such that,

$$\begin{aligned}
& o(1)\sum_{i=1}^{N}\|\nabla w_{j,n}\|_2 \\
& \geq I'(v_{1,n},\dots,v_{N,n})[(w_{1,n},\dots,w_{N,n})] \\
& = \sum_{j,k=1}^{N} a_{ij}\int_\Omega \nabla w_{j,n}\cdot\nabla w_{j,n}\mathrm{d}x + \lambda\sum_{j=1}^{N}\int_\Omega r_j \mathrm{e}^{u_{j,n}}(2r_j\mathrm{e}^{u_{j,n}} - r_{j-1}\mathrm{e}^{u_{j-1,n}} - r_{j+1}\mathrm{e}^{u_{j+1,n}})w_{j,n}\mathrm{d}x \\
& \quad - \lambda\sum_{j=1}^{N} r_j\int_\Omega \mathrm{e}^{u_{j,n}}w_{j,n}\mathrm{d}x \\
& \geq \beta_1\sum_{j=1}^{N}\|\nabla w_{j,n}\|_2^2 + \lambda\sum_{j=1}^{N}\int_\Omega 2r_j^2\mathrm{e}^{u_{j,n}}\left[\mathrm{e}^{u_j^0+c_{j,n}}(\mathrm{e}^{w_{j,n}}-1) + \mathrm{e}^{u_j^0+c_{j,n}}\right]w_{j,n}\mathrm{d}x \\
& \quad - \lambda\sum_{j=1}^{N}\int_\Omega r_j r_{j+1}\mathrm{e}^{u_{j,n}+u_{j+1,n}}(w_{j,n}+w_{j+1,n})\mathrm{d}x - \lambda\sum_{j=1}^{N} r_j\int_\Omega \mathrm{e}^{u_{j,n}}w_{j,n}\mathrm{d}x \\
& \geq \beta_1\sum_{j=1}^{N}\|\nabla w_{j,n}\|_2^2 - \beta_2\sum_{j=1}^{N}\|\nabla w_{j,n}\|_2 - \lambda\sum_{j=1}^{N} r_j r_{j+1}\int_\Omega \mathrm{e}^{u_{j,n}+u_{j+1,n}}(w_{j,n}+w_{j+1,n})\mathrm{d}x. \qquad (4.11)
\end{aligned}$$

At this point, our main effort will be to obtain a uniform estimate for the term:

$$\int_\Omega \mathrm{e}^{u_{j,n}+u_{j+1,n}}(w_{j,n}+w_{j+1,n})\mathrm{d}x, \ \text{for every } j=1,\dots,N.$$

We start by showing that $w_{j,n}$ is uniformly bounded in L^p, for any $p>1$.

To this purpose, for fixed $j\in\{1,\cdots,N\}$ we take:

$$\phi_k = \begin{cases} 0 & k\notin\{j-1,j,j+1\} \\ -\varphi & k=j-1,j+1 \\ 2\varphi & k=j \end{cases},$$

and from (4.3) we obtain:

$$\begin{aligned}
& \int_\Omega \nabla w_{j,n}\cdot\nabla\varphi\mathrm{d}x + \lambda\int_\Omega \Big[4r_j^2\mathrm{e}^{2u_{j,n}} - 2r_{j-1}^2\mathrm{e}^{2u_{j-1,n}} - 2r_{j+1}^2\mathrm{e}^{2u_{j+1,n}} + r_{j-1}r_{j-2}\mathrm{e}^{u_{j-1,n}+u_{j-2,n}} \\
& + r_{j+1}r_{j+2}\mathrm{e}^{u_{j+1,n}+u_{j+2,n}} - r_{j-1}r_j\mathrm{e}^{u_{j-1,n}+u_{j,n}} - r_j r_{j+1}\mathrm{e}^{u_{j,n}+u_{j+1,n}} + 2r_j\mathrm{e}^{u_{j,n}} - r_{j-1}\mathrm{e}^{u_{j-1,n}} \\
& - r_{j+1}\mathrm{e}^{u_{j+1,n}}\Big]\varphi\mathrm{d}x + \frac{1}{|\Omega|}(2b_j - b_{j-1} - b_{j+1})\int_\Omega \varphi\mathrm{d}x = o(1)\|\varphi\|, \quad \forall\varphi\in W^{1,2}(\Omega). \qquad (4.12)
\end{aligned}$$

For any $1<q<2$, by the Calderon–Zygmund inequality in L^p, see [51], we know that:

$$\|\nabla w_{j,n}\|_q \leq C\sup\left\{\int_\Omega \nabla w_{j,n}\cdot\nabla\varphi\mathrm{d}x, \quad \forall\varphi\in W^{1,p}(\Omega): \int_\Omega \varphi\mathrm{d}x = 0, \|\nabla\varphi\|_p\leq 1; \frac{1}{p}+\frac{1}{q}=1\right\} \qquad (4.13)$$

for a suitable constant $C>0$; and every φ in (4.13) satisfies: $\|\varphi\|+\|\varphi\|_\infty\leq C$, for suitable $C>0$.

Thus, from (4.12) and (4.13) we derive:

$$\|\nabla w_{j,n}\|_q \leq C_q, \ \text{for some } C_q>0.$$

As a consequence, for any $p\geq 1$, there exists $C_p>0$ such that:

$$\|w_{j,n}\|_p \leq C_p, \quad \forall n\in\mathbb{N} \ \text{and} \quad j=1,\dots,N. \qquad (4.14)$$

Next, in (4.3) we take $\phi_j = \varphi \in W^{1,2}(\Omega)$ for every $j = 1, \ldots, N$, and by simple calculations, we get,

$$\int_\Omega \nabla\Big(\sum_{j,k=1}^N a_{jk} w_{j,n}\Big) \cdot \nabla\varphi \mathrm{d}x + \lambda \sum_{j=0}^N \int_\Omega (r_j \mathrm{e}^{u_{j,n}} - r_{j+1}\mathrm{e}^{u_{j+1,n}})^2 \varphi \mathrm{d}x$$
$$+ \lambda \sum_{j=1}^N \left(-r_j \int_\Omega \mathrm{e}^{u_{j,n}} \varphi \mathrm{d}x + \frac{b_j}{|\Omega|} \int_\Omega \varphi \mathrm{d}x \right) = o(1)\|\varphi\|,$$

and since $\sum_{k=1}^N a_{jk} = r_j$, we find:

$$\int_\Omega \nabla\Big(\sum_{j=1}^N r_j w_{j,n}\Big) \cdot \nabla\varphi \mathrm{d}x + \lambda \int_\Omega \left[\sum_{j=0}^N (r_j \mathrm{e}^{u_{j,n}} - r_{j+1}\mathrm{e}^{u_{j+1,n}})^2 - \sum_{j=0}^N r_j \mathrm{e}^{u_{j,n}} \right] \varphi \mathrm{d}x$$
$$+ \sum_{j=1}^N \frac{b_j}{|\Omega|} \int_\Omega \varphi \mathrm{d}x = o(1)\|\varphi\|, \quad \forall \varphi \in W^{1,2}(\Omega). \tag{4.15}$$

Therefore, we can choose $\varphi = \big(\sum_{j=1}^N r_j w_{j,n}\big)^+$ in (4.15), and in view of (4.14) we find,

$$\Big\| \nabla\Big(\sum_{j=1}^N r_j w_{j,n}\Big)^+ \Big\|_2 \le C. \tag{4.16}$$

As usual, we have denoted by $f^+(x) = \max\{f(x), 0\}$ the positive part of $f = f(x)$.
More generally we define:

$$W_n^{(j)} = \sum_{k=j}^N r_k w_{k,n} \quad \text{and} \quad \hat{W}_n^{(j)} = \sum_{k=1}^{N+1-j} r_k w_{k,n},$$

with r_j in (2.3)–(2.4), and show the following:

Lemma 4.1. *If* $3 \le N \le 5$, *then*

$$\|\nabla(W_n^{(j)})^+\|_2 \le C \quad \textit{and} \quad \|\nabla(\hat{W}_n^{(j)})^+\|_2 \le C, \quad \forall j = 1, \ldots, N. \tag{4.17}$$

Proof. If $j = 1$ then (4.17) reduces to (4.16). Hence we take $j \ge 2$ and proceed by induction. Note that, by the symmetry:

$$r_j = r_{N+1-j}, \quad j \in \left\{1, \ldots, \left[\frac{N+1}{2}\right]\right\}. \tag{4.18}$$

It suffices to prove the uniform estimate for $W_n^{(j)}$ as the one for $\hat{W}_n^{(j)}$ follows similarly.
Next, we observe that,

$$\forall j = 1, \ldots, N, \quad (2 + r_j)^2 < 8(1 + r_j) \quad \text{if and only if} \quad 1 \le N \le 5, \tag{4.19}$$

and (4.19) is the exact reason for which we need the restriction on $N \in \{3, 4, 5\}$.
To check (4.19), observe that it is equivalent to:

$$r_j < 2(1 + \sqrt{2}) \quad \forall j = 1, \ldots, N, \tag{4.20}$$

and by (4.18) it suffices to check it only for $1 \le j \le [\frac{N+1}{2}]$. But for such j's the value of r_j is increasing with respect to j, and so (4.19) holds if and only if $r_{[\frac{N+1}{2}]} < 2(1 + \sqrt{2})$.

