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On the global Cauchy problem for the Hartree equation with rapidly decaying initial data

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Abstract

This paper is concerned with the Cauchy problem for the Hartree equation on $\mathbb{R}^n$, $n \in \mathbb{N}$ with the nonlinearity of type $(|\cdot|^{-\gamma} * |u|^2)u$, $0 < \gamma < n$. It is shown that a global solution with some twisted persistence property exists for data in the space $L^p \cap L^2$, $1 \le p \le 2$ under some suitable conditions on γ and spatial dimension $n \in \mathbb{N}$. It is also shown that the global solution u has a smoothing effect in terms of spatial integrability in the sense that the map $t \mapsto u(t)$ is well defined and continuous from $\mathbb{R} \setminus \{0\}$ to $L^{p'}$, which is well known for the solution to the corresponding linear Schrödinger equation. Local and global well-posedness results for hat L^p-spaces are also presented. The local and global results are proved by combining arguments by Carles–Mouzaoui with a new functional framework introduced by Zhou. Furthermore, it is also shown that the global results can be improved via generalized dispersive estimates in the case of one space dimension.

MSC: 35Q55

Keywords: Nonlinear Schrödinger equations; Hartree equation; Cauchy problem; Global well-posedness; L^p-Cauchy data; Rapidly decaying data

1. Introduction

In this paper, we consider the Cauchy problem for the Hartree equation

$$iu_t + \Delta u + (|\cdot|^{-\gamma} * |u|^2)u = 0, \qquad u(0,x) = \phi(x), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n \tag{1.1}$$

where $0 < \gamma < n$. When $\phi \in L^2$, the Cauchy problem is known to be globally well posed if $\gamma < \min(n, 2)$. More precisely, we have:

Theorem A. *(See e.g. [2, Proposition 2.3]) Assume that* $0 < \gamma < \min(2, n)$. *Then for any* $\phi \in L^2$ *there exists a unique global solution to (1.1) such that*

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$$u \in C(\mathbb{R} : L^2(\mathbb{R}^n)) \cap L^{\frac{8}{\gamma}}_{loc}(\mathbb{R} : L^{\frac{4n}{2n-\gamma}}(\mathbb{R}^n)).$$

Moreover, the solution u has the L^2-conservation

$$\|u(t,\cdot)\|_{L^2_x} = \|\phi\|_{L^2_x}, \quad \forall t \in \mathbb{R}$$

and

$$u \in L^q_{loc}(\mathbb{R} : L^r(\mathbb{R}^n))$$

for any Schrödinger admissible pair (q, r).

On the other hand, when Cauchy data ϕ are not in L^2, much less is known about the solvability of (1.1) and the Cauchy problem for other nonlinear dispersive equations. For examples, we refer to the works [5], [9], where the authors study well-posedness for the nonlinear Schrödinger equations in spaces that are not included in L^2. In particular, it is believed that (1.1) is ill posed in L^p if $p \neq 2$. Indeed, in general, the solution $u(\triangleq U(t)\phi)$ of the linear Schrödinger equation

$$iu_t + \Delta u = 0, \quad u(0) = \phi \in L^p \tag{1.2}$$

does not belong to L^p if $p \neq 2$. This implies that one may face great deal of difficulty in proving persistence property (i.e. the solution is a curve on the data space) of the solution when trying to solve nonlinear Schrödinger equations for data in L^p, $p \neq 2$. Note also that the persistency does not hold even if we assume $\phi \in L^p \cap L^2$:

Lemma B. *Let $1 \le p \le 2$ and $t_0 \neq 0$. Then $U(t_0)\phi_0 \notin L^p$ for any $\phi_0 \in (L^p \cap L^2) \setminus L^{p'}$.*

The lemma can easily be checked. We recall the well-known decay estimate for the free propagator

$$\|U(t)\phi\|_{L^{p'}} \le (4\pi|t|)^{-n(1/p-1/2)}\|\phi\|_{L^p}, \ t \neq 0, \quad 1 \le p \le 2. \tag{1.3}$$

In particular, $U(t)\phi \in L^{p'}$, $\forall t \neq 0$. If $U(t_0)\phi \in L^p$, then by (1.3) we have $\phi_0 = U(-t_0)(U(t_0)\phi) \in L^{p'}$, which is a contradiction. Nonetheless, in [18], Zhou studied the Cauchy problem for the one dimensional cubic NLS

$$iu_t + u_{xx} + |u|^2 u = 0 \tag{1.4}$$

and proved the existence of a local solution $u : [0, T] \times \mathbb{R} \to \mathbb{C}$, $T > 0$ to (1.4) such that

$$U(-t)u(t) \in C([0, T] : L^p) \tag{1.5}$$

for data in L^p, $1 < p < 2$. His results suggest that one may expect the existence of a solution with a property similar to (1.5) for other nonlinear dispersive equations instead of the usual persistence property. In particular, in [14] we proved the existence of a local solution u to (1.1) such that (1.5) holds for data in L^p, with

$$\max\left(\frac{2n}{n-\gamma+2}, \frac{2n}{n+\gamma}\right) < p < 2. \tag{1.6}$$

Once local solutions are established, one may further ask whether or not the solution extends to a global one such that $U(-t)u(t) \in L^p$ for any $t \in \mathbb{R}$. However, due to the lack of any conservation laws below L^2, constructing such global solutions is quite difficult and there have been no global existence results on nonlinear Schrödinger equations for data in the mere L^p-spaces. The aim of the present paper is to prove the existence of a global solution u to (1.1) for $\phi \in L^p$ such that

$$U(-t)u(t) \in C(\mathbb{R} : L^p) \tag{1.7}$$

under the additional assumption of $\phi \in L^2$. We also emphasize that the lower threshold in (1.6) can be pushed down to $p = 1$ if we assume $\phi \in L^2$. As mentioned below, it is interesting to pursue the existence of such global solutions, especially from the viewpoint of Fourier analysis. Indeed, it is known that the linear operator $U(t)$ has properties similar to the ones of the Fourier transform if $t \neq 0$. For example, (1.3) is very comparable with the Hausdorff–Young inequality,

$$\|\hat{f}\|_{L^{p'}} \le C\|f\|_{L^p}, \tag{1.8}$$

which is true if $1 \le p \le 2$. So one may wonder if similar, Fourier transform-like properties hold in the nonlinear setting. In fact, such a result is known. One typical example is the fact that the faster the data decays, the smoother the solution gets for some nonlinear Schrödinger equations including (1.1). In particular, if data decay exponentially, the solutions are real analytic. This is known as the analytic smoothing effect for the nonlinear Schrödinger equations. See [12] [13] and references therein for details. These properties are parallel to the relation between the rate of decay of a function f and smoothness of its Fourier transform $\hat{f}$. Here in this paper we will see that the global solutions to (1.1) satisfy

$$u(t) \in C(\mathbb{R} \setminus \{0\} : L^{p'}(\mathbb{R}^n)) \tag{1.9}$$

which can be viewed as a nonlinear analogue of the Hausdorff–Young like property (1.3) for the linear case. Note that this can also be regarded as a smoothing effect in terms of spatial integrability. Note also that it is very interesting that (1.9) follows immediately from the twisted persistence property (1.7).

Finally, another interesting aspect is that analysis of nonlinear Schrödinger equations in L^p space has close correlation with that in the weighed L^2-spaces $H^{0,\alpha}$ which is defined by

$$H^{0,\alpha} \triangleq \{\phi \mid (1+|x|^2)^{\frac{\alpha}{2}}\phi(x) \in L^2\}.$$

Indeed, in the setting of the weighted L^2-spaces, the solution $u : [0, T] \times \mathbb{R}^n \to \mathbb{C}$ does not have the usual persistence property but has a twisted one as observed in the L^p-setting. For example, it is known that there exists a unique global solution u to (1.1) such that

$$U(-t)u(t) \in H^{0,\alpha}(\mathbb{R}^n)$$

for any data $\phi \in H^{0,\alpha}$ if $\alpha = 1, 2$ (see [10], [11]). Moreover, one has the inclusion relation

$$H^{0,\alpha} \hookrightarrow L^p \cap L^2$$

for suitable choices of $\alpha > 0$ and $p < 2$. There are a lot of earlier works on existence, asymptotic behavior, decay estimates, etc, of the solutions to nonlinear dispersive equations for data $\phi \in H^{0,\alpha}$ and it is of interest to study various similar problems in the framework of L^p, $p < 2$, which is larger than weighted L^2-spaces. Here in this paper, as a first step to the study in this direction, we focus on establishing global solutions to (1.1), since there are very few earlier studies on the nonlinear dispersive equation for L^p-data and we first need to assure the existence of solutions.

Before stating our main results we summarize the notation used in this paper below.

Notations.

(i) Denote a' by the conjugate of $a \in [1, \infty]$: $1/a + 1/a' = 1$.

(ii) The spatial Fourier transform of f is denoted by $\hat{f}, \mathscr{F} f$:

$$\hat{f}(\xi) \triangleq \mathscr{F} f(\xi) \triangleq \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)dx.$$

Similarly, the inverse Fourier transform is denoted by $\mathscr{F}^{-1}$:

$$\mathscr{F}^{-1} f(x) \triangleq (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)d\xi.$$

(iii) Convolution with respect to the space variable is denoted by $*$. For instance

$$(\phi * \varphi)(x) = \int_{\mathbb{R}^n} \phi(x-y)\varphi(y)dy, \qquad (f * g)(t, x) = \int_{\mathbb{R}^n} f(t, x-y)g(t, y)dy.$$

(iv) Let A, B be two Banach spaces of functions in $\mathscr{S}'(\mathbb{R}^n)$. It is known (see e.g. [1]) that $A \cap B$ is a Banach space equipped with the norm $\|\cdot\|_{A\cap B} \triangleq \max(\|\cdot\|_A, \|\cdot\|_B)$.

(v) Let $q, r \in (0, \infty]$. The space $L^q(I : L^r)$ is defined by

$$\|u\|_{L^q(I:L^r)} \triangleq \left(\int_I \|u(t, \cdot)\|_{L^r}^q dt \right)^{\frac{1}{q}},$$

with the usual modification if $q = \infty$. When $I = \mathbb{R}$ we also use $L^q(\mathbb{R} : L^r) = L^q(L^r)$ and when $I = [0, T]$, $T > 0$ we write $L^q([0, T] : L^r) = L^q_T(L^r)$.

(vi) As mentioned above, $U(t)\phi$ denotes the solution of the linear Schrödinger equation (1.2) for data ϕ.

