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Hyperbolic ends with particles and grafting on singular surfaces

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Abstract

We prove that any 3-dimensional hyperbolic end with particles (cone singularities along infinite curves of angles less than π) admits a unique foliation by constant Gauss curvature surfaces. Using a form of duality between hyperbolic ends with particles and convex globally hyperbolic maximal (GHM) de Sitter spacetime with particles, it follows that any 3-dimensional convex GHM de Sitter spacetime with particles also admits a unique foliation by constant Gauss curvature surfaces. We prove that the grafting map from the product of Teichmüller space with the space of measured laminations to the space of complex projective structures is a homeomorphism for surfaces with cone singularities of angles less than π, as well as an analogue when grafting is replaced by "smooth grafting".

Keywords: Hyperbolic ends; Particles; Complex projective structures; Cone singularities; Constant Gauss curvature; Foliations

1. Introduction

Let $\theta = (\theta_1, ..., \theta_{n_0}) \in (0, \pi)^{n_0}$. In this paper we consider an oriented connected closed surface Σ of genus $\mathfrak{g}$ with n_0 marked points $p_1, ..., p_{n_0}$ and suppose that

$$2\pi(2 - 2\mathfrak{g}) + \sum_{i=1}^{n_0}(\theta_i - 2\pi) < 0. \qquad (1)$$

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This ensures that Σ can be equipped with a hyperbolic metric with cone singularities of angle θ_i at the marked points p_i for $i = 1, ..., n_0$ (see e.g. [25,33]). We denote by $\mathcal{T}_{\Sigma,\theta}$ the Teichmüller space of hyperbolic metrics on Σ with fixed cone angles, which is the space of hyperbolic metrics on Σ with cone singularities of angle θ_i at p_i, considered up to isotopy fixing each marked point (see more precisely Section 2.1). We also denote $\mathfrak{p} = (p_1, \cdots, p_{n_0})$, and let $\mathcal{ML}_{\mathfrak{p}}$ be the space of measured laminations on $\Sigma_{\mathfrak{p}} = \Sigma \setminus \{p_1, \cdots, p_{n_0}\}$. It is well-known that for all $g \in \mathcal{T}_{\Sigma,\theta}$, any $l \in \mathcal{ML}_{\mathfrak{p}}$ can be uniquely realized as a geodesic measured lamination on (Σ, g).

1.1. Hyperbolic ends with particles

We are interested in non-complete 3-dimensional hyperbolic manifolds homeomorphic to $\Sigma \times \mathbb{R}$, with cone singularities of angle θ_i along $\{p_i\} \times \mathbb{R}$, for all $i \in \{1, \cdots, n_0\}$. A relatively simple space of metrics of this type is provided by the *quasifuchsian metrics with particles* studied e.g. in [17,26]: complete hyperbolic cone-manifolds containing a non-empty, compact, convex subset.

Those quasifuchsian manifolds with particles contain a smallest non-empty convex subset, called their *convex core*. The complement of the convex core is the disjoint union of two non-complete manifolds, each homeomorphic to $\Sigma \times (0, +\infty)$, complete on the $+\infty$ side, but bounded on the 0 side by a concave pleated surface orthogonal to the particles. Moreover their boundary at infinity is endowed with a complex projective structure, with cone singularities of angle θ_i at the endpoint at infinity of the particle $\{p_i\} \times (0, +\infty)$.

Here we are interested in *non-degenerate hyperbolic ends with particles*: non-complete hyperbolic manifolds homeomorphic to $\Sigma \times (0, +\infty)$, with cone singularities of angle θ_i along $\{p_i\} \times (0, +\infty)$, complete on the $+\infty$ side, and bounded by a concave pleated surface orthogonal to the particles (see Definition 2.7 for more details). For instance, the complement of the convex core of a quasifuchsian manifold with particles is the disjoint union of two non-degenerate hyperbolic ends with particles. We call $\mathcal{HE}_\theta$ the space of those non-degenerate hyperbolic ends with particles, up to isotopy fixing each singular curve.

Our first result is a one-to-one correspondence between those hyperbolic ends and complex projective structures on Σ with cone singularities of prescribed angle at the p_i.