If $N=2k$ is even then $[\frac{N+1}{2}]=k$ and $r_k=\frac{k(k+1)}{2}$, while for $N=2k+1$ odd we have: $[\frac{N+1}{2}]=k+1$ and $r_{k+1}=\frac{(k+1)^2}{2}$. Hence (4.20) holds if and only if $k=0,1,2$, namely, $N=1,2,3,4,5$, as claimed.

Next, we illustrate the induction scheme for $j=2$, where we use (4.3) with $\phi_1=-r_2\varphi$, $\phi_2=(1+r_1)\varphi$ and $\phi_k=\varphi\ \forall k=3,\dots,N$. We obtain:

$$\begin{aligned}&\int_\Omega \nabla W_n^{(2)}\cdot\nabla\varphi \mathrm{d}x+\lambda\int_\Omega\left(-2r_2r_1^2\mathrm{e}^{2u_{1,n}}+(r_2-(1+r_1))\mathrm{e}^{u_{1,n}+u_{2,n}}\right)\varphi\mathrm{d}x\\&+\lambda\int_\Omega(2(1+r_1)r_2^2\mathrm{e}^{2u_{2,n}}-(2+r_1)r_2r_3\mathrm{e}^{u_{2,n}+u_{3,n}}+r_3^2\mathrm{e}^{2u_{3,n}})\varphi\mathrm{d}x\\&+\lambda\int_\Omega\left(\sum_{k=3}^N(r_k\mathrm{e}^{u_{k,n}}-r_{k+1}\mathrm{e}^{u_{k+1,n}})^2\right)\varphi\mathrm{d}x=o(1)\|\varphi\|.\end{aligned}\tag{4.21}$$

Thus, by using (4.19) with $j=1$, we let:

$$\mu_1=\left(2(1+r_1)-\frac{(2+r_1)^2}{4}\right)r_2^2>0$$

and rewrite (4.21) as follows:

$$\begin{aligned}&\int_\Omega \nabla W_n^{(2)}\cdot\nabla\varphi \mathrm{d}x+\lambda\int_\Omega\left(-2r_2r_1^2+\frac{(r_2-(1+r_1))^2}{4\mu_1}\right)\mathrm{e}^{2u_{1,n}}\varphi\mathrm{d}x\\&+\lambda\int_\Omega\left[\left(\frac{r_2-(1+r_1)}{2\sqrt{\mu_1}}\mathrm{e}^{u_{1,n}}+\sqrt{\mu_1}\mathrm{e}^{u_{2,n}}\right)^2+\left(\frac{2+r_1}{2}r_2\mathrm{e}^{u_{2,n}}-r_3\mathrm{e}^{u_{3,n}}\right)^2\right]\varphi\mathrm{d}x\\&+\lambda\int_\Omega\left(\sum_{k=3}^N(r_k\mathrm{e}^{u_{k,n}}-r_{k+1}\mathrm{e}^{u_{k+1,n}})^2\right)\varphi\mathrm{d}x=o(1)\|\varphi\|.\end{aligned}$$

Therefore, by choosing $\varphi=(W_n^{(2)})^+$ in (4.21) and by recalling (4.8), we obtain:

$$\begin{aligned}&\|\nabla(W_n^{(2)})^+\|_2^2\le C\left(\int_\Omega \mathrm{e}^{2w_{1,n}}(W_n^{(2)})^+\mathrm{d}x+1\right)\\&\le C\left(\int_{\Omega\cap\{w_{1,n}\ge0\}}\mathrm{e}^{\frac{2}{r_1}(r_1w_{1,n}+(W_n^{(2)})^+)}(W_n^{(2)})^+\mathrm{d}x+1\right)\\&\le C\left(\int_{\Omega\cap\{w_{1,n}\ge0\}}\mathrm{e}^{\frac{2}{r_1}(\sum_{j=1}^N r_jw_{j,n})^+}(W_n^{(2)})^+\mathrm{d}x+1\right)\\&\le C\left(\int_\Omega\mathrm{e}^{\frac{2}{r_1}(\sum_{j=1}^N r_jw_{j,n})^+}(W_n^{(2)})^+\mathrm{d}x+1\right).\end{aligned}$$

At this point, by virtue of (4.14) and (4.16), we can use the Moser–Trudinger inequality (see [5]), to conclude that,

$$\int_\Omega \mathrm{e}^{\frac{2}{r_1}(W_n^{(1)})^+}(W_n^{(2)})^+\mathrm{d}x\le C,$$

and we arrive at the desired conclusion.

Next, let us carry out our induction procedure, so for $j \geq 3$ we assume that,

$$\|\nabla(W_n^{(j-1)})^+\|_2 \leq C, \tag{4.22}$$

and we are left to prove that a similar estimate holds for $(W_n^{(j)})^+$.

To this purpose, we are going to use (4.3) with $\phi_{j-1} = -r_j\varphi$, $\phi_j = (1 + r_{j-1})\varphi$, $\phi_k = 0$ for $1 \leq k \leq j-2$ and $\phi_k = \varphi$ for $k \geq j+1$, and obtain the following:

$$\begin{aligned}
&\int_\Omega \nabla W_n^{(j)} \cdot \nabla\varphi \mathrm{d}x + \lambda \int_\Omega \big(-2r_j r_{j-1}^2 \mathrm{e}^{2u_{j-1,n}} + (r_j - (1+r_{j-1}))\mathrm{e}^{u_{j-1,n}+u_{j,n}} \big)\varphi \mathrm{d}x \\
&+ \lambda \int_\Omega \big(2(1+r_{j-1})r_j^2 \mathrm{e}^{2u_{j,n}} - (2+r_{j-1})r_j r_{j+1} \mathrm{e}^{u_{j,n}+u_{j+1,n}} + r_{j+1}^2 \mathrm{e}^{2u_{j+1,n}} \big)\varphi \mathrm{d}x \\
&+ \lambda \int_\Omega \left(\sum_{k=j+1}^{N} (r_k \mathrm{e}^{u_{k,n}} - r_{k+1}\mathrm{e}^{u_{k+1,n}})^2 \right) \varphi \mathrm{d}x = o(1)\|\varphi\|.
\end{aligned}$$

Exactly as above, by using (4.19) and by letting:

$$\mu_{j-1} = \left(2(1+r_{j-1}) - \frac{(2+r_{j-1})^2}{4} \right) r_2^2 > 0, \quad \sigma_j = \frac{(r_j - (1+r_{j-1}))^2}{4\mu_{j-1}} + 2r_j r_{j-1}^2 > 0, \tag{4.23}$$

we find:

$$\begin{aligned}
&\int_\Omega \nabla W_n^{(j)} \cdot \nabla\varphi \mathrm{d}x + \lambda \int_\Omega \left(-\sigma_j \mathrm{e}^{2u_{j-1,n}} + \left(\frac{r_j - (1+r_{j-1})}{2\sqrt{\mu_{j-1}}} \mathrm{e}^{u_{j-1,n}} + \sqrt{\mu_{j-1}}\mathrm{e}^{u_{j,n}} \right)^2 \right) \varphi \mathrm{d}x \\
&+ \lambda \int_\Omega \left(\frac{2+r_{j-1}}{2} r_j \mathrm{e}^{u_{j,n}} - r_{j+1}\mathrm{e}^{u_{j+1,n}} \right)^2 \varphi \mathrm{d}x \\
&+ \lambda \int_\Omega \left(\sum_{k=j+1}^{N} \big(r_k \mathrm{e}^{u_{k,n}} - r_{k+1}\mathrm{e}^{u_{k+1,n}}\big)^2 \right) \varphi \mathrm{d}x = o(1)\|\varphi\|.
\end{aligned}$$

Therefore, by taking $\varphi = (W_n^{(j)})^+$ we derive:

$$\begin{aligned}
\|\nabla(W_n^{(j)})^+\|_2^2 &\leq C \left(\int_\Omega \mathrm{e}^{2w_{j-1,n}} (W_n^{(j)})^+ \mathrm{d}x + 1 \right) \\
&\leq C \left(\int_{\Omega\cap\{w_{j-1,n}\geq 0\}} \mathrm{e}^{\frac{2}{r_{j-1}}(r_{j-1}w_{j-1,n} + (W_n^{(j)})^+)} (W_n^{(j)})^+ \mathrm{d}x + 1 \right) \\
&\leq C \left(\int_\Omega \mathrm{e}^{\frac{2}{r_{j-1}}(W_n^{(j-1)})^+} (W_n^{(j)})^+ \mathrm{d}x + 1 \right) \leq C,
\end{aligned}$$

and the last estimate follows as above, from the induction hypothesis (4.21), (4.14) and the Moser–Trudinger inequality [5].