(vii) A pair of exponents (q, r) is Schrödinger admissible if

$$2 \le r \begin{cases} \le \infty & \text{if } n = 1 \\ < \infty & \text{if } n = 2 \\ \le \frac{2n}{n-2} & \text{if } n \ge 3 \end{cases}$$

and

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2}.$$

It is well known that the Strichartz estimate

$$\|U(t)\phi\|_{L^q(L^r)} \le C\|\phi\|_{L^2}$$

holds for any admissible pair (q, r).

(viii) C, c are positive constants which may vary from line to line. We also use the symbols $C_{A,B,C,\cdots}, c_{A,B,C,\cdots}$ to denote constants that may depend on the parameters $A, B, C, \cdots$.

Our main results are as follows:

Theorem 1.1. *Assume that* $0 < \gamma < 1$. *Then for any* $\phi \in L^1 \cap L^2$, *there exist* $T > 0$ *and a unique local solution* u *to (1.1) such that*

$$U(-t)u(t) \in C([0, T] : L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)). \tag{1.10}$$

Moreover, the continuous dependence on data holds in the sense that the map $\phi \mapsto U(-t)u(t)$ *is locally Lipschitz from* $L^1 \cap L^2$ *to* $C([0, T] : L^1 \cap L^2)$.

In particular, it follows that $u(t) \in C([0, T] : L^2)$ *and that the map* $\phi \mapsto u(t)$ *is locally Lipschitz from* $L^1 \cap L^2$ *to* $C([0, T] : L^2)$.

The local solution can be extended globally, under an additional assumption on γ:

Theorem 1.2. *Assume that* $0 < \gamma < \min(1, n/2)$. *Then the local solution to (1.1) given by Theorem 1.1 extends to a global one such that*

$$U(-t)u(t) \in C(\mathbb{R} : L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)).$$

Moreover, it follows that $u(t) \in C(\mathbb{R} : L^2)$ *and that the global solution enjoys the following smoothing effect in terms of spatial integrability:*

$$u|_{(\mathbb{R}\setminus\{0\})\times\mathbb{R}^n} \in C(\mathbb{R} \setminus \{0\} : L^\infty(\mathbb{R}^n)). \tag{1.11}$$

The proof of Theorem 1.1 relies on the fact that the Hartree-type nonlinearity becomes a closed operation on $L^1 \cap L^2$ for $t \ne 0$ after a certain linear transformation, and that the singular factor $t^{-\gamma}$ appearing in the nonlinearity of the transformed equation is locally integrable when $\gamma < 1$. For that reason, the argument in the proof of Theorem 1.1 cannot be applied to the cases of intermediate spaces $\phi \in L^p \cap L^2$, $p > 1$ or $\gamma > 1$. This difficulty can be overcome by solving the equation in smaller function spaces $Y^p_{q,\theta}$, whose definition is given at the end of this section.

Theorem 1.3. *Let $1 < p \le 2$. Assume that $0 < \gamma < \min(p, n)$.*
Then for any $\phi \in L^p \cap L^2$, there exist $T > 0$ and a unique local solution $u \in Y^2_{\frac{2}{\gamma},0}(T) \cap Y^p_{q,\theta}(T)$ to (1.1) such that

$$U(-t)u(t) \in C([0,T] : L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)), \tag{1.12}$$

where $\theta = \gamma(2/p - 1)$ and $q = \gamma p'$. Moreover, the continuous dependence on data holds in the sense that the map $\phi \mapsto U(-t)u(t)$ is locally Lipschitz from $L^p \cap L^2$ to $C([0,T] : L^p \cap L^2)$.

In particular, it follows that $u(t) \in C([0,T] : L^2)$ and that the map $\phi \mapsto u(t)$ is locally Lipschitz from $L^p \cap L^2$ to $C([0,T] : L^2)$.

As in Theorem 1.2 the local solution can be extended globally:

Theorem 1.4. *Assume that $0 < \gamma < \min(1, n/2)$. Then the local solution to (1.1) given by Theorem 1.3 extends to a global one such that*

$$U(-t)u(t) \in C(\mathbb{R} : L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)).$$

Moreover, it follows that $u(t) \in C(\mathbb{R} : L^2)$ and that the global solution enjoys the following smoothing effect in terms of spatial integrability:

$$u|_{(\mathbb{R}\setminus\{0\})\times\mathbb{R}^n} \in C(\mathbb{R} \setminus \{0\} : L^{p'}(\mathbb{R}^n)). \tag{1.13}$$

An improvement. In view of Theorem A and the above local results, the extra assumption $\gamma < n/2$ in Theorem 1.2 and 1.4 is expected to be removed or relaxed. Indeed, in the present paper, we demonstrate that the global existence holds without this condition if $n = 1$ and $4/3 < p \le 2$. This is achieved by establishing an L^p-estimate of the Duhamel term by means of generalized Strichartz type estimates in place of the direct L^p-estimate of the Hartree nonlinearity.

Theorem 1.5. *Assume that $n = 1$, $0 < \gamma < 1$, and $4/3 < p \le 2$. Then the local solution to (1.1) given by Theorem 1.3 extends to a global one such that*

$$U(-t)u(t) \in C(\mathbb{R} : L^p(\mathbb{R}) \cap L^2(\mathbb{R})).$$

Moreover, it follows that $u(t) \in C(\mathbb{R} : L^2(\mathbb{R}))$ and that the global solution enjoys the following smoothing effect in terms of spatial integrability:

$$u|_{(\mathbb{R}\setminus\{0\})\times\mathbb{R}} \in C(\mathbb{R} \setminus \{0\} : L^{p'}(\mathbb{R})). \tag{1.14}$$

Remark 1.6. As mentioned earlier, local results with the property (1.5) for the Hartree equation in the mere L^p-space have been obtained if

$$\max\left(\frac{2n}{n-\gamma+2}, \frac{2n}{n+\gamma}\right) < p \le 2.$$

The first exponent in the lower threshold $p > \frac{2n}{n-\gamma+2}$ is the well-known scaling limit. The second condition $p > \frac{2n}{n+\gamma}$ looks unfamiliar, but perhaps this is also essential, which may stem from the singularity at zero frequency in the Hartree nonlinearity $(|x|^{-\gamma} * |u|^2)u = c[D_x^{-(n-\gamma)}|u|^2]u$. For details, see [2], [16].

Function spaces and embeddings. We give the definition of function space $Y^p_{q,\theta}(T)$ appearing in the statement of Theorem 1.3–1.4 along with related spaces. Such spaces were firstly introduced by Zhou in [18] to obtain local existence results for the 1D cubic NLS for Cauchy data in L^p. Let $T > 0$ and let $1 \le p, q \le \infty$ and $\theta > 0$. We first introduce the space $\tilde{X}^p_{q,\theta}(T)$ and $\tilde{Y}^p_{q,\theta}(T)$ by

$$\tilde{X}^p_{q,\theta}(T) \triangleq \{v : [0,T] \times \mathbb{R}^n \to \mathbb{C} \mid \|v\|_{\tilde{X}^p_{q,\theta}(T)} < \infty\},$$

where

$$\|v\|_{\tilde{X}^p_{q,\theta}(T)} \triangleq \left(\int_0^T s^{q\theta}\|(\partial_s v)(s,\cdot)\|^q_{L^p}ds\right)^{\frac{1}{q}},$$

and

$$\tilde{Y}^p_{q,\theta}(T) \triangleq \{v \in \tilde{X}^p_{q,\theta}(T) \mid v(0) \in L^p\},$$

endowed with the norm

$$\|v\|_{\tilde{Y}^p_{q,\theta}(T)} \triangleq \|v(0)\|_{L^p} + \|v\|_{\tilde{X}^p_{q,\theta}(T)}.$$

Given these spaces, we now define the space $Y^p_{q,\theta}(T)$ by

$$Y^p_{q,\theta}(T) \triangleq \{u : [0,T] \times \mathbb{R}^n \to \mathbb{C} \mid U(-t)u(t) \in \tilde{Y}^p_{q,\theta}(T)\}.$$

The following embedding results are elementary but important.

Lemma 1.7. *(See e.g. [15, Lemma 2.1])*

(i) Let $T > 0$. Assume that $\theta q' < 1$. Then the embedding

$$\tilde{Y}^p_{q,\theta}(T) \hookrightarrow C([0,T] : L^p)$$

holds. As an immediate consequence, any function u in $Y^p_{q,\theta}(T)$ satisfies

$$U(-t)u(t) \in C([0,T] : L^p).$$

(ii) Let $T > 0$ and let q, r be such that the estimate

$$\|U(t)\phi\|_{L^q_T(L^r)} \le C_T\|\phi\|_{L^p}$$

holds true for all $\phi \in L^p$. Then

$$Y^p_{q,\theta}(T) \hookrightarrow L^q_T(L^r). \tag{1.15}$$

In particular, for all fixed $T > 0$, $\theta > 0$, $1 \le q \le \infty$ the embeddings

$$Y^p_{q,\theta}(T) \subset Y^p_{1,0}(T), \quad Y^p_{1,0}(T) \subset L^q_T(L^r)$$

hold true.

Local and global existence result in hat L^p-spaces. While there are very few well-posedness results in the mere L^p-spaces, several attempts have been made to establish a local solution of nonlinear dispersive equations in alternative data spaces which scale like L^p. One typical example of such spaces is the hat L^p-space $\widehat{L}^p$ defined by

$$\widehat{L}^p(\mathbb{R}^n) \triangleq \{f \in \mathscr{S}'(\mathbb{R}^n) \mid \mathscr{F}f \in L^{p'}(\mathbb{R}^n)\}$$

equipped with the norm

$$\|f\|_{\widehat{L}^p} \triangleq \|\mathscr{F}f\|_{L^{p'}}.$$

Note that by the Hausdorff–Young inequality the inclusions

$$L^p \subset \widehat{L}^p \quad \text{if } p \le 2, \qquad \widehat{L}^p \subset L^p \quad \text{if } p \ge 2$$

hold. Moreover, the space $\widehat{L}^p$ has the unitarity property

$$\|U(-t)\phi\|_{\widehat{L}^p} = \|\phi\|_{\widehat{L}^p},\ \forall t \in \mathbb{R}. \tag{1.16}$$

Therefore, when studying nonlinear Schrödinger equation in the L^p-based framework, $\widehat{L}^p$ is a good substitute for L^p. In fact, several results on the solvability of various nonlinear dispersive equations in hat L^p-spaces have been reported (see e.g. [4], [9], [8]).

By means of the functional framework similar to $Y^p_{q,\theta}$-type spaces above, we may show local and global existence results for (1.1) in $L^2 \cap \widehat{L}^p$ with $p \ge 2$.