Theorem 1.1. *For each hyperbolic end $M \in \mathcal{HE}_\theta$, the boundary at infinity $\partial_\infty M$ is equipped with a complex projective structure with cone singularities of angle θ_i at the p_i. Conversely, any complex projective structure on Σ with cone singularities of angle θ_i at the p_i is obtained at infinity from a unique hyperbolic end $M \in \mathcal{HE}_\theta$.*

We will denote by $\mathcal{CP}_\theta$ the space of complex projective structures on Σ with cone singularities of angle θ_i at the p_i, considered up to isotopy fixing the marked points.

1.2. Grafting on hyperbolic surfaces with cone singularities

Given a hyperbolic end $M \in \mathcal{HE}_\theta$, its concave pleated boundary is equipped with a hyperbolic metric m with cone singularities of angle θ_i at the p_i. Moreover, it is pleated along a measured geodesic lamination l. We prove in Section 3.9 that its complex projective structure at infinity σ is obtained by a grafting operation, applied along l to the Fuchsian complex projective structure associated to (Σ, m). Moreover, we will show that it follows from Theorem 1.1 that any complex projective structure $\sigma \in \mathcal{CP}_\theta$ is obtained uniquely in this manner. The following statement, extending a classical result of Thurston (see e.g. [15, Theorem 4.1]) to hyperbolic surfaces with cone singularities, will be a consequence.

Theorem 1.2. *The grafting map defined for non-singular hyperbolic surfaces extends to a map $Gr_\theta : \mathcal{T}_{\Sigma,\theta} \times \mathcal{ML}_{\mathfrak{p}} \to \mathcal{CP}_\theta$. This map is a homeomorphism.*

1.3. Foliations of hyperbolic ends with particles by K-surfaces

We also prove that our non-degenerate hyperbolic ends with particles have a unique foliation by surfaces of constant (Gauss) curvature, extending a result of Labourie [23, Theorem 1].

Theorem 1.3. *Let $M \in \mathcal{HE}_\theta$ be a non-degenerate hyperbolic end with particles. There is a unique foliation of M by surfaces of constant curvature K with K varying from -1 near the concave pleated boundary to 0 near the boundary at infinity. Moreover, for each $K \in (-1, 0)$, M contains a unique closed surface of constant curvature K.*

1.4. De Sitter spacetimes with particles

Given a non-singular hyperbolic end M, there is a "dual" future-complete globally hyperbolic maximal de Sitter spacetime M^d. There are several ways to describe this duality, but one way is by noting that future-complete globally hyperbolic maximal de Sitter spacetimes are equipped with a complex projective structure at infinity (see [1,24,27]) that uniquely determines them. The complex projective structure defined at infinity by M and M^d are identical.

We extend this point of view to future-complete convex globally hyperbolic maximal (GHM) de Sitter spacetimes with particles, as defined in Section 2.4.

We denote by $\mathcal{DS}_\theta$ the space of future-complete convex GHM de Sitter spacetimes homeomorphic to $\Sigma \times (0, +\infty)$, with cone singularities of angle θ_i along $\{p_i\} \times (0, +\infty)$, up to isotopy fixing each singular line.

Theorem 1.4. *Any future-complete convex GHM de Sitter spacetime $M^d \in \mathcal{DS}_\theta$ determines on Σ a complex projective structure with cone singularities of angle θ_i at the p_i. Any complex projective structure $\sigma \in \mathcal{CP}_\theta$ is obtained from a unique $M^d \in \mathcal{DS}_\theta$.*

This result, along with Theorem 1.1, determines a natural map from $\mathcal{HE}_\theta$ to $\mathcal{DS}_\theta$ sending a hyperbolic end with particles to the unique future-complete convex GHM de Sitter spacetime with the same complex projective structure at infinity.

This duality extends to closed strictly concave surfaces (orthogonal to the particles) in those hyperbolic ends and closed strictly future-convex surfaces (orthogonal to the particles) in the corresponding de Sitter spacetimes.