Again by the symmetry (4.18), we can argue similarly simply by replacing in the above arguments each index involved, say k, with $N+1-k$ and obtain:

$$\|\nabla(\hat{W}_n^{(j)})^+\|_2 \leq C, \quad \forall j = 1, \ldots, N$$

and the proof is completed. □

At this point, we observe that: $r_1 w_{1,n} + r_2 w_{2,n} = \hat{W}_n^{(N-1)}$ and $r_1 w_{1,n} = \hat{W}_n^{(N)}$; while, $r_{N-1} w_{N-1,n} + r_N w_{N,n} = W_n^{(N-1)}$ and $r_N w_{N,n} = W_n^{(N)}$. Thus, from Lemma 4.1, we obtain in particular that,

$$\|\nabla(r_1 w_{1,n} + r_2 w_{2,n})^+\|_2 + \|\nabla(r_{N-1} w_{N-1,n} + r_N w_{N,n})^+\|_2 \le C, \tag{4.24}$$

and

$$\|\nabla w_{1,n}^+\|_2 + \|\nabla w_N^+\|_2 \le C. \tag{4.25}$$

More generally there holds:

Lemma 4.2. *For $N = 3, 4, 5$ we have:*

$$\|(r_j w_{j,n} + r_{j+1} w_{j+1,n})^+\| \le C, \quad \forall j = 1, \ldots, N. \tag{4.26}$$

Proof. For $N = 3$, (4.26) follows already from (4.14), (4.24) and (4.25). So we let, $N \ge 4$ and set,

$$Z_n = \sum_{k=2}^{N-1} r_k w_{k,n}. \tag{4.27}$$

Claim 1.

$$\|\nabla(Z_n)^+\|_2 \le C. \tag{4.28}$$

To establish (4.28) we apply (4.3) with $\phi_1 = -r_2\varphi$, $\phi_2 = (1 + r_1)\varphi$, $\phi_{N-2} = (1 + r_{N-3})\varphi$, $\phi_{N-1} = (1 + r_N)\varphi$ and $\phi_N = -r_{N-1}\varphi$. Note that for $N = 4$ the definition above is consistent since $\phi_2 = (1 + r_1)\varphi = (1 + r_{N-3})\varphi = \phi_{N-2}$.

For $N = 4$ we have that, $r_1 = r_4 = 2$ and $r_2 = r_3 = 3$, so we obtain:

$$\begin{aligned}
&\int_\Omega \nabla Z_n \cdot \nabla\varphi \mathrm{d}x + \lambda \int_\Omega \Big[-2r_1^2 r_2 \mathrm{e}^{2u_{1,n}} - (1 + r_1 - r_2) r_1 r_2 \mathrm{e}^{u_{1,n}+u_{2,n}} + 2(1 + r_1)(r_2 \mathrm{e}^{u_{2,n}} - r_3 \mathrm{e}^{u_{3,n}})^2 \\
&- 2r_3 r_4^2 \mathrm{e}^{2u_{4,n}} + (1 + r_4 - r_3) r_3^2 r_4 \mathrm{e}^{u_{3,n}+u_{4,n}} \Big] \varphi \mathrm{d}x \\
&= \int_\Omega \nabla Z_n \cdot \nabla\varphi \mathrm{d}x - 24\lambda \int_\Omega (\mathrm{e}^{2u_{1,n}} + \mathrm{e}^{2u_{4,n}})\varphi \mathrm{d}x + 6\lambda \int_\Omega (r_2 \mathrm{e}^{u_{2,n}} - r_3 \mathrm{e}^{u_{3,n}})^2 \varphi \mathrm{d}x \\
&= o(1)\|\varphi\|.
\end{aligned}$$

Hence, by taking $\varphi = (Z_n)^+$, and by using (4.25) and the Moser–Trudinger inequality as above, we derive:

$$\|\nabla(Z_n)^+\|_2^2 \le C \int_\Omega (\mathrm{e}^{2w_{1,n}} + \mathrm{e}^{2w_{4,n}})(Z_n)^+ \mathrm{d}x \le C \int_\Omega (\mathrm{e}^{2w_{1,n}^+} + \mathrm{e}^{2w_{4,n}^+})(Z_n)^+ \mathrm{d}x \le C$$

and (4.28) follows for $N = 4$.

For $N = 5$, we have:

$$r_1 = r_5 = \frac{5}{2},\ r_2 = r_4 = 4,\ r_3 = \frac{9}{2}, \tag{4.29}$$

and in this case from (4.3) we obtain:

$$\begin{aligned}
&\int_\Omega \nabla Z_n \cdot \nabla\varphi \mathrm{d}x + \lambda \int_\Omega \big[-2r_1^2 r_2 \mathrm{e}^{2u_{1,n}} - (1 + r_1 - r_2) r_1 r_2 \mathrm{e}^{u_{1,n}+u_{2,n}} \big] \varphi \mathrm{d}x \\
&+ \lambda \int_\Omega \Big[2(1 + r_1) r_2^2 \mathrm{e}^{2u_{2,n}} - (2 + r_1 + r_2) r_2 r_3 \mathrm{e}^{u_{2,n}+u_{3,n}} + 2(1 + r_2) r_3^2 \mathrm{e}^{2u_{3,n}} \Big] \varphi \mathrm{d}x
\end{aligned}$$

$$+\lambda\int_\Omega\left[2(1+r_5)r_4^2e^{2u_{4,n}}-(2+r_2+r_5)r_3r_4e^{u_{3,n}+u_{4,n}}-2r_4r_5^2e^{2u_{5,n}}+r_4r_5(r_4-(1+r_5))e^{u_{4,n}+u_{5,n}}\right]\varphi dx$$
$$=\int_\Omega\nabla Z_n\cdot\nabla\varphi dx-18\lambda\int_\Omega(e^{2u_{1,n}}+e^{2u_{5,n}})\varphi dx+5\lambda\int_\Omega(e^{u_{1,n}+u_{2,n}}+e^{u_{4,n}+u_{5,n}})\varphi dx$$
$$+\lambda\int_\Omega[2(1+r_1)r_2^2e^{2u_{2,n}}-(2+r_1+r_2)r_2r_3e^{u_{2,n}+u_{3,n}}+(1+r_2)r_3^2e^{2u_{3,n}}]\varphi dx$$
$$+\lambda\int_\Omega[2(1+r_1)r_2^2e^{2u_{4,n}}-(2+r_1+r_2)r_2r_3e^{u_{3,n}+u_{4,n}}+(1+r_2)r_3^2e^{2u_{3,n}}]\varphi dx$$
$$=o(1)\|\varphi\|.$$

At this point, we observe that, $(2+r_1+r_2)^2\le 8(1+r_1)(1+r_2)$, and so we can check that the terms within the brackets in the last two integrals above are positive.

As a consequence for $\varphi=(Z_n)^+$, arguing as above we find:

$$\|\nabla(Z_n)^+\|_2^2\le C\int_\Omega(e^{2w_{1,n}^+}+e^{2w_{5,n}^+})(Z_n)^+dx\le C,$$

and Claim 1 is established for $N=5$ as well.

Claim 2. If $N=5$ then,

$$\|\nabla(r_jw_{j,n}+r_{j+1}w_{j+1,n})^+\|_2\le C,\quad j=2,3. \tag{4.30}$$

To establish (4.30), we observe that, from Lemma 4.1 and (4.28), we have:

$$\|\nabla(r_jw_{j,n}+r_{j+1}w_{j+1,n}+r_{j+2}w_{j+2,n})^+\|_2\le C,\quad j=1,2,3. \tag{4.31}$$

So, for $j=2,3$, we can take $\phi_{j-1}=-r_j\varphi$, $\phi_j=(1+r_{j-1})\varphi$, $\phi_{j+1}=(1+r_{j+2})\varphi$, $\phi_{j+2}=-r_{j+1}\varphi$, in (4.3) and obtain:

$$\begin{aligned}&\int_\Omega\nabla(r_jw_{j,n}+r_{j+1}w_{j+1,n})\cdot\nabla\varphi dx+\lambda\int_\Omega\Big[-2r_{j-1}^2r_je^{2u_{j-1,n}}\\&-(1+r_{j-1}-r_j)r_{j-1}r_je^{u_{j-1,n}+u_{j,n}}+2(1+r_{j-1})r_j^2e^{2u_{j,n}}-(2+r_{j-1}+r_{j+1})r_jr_{j+1}e^{u_{j,n}+u_{j+1,n}}\\&+2(1+r_{j+2})r_{j+1}^2e^{2u_{j+1,n}}-(1+r_{j+2}-r_{j+1})r_{j+1}r_{j+2}e^{u_{j+1,n}+u_{j+2,n}}\\&-2r_{j+1}r_{j+2}^2e^{2u_{j+2,n}}+r_{j+1}r_{j+2}r_{j+3}e^{u_{j+2,n}+u_{j+3,n}}\Big]\varphi dx=o(1)\|\varphi\|.\end{aligned} \tag{4.32}$$