Theorem 1.8. *Let $0 < \gamma < 2$ and $2 \le p \le \infty$. Then (1.1) is locally well posed in $L^2 \cap \widehat{L}^p$ in the usual sense, that is: for any $\phi \in L^2 \cap \widehat{L}^p$ there are $T > 0$ and a unique local solution u to (1.1) such that*

$$u \in Z^2_{\frac{2}{\gamma}}(T) \cap Z^p_q(T) \hookrightarrow C([0,T] : L^2 \cap \widehat{L}^p),$$

where

$$q = \frac{p}{\gamma(p-1)}$$

and the definition of $Z^2_{\frac{2}{\gamma}}$, Z^p_q is given in section 6. Moreover, the map $\phi \mapsto u(t)$ is locally Lipschitz from $L^2 \cap \widehat{L}^p$ to $C([0,T] : L^2 \cap \widehat{L}^p)$.

The global result is as follows:

Theorem 1.9. *Assume that $0 < \gamma < \min(2, n/2)$. Then the local solution u to (1.1) for $\phi \in L^2 \cap \widehat{L}^p$ given by Theorem 1.8 extends to a global one such that*

$$u \in C(\mathbb{R} : L^2 \cap \widehat{L}^p).$$

Remark 1.10.

(i) Carles and Mouzaoui [2] proved local and global well-posedness results in $L^2 \cap \widehat{L}^\infty$.
(ii) If $\phi \notin L^2$, the well-posedness in $\widehat{L}^p$ holds up to the exponent $p \le \frac{2n}{n-\gamma}$. See [16].

As in the L^p-setting, we may improve the global well-posedness result if $n = 1$ and $2 \le p < 4$:

Theorem 1.11. *Assume that $n = 1$, $0 < \gamma < 1$, and $2 \le p < 4$. Then the local solution u to (1.1) for $\phi \in L^2 \cap \widehat{L}^p$ given by Theorem 1.8 extends to a global one such that*

$$u \in C(\mathbb{R} : L^2 \cap \widehat{L}^p).$$

2. Key lemmata

2.1. Transformation of the Hartree nonlinearity via factorization of $U(-t)$

Trilinear forms. We introduce several trilinear forms to estimate the nonlinear term. Let f, g, h be three space variable functions. We define the trilinear operator $H_\gamma(f, g, h)$ associated with the Hartree type nonlinearity:

$$H_\gamma(f,g,h) \triangleq [|\cdot|^{-\gamma} * (f\bar{g})]h. \tag{2.1}$$

We also define $\widehat{H}_\gamma$ by

$$\widehat{H}_\gamma(f,g,h) \triangleq [|\cdot|^{-(n-\gamma)}(f * \bar{g})] * h. \tag{2.2}$$

Following [2], we introduce two cutoff functions k_1, k_2 by

$$k_1(x) \triangleq \mathbf{1}_{\{|x|\le 1\}}(x) \cdot |x|^{-(n-\gamma)}, \quad k_2(x) \triangleq \mathbf{1}_{\{|x|>1\}} \cdot |x|^{-(n-\gamma)}.$$

Note that

$$k_1 \in \bigcap_{q\in[1,\frac{n}{n-\gamma})} L^q(\mathbb{R}^n), \qquad k_2 \in \bigcap_{q\in(\frac{n}{n-\gamma},\infty]} L^q(\mathbb{R}^n).$$

The trilinear forms $\widehat{H}_\gamma^j(f,g,h)$, $j=1,2$ are defined as the cut-off of $\widehat{H}_\gamma$ by k_j:

$$\widehat{H}_\gamma^j(f,g,h) \triangleq [k_j(f*\bar{g})]*h.$$

Next we need several operators to handle the Hartree nonlinearity. Fix $t\neq 0$. We define the multiplication operator M_t by

$$M_t : w \mapsto e^{i\frac{|x|^2}{4t}} w.$$

The dilation operator D_t, $t\neq 0$ is given by

$$(D_t w)(x) \triangleq (4\pi i t)^{-\frac{n}{2}} w\left(\frac{x}{4\pi i t}\right).$$

The reflection operator R is defined by $(Rw)(x)\triangleq w(-x)$. Using these operators we get the following factorization formula (see [3]) for $U(t)$ and $U(-t)$:

$$U(t)\varphi = M_t D_t \mathscr{F} M_t \varphi, \quad U(-t)\varphi = M_t^{-1}\mathscr{F}^{-1}D_t^{-1}M_t^{-1}.$$

Using the factorization of $U(-t)$, we get the following key identity for $t\neq 0$:

Lemma 2.1. *There exist $c>0$ such that the following equality holds:*

$$U(-t)H_\gamma(u_1,u_2,u_3) = c|t|^{-\gamma}M_t^{-1}\widehat{H}_\gamma(M_t v_1(t), RM_t v_2(t), M_t v_3(t)), \quad t\neq 0, \tag{2.3}$$

where $v_j(t)\triangleq U(-t)u_j(t)$, $j=1,2,3$.

Proof. It is easy to check that

$$D_t^{-1}(fg) = (4\pi i t)^{-\frac{n}{2}}(D_t^{-1}f)(D_t^{-1}g) \tag{2.4}$$

$$D_t^{-1}\left(|\cdot|^{-\gamma}*(fg)\right) = (4\pi i t)^{\frac{n}{2}}(4\pi|t|)^{-\gamma}\left(|\cdot|^{-\gamma}*(D_t^{-1}f)(D_t^{-1}g)\right) \tag{2.5}$$

$$\mathscr{F}^{-1}D_t^{-1} = cRD_t\mathscr{F}. \tag{2.6}$$

Note also that

$$U(t)\bar{u} = \overline{U(-t)u}.$$

It follows from these equalities that

$$\begin{aligned}
M_tU(-t)H_\gamma(u_1,u_2,u_3) &= \mathscr{F}^{-1}D_t^{-1}\left(|\cdot|^{-\gamma}*(M_t^{-1}u_1)(M_t\overline{u_2})\right)M_t^{-1}u_3\\
&= (4\pi|t|)^{-\gamma}\mathscr{F}^{-1}\left(|\cdot|^{-\gamma}*(D_t^{-1}M_t^{-1}u_1)(D_t^{-1}M_t\overline{u_2})\right)\cdot D_t^{-1}M_t^{-1}u_3\\
&= c(4\pi|t|)^{-\gamma}\\
&\quad\times\left[|x|^{-(n-\gamma)}(\mathscr{F}^{-1}D_t^{-1}M_t^{-1}u_1)*(\mathscr{F}^{-1}D_t^{-1}M_t\overline{u_2})\right]*\mathscr{F}^{-1}D_t^{-1}M_t^{-1}u_3\\
&= c(4\pi|t|)^{-\gamma}\\
&\quad\times\left[|x|^{-(n-\gamma)}(\mathscr{F}^{-1}D_t^{-1}M_t^{-1}u_1)*(RD_t\mathscr{F}M_t^{-1}\overline{u_2})\right]*\mathscr{F}^{-1}D_t^{-1}M_t^{-1}u_3\\
&= c|t|^{-\gamma}\left[|x|^{-(n-\gamma)}(M_tU(-t)u_1(t))*(RM_t^{-1}U(t)\overline{u_2})\right]*M_tU(-t)u_3(t)\\
&= c|t|^{-\gamma}\left[|x|^{-(n-\gamma)}(M_tU(-t)u_1(t))*(R\overline{M_tU(-t)u_2})\right]*M_tU(-t)u_3(t).
\end{aligned}$$

This completes the proof. □

The following lemma indicates that $\hat{H}_\gamma$ is a closed operation on $L^1 \cap L^2$. This was essentially observed by R. Carles and Mouzaoui in [2].

Lemma 2.2. *([2])*

(i) The following estimates hold:

$$\|\widehat{H}_\gamma^1(f_1, f_2, f_3)\|_{L^1} \le C\|f_1\|_{L^2}\|f_2\|_{L^2}\|f_3\|_{L^1}, \tag{2.7}$$
$$\|\widehat{H}_\gamma^1(f_1, f_2, f_3)\|_{L^2} \le C\|f_1\|_{L^2}\|f_2\|_{L^2}\|f_3\|_{L^2}, \tag{2.8}$$
$$\|\widehat{H}_\gamma^2(f_1, f_2, f_3)\|_{L^1} \le C\|f_1\|_{L^1}\|f_2\|_{L^1}\|f_3\|_{L^1}, \tag{2.9}$$
$$\|\widehat{H}_\gamma^2(f_1, f_2, f_3)\|_{L^2} \le C\|f_1\|_{L^1}\|f_2\|_{L^1}\|f_3\|_{L^2}. \tag{2.10}$$

(ii) The following estimate holds:

$$\|\widehat{H}_\gamma(f_1, f_2, f_3)\|_{L^1\cap L^2} \le C\prod_{j=1}^{3}\|f_j\|_{L^1\cap L^2}.$$

Proof. By the Hölder and Hausdorff–Young inequalities, the Young inequality for convolution, and the Plancherel identity for the Fourier transform we have:

$$\begin{aligned}
\|k_1(f_1 * f_2) * f_3\|_{L^1} &\le \|k_1(f_1 * f_2)\|_{L^1}\|f_3\|_{L^1}\\
&\le \|k_1\|_{L^1}\|f_1 * f_2\|_{L^\infty}\|f_3\|_{L^1}\\
&\le \|k_1\|_{L^1}\|\hat{f}_1\hat{f}_2\|_{L^1}\|f_3\|_{L^1}\\
&\le \|k_1\|_{L^1}\|f_1\|_{L^2}\|f_2\|_{L^2}\|f_3\|_{L^1},\\
\|k_1(f_1 * f_2) * f_3\|_{L^2} &\le \|k_1(f_1 * f_2)\|_{L^1}\|f_3\|_{L^2}\\
&\le \|k_1\|_{L^1}\|f_1 * f_2\|_{L^\infty}\|f_3\|_{L^2}\\
&\le \|k_1\|_{L^1}\|f_1\|_{L^2}\|f_2\|_{L^2}\|f_3\|_{L^2},\\
\|k_2(f_1 * f_2) * f_3\|_{L^1} &\le \|k_2(f_1 * f_2)\|_{L^1}\|f_3\|_{L^1}\\
&\le \|k_2\|_{L^\infty}\|f_1 * f_2\|_{L^1}\|f_3\|_{L^1}\\
&\le \|k_2\|_{L^\infty}\|f_1\|_{L^1}\|f_2\|_{L^1}\|f_3\|_{L^1},
\end{aligned}$$

and

$$\begin{aligned}
\|k_2(f_1 * f_2) * f_3\|_{L^2} &\le \|k_2(f_1 * f_2)\|_{L^1}\|f_3\|_{L^2}\\
&\le \|k_2\|_{L^\infty}\|f_1 * f_2\|_{L^1}\|f_3\|_{L^2}\\
&\le \|k_2\|_{L^\infty}\|f_1\|_{L^1}\|f_2\|_{L^1}\|f_3\|_{L^2}.
\end{aligned}$$

This proves (i). Collecting these estimates we get (ii). □

Remark 2.3. As observed above, one key point is the fact that the Hartree nonlinearity becomes closed on $L^1 \cap L^2$ after the linear transformation $u(t) \mapsto v(t) \triangleq U(-t)u(t)$. However, we emphasize that even the bounded oscillating term M_t^{-1} before $\widehat{H}_\gamma$ in (2.3) plays an important role in this paper. See section 7.