Theorem 1.5. *Let $M \in \mathcal{HE}_\theta$ be a non-degenerate hyperbolic end with particles, and let $M^d \in \mathcal{DS}_\theta$ be the dual future-complete convex GHM de Sitter spacetime with particles. Given a closed, strictly concave surface $S \subset M$, there is a unique strictly future-convex spacelike surface S^d and a unique diffeomorphism $u : S \to S^d$ such that $u^*I^d = III$ and $u^*III^d = I$, where I, III are the induced metric and third fundamental form on S, and I^d and III^d are the induced metric and third fundamental form on S^d.*

Conversely, given any space-like, strictly future-convex surface S^d in M^d, there is a unique strictly concave surface S in M such that S^d is the dual of S in the sense of Theorem 1.5.

Proposition 1.6. *Let S be a strictly concave surface in M, and let S^d be the dual surface in M^d. Then S has constant curvature $K \in (-1, 0)$ if and only if S^d has constant curvature $K^d = K/(K+1) \in (-\infty, 0)$.*

1.5. Foliations of de Sitter spacetimes with particles by K-surfaces

As a consequence of Proposition 1.6, each foliation of a non-degenerate hyperbolic end with particles has a dual foliation of the dual future-complete convex GHM de Sitter space-time. We therefore obtain the following.

Corollary 1.7. *Let $M^d \in \mathcal{DS}_\theta$ be a future-complete convex GHM de Sitter spacetime with particles. There is a unique foliation of M^d by surfaces of constant curvature K^d with K^d varying from $-\infty$ near the initial singularity to 0 near the boundary at infinity. Moreover, for each $K^d \in (-\infty, 0)$, M^d contains a unique closed surface of constant curvature K^d.*

This gives an affirmative answer to Question 6.4 in [18], and generalizes a result about constant Gauss curvature foliation of future-complete globally hyperbolic maximal compact de Sitter spacetimes (see [4, Theorem 2.1]) to the case with particles.

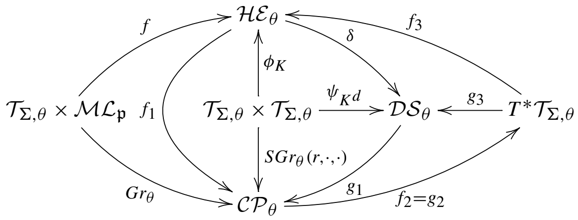

Fig. 1. A diagram showing the relations among all the spaces.

1.6. Smooth grafting on hyperbolic surfaces with cone singularities

Constant Gauss curvature surfaces in hyperbolic ends are related to the "smooth grafting" map $SGr : (0,1) \times \mathcal{T} \times \mathcal{T} \to \mathcal{CP}$, see [8, Section 1.2]. The properties of K-surfaces in hyperbolic ends with particles as described here show that this "smooth grafting" map is still well-defined on hyperbolic surfaces with cone singularities of angles less than π, as a map SGr_θ from $(0,1) \times \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$ to $\mathcal{CP}_\theta$.

For each $K \in (-1, 0)$, we construct a map $\phi_K : \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta} \to \mathcal{HE}_\theta$ sending a pair of hyperbolic metrics $(h, h') \in \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$ with cone singularities to the unique hyperbolic end with particles containing a constant curvature K surface with induced metric homothetic to h and third fundamental form homothetic to h'. We prove that ϕ_K is a homeomorphism (see Proposition 5.3) and that $\mathcal{HE}_\theta$ is parameterized by a homeomorphism $f_1 : \mathcal{HE}_\theta \to \mathcal{CP}_\theta$ (see Proposition 3.11). For each $r \in (0,1)$, we define $SGr_\theta(r,\cdot,\cdot)$ to be $f_1 \circ \phi_K : \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta} \to \mathcal{CP}_\theta$, where $K = -4r/(1+r)^2$. The applications of constant Gauss curvature foliations in hyperbolic ends with particles and smooth grafting on hyperbolic surfaces with cone singularities are outlined in Section 5.6.

This implies that for all $r \in (0,1)$, the map $SGr_\theta(r,\cdot,\cdot)$ is a homeomorphism from $\mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$ to $\mathcal{CP}_\theta$. We do not elaborate on this point here, since it follows from the same arguments as in the non-singular case, see [8]. The relations among all the spaces we consider throughout this paper are presented in Fig. 1, which is a combination of Fig. 2, Fig. 3 and Fig. 4, in Section 3.8, Section 4.2 and Section 6.2 respectively.