Now notice that, for $j=2,3$ we have:

$$1+r_{j-1}-r_j=\frac{2j-N}{2}=-(1+r_{j+2}-r_{j+1})$$

and by (4.29) we can check directly that,

$$(2+r_{j-1}+r_{j+2})^2<16(1+r_{j-1})(1+r_{j+2}). \tag{4.33}$$

Hence (4.32) can be expressed as follows:

$$\int_\Omega\nabla(r_jw_{j,n}+r_{j+1}w_{j+1,n})\cdot\nabla\varphi dx-2\lambda\int_\Omega(r_{j-1}^2r_je^{2u_{j-1,n}}+r_{j+1}r_{j+2}^2e^{2u_{j+2,n}})\varphi dx$$
$$+\lambda\frac{N-2j}{2}\int_\Omega(r_{j-1}r_je^{u_{j-1,n}+u_{j,n}}-r_{j+1}r_{j+2}e^{u_{j+1,n}+u_{j+2,n}})\varphi dx$$

$$+\lambda\int_\Omega\Big[2(1+r_{j-1})r_j^2\mathrm{e}^{2u_{j,n}}-(2+r_{j-1}+r_{j+2})r_jr_{j+1}\mathrm{e}^{u_{j,n}+u_{j+1,n}}+2(1+r_{j+2})r_{j+1}^2\mathrm{e}^{2u_{j+1,n}}\Big]\varphi\mathrm{d}x$$

$$+\lambda r_{j+1}r_{j+2}r_{j+3}\int_\Omega \mathrm{e}^{u_{j+2,n}+u_{j+3,n}}\varphi\mathrm{d}x=o(1)\|\varphi\|.$$

Consequently, for $j=2$, we set:

$$\varepsilon_1=\Big(2(1+r_4)-\frac{(2+r_1+r_4)^2}{8(1+r_1)}\Big)r_3^2>0$$

(recall (4.33)) and find:

$$\int_\Omega\nabla(r_2w_{2,n}+r_3w_{3,n})\cdot\nabla\varphi\mathrm{d}x-2\lambda\int_\Omega\Big(r_1^2r_2\mathrm{e}^{2u_{1,n}}+r_3r_4^2\Big(1+\frac{r_3}{32\varepsilon_1}\Big)\mathrm{e}^{2u_{4,n}}\Big)\varphi\mathrm{d}x$$

$$+\lambda\int_\Omega\Big(\frac{1}{2}r_1r_2\mathrm{e}^{u_{1,n}+u_{2,n}}+r_3r_4r_5\mathrm{e}^{u_{4,n}+u_{5,n}}\Big)\varphi\mathrm{d}x$$

$$+\lambda\int_\Omega\Big(\sqrt{2(1+r_1)}r_2\mathrm{e}^{u_{2,n}}-\frac{2+r_1+r_4}{2\sqrt{2(1+r_1)}}r_3\mathrm{e}^{u_{3,n}}\Big)^2\varphi\mathrm{d}x$$

$$+\lambda\int_\Omega\Big(\sqrt{\varepsilon}_1\mathrm{e}^{u_{3,n}}-\frac{r_3r_4}{4\sqrt{\varepsilon}_1}\mathrm{e}^{u_{4,n}}\Big)^2\varphi\mathrm{d}x=o(1)\|\varphi\|.$$

Thus, by taking $\varphi=(r_2w_{2,n}+r_3w_{3,n})^+$, we can estimate:

$$\|\nabla(r_2w_{2,n}+r_3w_{3,n})^+\|_2^2$$
$$\le C\left(\int_\Omega \mathrm{e}^{2w_{1,n}}(r_2w_{2,n}+r_3w_{3,n})^+\mathrm{d}x+\int_\Omega \mathrm{e}^{2w_{4,n}}(r_2w_{2,n}+r_3w_{3,n})^+\mathrm{d}x\right)+C_1$$
$$\le C\left(\int_{\{w_{1,n}\ge 0\}} \mathrm{e}^{r_1w_{1,n}}(r_2w_{2,n}+r_3w_{3,n})^+\mathrm{d}x+\int_{\{w_{4,n}\ge 0\}} \mathrm{e}^{r_4w_{4,n}}(r_2w_{2,n}+r_3w_{3,n})^+\mathrm{d}x\right)+C_2$$
$$\le C\left(\int_\Omega \big(\mathrm{e}^{(r_1w_{1,n}+r_2w_{2,n}+r_3w_{3,n})^+}+\mathrm{e}^{(r_2w_{2,n}+r_3w_{3,n}+r_4w_{4,n})^+}\big)(r_2w_{2,n}+r_3w_{3,n})^+\mathrm{d}x\right)+C_2$$
$$\le C$$

as it follows from (4.31) and again by using (4.14) together with the Moser–Trudinger inequality. Thus (4.30) is established for $j=2$.

For $j=3$, we argue similarly and for

$$\varepsilon_2=\Big(2(1+r_2)-\frac{(2+r_2+r_5)^2}{8(1+r_5)}\Big)r_3^2>0$$

we obtain:

$$\int_\Omega\nabla(r_3w_{3,n}+r_4w_{4,n})\cdot\nabla\varphi\mathrm{d}x-2\lambda\int_\Omega\Big[r_2^2r_3\Big(1+\frac{r_3}{32\varepsilon_2}\Big)\mathrm{e}^{2u_{2,n}}+r_4r_5^2\mathrm{e}^{2u_{5,n}}\Big]\varphi\mathrm{d}x$$

$$+\frac{\lambda}{2}r_4r_5\int_\Omega \mathrm{e}^{u_{4,n}+u_{5,n}}\varphi\mathrm{d}x+\lambda\int_\Omega\Big(r_3\frac{2+r_2+r_5}{2\sqrt{2(1+r_5)}}\mathrm{e}^{u_{3,n}}-\sqrt{2(1+r_5)}r_4\mathrm{e}^{u_{4,n}}\Big)^2\varphi\mathrm{d}x$$

$$+\lambda\int_\Omega(\sqrt{\varepsilon_2}\mathrm{e}^{u_{3,n}}-\frac{r_2r_3}{4\sqrt{\varepsilon_2}}\mathrm{e}^{u_{2,n}})^2\varphi\mathrm{d}x=o(1)\|\varphi\|.$$

As above, for $\varphi = (r_3 w_{3,n} + r_4 w_{4,n})^+$ we obtain the following estimate:

$$\|\nabla(r_3 w_{3,n} + r_4 w_{4,n})^+\|_2^2 \le C \int_\Omega (e^{2w_{2,n}} + e^{2w_{5,n}})(r_3 w_{3,n} + r_4 w_{4,n})^+ dx + C_1$$
$$\le C \left(\int_{\{w_{2,n}\ge 0\}} e^{r_2 w_{2,n}}(r_3 w_{3,n} + r_4 w_{4,n})^+ dx + \int_{\{w_{5,n}\ge 0\}} e^{r_5 w_{5,n}}(r_3 w_{3,n} + r_4 w_{4,n})^+ dx \right) + C_2$$
$$\le C \left(\int_\Omega (e^{(r_2 w_{2,n} + r_3 w_{3,n} + r_4 w_{4,n})^+} + e^{(r_3 w_{3,n} + r_4 w_{4,n} + r_5 w_{5,n})^+})(r_3 w_{3,n} + r_4 w_{4,n})^+ dx \right) + C_2$$
$$\le C,$$

and (4.30) is established. Thus, the proof of (4.26) is completed. □

Proof of Proposition 3.2. According to (4.8) and Lemma (4.2), for $j = 1, \dots, N$, we can estimate:

$$\int_\Omega e^{u_{j,n} + u_{j+1,n}}(w_{j,n} + w_{j+1,n})dx \le C_1 \Big(\int_{\{w_{j,n}\ge 0\}\}\cap\{w_{j+1,n}\ge 0\}} e^{w_{j,n} + w_{j+1,n}}(w_{j,n} + w_{j+1,n})dx \Big) + C_2$$
$$\le C_1 \Big(\int_{\{w_{j,n}\ge 0\}\}\cap\{w_{j+1,n}\ge 0\}} e^{(r_j w_{j,n} + r_{j+1} w_{j+1,n})^+}(w_{j,n} + w_{j+1,n})dx \Big) + C_2 \le C,$$

where the last estimate follows as above, by (4.14), Lemma 4.2 and the Moser–Trudinger inequality.

At this point, by virtue of (4.11) we may conclude also that,

$$\sum_{j=1}^N \|\nabla w_{j,n}\|_2^2 \le C.$$

In particular, along a subsequence, as $n \to +\infty$, we may conclude that,

$$w_{j,n} \to w_j \quad \text{weakly in } W^{1,2}(\Omega) \text{ and strongly in } L^p(\Omega),$$
$$e^{w_{j,n}} \to e^{w_j} \quad \text{strongly in} \quad L^p(\Omega), \quad \text{for } p > 1.$$

Moreover, from (4.4), we derive that, (along a subsequence) there holds:

$$e^{2c_{j,n}} \int_\Omega e^{2u_j^0 + 2w_{j,n}} dx \to L_j > 0, \quad \text{as} \quad n \to +\infty,$$

with a suitable $L_j > 0$.

Consequently, by setting:

$$c_j = \frac{1}{2} log\Big(\frac{L_j}{\int_\Omega e^{2u_j^0 + 2w_j} dx}\Big) \text{ and } v_j = w_j + c_j,$$

as $n \to +\infty$, we have:

$$v_{j,n} \to v_j, \quad e^{u_j^0 + v_{j,n}} \to e^{u_j^0 + v_j} \text{ strongly in } L^p(\Omega) \text{ for } p > 1, \quad \forall j = 1, \dots, N.$$

Therefore, from (4.15), we get:

$$\sum_{j,k=1}^N a_{jk} \int_\Omega \nabla(w_{j,n} - w_j) \cdot \nabla(w_{k,n} - w_k) dx$$
$$= I'(v_{1,n}, \dots, v_{N,N})(v_{1,n} - v_1, \dots, v_{N,n} - v_N) + o(1)$$

$$= o(1)\left(\sum_{j=1}^{N}\|\nabla(w_{j,n}-w_j)\|_2+1\right), \quad \text{as } n\to+\infty$$

which implies that,

$$w_{j,n}\to w_j \quad \text{strongly in} \quad W^{1,2}(\Omega), \quad \text{as } n\to+\infty, \quad \forall j=1,\dots,N,$$

and the proof is completed. □

Conflict of interest statement

There is no conflict of interest.