2.2. *Hausdorff–Young like property*

As mentioned in introduction, the twisted property (1.5) implies the Hausdorff–Young like property $u(t) \in C(]0, T] : L^{p'})$ for $1 \le p \le 2$. For convenience, we prove this here.

Proposition 2.4. *Let $1 \le p \le 2$. Let $I \subset \mathbb{R}$ and let u be such that*

$$U(-t)u(t) \in C(I : L^p(\mathbb{R}^n)).$$

Then

$$u|_{(I\setminus\{0\})\times\mathbb{R}^n} \in C(I \setminus \{0\} : L^{p'}(\mathbb{R}^n)).$$

In order to prove Proposition 2.4 we need:

Lemma 2.5. *Let $\phi \in L^p(\mathbb{R}^n)$ with $1 \le p \le 2$. Then the map $t \mapsto U(t)\phi$ is continuous from $\mathbb{R} \setminus \{0\}$ to $L^{p'}(\mathbb{R}^n)$.*

Proof. We prove the lemma for completeness though the assertion is known (see e.g. [7]). Assume first that $\phi \in C_0^\infty(\mathbb{R}^n)$. Then, the assertion is obvious since

$$\|U(t_1)\phi - U(t_2)\phi\|_{L^{p'}} \le C|t_1 - t_2| \times \left\| |\cdot|^2 \hat{\phi} \right\|_{L^p}$$

for $t_1, t_2 \in \mathbb{R}$, where we used the Hausdorff–Young inequality and the elementary estimate $|e^{ix} - e^{iy}| \le |x - y|$. The general case follows from the standard $\varepsilon/3$-argument: write

$$\begin{aligned}\|U(t_1)\phi - U(t_2)\phi\|_{L^{p'}} &\le \|U(t_1)\phi - U(t_1)\tilde{\phi}\|_{L^{p'}} + \|U(t_1)\tilde{\phi} - U(t_2)\tilde{\phi}\|_{L^{p'}} \\ &\quad + \|U(t_2)\tilde{\phi} - U(t_2)\phi\|_{L^{p'}} \\ &\le (4\pi|t_1|)^{-n(\frac{1}{p}-\frac{1}{2})}\|\phi - \tilde{\phi}\|_{L^p} + \|U(t_1)\tilde{\phi} - U(t_2)\tilde{\phi}\|_{L^{p'}} \\ &\quad + (4\pi|t_2|)^{-n(\frac{1}{p}-\frac{1}{2})}\|\phi - \tilde{\phi}\|_{L^p},\end{aligned}$$

for $t_1, t_2 \ne 0$ and approximate $\phi \in L^p$ by $\tilde{\phi} \in C_0^\infty(\mathbb{R}^n)$. □

Proof of Proposition 2.4. In order to show continuity at $t_0 \in I \setminus \{0\}$, we write

$$\begin{aligned}\|u(t) - u(t_0)\|_{L^{p'}} &\le \|U(t)(U(-t)u(t) - U(-t_0)u(t_0))\|_{L^{p'}} \\ &\quad + \|U(t)U(-t_0)u(t_0) - U(t_0)U(-t_0)u(t_0)\|_{L^{p'}} \\ &\le (4\pi|t|)^{-d(\frac{1}{p}-\frac{1}{2})}\|U(-t)u(t) - U(-t_0)u(t_0)\|_{L^p} \\ &\quad + \|U(t)U(-t_0)u(t_0) - U(t_0)U(-t_0)u(t_0)\|_{L^{p'}},\end{aligned}$$

for $t \in I \setminus \{0\}$. Letting $t \to t_0$, the first term in the right hand side tends to 0 by the assumption. The second term also tends to 0 thanks to Lemma 2.5. □

3. Proof of Theorem 1.1

The integral equation corresponding to (1.1) is given by

$$u(t) = U(t)\phi + i\int_0^t U(t-s)H_\gamma(u(s), u(s), u(s))ds. \tag{3.1}$$

Following [18] we put

$$v(t) \triangleq U(-t)u(t).$$

Then by Lemma 2.1, we see that the solution u of (3.1) is given by $u(t) = U(t)v(t)$ where v solves

$$v(t) = \phi + ci\int_0^t M_s^{-1}s^{-\gamma}\widehat{H}_\gamma(M_s v(s), RM_s v(s), M_s v(s))ds. \tag{3.2}$$

We seek a solution of the transformed integral equation (3.2). To this end, we define the operator Φ by

$$(\Phi v)(t) \triangleq \phi + ci \int_0^t M_s^{-1} s^{-\gamma} \widehat{H}_\gamma(M_s v(s), RM_s v(s), M_s v(s)) ds$$

and define the closed set $\mathscr{V}(a)$ and the distance on it by

$$\mathscr{V}(a) \triangleq \{v \in L_T^\infty(L^1 \cap L^2) : \|v\|_{L_T^\infty(L^1\cap L^2)} \le a\}, \quad T > 0,$$

and

$$d_{\mathscr{V}(a)}(v_1, v_2) = \|v_1 - v_2\|_{L_T^\infty(L^1\cap L^2)}.$$

Then for $v \in \mathscr{V}(a)$ we have

$$\|\Phi v\|_{L_T^\infty(L^1\cap L^2)} \le \|\phi\|_{L^1\cap L^2} + \int_0^T s^{-\gamma} \|\hat{H}_\gamma(M_s v(s), RM_s v(s), M_s v(s)\|_{L^1\cap L^2} ds.$$

Now Lemma 2.2 (ii) tells us that the right hand side is estimated by above by

$$\begin{aligned} \|\phi\|_{L^1\cap L^2} + C\int_0^T s^{-\gamma}\|v(s)\|_{L^1\cap L^2}^3 ds &\le \|\phi\|_{L^1\cap L^2} + C\left(\int_0^T s^{-\gamma} ds\right) \times \|v\|_{L_T^\infty(L^1\cap L^2)}^3 \\ &\le \|\phi\|_{L^1\cap L^2} + CT^{1-\gamma}a^3. \end{aligned}$$

If we choose a so that $\|\phi\|_{L^1\cap L^2} = a/2$ and $CT^{1-\gamma}a^3 \le a/2$, then Φ maps $\mathscr{V}(a)$ to itself.

Similarly, we have the difference estimate:

$$\begin{aligned} d_{\mathscr{V}(a)}(\Phi v_1, \Phi v_2) \le &\int_0^T s^{-\gamma} \|\widehat{H}_\gamma(M_s(v_1(s) - v_2(s)), \overline{RM_s v_1(s)}, M_s v_1(s))\|_{L^1\cap L^2} ds \\ &+ \int_0^T s^{-\gamma} \|\widehat{H}_\gamma(M_s(v_2(s)), RM_s(v_1(s) - v_2(s)), M_s v_1(s))\|_{L^1\cap L^2} ds \\ &+ \int_0^T s^{-\gamma} \|\widehat{H}_\gamma(M_s(v_2(s)), RM_s(v_2(s)), M_s(v_1(s) - v_2(s))\|_{L^1\cap L^2} ds \\ \le\, &CT^{1-\gamma}(\|v_1\|_{L_T^\infty(L^1\cap L^2)}^2 + \|v_1\|_{L_T^\infty(L^1\cap L^2)}\|v_2\|_{L_T^\infty(L^1\cap L^2)} + \|v_2\|_{L_T^\infty(L^1\cap L^2)}^2) \\ &\times \|v_1 - v_2\|_{L_T^\infty(L^1\cap L^2)} \\ \le\, &CT^{1-\gamma}a^2 d_{\mathscr{V}(a)}(v_1, v_2). \end{aligned}$$

The above estimate indicates that $\Phi : \mathscr{V}(a) \to \mathscr{V}(a)$ is a contraction mapping if $CT^{1-\gamma}a^2 \le 1/2$. Hence by the fixed point theorem, we get a solution $v \in C([0, T] : L^1 \cap L^2)$ of the transformed integral equation (3.2) for

$$T \sim \|\phi\|_{L^1\cap L^2}^{-\frac{2}{1-\gamma}},$$

which then implies that a solution $u(t) = U(t)v(t)$ of the original equation (3.1) exists. Uniqueness and stability property can be proved in a similar way.

4. Proof of Theorem 1.3

4.1. Key nonlinear estimates

As in section 3, we seek a solution v to the transformed integral equation (3.2). Observe first that the argument in the previous section cannot be applied since we cannot integrate the singular factor $t^{-\gamma}$ when $\gamma \geq 1$ and since the nonlinear operation $\widehat{H}_\gamma$ is not closed on $L^p \cap L^2$ if $p > 1$. In this section we establish a local solution of (3.2) in the smaller function space $\tilde{Y}^2_{\frac{2}{\gamma},0}(T) \cap \tilde{Y}^p_{q,\theta}(T)$, which was originally introduced by Zhou [18]. To this end, we define the trilinear Duhamel type operators $\mathscr{D}_\gamma, \mathscr{D}^j_\gamma,\ j = 1, 2$ by

$$\mathscr{D}_\gamma(v_1, v_2, v_3) \triangleq \int_0^t M_s^{-1} s^{-\gamma} \widehat{H}_\gamma(M_s v_1(s), RM_s v_2(s), M_s v_3(s))ds$$

and

$$\mathscr{D}^j_\gamma(v_1, v_2, v_3) \triangleq \int_0^t M_s^{-1} s^{-\gamma} \widehat{H}^j_\gamma(M_s v_1(s), RM_s v_2(s), M_s v_3(s))ds, \quad j = 1, 2.$$

The key estimates to the local existence is:

Proposition 4.1. *Assume that* $0 < \gamma < 2$ *and* $1 \leq p \leq 2$. *Then*

$$\|\mathscr{D}^1_\gamma(v_1, v_2, v_3)\|_{\tilde{X}^p_{q,\theta}(T)} \leq C\|v_1\|_{\tilde{Y}^2_{1,0}(T)}\|v_2\|_{\tilde{Y}^2_{1,0}(T)}\|v_3\|_{\tilde{Y}^p_{1,0}(T)} \tag{4.1}$$

and

$$\|\mathscr{D}^2_\gamma(v_1, v_2, v_3)\|_{\tilde{X}^p_{q,\theta}(T)} \leq C\prod_{l=1}^{3}\|v_l\|_{\tilde{Y}^p_{1,0}(T)}, \tag{4.2}$$

where

$$q = \frac{p}{\gamma(p-1)}, \quad \theta = \gamma(\frac{2}{p} - 1).$$

To prove Proposition 4.1, we need:

Lemma 4.2. *Assume that* $0 < \gamma < n$. *Then*

$$\|\mathscr{D}^1_\gamma(v_1, v_2, v_3)\|_{\tilde{X}^1_{\infty,\gamma}(T)} \leq C\|v_1\|_{\tilde{Y}^2_{1,0}(T)}\|v_2\|_{\tilde{Y}^2_{1,0}(T)}\|v_3\|_{\tilde{Y}^1_{1,0}(T)} \tag{4.3}$$

and

$$\|\mathscr{D}^2_\gamma(v_1, v_2, v_3)\|_{\tilde{X}^1_{\infty,\gamma}(T)} \leq C\prod_{l=1}^{3}\|v_l\|_{\tilde{Y}^1_{1,0}(T)}. \tag{4.4}$$

Proof. By definition

$$\begin{aligned}\|\mathscr{D}^1_\gamma(v_1, v_2, v_3)\|_{\tilde{X}^1_{\infty,\gamma}(T)} &= \sup_{t\in[0,T]} t^\gamma \|\partial_t \left(\mathscr{D}^1(v_1, v_2, v_3)\right)(t,\cdot)\|_{L^1} \\ &\leq \sup_{t\in[0,T]} \|\widehat{H}^1_\gamma(M_t v_1(t), RM_t v_2(t), M_t v_3(t))\|_{L^1} \\ &\leq C \sup_{t\in[0,T]} \|v_1(t)\|_{L^2}\|v_2(t)\|_{L^2}\|v_3(t)\|_{L^1},\end{aligned}$$

where the last inequality follows from (2.7). Now we write

$$v_l(t) = v_l(0) + \int_0^t (\partial_s v_l)(s) ds \tag{4.5}$$

and taking L^1 and L^2-norm of both sides, we get the first estimate. The proof of the second estimate is similar. We use (2.9) in place of (2.7). □

Proof of Proposition 4.1. We first prove the case of $p = 2$, that is:

$$\|\mathscr{D}_\gamma^j(v_1, v_2, v_3)\|_{\tilde{X}^2_{\frac{2}{\gamma},0}(T)} \le C \prod_{l=1}^3 \|v_l\|_{\tilde{Y}^2_{1,0}(T)}, \tag{4.6}$$

for $j = 1, 2$.

We have

$$\begin{aligned}\|\partial_t(\mathscr{D}_\gamma^j(v_1, v_2, v_3))\|_{L_T^q(L^2)} &= c\|t^{-\gamma}\widehat{H}_\gamma^j(v_1, v_2, v_3)\|_{L_T^q(L^2)}\\ &\le c\|t^{-\gamma}\widehat{H}_\gamma(v_1, v_2, v_3)\|_{L_T^q(L^2)}\\ &= \|\partial_t(\mathscr{D}_\gamma(v_1, v_2, v_3))\|_{L_T^q(L^2)}\\ &= \|U(-t)H_\gamma(U(t)v_1(t), U(t)v_2(t), U(t)v_3(t))\|_{L_T^q(L^2)}\\ &= \|H_\gamma(U(t)v_1(t), U(t)v_2(t), U(t)v_3(t))\|_{L_T^q(L^2)}.\end{aligned}$$

By Hölder and Hardy–Littlewood–Sobolev estimates, the right hand side is estimated by

$$\begin{aligned}&\left\| \left\| |\cdot|^{-\gamma} * \left[(U(t)v_1(t))(\overline{U(t)v_2(t)})\right]\right\|_{L^{\frac{3n}{\gamma}}} \|U(t)v_3(t)\|_{L^{\frac{6n}{3n-2\gamma}}}\right\|_{L_T^q}\\ &\le C\left\|\prod_{l=1}^3 \|U(t)v_l(t)\|_{L^{\frac{6n}{3n-2\gamma}}}\right\|_{L_T^q} \le C\prod_{l=1}^3 \|U(t)v_l(t)\|_{L_T^{3q}(L^{\frac{6n}{3n-2\gamma}})}.\end{aligned}$$

Note that the pair $(\frac{6}{\gamma}, \frac{6n}{3n-2\gamma})$ is admissible if $\gamma < 3$. By (4.5), we may write

$$U(t)v_l(t) = U(t)v_l(0) + \int_0^t U(t)(\partial_s v_l)(s) ds.$$

Taking $L_T^{3q}(L^{\frac{6n}{3n-2\gamma}})$-norm of both sides (here $p = 2$, i.e. $q = 2/\gamma$), we get

$$\begin{aligned}\|U(t)v_l(t)\|_{L_T^{3q}(L^{\frac{6n}{3n-2\gamma}})} &\le \|U(t)v_l(0)\|_{L_T^{3q}(L^{\frac{6n}{3n-2\gamma}})} + \left\|\int_0^t U(t)(\partial_s v_l)(s) ds\right\|_{L_T^{3q}(L^{\frac{6n}{3n-2\gamma}})}\\ &\le \|U(t)v_l(0)\|_{L_T^{3q}(L^{\frac{6n}{3n-2\gamma}})} + \left\|\int_0^t \|U(t)(\partial_s v_l)(s)\|_{L^{\frac{6n}{3n-2\gamma}}} ds\right\|_{L_T^{3q}}\\ &\le \|U(t)v_l(0)\|_{L_T^{3q}(L^{\frac{6n}{3n-2\gamma}})} + \left\|\int_0^T \|U(t)(\partial_s v_l)(s) ds\|_{L^{\frac{6n}{3n-2\gamma}}}\right\|_{L_T^{3q}}\\ &\le \|U(t)v_l(0)\|_{L_T^{3q}(L^{\frac{6n}{3n-2\gamma}})} + \int_0^T \|U(t)(\partial_s v_l)(s)\|_{L_T^{3q}(L^{\frac{6n}{3n-2\gamma}})} ds.\end{aligned}$$

By the Standard Strichartz estimates, this is estimated by

$$C\|v_l(0)\|_{L^2} + C\int_0^T \|(\partial_s v_l)(s)\|_{L^2} ds = C\|v_l\|_{\tilde{Y}^2_{1,0}(T)},$$

which proves (4.6). Finally, we get (4.1) by interpolation between (4.3) and (4.6) after some suitable change of measure (see [18]). Similarly, interpolating (4.4) and (4.6), we get (4.2). □

4.2. *Proof of Theorem 1.3*

Now we prove Theorem 1.3. Let q, θ be as in Proposition 4.1. We define a closed subset of $\tilde{Y}^2_{\frac{2}{\gamma},0}(T) \cap \tilde{Y}^p_{q,\theta}(T)$ by

$$\mathscr{V}(a) \triangleq \{v \in \tilde{Y}^2_{\frac{2}{\gamma},0}(T) \cap \tilde{Y}^p_{q,\theta}(T) \mid \|v\|_{\tilde{X}^2_{\frac{2}{\gamma},0}(T)\cap \tilde{X}^p_{q,\theta}(T)} \le a, \quad v(0) = \phi\}.$$

The distance on $\mathscr{V}(a)$ is given by

$$d_{\mathscr{V}(a)}(v_1, v_2) \triangleq \|v_1 - v_2\|_{\tilde{X}^2_{\frac{2}{\gamma},0}(T)\cap \tilde{X}^p_{q,\theta}(T)}.$$

Assume that $v \in \mathscr{V}(a)$. Define

$$(\Phi v)(t) \triangleq \phi + i\mathscr{D}_\gamma(v, v, v)$$

and we show that $\Phi : \mathscr{V}(a) \to \mathscr{V}(a)$ is well defined and is a contraction mapping for sufficiently small $T > 0$. Assume that $v \in \mathscr{V}(a)$. Then we have by (4.6)

$$\begin{aligned}
\|\Phi v\|_{\tilde{X}^2_{\frac{2}{\gamma},0}(T)} &\le \|\mathscr{D}_\gamma(v, v, v)\|_{\tilde{X}^2_{\frac{2}{\gamma},0(T)}} \\
&\le C\|v\|^3_{\tilde{Y}^2_{1,0}(T)} \\
&= (\|\phi\|_{L^2} + T^{1-\frac{\gamma}{2}}\|v\|_{\tilde{X}^2_{\frac{2}{\gamma},0}(T)})^3 \\
&\le 8\|\phi\|^3_{L^2} + 8T^{3(1-\frac{\gamma}{2})}\|v\|^3_{X^2_{\frac{2}{\gamma},0}(T)} \\
&\le 8\|\phi\|^3_{L^2} + 8T^{3(1-\frac{\gamma}{2})}a^3.
\end{aligned}$$

Similarly, by (4.1) and (4.2), we get

$$\begin{aligned}
\|\Phi v\|_{\tilde{X}^p_{q,\theta}(T)} &\le C\|v\|^2_{\tilde{Y}^2_{1,0}(T)}\|v\|_{\tilde{Y}^p_{1,0}} + C\|v\|^3_{\tilde{Y}^p_{1,0}} \\
&\le C(\|\phi\|_{L^2} + T^{1-\frac{\gamma}{2}}\|v\|_{\tilde{X}^2_{\frac{2}{\gamma},0}(T)})^2(\|\phi\|_{L^p} + T^{1-\frac{\gamma}{p}}\|v\|_{\tilde{X}^p_{q,\theta}(T)}) \\
&\quad + (\|\phi\|_{L^p} + T^{1-\frac{\gamma}{p}}\|v\|_{\tilde{X}^p_{q,\theta}(T)})^3 \\
&\le 8\|\phi\|^3_{L^p\cap L^2} + \frac{8}{3}CT^{3(1-\frac{\gamma}{2})}a^3 + \frac{28}{3}CT^{\frac{3(p-\gamma)}{p}}a^3,
\end{aligned}$$

where we have used the elementary inequality

$$(\alpha + \beta)^2(\alpha + \gamma) \le 4\alpha^3 + \frac{8}{3}\beta^3 + \frac{4}{3}\gamma^3, \quad \alpha, \beta, \gamma \ge 0.$$

Note that this follows from a repeated use of the Young inequality

$$kl \le \frac{k^3}{3} + \frac{2}{3}l^{\frac{3}{2}}, \quad k, l \ge 0.$$

Now, in view of these estimates, we set

$$a = 32C\|\phi\|^3_{L^p\cap L^2}, \quad T = K_1 \min(\|\phi\|_{L^2}^{-\frac{4}{2-\gamma}}, \|\phi\|_{L^p\cap L^2}^{-\frac{2p}{p-\gamma}}), \tag{4.7}$$

where K_1 is a positive constant.