1.7. Outline of the paper

Section 2 contains the background material on various notions used in the paper.

In Section 3 we analyse the complex projective structure at infinity of a hyperbolic end with particles, and show that a hyperbolic end with particles is uniquely determined by either a complex projective structure with cone singularities, or a meromorphic quadratic differential (with respect to a conformal structure) on Σ with at worst simple poles at the marked points. We also describe the induced metric and pleating data on the "compact" boundary of a hyperbolic end with particles, and show that a hyperbolic end with particles is uniquely determined by a hyperbolic metric with cone singularities along with a measured lamination. As a consequence, we obtain at the end of Section 3 the proof of Theorem 1.2, on the grafting map for surfaces with cone singularities.

The same analysis is conducted in Section 4 for convex GHM de Sitter spacetimes with particles. The two constructions, taken together, allow for the definition of the duality between hyperbolic ends with particles and convex GHM de Sitter spacetimes with particles, and some key properties of this duality are developed.

We then turn in Section 5 to constant Gauss curvature surfaces in hyperbolic ends with particles, and show how a pair of hyperbolic metrics with cone singularities uniquely determine a hyperbolic end with particles.

Finally, Section 6 deals with convex GHM de Sitter spacetimes with particles, develops the duality relation between hyperbolic ends with particles and convex GHM de Sitter spacetimes with particles, and obtains the results on constant Gauss curvature surfaces in those de Sitter spacetimes.

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2. Background material

2.1. Hyperbolic surfaces with cone singularities

First we recall the local model of a hyperbolic metric with a cone singularity of angle θ_0 on surfaces.

Let $\mathbb{H}^2$ be the Poincaré model of the hyperbolic plane. Denote by $\mathbb{H}^2_{\theta_0}$ the space obtained by taking a wedge of angle θ_0 bounded by two half-lines intersecting at the center 0 of $\mathbb{H}^2$ and gluing the two half-lines by a rotation fixing 0. We call $\mathbb{H}^2_{\theta_0}$ the *hyperbolic disk with cone singularity of angle* θ_0, which is a punctured disk with the induced metric

$$g_{\theta_0} = dr^2 + \sinh^2(r)d\alpha^2,$$

where $(r,\alpha) \in \mathbb{R}_{>0} \times \mathbb{R}/\theta_0\mathbb{Z}$ is a polar coordinate of $\mathbb{H}^2_{\theta_0}$.

Note that the hyperbolic metrics near the cone singularities throughout this paper are assumed to satisfy a regularity condition. This ensures the existence of harmonic maps from Riemann surfaces with marked points to hyperbolic surfaces with cone singularities at the marked points (see [16, Theorem 2]), so that we can relate minimal Lagrangian maps (see Definition 2.12) to harmonic maps, and apply the result in [12, Lemma 3.19] to show the continuity of the parametrization map ϕ_K of $\mathcal{HE}_\theta$ (see Section 5.3). This regularity condition is defined by using the weighted Hölder spaces (see [16, Section 2.2] and [31, Definition 2.1]).

Definition 2.1. For $R > 0$, let $D(R) := \{z \in \mathbb{C}, 0 < |z| \leq R\}$. A function $f : D(R) \to \mathbb{C}$ is said to be in $\chi_b^{0,\gamma}(D(R))$ with $\gamma \in (0,1)$ if

$$||f||_{\chi_b^{0,\gamma}} := \sup_{z\in D(R)} |f(z)| + \sup_{z,z'\in D(R)} \frac{|f(z) - f(z')|}{|\alpha - \alpha'|^\gamma + |\frac{r-r'}{r+r'}|^\gamma} < \infty,$$

where $z = re^{i\alpha}$ and $z' = r'e^{i\alpha'}$. Let $k \in \mathbb{N}$, we say that $f \in \chi_b^{k,\gamma}(D(R))$ if $(r\partial_r)^i \partial_\alpha^j f$ is in $\chi_b^{0,\gamma}(D(R))$ for all $i + j \leq k$. In particular, this implies that $f \in \mathcal{C}^k(D(R))$.