Appendix A. Appendix of linear algebra

For $N\ge 2$ and for $j=1,\dots,N$ we let,

$$\beta_{j,1}\in\mathbb{R} \quad \text{for } j=2,\dots,N, \quad \beta_{j,2}\in\mathbb{R} \quad \text{for } j=1,\dots,N-1 \text{ and set } \beta_{1,1}=0=\beta_{N,2}. \tag{A.1}$$

Given $1\le k\le l\le N$, we define the square $(l-k+1)\times(l-k+1)$ matrix $T_l^{(k)}$ as follows:

$$T_l^{(k)}=(t_{j,s})_{j,s=k,\dots,l} \quad \text{with} \quad t_{j,s}=\beta_{j,1}\delta_{j-1}^s+\delta_j^s+\beta_{j,2}\delta_{j+1}^s \tag{A.2}$$

where,

$$\delta_p^s=\begin{cases}1, & s=p\\ 0, & s\ne p\end{cases}$$

is the usual Kronecker symbol.

Notice that, if $k=l$ then $T_l^{(l)}=1$, while in general the matrix $T_l^{(k)}$ is expressed in terms of the quantities:

$$(\beta_{k,2},\beta_{k+1,1},\beta_{k+1,2},\dots,\beta_{l-1,1},\beta_{l-1,2},\beta_{l,1}). \tag{A.3}$$

We wish to identify the determinant of $T_l^{(k)}$, i.e. $\det T_l^{(k)}$.

To this purpose we define the quantities $F_l^{(k)}$ via a recursive formula, starting with the case $k=l$, where we set,

$$F_l^{(l)}=1. \tag{A.4}$$

For $k>l$ we define $F_l^{(k)}$ as follows:

$$F_l^{(l+1)}=1 \quad \text{and} \quad F_l^{(k)}=0 \quad \forall k\ge l+2. \tag{A.5}$$

More importantly, for $1\le k\le l$ recursively we set:

$$\begin{aligned} F_l^{(k)} &= 1-\sum_{j=k}^{l}\beta_{j+1,1}\beta_{j,2}F_l^{(j+2)}\\ &= 1-\sum_{j=k}^{l}\beta_{j+1,1}\beta_{j,2}\left(1-\sum_{k=j+2}^{l}\beta_{k+1,1}\beta_{k,2}\left(1-\sum_{s=k+2}^{l}\beta_{s+1,1}\beta_{s,2}(1-\cdots)\right)\right)\end{aligned} \tag{A.6}$$

so that,

$$F_l^{(l-1)}=1-\beta_{l,1}\beta_{l-1,2}, \quad F_l^{(l-2)}=1-\beta_{l-2,2}\beta_{l-1,1}-\beta_{l-1,2}\beta_{l,1}$$

and so on.

Remark A.1. In view of (A.5), it suffices to take the summation in (A.6) up to the index $k=l-1$, instead of l as indicated there.

Notice that the value of $F_l^{(k)}$ depends on the same terms of the matrix $T_l^{(k)}$, as specified in (A.3). In fact, the following holds:

Lemma A.1. *For $1 \le k \le l \le N$ we have:*

$$\det T_l^{(k)} = F_l^{(k)}. \tag{A.7}$$

Proof. We proceed by induction on k. In fact for $k = l$, in view of (A.5), we see that (A.7) is obviously satisfied. Thus, we assume that for $1 \le k \le l-1$, there holds:

$$\det T_l^{(j)} = F_l^{(j)} \quad \forall\, j \in \{k+1, \dots, l\}.$$

To check (A.7), we observe that,

$$\det T_l^{(k)} = \det T_l^{(k+1)} - \beta_{k+1,1}\beta_{k,2} \det T_l^{(k+2)}.$$

Thus, by the induction hypothesis, we find that,

$$\det T_l^{(k)} = F_l^{(k+1)} - \beta_{k+1,1}\beta_{k,2} F_l^{(k+2)} = F_l^{(k)}$$

and the proof is complete. □

To proceed further, we need to be more specific about our choice of β's in (A.1). More precisely we let,

$$\beta_{j,1} = (-1)^{\varepsilon_{j,1}}\alpha_{j,1},\ \varepsilon_{j,1} \in \{0,1\}, \alpha_{j,1} \ge 0,\ j = 2, \dots, N, \tag{A.8}$$

$$\beta_{j,2} = (-1)^{\varepsilon_{j,2}}\alpha_{j,2},\ \varepsilon_{j,2} \in \{0,1\}, \alpha_{j,2} \ge 0,\ j = 1, \dots, N-1,\ \text{and}\ \alpha_{1,1} = 0 = \alpha_{N,2}. \tag{A.9}$$

Lemma A.2. *Assume* (A.8) *and* (A.9)*. Then, for given $1 \le k \le l \le N$ there holds:*

$$\frac{\partial F_l^{(k)}}{\partial \alpha_{j+1,1}} = -(-1)^{\varepsilon_{j,2}+\varepsilon_{j+1,2}}\alpha_{j,2} F_l^{(j+2)} F_{j-1}^{(k)}, \tag{A.10}$$

$$\frac{\partial F_l^{(k)}}{\partial \alpha_{j,2}} = -(-1)^{\varepsilon_{j,2}+\varepsilon_{j+1,1}}\alpha_{j+1,1} F_l^{(j+2)} F_{j-1}^{(k)}. \tag{A.11}$$

Proof. First of all, by virtue of (A.5), we can check that, for $1 \le j < k$ we have: $F_{j-1}^{(k)} = 0$. Similarly, for $l \le j \le N$ we have $F_l^{j+2} = 0$. So, for such choice of indices, we have: $\frac{\partial F_l^{(k)}}{\partial \alpha_{j+1,1}} = 0 = \frac{\partial F_l^{(k)}}{\partial \alpha_{j,2}}$ consistently with the definition of $F_l^{(k)}$. Hence we let, $1 \le k < l$ and for $j \in \{k, \dots, l-1\}$ we are going to verify (A.10) and (A.11) by an induction argument on k. Actually, we provide the details only for (A.10), as (A.11) follows similarly.

For $k = l-1$ we see that,

$$F_{l-1}^{(l-1)} = 1 - (-1)^{\varepsilon_{l-1,1}+\varepsilon_{l,1}}\alpha_{l-1,2}\alpha_{l,1},$$

and in this case we have only the choice of $j = k = l-1$. Hence,

$$\frac{\partial F_l^{(k)}}{\partial \alpha_{j+1,1}} = \frac{\partial F_{l-1}^{(l-1)}}{\partial \alpha_{l,1}} = (-1)^{\varepsilon_{l-1,1}+\varepsilon_{l,1}}\alpha_{l-1,2},$$

which gives exactly (A.10), since in this case we have:

$$F_l^{(j+2)} = F_l^{(l+2)} = 1 \quad \text{and} \quad F_{j-1}^{(k)} = F_{l-2}^{(l-1)} = 1.$$

Next, we take $k \in \{1, \dots, l-2\}$ and by induction we assume that, for $j \in \{k+1, \dots, l-1\}$ the identity (A.10) holds for $\frac{\partial F_l^{(k)}}{\partial \alpha_{j+1,1}}$.

For $k \le j < l$, we write,

$$F_l^{(k)} = 1 - \sum_{s=k}^{j-1} (-1)^{\varepsilon_{s,2}+\varepsilon_{s+1,1}} \alpha_{s,2}\alpha_{s+1,1} F_l^{(s+2)} - (-1)^{\varepsilon_{j,2}+\varepsilon_{j+1,1}} \alpha_{j,2}\alpha_{j+1,1} F_l^{(j+2)}$$
$$- \sum_{s=j+1}^{l} (-1)^{\varepsilon_{s,2}+\varepsilon_{s+1,1}} \alpha_{s,2}\alpha_{s+1,1} F_l^{(s+2)}$$

with the understanding that, when $j = k$ then the first summation term above is dropped. We compute:

$$\frac{\partial F_l^{(k)}}{\partial \alpha_{j+1,1}} = -\sum_{s=k}^{j-1} (-1)^{\varepsilon_{s,2}+\varepsilon_{s+1,1}} \alpha_{s,2}\alpha_{s+1,1} \frac{\partial F_l^{(s+2)}}{\partial \alpha_{j+1,1}} - (-1)^{\varepsilon_{j,2}+\varepsilon_{j+1,1}} \alpha_{j,2} F_l^{(j+2)}.$$

Hence, by virtue of the induction assumption we find:

$$\frac{\partial F_l^{(k)}}{\partial \alpha_{j+1,1}} = -\left(1 - \sum_{s=k}^{j-1} (-1)^{\varepsilon_{s,2}+\varepsilon_{s+1,1}} \alpha_{s,2}\alpha_{s+1,1} F_{j-1}^{(s+2)}\right) (-1)^{\varepsilon_{j,2}+\varepsilon_{j+1,1}} \alpha_{j,2} F_l^{(j+2)}$$
$$= -(-1)^{\varepsilon_{j,2}+\varepsilon_{j+1,1}} \alpha_{j,2} F_l^{(j+2)} F_{j-1}^{(k)}$$

as claimed. □

Remark A.2. Note that the term $F_l^{(j+2)} F_{j-1}^{(k)}$ on the right-hand side of (A.10) and (A.11) is independent of $\alpha_{j,2}$ and $\alpha_{j+1,1}$. Therefore if such term vanishes then $F_l^{(k)}$ is independent of both $\alpha_{j,2}$ and $\alpha_{j+1,1}$.