Then if we take K sufficiently small, we see that

$$\|\Phi v\|_{\tilde{X}^2_{\frac{2}{\gamma},0}(T)\cap \tilde{X}^p_{q,\theta}(T)} \le a$$

and thus Φ is well defined. We then show that $\Phi : \mathscr{V}(a) \to \mathscr{V}(a)$ is a contraction mapping. Assume that $v_1, v_2 \in \mathscr{V}(a)$. Then we have

$$\begin{aligned}
\|\Phi v_1 - \Phi v_2\|_{\tilde{X}^2_{\frac{2}{\gamma},0}(T)} &\le \|\mathscr{D}_\gamma(v_1, v_1, v_1) - \mathscr{D}_\gamma(v_2, v_2, v_2)\|_{\tilde{X}^2_{\frac{2}{\gamma},0}(T)} \\
&\le \|\mathscr{D}_\gamma(v_1 - v_2, v_1, v_1)\|_{\tilde{X}^2_{\frac{2}{\gamma},0}(T)} + \|\mathscr{D}_\gamma(v_2, v_1 - v_2, v_1)\|_{\tilde{X}^2_{\frac{2}{\gamma},0}(T)} \\
&\quad + \|\mathscr{D}_\gamma(v_2, v_2, v_1 - v_2)\|_{\tilde{X}^2_{\frac{2}{\gamma},0}(T)} \\
&\le C\|v_1 - v_2\|_{\tilde{Y}^2_{1,0}(T)} \sum_{1\le j,k\le 2} \|v_j\|_{\tilde{Y}^2_{1,0}(T)} \|v_k\|_{\tilde{Y}^2_{1,0}(T)} \\
&\le CT^{1-\frac{\gamma}{2}} \|v_1 - v_2\|_{\tilde{Y}^2_{\frac{2}{\gamma},0}(T)} \\
&\quad \times \sum_{1\le j,k\le 2} (\|\phi\|_{L^2} + T^{1-\frac{\gamma}{2}} \|v_j\|_{\tilde{Y}^2_{\frac{2}{\gamma},0}(T)})(\|\phi\|_{L^2} + T^{1-\frac{\gamma}{2}} \|v_j\|_{\tilde{Y}^2_{\frac{2}{\gamma},0}(T)}).
\end{aligned}$$

Then if we take K_1 in (4.7) small enough, we see that

$$\|\Phi v_1 - \Phi v_2\|_{\tilde{X}^2_{\frac{2}{\gamma},0}(T)} \le \frac{1}{2}\|v_1 - v_2\|_{\tilde{X}^2_{\frac{2}{\gamma},0}(T)}$$

Similarly, we get

$$\|\Phi v_1 - \Phi v_2\|_{\tilde{X}^p_{q,\theta}(T)} \le \frac{1}{2}\|v_1 - v_2\|_{\tilde{X}^p_{q,\theta}(T)}.$$

Therefore,

$$d_{\mathscr{V}(a)}(\Phi v_1, \Phi v_2) \le \frac{1}{2} d_{\mathscr{V}(a)}(v_1, v_2).$$

Consequently, by the standard fixed point argument we see that there exists a local solution of the integral equation (3.2) in the space $\tilde{Y}^2_{\frac{2}{\gamma},0}(T) \cap \tilde{Y}^p_{q,\theta}(T) \hookrightarrow C([0,T] : L^p \cap L^2)$. This implies that we get a solution of the original Cauchy problem (1.1) in the form of $u(t) = U(t)v(t) \in Y^2_{\frac{2}{\gamma},0}(T) \cap Y^p_{q,\theta}(T)$.

5. Proof of Theorem 1.2 and 1.4

Global existence. We extend the local solution established in previous sections globally. We follow the argument by Carles and Mouzaoui [2]. Let $p \in [1, 2]$ and let $T_0 > 0$. Let $v : [0, T_0] \times \mathbb{R}^n \to \mathbb{C}$ be a local solution to the transformed integral equation (3.2) such that

$$v|_{[0,T]\times\mathbb{R}^n} \in \begin{cases} C([0,T] : L^1 \cap L^2) & \text{if} \quad p = 1 \\ \tilde{Y}^p_{q,\theta}(T) \cap \tilde{Y}^2_{\frac{2}{\gamma},0}(T) \text{ for some } q, \theta \text{ with } q'\theta < 1 & \text{if} \quad 1 < p \le 2 \end{cases}$$

for any $T \in (0, T_0)$. The key to the global well-posedness result is:

Proposition 5.1. *Assume that* $0 < \gamma < n/2$. *Then*

$$\sup_{t\in[0,T_0)} \|v(t)\|_{L^p} < \infty. \tag{5.1}$$

Throughout this section we use the convention that $u(t) \triangleq U(t)v(t)$ and $v(t) = U(-t)u(t)$. Note that u solves the original Cauchy problem (3.1). To prove the proposition, we prepare a few lemmata.

Lemma 5.2. *On the time interval* $[0, T_0)$, $u(t) = U(t)v(t)$ *coincides with the global* L^2*-solution for the initial datum* $\phi = U(0)v(0)$ *given by Theorem A.*

Proof. If $p = 1$, the assertion is obvious, since the uniqueness holds in the space $\{u|U(-t)u(t) \in C([0, T]; L^1 \cap L^2)\}$. If $1 < p \le 2$, we have by Lemma 1.7

$$u \in Y^2_{\frac{2}{\gamma},0}(T) \subset L^\rho([0, T] : L^r)$$

for any Schrödinger admissible pair (ρ, r). The assertion follows from the uniqueness of the solution given by Theorem A in $L^{\frac{8}{\gamma}}([0, T] : L^{\frac{4n}{2n-\gamma}})$. □

Lemma 5.3. *Set*

$$(\Theta u)(t) \triangleq (M_t U(-t)u(t)) * (R\overline{M_t U(-t)u(t)}). \tag{5.2}$$

Then for any $0 < \rho < \infty$,

$$\|\widehat{(\Theta u)(t)}\|_{L^\rho} = c|t|^{n(1-\frac{1}{\rho})}\|u(t)\|^2_{L^{2\rho}},$$

where c is a positive constant independent of t.

Proof. Noting the relation

$$\mathscr{F} R\bar{f} = \overline{\mathscr{F} f}, \quad f \in \mathscr{S}$$

and the factorization formula, we see that

$$\begin{aligned}\widehat{(\Theta u)(t)} &= \{\mathscr{F} M_t U(-t)u(t)\} \times \{\mathscr{F} R\overline{M_t U(-t)u(t)}\} \\ &= \{\mathscr{F} M_t U(-t)u(t)\} \times \{\overline{\mathscr{F} M_t U(-t)u(t)}\} \\ &= |D_t^{-1} M_t^{-1} u(t)|^2.\end{aligned}$$

The assertion of the lemma follows by integrating this equality. □

Proof of Proposition 5.1. We write

$$v(t) = \phi + ci \sum_{j=1}^{2} \int_0^t M_s^{-1} s^{-\gamma} \widehat{H}^j_\gamma(M_s v(s), RM_s v(s), M_s v(s))ds. \tag{5.3}$$

We fix $T \in (0, T_0)$ and $t \in [0, T]$. Taking the L^p-norm of (5.3) we have

$$\begin{aligned}\|v(t)\|_{L^p} &\le \|\phi\|_{L^p} + c\int_0^t s^{-\gamma} \left\|\widehat{H}^1_\gamma(M_s v(s), RM_s v(s), M_s v(s))\right\|_{L^p} ds \\ &\quad + \int_0^t s^{-\gamma} \left\|\widehat{H}^2_\gamma(M_s v(s), RM_s v(s), M_s v(s))\right\|_{L^p} ds \\ &\triangleq \|\phi\|_{L^p} + I_1 + I_2.\end{aligned}$$

By (2.7) and (2.8), we have

$$I_1 \le c\int_0^t s^{-\gamma} \|v(s)\|^2_{L^2} \|v(s)\|_{L^p} ds. \tag{5.4}$$

Since $u = U(t)v(t)$ coincides with the standard L^2 solution of the original Cauchy problem (1.1), we have the L^2 conservation law

$$\|v(t)\|_{L^2} = \|U(-t)u(t)\|_{L^2} = \|u(t)\|_{L^2} = \|\phi\|_{L^2}.$$

By Hölder's inequality and the above equality, we see that the right hand side of (5.4) is smaller than

$$C\|\phi\|_{L^2}^2 \int_0^t s^{-\gamma}\|v(s)\|_{L^p} ds \le C\left(\int_0^t s^{-q_1'\gamma} ds\right)^{\frac{1}{q_1{}'}} \left(\int_0^t \|v(s)\|_{L^1}^{q_1} ds\right)^{\frac{1}{q_1}}$$
$$= CT_0^{1-\gamma-\frac{1}{q_1}} \|v\|_{L^{q_1}([0,t]:L^p)}$$

for any q_1 satisfying

$$q_1 > \frac{1}{1-\gamma}.$$

Next we consider I_2. Using the expression Θu given by (5.2), we may write

$$I_2 = ci \int_0^t M_s^{-1} s^{-\gamma} \left(k_1(\Theta u)(s)\right) * (M_s v(s)) ds. \tag{5.5}$$

Taking the L^p-norm, we get

$$\|I_2\|_{L^p} \le c \int_0^t s^{-\gamma}\|k_1(\Theta u)(s)\|_{L^1}\|v(s)\|_{L^p} ds$$
$$\le c \int_0^t s^{-\gamma}\|k_1\|_\rho \|\Theta u(s)\|_{L^{\rho'}}\|v(s)\|_{L^p} ds.$$

Note that $\frac{n}{n-\gamma} < 2$ since $\gamma < n/2$. Thus, we can choose ρ so that

$$\frac{n}{n-\gamma} < \rho < 2 \tag{5.6}$$

and we take one such ρ. Then, the Haudorff–Young inequality can be applied to the $L^{\rho'}$-norm and we have

$$\|I_2\|_{L^p} \le c \int_0^t s^{-\gamma}\|\widehat{\Theta u(s)}\|_{L^\rho}\|v(s)\|_{L^p} ds$$
$$\le \int_0^t s^{n-\gamma-\frac{n}{\rho}} \|u(s)\|_{L^{2\rho}}^2 \|v(s)\|_{L^p} ds,$$

where we have also used Lemma 5.3 and observe that $n - \gamma - n/\rho > 0$ by (5.6). Next we set

$$\frac{1}{q_2} = \frac{n}{4}\left(1 - \frac{1}{\rho}\right), \quad \frac{1}{q_3} = 1 - \frac{n}{2}\left(1 - \frac{1}{\rho}\right).$$