The space $\chi_b^{k,\gamma}(D(R))$ has an alternative characterization (see [16, Section 2.2]), that is, the space of functions from $D(R)$ to $\mathbb{C}$ with $\mathcal{C}^{k,\gamma}$ norm uniformly bounded on balls of uniform size with respect to the metric $(dr^2 + r^2d\alpha^2)/r^2$. Under the transformation $w(z) = \log z$ and let $\Omega = w(D(R))$, this space has a simpler description, denoted by $\mathcal{C}^{k,\gamma}(\Omega)$, which consists of functions from Ω to $\mathbb{C}$ with $\mathcal{C}^{k,\gamma}$ norm uniformly bounded on balls of uniform size with respect to the Euclidean metric $d\rho^2 + d\beta^2$, where $w = \rho + i\beta$.

Definition 2.2. Let Σ be a Riemann surface, let $\mathfrak{p} = (p_1, ..., p_{n_0}) \in \Sigma^{n_0}$, and let $\theta = (\theta_1, ..., \theta_{n_0}) \in (0,\pi)^{n_0}$. Let $\gamma_0 \in (0,1)$. A hyperbolic metric on Σ with cone singularities of angle θ at $\mathfrak{p}$ is a (singular) metric g on $\Sigma_\mathfrak{p}$ with the property that for each compact subset $K \subset \Sigma_\mathfrak{p}$, $g|_K$ is $\mathcal{C}^2$ and has constant curvature -1, and for each marked point p_i, there exists a neighborhood $U_i \subset \Sigma$ with local conformal coordinates z centered at p_i and a local diffeomorphism $\psi \in \chi_b^{2,\gamma_0}(U_i \setminus \{p_i\})$ such that $g|_{U_i\setminus\{p_i\}}$ is the pull back by ψ of the metric g_{θ_i}. We denote by $\mathfrak{M}^\theta_{-1}$ the space of hyperbolic metrics on Σ with cone singularities of angle θ at $\mathfrak{p}$.

We say that f is a *diffeomorphism* of $\Sigma_\mathfrak{p}$ if for each compact subset $K \subset \Sigma_\mathfrak{p}$, $f|_K$ is of class $\mathcal{C}^3$ and for each marked point p_i, there exists a neighborhood $U_i \subset \Sigma$ of p_i such that $f|_{U_i\setminus\{p_i\}} \in \chi_b^{2,\gamma_0}(U_i \setminus \{p_i\})$. Denote by $\mathfrak{Diff}_0(\Sigma_\mathfrak{p})$ the space of diffeomorphisms on $\Sigma_\mathfrak{p}$ which are isotopic to the identity (fixing each marked point). They act by pull-back on $\mathfrak{M}^\theta_{-1}$. We say that two metrics $h_1, h_2 \in \mathfrak{M}^\theta_{-1}$ are *isotopic* if there exists a map $f \in \mathfrak{Diff}_0(\Sigma_\mathfrak{p})$ such that h_1 is the pull back by f of h_2.

Note that $\mathfrak{Diff}_0(\Sigma_\mathfrak{p})$ is indeed a group. Since $\mathcal{C}^{k,\gamma}(\Omega) \cap \mathfrak{Diff}(\Omega)$ is closed under composition and inverse for any fixed $k \geq 1$ and $\gamma \in (0,1)$ (see e.g. [6, Theorem 2.1, Lemma 2.3]), then the same holds for $\chi_b^{k,\gamma}(D(R)) \cap \mathfrak{Diff}(D(R))$, where $\mathfrak{Diff}(\Omega)$ is the space of self-diffeomorphisms of Ω and similarly for $\mathfrak{Diff}(D(R))$. Combined with the fact that all the maps in $\mathfrak{Diff}_0(\Sigma_\mathfrak{p})$ fix each marked point p_i, this shows that $\mathfrak{Diff}_0(\Sigma_\mathfrak{p})$ is a group.

Denote by $\mathcal{T}_{\Sigma,\theta}$ the space of isotopy classes of hyperbolic metrics on Σ with cone singularities of angle θ at $\mathfrak{p}$. Note that $\mathcal{T}_{\Sigma,\theta} = \mathfrak{M}^\theta_{-1}/\mathfrak{Diff}_0(\Sigma_\mathfrak{p})$ and $\mathfrak{M}^\theta_{-1}$ is a differentiable submanifold of the manifold consisting of all $\mathcal{H}^2$ symmetric

(0,2)-type tensor fields (see e.g. [32] for non-singular case). $\mathcal{T}_{\Sigma,\theta}$ is a finite-dimensional differentiable manifold which inherits a natural quotient topology.