Proposition A.1. *Let $i \in \{1, \dots, N-1\}$ and let $k, l \in \mathbb{N}$ be such that $k \le i \le l \le N$. We have:*

$$\text{i) if} \quad \varepsilon_{i,2} + \varepsilon_{i+1,1} = 0 \ (\text{mod}\, 2) \quad \text{then} \quad F_l^{(k)} = F_l^{(k)}|_{\varepsilon_{i,2}=0=\varepsilon_{i+1,1}}; \tag{A.12}$$
$$\text{ii) if} \quad \varepsilon_{i,2} + \varepsilon_{i+1,1} = 1 \quad \text{and} \quad F_l^{(i+2)} F_{i-1}^{(k)} \ge 0 \quad \text{then} \quad F_l^{(k)} \ge F_l^{(k)}|_{\varepsilon_{i,2}=0=\varepsilon_{i+1,1}}. \tag{A.13}$$

Although it is intuitively clear, we wish to clarify the notation adopted in (A.12) and (A.13) before presenting the proof of Proposition A.1. We have set,

$$F_l^{(k)}\Big|_{\varepsilon_{i,2}=0=\varepsilon_{i+1,1}} = \det(\overline{T}_{l,i}^{(k)}), \tag{A.14}$$

where,

$$\overline{T}_{l,i}^{(k)} = (\overline{t}_{j,s}^{\,i})_{j,s=k,\dots,l} \quad \text{with} \tag{A.15}$$
$$\overline{t}_{j,s}^{\,i} = t_{j,s} \text{ (defined in (A.2)) for} \quad j \notin \{i, i+1\}, \tag{A.16}$$
$$\overline{t}_{i,s}^{\,i} = (-1)^{\varepsilon_{i,1}} \alpha_{i,1} \delta_{i-1}^s + \delta_i^s + \alpha_{i,2} \delta_{i+1}^s, \tag{A.17}$$
$$\overline{t}_{i+1,s}^{\,i} = \alpha_{i+1,1} \delta_i^s + \delta_{i+1}^s + (-1)^{\varepsilon_{i+1,2}} \alpha_{i+1,2} \delta_{i+2}^s. \tag{A.18}$$

Proposition A.1. In order to establish (A.12), we proceed by induction on k. Indeed, if $k = i$ then

$$F_l^{(i)} = F_l^{(i+1)} - (-1)^{\varepsilon_{i,2}+\varepsilon_{i+1,1}} \alpha_{i,2}\alpha_{i+1,1} F_l^{(i+2)} = F_l^{(i+1)} - \alpha_{i,2}\alpha_{i+1,1} F_l^{(i+2)},$$

as follows by the assumption: $\varepsilon_{i,2} + \varepsilon_{i+1,1} = 0 \ (\text{mod}\, 2)$.

Since neither $F_l^{(i+1)}$ nor $F_l^{(i+2)}$ depends on the terms $\alpha_{i,2}$ and $\alpha_{i+1,1}$ (recall (A.4)), we see that,

$$F_l^{(i)} = F_l^{(i)}|_{\varepsilon_{i,2}=0=\varepsilon_{i+1,1}}.$$

Next, for $1 \le k < i$, suppose that

$$F_l^{(s)} = F_l^{(s)}|_{\varepsilon_{i,2}=0=\varepsilon_{i+1,1}}, \quad \forall s \in \{k+1, \dots, l\}.$$

To establish that the same identity also holds for $s = k$, we observe that,

$$F_l^{(k)} = F_l^{(k+1)} - (-1)^{\varepsilon_{k,2}+\varepsilon_{k+1,1}} \alpha_{k,2}\alpha_{k+1,1} F_l^{(k+2)}$$
$$= F_l^{(k+1)}\big|_{\varepsilon_{i,2}=0=\varepsilon_{i+1,1}} - \alpha_{k,2}\alpha_{k+1,1} F_l^{(k+2)}\big|_{\varepsilon_{i,2}=0=\varepsilon_{i+1,1}} = F_l^{(k)}\big|_{\varepsilon_{i,2}=0=\varepsilon_{i+1,1}},$$

where the last identity is a consequence of the definition in (A.14)–(A.18).

To prove (A.13), we use the derivation formulae (A.10) and (A.11) which under the given assumptions imply that $F_l^{(k)}$ is increasing separately with respect to $\alpha_{i,2}$ and $\alpha_{i+1,1}$. In other words, in case $\varepsilon_{i,2} = 0$ and $\varepsilon_{i+1} = 1$ we have that:

$$F_l^{(k)}((-1)^{\varepsilon_{k,2}}\alpha_{k,2}, \ldots, \alpha_{i,2}, -t, (-1)^{\varepsilon_{i+1,2}}\alpha_{i+1,2}, \ldots, (-1)^{\varepsilon_{l,1}}\alpha_{l,1})$$
$$\geq F_l^{(k)}((-1)^{\varepsilon_{k,2}}\alpha_{k,2}, \ldots, \alpha_{i,2}, t, (-1)^{\varepsilon_{i+1,2}}\alpha_{i+1,2}, \ldots, (-1)^{\varepsilon_{l,1}}\alpha_{l,1})$$

and (A.13) follows by taking $t = \alpha_{j+1,1}$. Similarly, if $\varepsilon_{i,2} = 1$ and $\varepsilon_{i+1} = 0$ then,

$$F_l^{(k)}((-1)^{\varepsilon_{k,2}}\alpha_{k,2}, \ldots, \alpha_{i,1}, -t, \alpha_{i+1,1}, \ldots, \alpha_{l,1})$$
$$\geq F_l^{(k)}((-1)^{\varepsilon_{k,2}}\alpha_{k,2}, \ldots, \alpha_{i,1}, t, \alpha_{i+1,1}, \ldots, \alpha_{l,1}), \quad \forall t \in \mathbb{R},$$

and in this case (A.13) follows by taking $t = \alpha_{i,2}$. □

The main purpose of this Appendix is to establish the following result:

Theorem A.1. *Let* $1 \leq k \leq l \leq N$ *and assume that* (A.8) *and* (A.9) *hold.*

For given $\tau_j \in [0, 1] \quad j = 1, \ldots, l$, *we suppose that,*

$$0 \leq \alpha_{j,2} < 1 - \tau_j \quad \text{and} \quad 0 \leq \alpha_{j+1,1} < \tau_{j+1} \quad j = k, \ldots, l-1, \tag{A.19}$$

then

$$\det T_l^{(k)} = F_l^{(k)} > 0. \tag{A.20}$$

The proof will be given in several steps. Firstly, we proceed to prove (A.20) in case:

$$\varepsilon_{j,2} = 0 = \varepsilon_{j+1,1} \quad \forall, j = k, \ldots, l-1. \tag{A.21}$$

Thus, we let $T_{l,0}^{(k)}$ be the matrix defined in (A.2)–(A.3) with $\varepsilon_{j,i}$ satisfying (A.21), namely

$$T_{l,0}^{(k)} = \begin{pmatrix} 1 & \alpha_{k,2} & 0 & 0 & \cdots & 0 \\ \alpha_{k+1,1} & 1 & \alpha_{k+1,2} & 0 & \cdots & 0 \\ 0 & \alpha_{k+2,1} & 1 & \alpha_{k+2,2} & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & \alpha_{l-1,1} & 1 & \alpha_{l-1,2} \\ 0 & \cdots & \cdots & 0 & \alpha_{l,1} & 1 \end{pmatrix}, \tag{A.22}$$

and set

$$F_{l,0}^{(k)} = \det T_{l,0}^{(k)}. \tag{A.23}$$

Furthermore, for $0 \leq \tau_j \leq 1$, and $j = k, \ldots, l$, we introduce the matrix:

$$\overline{T}_l^{(k)} = (\bar{t}_{js})_{j,s=k,\ldots,l} \tag{A.24}$$

with

$$\bar{t}_{js} = \tau_j \delta_{j-1}^s + \delta_j^s + (1 - \tau_j)\delta_{j+1}^s \tag{A.25}$$

and set,

$$\overline{F}_l^{(k)} = \det \overline{T}_l^{(k)} = 1 - \sum_{j=k}^{l} \tau_{j+1}(1-\tau_j)\overline{F}_l^{(j+2)}. \tag{A.26}$$

Concerning $\overline{F}_l^{(k)}$ we have the following:

Lemma A.3. *Let* $1 \le k \le l \le N$ *then*

$$\overline{F}_l^{(k)}(\tau_k, \dots, \tau_l) \ge 0, \quad \forall \tau_j \in [0,1],\ j = k, \dots, l. \tag{A.27}$$

Moreover,

$$\overline{F}_l^{(k)}(0, \tau_{k+1}, \dots, \tau_{l-1}, 1) = 0. \tag{A.28}$$

Proof. We can establish (A.28) simply by observing that $\overline{F}_l^{(k)}(0, \tau_{k+1}, \dots, \tau_{l-1}, 1)$ coincides with the determinant of the matrix,

$$\begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ \tau_{k+1} & 1 & 1-\tau_{k+1} & 0 & \cdots & 0 \\ 0 & \tau_{k+2} & 1 & 1-\tau_{k+2} & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & \tau_{l-1} & 1 & 1-\tau_{l-1} \\ 0 & \cdots & & 0 & 1 & 0 \end{pmatrix},$$

which is clearly singular. Indeed, the sum of the odd columns coincides with the sum of the even columns and it is given by the column with all entries equal to 1. Hence we have the linear dependence of the column-vectors and (A.28) follows.