It is easy to see that $(q_2, 2\rho)$ is admissible and therefore, by Lemma 5.2 and Theorem A, we observe that

$$\|u\|_{L^{q_2}_{[0,t]}(L^{2\rho})} \le \|u\|_{L^{q_2}_{[0,T_0]}(L^{2\rho})} < \infty.$$

By Hölder's inequality in time, we have

$$I_2 \le T_0^{n-\gamma-\frac{n}{\rho}} \|u\|^2_{L^{q_2}_{T_0}(L^{2\rho})} \|v\|_{L^{q_3}_{[0,t]}(L^p)} \le C_{T_0} \|v\|_{L^{q_3}_{[0,t]}(L^p)}.$$

Consequently, we arrive at the estimate

$$\|v(t)\|_{L^p} \le C_1 + C_{T_0} \|v\|_{L^q_{[0,t]}(L^p)}, \quad \forall t \in [0,T] \tag{5.7}$$

for $q > \max(q_1, q_3)$. The wanted estimate follows by applying Gronwall's lemma after writing

$$\|v(t)\|^q_{L^p} \le C_1 + C_{T_0} \int_0^t \|v(s)\|^q_{L^p} ds, \quad \forall t \in [0,T]. \quad \square \tag{5.8}$$

Now the global well-posedness results follow from the standard argument of blow-up alternative. Let $p \in [1,2]$ and let $u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be a global L^2-solution given by Theorem A. We define

$$T_{\max}(u) \triangleq \sup\{T > 0;\ v(t) \triangleq U(-t)u(t)|_{[0,T]\times\mathbb{R}^n} \in C([0,T] : L^p)\}.$$

Theorem 1.1 and 1.3, and Lemma 5.2 tell us that $T_{\max}(u) > 0$ if $\phi \in L^p \cap L^2$. Our goal in this section is to show $T_{\max}(u) = \infty$. This is an immediate consequence of Proposition 5.1 and the following lemma.

Lemma 5.4. *Assume that* $0 < T_{\max}(u) < \infty$. *Then*

$$\lim_{t \nearrow T_{\max}(u)} \|U(-t)u(t)\|_{L^p} = \infty.$$

Proof. Suppose, to the contrary, that there are $M > 0$ and $\{t_k\}_{k=1}^\infty$ such that

$$t_k \nearrow T_{\max}(u), \ (k \to \infty), \qquad \|v(t_k)\|_{L^p} \le M.$$

Recall that the life span of the local solution in Theorem 1.1 and 1.3 depends on the norm of data (see Section 3 and 4). Therefore, there is $T = T(M) > 0$ such that for each $k \in \mathbb{N}$, the solution v of the integral equation

$$v(t) = v(t_k) + ci \int_{t_k}^t M_s^{-1} s^{-\gamma} \hat{H}_\gamma(M_s v(s), RM_s v(s), M_s v(s)) ds$$

can be established on the time interval $[t_k, t_k + T(M)]$. By uniqueness, $U(t)v(t)$ coincides with the standard global L^2-solution on this interval, which implies

$$U(-t)u(t)|_{[0,T_{\max}+\varepsilon]\times\mathbb{R}^n} \in C([0, T_{\max}(u)+\varepsilon] : L^p)$$

for some $\varepsilon \in (0, T(M))$. A contradiction. $\square$

6. Proof of Theorem 1.8 and 1.9

In this section we prove the local and global results for data in hat spaces $L^2 \cap \widehat{L}^p$. Note first that

$$\|H_\gamma(f,g,h)\|_{\widehat{L}^p} = \|\mathscr{F}[H_\gamma(f,g,h)]\|_{L^{p'}} = c\|\widehat{H}_\gamma(\hat{f}, \hat{g}, \hat{h})\|_{L^{p'}},$$

where $\widehat{H}_\gamma$ is the trilinear operator introduced in section 2. Therefore, it is not difficult to see that analysis of the original integral equation (3.1) in $\widehat{L}^p$-space is very similar to the one of the transformed integral equation (3.2) in $L^{p'}$-space. In fact, the local and global results for $\phi \in L^2 \cap \widehat{L}^p$ can be proved by arguments similar to the ones used in Section 4 and 5. For that reason, we only give the notation of function spaces suitable for $\widehat{L}^p$-setting and present key trilinear estimates in these spaces, and we omit the detailed proof of Theorem 1.8 and 1.9.

6.1. Function spaces

In this section we prove the local and global well-posedness results in the hat space $L^2 \cap \widehat{L}^p$. We first introduce function spaces. We define

$$\tilde{W}_q^p(T) \triangleq \{v : [0,T] \times \mathbb{R}^n \to \mathbb{C} \mid \|v\|_{\tilde{W}_q^p(T)} < \infty\},$$

where

$$\|v\|_{\tilde{W}_q^p(T)} \triangleq \left(\int_0^T \|(\partial_s v)(s,\cdot)\|_{\widehat{L}^p}^q ds \right)^{\frac{1}{q}}$$

and

$$\tilde{Z}_q^p(T) \triangleq \{v \in \tilde{W}_q^p \mid v(0) \in \widehat{L}^p\}$$

equipped with the norm

$$\|v\|_{\tilde{Z}_q^p(T)} \triangleq \|v(0)\|_{\widehat{L}^p} + \|v\|_{\tilde{W}_q^p(T)}.$$

Given these spaces we define

$$Z_q^p(T) \triangleq \{u : [0,T] \times \mathbb{R}^n \to \mathbb{C} \mid U(-t)u(t) \in \tilde{Z}_q^p(T)\}.$$

Arguing as in the proof of Lemma 1.7 noting that the generalized Strichartz estimate

$$\|U(t)\phi\|_{L^\rho(L^r)} \le C\|\phi\|_{\widehat{L}^p}$$

holds for any $2 < \rho, r, p \le \infty$ with $2/\rho + n/r = n/p$ (see [17, Theorem 3.4]), we get the following basic embedding results.

Lemma 6.1.

(i) For any $1 \le q < \infty$, we have

$$Z_q^p(T) \hookrightarrow C([0,T] : \widehat{L}^p).$$

(ii) For any $2 \le q, r, p \le \infty$ with

$$\frac{2}{\rho} + \frac{n}{r} = \frac{n}{p}$$

the embedding

$$Z_q^p(T) \hookrightarrow L_T^\rho(L^r)$$

holds true.

6.2. Key nonlinear estimates and well-posedness

In view of Lemma 6.1, it suffices to establish a local solution to (1.1) in $Z_{\frac{2}{\gamma}}^p(T) \cap Z_q^2(T)$ for some q. To this end we construct key trilinear estimates for the Duhamel type operator as in Section 4 in order to obtain a solution v to the transformed integral equation (3.2). Observe that thanks to the unitarity property (1.16), $v \in C([0,T] : \widehat{L}^p)$ implies $u(t) \triangleq U(t)v(t) \in C([0,T] : \widehat{L}^p)$.

Let $\mathscr{D}_\gamma, \mathscr{D}_\gamma^j$, $j = 1, 2$ be as in Section 4. Arguing as in the proof of (4.3) and (4.4), we get:

Lemma 6.2. *Let* $0 < \gamma < n$. *Then*

$$\|\mathscr{D}^1_\gamma(v_1, v_2, v_3)\|_{\tilde{W}^\infty_\infty(T)} \le C\|v_1\|_{\tilde{Z}^2_1(T)}\|v_2\|_{\tilde{Z}^2_1(T)}\|v_3\|_{\tilde{Z}^\infty_1(T)}$$

and

$$\|\mathscr{D}^2_\gamma(v_1, v_2, v_3)\|_{\tilde{W}^\infty_\infty(T)} \le C\prod_{l=1}^{3}\|v_l\|_{\tilde{Z}^\infty_1(T)}.$$

Since $\widehat{L}^2 = L^2$, we have $\tilde{W}^2_q = \tilde{X}^2_{q,0}$ and $\tilde{Z}^2_q = \tilde{Y}^2_{q,0}$. Therefore, by the L^2 estimates established in Section 4, we have

$$\|\mathscr{D}^j_\gamma(v_1, v_2, v_3)\|_{\tilde{W}^2_{\frac{2}{\gamma}}(T)} \le C\prod_{l=1}^{3}\|v_l\|_{\tilde{Z}^2_{1,0}(T)} \tag{6.1}$$

for $j = 1, 2$ and $0 < \gamma < 2$. These estimates yield our key estimates for the well-posedness results in $\widehat{L}^p$:

Proposition 6.3. *Assume that* $0 < \gamma < 2$ *and* $2 \le p \le \infty$. *Then*

$$\|\mathscr{D}^1_\gamma(v_1, v_2, v_3)\|_{\tilde{W}^p_q(T)} \le C\|v_1\|_{\tilde{Z}^2_1(T)}\|v_2\|_{\tilde{Z}^2_1(T)}\|v_3\|_{\tilde{Z}^p_1(T)} \tag{6.2}$$

and

$$\|\mathscr{D}^2_\gamma(v_1, v_2, v_3)\|_{\tilde{W}^p_q(T)} \le C\prod_{l=1}^{3}\|v_l\|_{\tilde{Z}^p_1(T)}, \tag{6.3}$$

where

$$q = \frac{p}{\gamma(p-1)}.$$

Now, arguing as in subsection 4.2, we can establish a local solution u to (1.1) in $Z^2_{\frac{2}{\gamma}}(T) \cap Z^p_q(T) \hookrightarrow C([0, T] : L^2 \cap \widehat{L}^p)$ for a suitable $T > 0$. Proceeding as in section 5, we can extend the local solution to global one.

7. Proof of Theorem 1.5 and 1.11

This section is devoted to the proof of the two improved global well-posedness results in one space dimension. As seen in Section 5, it is enough to show the following estimates.