To establish (A.27) we proceed by induction on $n \in \mathbb{N}$, with $0 \le l-k \le n$. Indeed, for $n = 1$ then either $k = l$ and $\overline{F}_l^{(l)} = 1$ or $k = l-1$ and

$$\overline{F}_l^{(l-1)}(\tau_{l-1}, \tau_l) = 1 - \tau_l(1-\tau_{l-1}) \ge 0.$$

Thus, we assume that $n > 1$ and suppose that,

$$\overline{F}_m^{(s)}(\tau_m, \dots, \tau_s) \ge 0 \quad \forall m, s \in \mathbb{N} : 0 \le m - s \le n-1. \tag{A.29}$$

Hence, for $l \ge 3$ and $l - k = n$, we need to prove that $\overline{F}_l^{(k)} \ge 0$. To this purpose we observe that,

$$\overline{F}_l^{(k)} = \overline{F}_l^{(k+1)} - \tau_{k+1}(1-\tau_k)\overline{F}_l^{(k+2)} = \overline{F}_{l-1}^{(k+1)} - \tau_l(1-\tau_{l-1})\overline{F}_{l-2}^{(k+1)} - \tau_{k+1}(1-\tau_k)\overline{F}_l^{(k+2)}.$$

According to our induction assumption (A.29), we see that $\overline{F}_{l-2}^{(k+1)} \ge 0$ and $\overline{F}_l^{(k+2)} \ge 0$ and therefore,

$$\overline{F}_l^{(k)} \ge \overline{F}_{l-1}^{(k+1)} - \tau_{k+1}\overline{F}_l^{(k+2)} - (1-\tau_{l-1})\overline{F}_{l-2}^{(k+1)} = \overline{F}_l^{(k)}(0, \tau_{k+1}, \dots, \tau_{l-1}, 1) = 0$$

and (A.27) is established. □

Lemma A.4. *Let* $1 \le k < l \le N$ *and assume* (A.19). *Then*

$$F_l^{(k)} \ge F_{l,0}^{(k)} > \overline{F}_l^{(k)} \ge 0. \tag{A.30}$$

Proof. Again we proceed by induction on $n \in \mathbb{N}$, such that $1 \le l - k \le n$. Hence for $n = 1$, by direct inspection we easily check that (A.30) holds for $k = l-1$. Hence we let $n > 1$, and by induction we assume that,

$$F_m^{(s)} \ge F_{m,0}^{(s)} > \overline{F}_m^{(s)} \ge 0 \quad \forall m, s \in \mathbb{N} : 1 \le m - s \le n-1, \tag{A.31}$$

and we are left to show that (A.30) holds also when $l \ge 3$ and

$$l - k = n. \tag{A.32}$$

From (A.31) and (A.32) we see that,

$$F_l^{(j)} > 0 \quad F_{j-1}^{(k)} > 0, \quad \forall k+1 \le j \le l, \tag{A.33}$$

so we can use Proposition A.1 to conclude that $F_l^{(k)} \ge F_{l,0}^{(k)}$.

To establish the second (strict) inequality in (A.30), we use the derivation formulae (A.10)–(A.11). In view of (A.33) we find:

$$\frac{\partial F_{l,0}^{(k)}}{\partial \alpha_{k,2}} = -\alpha_{k+1,1} F_l^{(k+2)} F_{k-1}^{(k)} = -\alpha_{k+1,1} F_l^{(k+2)} < 0$$

so that,

$$F_{l,0}^{(k)}(\alpha_{k,2}, \alpha_{k+1,1}, \dots, \alpha_{l,1}) > F_{l,0}^{(k)}(1-\tau_k, \alpha_{k+1,1}, \dots, \alpha_{l,1}).$$

Furthermore,

$$\frac{\partial}{\partial \alpha_{k+1,1}} \Big(F_{l,0}^{(k)}(1-\tau_k, \alpha_{k+1,1}, \dots, \alpha_{l,1}) \Big) = -(1-\tau_k) F_l^{(k+2)} < 0,$$

which implies,

$$F_{l,0}^{(k)}(1-\tau_k, \alpha_{k+1,1}, \alpha_{k+1,2} \dots, \alpha_{l,1}) > F_{l,0}^{(k)}(1-\tau_k, \tau_{k+1}, \alpha_{k+1,2} \dots, \alpha_{l,1}).$$

Thus, by observing that, for $k < j \le l$ we have:

$$\begin{aligned}
&\frac{\partial}{\partial \alpha_{j,2}} \Big(F_{l,0}^{(k)}(1-\tau_k, \tau_{k+1}, 1-\tau_{k+1}, \dots, \tau_j, \alpha_{j,2}, \dots, \alpha_{l,1}) \Big) \\
&= \frac{\partial F_{l,0}^{(k)}}{\partial \alpha_{j,2}} \big(1-\tau_k, \tau_{k+1}, 1-\tau_{k+1}, \dots, \tau_j, \alpha_{j,2}, \dots, \alpha_{l,1}\big) \\
&= -\alpha_{j+1,1} F_l^{(j+2)} \overline{F}_{j-1}^{(k)} < 0
\end{aligned}$$

then we can proceed inductively as above, to conclude that,

$$\begin{aligned}
&F_{l,0}^{(k)}(\alpha_{k,2}, \alpha_{k+1,1}, \alpha_{k+1,2}, \dots, \alpha_{l,1}) > F_{l,0}^{(k)}(1-\tau_k, \alpha_{k+1,1}, \alpha_{k+1,2}, \dots, \alpha_{l,1}) \\
&> F_{l,0}^{(k)}(1-\tau_k, \tau_{k+1}, \alpha_{k+1,2}, \dots, \alpha_{l,1}) > F_{l,0}^{(k)}(1-\tau_k, \tau_{k+1}, 1-\tau_{k+1}, \dots, \alpha_{l,1}) \\
&> \dots > F_l^{(k)}(1-\tau_k, \tau_{k+1}, 1-\tau_{k+1}, \dots, \tau_{l-1}, 1-\tau_{l-1}, \tau_l) = \overline{F}_l^{(k)}(\tau_k, \dots, \tau_l) \ge 0
\end{aligned}$$

and (A.30) is established. □

Proof of Theorem A.1. The property (A.20) is a direct consequence of Lemma A.4. □

Remark A.3. By the tri-diagonal structure of the matrix $T_N^{(1)}$, Lemma A.4 and Sylvester's theorem, we can conclude that $T_N^{(1)}$ is actually positive definite, when (A.8), (A.9) and (A.10) hold.

References

[1] A. Ambrosetti, P. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14 (1973) 349–381.
[2] W. Ao, C.-S. Lin, J. Wei, On non-topological solutions of the A_2 and B_2 Chern–Simons system, Mem. Am. Math. Soc. 239 (2016) 1132.
[3] W. Ao, C.-S. Lin, J.C. Wei, On non-topological solutions of the G_2 Chern–Simons system, Commun. Anal. Geom. 24 (2016) 717–752.
[4] M.F. Atiyah, Topological quantum field theory, Publ. Math. Inst. Hautes Etudes Sci. Paris 68 (1989) 175–186.
[5] T. Aubin, Nonlinear Analysis on Manifolds: Monge–Ampére Equations, Springer, Berlin, New York, 1982.
[6] D. Bazeia, E. da Hora, C. dos Santos, R. Menezes, Generalized self-dual Chern–Simons vortices, Phys. Rev. D 81 (2010) 125014.
[7] A. Bezryadina, E. Eugenieva, Z. Chen, Self-trapping and flipping of double-charged vortices in optically induced photonic lattices, Opt. Lett. 31 (2006) 2456–2458.

[8] E.B. Bogomol'nyi, The stability of classical solutions, Sov. J. Nucl. Phys. 24 (1976) 449–454.
[9] J. Burzlaff, A. Chakrabarti, D.H. Tchrakian, Generalized self-dual Chern–Simons vortices, Phys. Lett. B 293 (1992) 127–131.
[10] L.A. Caffarelli, Y. Yang, Vortex condensation in the Chern–Simons Higgs model: an existence theorem, Commun. Math. Phys. 168 (1995) 321–336.
[11] D. Chae, O.Y. Imanuvilov, The existence of non-topological multivortex solutions in the relativistic self-dual Chern–Simons theory, Commun. Math. Phys. 215 (2000) 119–142.
[12] S.S. Chern, J. Simons, Some cohomology classes in principal fiber bundles and their application to Riemannian geometry, Proc. Natl. Acad. Sci. USA 68 (1971) 791–794.
[13] S.S. Chern, J. Simons, Characteristic forms and geometric invariants, Ann. Math. 99 (1974) 48–69.
[14] K. Choe, N. Kim, C.-S. Lin, Self-dual symmetric non-topological solutions in the $SU(3)$ model in $\mathbb{R}^2$, Commun. Math. Phys. 334 (2015) 1–37.
[15] K. Choe, N. Kim, C.-S. Lin, Existence of mixed type solutions in the SU(3) Chern–Simons theory in $\mathbb{R}^2$, Calc. Var. Partial Differ. Equ. 56 (2017).
[16] K. Choe, N. Kim, Y. Lee, C.-S. Lin, Existence of mixed type solutions in the Chern–Simons gauge theory of rank two in $\mathbb{R}^2$, J. Funct. Anal. 273 (2017) 1734–1761.
[17] K. Choe, N. Kim, Y. Lee, C.-S. Lin, New type of non-topological bubbling solutions in the $SU(3)$ Chern–Simons model in $\mathbb{R}^2$, J. Funct. Anal. 270 (2016) 1–33.
[18] L.F. Cugliandolo, G. Lozano, M.V. Manias, F.A. Schaposnik, Bogomolny equations for non-Abelian Chern–Simons Higgs theories, Mod. Phys. Lett. A 6 (1991) 479.
[19] K. Dasgupta, V. Errasti Díez, P. Ramadevi, R. Tatar, Knot invariants and M-theory: Hitchin equations, Chern–Simons actions, and surface operators, Phys. Rev. D 95 (2017) 026010.
[20] G. Dunne, Self-Dual Chern–Simons Theories, Lecture Notes in Physics, vol. 36, Springer, Berlin, 1995.
[21] G. Dunne, Mass degeneracies in self-dual models, Phys. Lett. B 345 (1995) 452–457.
[22] G. Dunne, R. Jackiw, S.-Y. Pi, C. Trugenberger, Self-dual Chern–Simons solitons and two-dimensional nonlinear equations, Phys. Rev. D 43 (1991) 1332–1345.
[23] P.K. Ghosh, Bogomol'nyi equations of Maxwell–Chern–Simons vortices from a generalized Abelian Higgs model, Phys. Rev. D 49 (1993) 5458–5468.
[24] N. Ghoussoub, Duality and Perturbation Methods in Critical Point Theory, Cambridge University Press, Cambridge, 1993.
[25] X. Han, C.-S. Lin, Y. Yang, Resolution of Chern–Simons–Higgs vortex equations, Commun. Math. Phys. 343 (2016) 701–724.
[26] X. Han, G. Tarantello, Doubly periodic self-dual vortices in a relativistic non-Abelian Chern–Simons model, Calc. Var. Partial Differ. Equ. 49 (2014) 1149–1176.
[27] X. Han, Y. Yang, Relativistic Chern–Simons–Higgs vortex equations, Trans. Am. Math. Soc. 368 (2016) 3565–3590.
[28] J. Hong, Y. Kim, P.-Y. Pac, Multivortex solutions of the Abelian Chern–Simons–Higgs theory, Phys. Rev. Lett. 64 (1990) 2330–2333.
[29] P.A. Horvathy, P. Zhang, Vortices in (Abelian) Chern–Simons gauge theory, Phys. Rep. 481 (2009) 83–142.
[30] G. Huang, C.-S. Lin, The existence of non-topological solutions for a skew-symmetric Chern–Simons system, Indiana Univ. Math. J. 65 (2016) 453–491.
[31] H.-Y. Huang, Y. Lee, C.-S. Lin, Uniqueness of topological multi-vortex solutions for a skew-symmetric Chern–Simons system, J. Math. Phys. 56 (2015) 041501.
[32] S. Inouye, S. Gupta, T. Rosenband, A.P. Chikkatur, A. Görlitz, T.L. Gustavson, A.E. Leanhardt, D.E. Pritchard, W. Ketterle, Observation of vortex phase singularities in Bose–Einstein condensates, Phys. Rev. Lett. 87 (2001) 080402.
[33] R. Jackiw, K.M. Lee, E.J. Weinberg, Self-dual Chern–Simons solitons, Phys. Rev. D 42 (1990) 3488–3499.
[34] R. Jackiw, E.J. Weinberg, Self-dual Chern–Simons vortices, Phys. Rev. Lett. 64 (1990) 2334–2337.
[35] H.-C. Kao, K. Lee, Self-dual $SU(3)$ Chern–Simons Higgs systems, Phys. Rev. D 50 (1994) 6626–6632.
[36] A. Khare, Chern–Simons term and charged vortices in Abelian and non-Abelian gauge theories, Proc. Indian Natl. Sci. Acad., A, Phys. Sci. 61 (1995) 161–178.
[37] D.I. Khomskii, A. Freimuth, Charged vortices in high temperature superconductors, Phys. Rev. Lett. 75 (1995) 1384–1386.
[38] C. Kim, Self-dual vortices in the generalized Abelian Higgs model with independent Chern–Simons interaction, Phys. Rev. D 47 (1993) 673–684.
[39] C.N. Kumar, A. Khare, Charged vortex of finite energy in nonabelian gauge theories with Chern–Simons term, Phys. Lett. B 178 (1986) 95–399.
[40] K.M. Lee, Relativistic non-Abelian self-dual Chern–Simons systems, Phys. Lett. B 255 (1991) 381.
[41] C.-S. Lin, S. Yan, Bubbling solutions for the $SU(3)$ Chern–Simons model on a torus, Commun. Pure Appl. Math. 66 (2013) 991–1027.
[42] M. Marino, Chern–Simons theory and topological strings, Rev. Mod. Phys. 77 (2005) 675–720.
[43] Y. Matsuda, K. Nozakib, K. Kumagaib, Charged vortices in high temperature superconductors probed by nuclear magnetic resonance, J. Phys. Chem. Solids 63 (2002) 1061–1063.
[44] C. Nayak, S.H. Simon, A. Stern, M. Freedman, S.D. Sarma, Non-Abelian anyons and topological quantum computation, Rev. Mod. Phys. 80 (2008) 1083–1159.
[45] M. Nolasco, G. Tarantello, Double vortex condensates in the Chern–Simons–Higgs theory, Calc. Var. Partial Differ. Equ. 9 (1999) 31–94.
[46] M. Nolasco, G. Tarantello, Vortex condensates for the SU(3) Chern–Simons theory, Commun. Math. Phys. 213 (2000) 599–639.
[47] P. Oh, Classical and quantum mechanics of non-Abelian Chern–Simons particles, Nucl. Phys. B 462 (1996) 551–570.
[48] M.K. Prasad, C.M. Sommerfield, Exact classical solutions for the 't Hooft monopole and the Julia–Zee dyon, Phys. Rev. Lett. 35 (1975) 760–762.

[49] X.-L. Qi, S.-C. Zhang, Topological insulators and superconductors, Rev. Mod. Phys. 83 (2011) 1057.
[50] G.W. Rayfield, F. Reif, Evidence for the creation and motion of quantized vortex rings in superfluid helium, Phys. Rev. Lett. 11 (1963) 305.
[51] C.G. Simader, On Dirichlet's Boundary Value Problem, Lecture Notes Math., vol. 268, Springer, Berlin, Heidelberg, New York, 1972.
[52] J. Spruck, Y. Yang, Topological solutions in the self-dual Chern–Simons theory: existence and approximation, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 12 (1995) 75–97.
[53] J. Spruck, Y. Yang, The existence of non-topological solitons in the self-dual Chern–Simons theory, Commun. Math. Phys. 149 (1992) 361–376.
[54] G. Tarantello, Multiple condensate solutions for the Chern–Simons–Higgs theory, J. Math. Phys. 37 (1996) 3769–3796.
[55] G. Tarantello, Self-Dual Gauge Field Vortices, an Analytic Approach, Progress in Nonlinear Differential Equations and Their Applications, vol. 72, Birkhäuser, Boston, Basel, Berlin, 2008.
[56] M. Tinkham, Introduction to Superconductivity, McGraw-Hill, New York, 1996.
[57] R. Wang, The existence of Chern–Simons vortices, Commun. Math. Phys. 137 (1991) 587–597.
[58] F. Wilczek, Fractional Statistics and Anyon Superconductors, World Scientific, Singapore, 1990.
[59] E. Witten, Quantum field theory and the Jones polynomial, Commun. Math. Phys. 121 (1989) 351–399.
[60] E. Witten, Chern–Simons theory as a string theory, Prog. Math. 133 (1995) 637–678.
[61] Y. Yang, The relativistic non-Abelian Chern–Simons equations, Commun. Math. Phys. 186 (1997) 199–218.
[62] Y. Yang, Solitons in Field Theory and Nonlinear Analysis, Springer, New York, 2001.