Proposition 7.1. *Assume that* $4/3 < p \le 2$ *and* $1/2 \le \gamma < 1$. *Let* $T_0 > 0$ *and let* $v : [0, T_0] \times \mathbb{R} \to \mathbb{C}$ *be a local solution to the transformed integral equation (3.2) such that*

$$v|_{[0,T]\times\mathbb{R}} \in \tilde{Y}^p_{q,\theta}(T) \cap \tilde{Y}^2_{\frac{2}{\gamma},0}(T)$$

for any $T \in (0, T_0)$. *Then*

$$\sup_{t\in[0,T_0)} \|v(t)\|_{L^p} < \infty.$$

Proposition 7.2. *Assume that* $2 \le p < 4$ *and* $1/2 \le \gamma < 1$. *Let* $T_0 > 0$ *and let* $u : [0, T_0] \times \mathbb{R} \to \mathbb{C}$ *be a local solution to the original Cauchy problem (3.1) such that*

$$u|_{[0,T]\times\mathbb{R}} \in Z^p_{\frac{2}{\gamma}}(T) \cap Z^2_q(T)$$

for any $T \in (0, T_0)$. *Then*

$$\sup_{t\in[0,T_0)} \|u(t)\|_{\widehat{L}^p} < \infty.$$

7.1. Generalized Strichartz estimates

In Section 5, we estimated the norm $\|\widehat{H}_\gamma(v,v,v)\|_{L^p}$ (resp. $\|H_\gamma(u,u,u)\|_{\widehat{L}^p}$) to obtain Theorem 1.2 and 1.4 (resp. Theorem 1.9), which is essentially due to Carles and Mouzaoui [2]. In this section we use generalized dispersive estimates to control the Duhamel term of the integral equations rather than the direct L^p (resp. $\widehat{L}^p$) estimates of the Hartree nonlinearity. The key estimate to Proposition 7.1–7.2 is as follows.

Lemma 7.3. *Assume that* $4/3 < p \le 2$*. Then the estimate*

$$\|U(t)\phi\|_{L^{3p}_{xt}(\mathbb{R}\times\mathbb{R})} \le C\|\phi\|_{\widehat{L}^p} \tag{7.1}$$

holds true.

Note that the generalized Strichartz estimate (7.1) is useful in proving the existence and well-posedness results for nonlinear Schrödinger equations in $\widehat{L}^p$ or similar spaces. See e.g. [4], [8], [9]. Note also that estimates of this type go back to Fefferman and Stein, see [6]. Here in this paper, we exploit this estimate to handle the Duhamel term of the integral equation. By the standard duality argument, we get:

Corollary 7.4. *Assume that* $2 \le p < 4$*. Let* $J \subset \mathbb{R}$*. Then the estimate*

$$\sup_{I\in\mathscr{I}(J)} \left\| \int_I U(-s)F(s)ds \right\|_{\widehat{L}^p(\mathbb{R})} \le \|F\|_{L^r_{xt}(J\times\mathbb{R})} \tag{7.2}$$

holds true, where

$$r = (3p')' = \left(\frac{2}{3}+\frac{1}{3p}\right)^{-1} \tag{7.3}$$

and

$$\mathscr{I}(J) \triangleq \{I \subset \mathbb{R} : \text{interval} \mid I \subset J\}.$$

7.2. Proof of Proposition 7.2

We first prove Proposition 7.1 which is relatively simpler. We estimate the right hand side of the corresponding integral equation

$$u(t) = U(t)u(0) + i\int_0^t U(t-s)H_\gamma(u(s),u(s),u(s))ds.$$

We fix $T \in (0,T_0)$. By Corollary 7.4, we have

$$\left\| \int_0^t U(t-s)H_\gamma(u(s),u(s),u(s))ds \right\|_{\widehat{L}^p} \le C\|H_\gamma(u,u,u)\|_{L^r([0,t]\times\mathbb{R})}$$

for any $t \in [0,T]$, where r is given by (7.3).

By the Hölder, Hausdorff–Young, and Hardy–Littlewood–Sobolev inequalities, we have

$$\begin{aligned}\|H_\gamma(u(s),u(s),u(s))\|_{L^r_x} &\le \||\cdot|^{-\gamma} * (u(s)\overline{u(s)})\|_{L^{\tilde{R}}} \times \|u(s)\|_{\widehat{L}^p}\\ &\le \|u(s)\overline{u(s)}\|_{L^R} \times \|u(s)\|_{\widehat{L}^p}\\ &= \|u(s)\|^2_{L^{2R}}\|u(s)\|_{\widehat{L}^p},\end{aligned}$$

for $s \in [0,t]$, where

$$R=\left(\frac{5}{3}-\gamma-\frac{2}{3p}\right)^{-1}, \quad \tilde{R}=\frac{3p'}{2}.$$

Now taking $L^r([0,t])$-norm of both sides, we get

$$\left\| \int_0^t U(t-s)H_\gamma(u(s),u(s),u(s))ds \right\|_{\widehat{L}^p} \le C\|u\|^2_{L^Q_{T_0}(L^{2R})}\|u\|_{L^{\frac{2}{2-\gamma}}([0,t];\widehat{L}^p)},$$

where

$$Q=\left(\frac{\gamma}{4}+\frac{1}{6p}-\frac{1}{6}\right)^{-1}$$

and observe that $(Q,2R)$ is admissible. Consequently, we see that

$$\|u(t)\|_{\widehat{L}^p} \le C + C_{T_0}\|u\|_{L^{\frac{2}{2-\gamma}}([0,t];\widehat{L}^p)}$$

for any $t\in[0,T]$. Arguing as in Section 5, we obtain the wanted estimate.

7.3. Proof of Proposition 7.1

We estimate the L^p-norm of the solution v to the transformed integral equation (3.2). Denote $[Iv](t)$ by the Duhamel contribution of the integral equation (3.2), that is,

$$[Iv](t) \triangleq \int_0^t s^{-\gamma}M_s^{-1}\widehat{H}_\gamma(M_s v(s), RM_t v(t), M_t v(t)).$$

As earlier, we use the convention that $u(t)\triangleq U(t)v(t),\ v(t)=U(-t)u(t)$. We also write

$$\mathscr{V}(t)\triangleq M_t v(t).$$

Using the symbol Θu given by (5.2), we may write

$$\widehat{H}_\gamma(M_s v(t), RM_t v(t), M_t v(t)) = \left[|x|^{-(n-\gamma)}\Theta u(t)\right]*(M_t v(t)).$$

The key is to notice that $\mathscr{F}M_t\mathscr{F}^{-1}=cU(-t^{-1})$ for $t\neq 0$ and to exploit the first oscillating term in (2.3), which was ignored in the proofs in Section 3–5. Fix $t\in(0,T]$. Then we have

$$\begin{aligned}
\|Iv(t)\|_{L^p} &= \|\mathscr{F}^{-1}I(v)\|_{\widehat{L}^{p'}} = \|\overline{\mathscr{F}\overline{Iv(t)}}\|_{\widehat{L}^{p'}} = \|\mathscr{F}\overline{Iv(t)}\|_{\widehat{L}^{p'}} \\
&= \left\| \int_0^t s^{-\gamma}\left(\mathscr{F}M_s\mathscr{F}^{-1}\right)\mathscr{F}\overline{\widehat{H}_\gamma(\mathscr{V}(s),\mathscr{V}(s),\mathscr{V}(s))}ds \right\|_{\widehat{L}^{p'}} \\
&= \left\| \int_0^t s^{-\gamma}U(-1/s)\mathscr{F}\overline{\widehat{H}_\gamma(\mathscr{V}(s),\mathscr{V}(s),\mathscr{V}(s))}ds \right\|_{\widehat{L}^{p'}} \\
&= \left\| \int_{t^{-1}}^\infty \tau^{\gamma-2}U(-\tau)\mathscr{F}\overline{\widehat{H}_\gamma(\mathscr{V}(1/\tau),\mathscr{V}(1/\tau),\mathscr{V}(1/\tau))}d\tau \right\|_{\widehat{L}^{p'}}.
\end{aligned}$$

Now applying Corollary 7.4 to the last term, we see that it is controlled by

$$C\left\|\tau^{\gamma-2}\mathscr{F}\widehat{H}_\gamma(\mathscr{V}(1/\tau),\mathscr{V}(1/\tau),\mathscr{V}(1/\tau))\right\|_{L^{\tilde{r}}_{x\tau}([t^{-1},\infty)\times\mathbb{R})},$$

where

$$\tilde{r} = \left(1 - \frac{1}{3p}\right)^{-1}.$$

After the change of variable, this is equal to

$$\|s^{2-\gamma-\frac{2}{\tilde{r}}}\mathscr{F}\widehat{H}_\gamma(\mathscr{V}(s),\mathscr{V}(s),\mathscr{V}(s))\|_{L^{\tilde{r}}_{xs}([0,t]\times\mathbb{R})}.$$

Then, by the Hölder, Hausdorff–Young, and Hardy–Littlewood–Sobolev inequalities, we have

$$\begin{aligned}\|\mathscr{F}\widehat{H}_\gamma(\mathscr{V}(s),\mathscr{V}(s),\mathscr{V}(s))\|_{L^{\tilde{r}}_x(\mathbb{R})} &\le \left\|\mathscr{F}\left[|\cdot|^{-(n-\gamma)}\Theta u(s)\right]\right\|_{L^{\frac{3p}{2}}}\|\mathscr{F}M_s v(s)\|_{L^{p'}}\\ &\le C\||\cdot|^{-\gamma} * \widehat{\Theta u(s)}\|_{L^{\frac{3p}{2}}}\|v(s)\|_{L^p}\\ &\le C\|\widehat{\Theta u(s)}\|_{L^{\tilde{R}}}\|v(s)\|_{L^p},\end{aligned}$$

where

$$\tilde{R} = \left(1 + \frac{2}{3p} - \gamma\right)^{-1}.$$

Applying Lemma 5.3, we see that the first norm of the right hand side is equal to

$$Cs^{1-\frac{1}{\tilde{R}}}\|u(s)\|^2_{L^{2\tilde{R}}}.$$

Finally, taking $L^{\tilde{r}}([0,T], t^{\tilde{r}(\gamma-2)-2}dt)$-norm and applying Hölder's inequality, we have

$$\|[Iv](t)\|_{L^p} \le C\|u\|^2_{L^{\tilde{Q}}_{T_0}(L^{2\tilde{R}})}\|v\|_{L^{\frac{2}{2-\gamma}}([0,t]:L^p)}, \qquad \forall t \in [0,T],$$

where

$$\tilde{Q} = \left(\frac{\gamma}{4} - \frac{1}{6p}\right)^{-1}$$

and note that $(\tilde{Q}, 2\tilde{R})$ is admissible. Consequently, we have

$$\|v(t)\|_{L^p} \le C + C_{T_0}\|v\|_{L^{\frac{2}{2-\gamma}}([0,t]:L^p)}, \qquad \forall t \in [0,T]$$

from which we deduce the wanted estimate by arguing as in Section 5.

Conflict of interest statement

There is no conflict of interest.

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