2.2. Hyperbolic 3-dimensional manifolds with particles

First we recall the related notations and terminology in order to define hyperbolic manifolds with particles.

2.2.1. Hyperbolic 3-space

Let $\mathbb{R}^{3,1}$ be $\mathbb{R}^4$ with the quadratic form $q(x) = x_1^2 + x_2^2 + x_3^2 - x_4^2$. The *hyperbolic 3-space* is defined as the quadric:

$$\mathbb{H}^3 = \{x \in \mathbb{R}^{3,1} : q(x) = -1,\ x_4 > 0\}.$$

It is a 3-dimensional Riemannian symmetric space of constant curvature -1, diffeomorphic to a 3-dimensional open ball B^3. The group $\mathrm{Isom}_0(\mathbb{H}^3)$ of orientation preserving isometries of $\mathbb{H}^3$ is $SO^+(3,1) \cong PSL_2(\mathbb{C})$.

2.2.2. The singular hyperbolic 3-space

Let $\theta_0 > 0$. We define the *singular hyperbolic 3-space with cone singularities of angle* θ_0 as the space

$$\mathbb{H}^3_{\theta_0} := \{(\rho, r, \alpha) \in \mathbb{R} \times \mathbb{R}_{>0} \times \mathbb{R}/\theta_0\mathbb{Z}\}$$

with the metric

$$d\rho^2 + \cosh^2(\rho)(dr^2 + \sinh^2(r)d\alpha^2).$$

The set $\{r = 0\}$ is called the *singular line* in $\mathbb{H}^3_{\theta_0}$ and θ_0 is called the *total angle* around this singular line.

A direct computation shows that $\mathbb{H}^3_{\theta_0}$ has constant curvature -1 outside the singular line. Indeed, it is obtained from the hyperbolic plane with a cone singularity of angle θ_0 by taking a warped product with $\mathbb{R}$ (see e.g. [18]).

2.2.3. Hyperbolic manifold with particles

A *hyperbolic manifold with particles* is a 3-manifold endowed with a metric for which each point has a neighborhood isometric to a subset of $\mathbb{H}^3_{\theta_0}$ for some $\theta_0 \in (0, \pi)$.

In a hyperbolic manifold M with particles, those points which have a neighborhood isometric to a neighborhood of a point of some $\mathbb{H}^3_{\theta_0}$ outside the singular line are called *regular points*, while the others are called *singular points*. We denote by M_r the set of regular points and by M_s the set of singular points. By definition, M_s is a disjoint union of curves. To each of those curves is associated a number, which is equal at each point to the number θ_0 in the definition, called the *total angle* around the singular curve (see e.g. [17–19]).

Definition 2.3. We say that B is a regular half-ball in $\mathbb{H}^3_{\theta_0}$ if it is isometric to the interior of a hyperbolic half-ball in $\mathbb{H}^3$. We say that B is a singular half-ball in $\mathbb{H}^3_{\theta_0}$ if it can be written as the subset $\{x \in \mathbb{H}^3_{\theta_0} : \rho > 0, d(x, O) < r_0\}$ for some $r_0 > 0$, where $O = (0, 0, 0) \in \mathbb{H}^3_{\theta_0}$ and d is the hyperbolic distance induced by the metric on $\mathbb{H}^3_{\theta_0}$.

Definition 2.4. Let $S \subset \mathbb{H}^3_{\theta_0}$ be a surface which intersects the singular line at a point x. S is orthogonal to the singular line at x if the distance from a point y of S to the totally geodesic plane P orthogonal to the singular line at x satisfies:

$$\lim_{y \in S, y \to x} \frac{d(y, P)}{d_S(x, y)} = 0,$$

where $d_S(x, y)$ is the distance between x and y with respect to the induced metric on S.

If now S is a surface in a hyperbolic manifold M with particles which intersects a singular curve l at a point x', S is said to be orthogonal to l at x' if there exists a neighborhood U of x' in M which is isometric to a neighborhood of a singular point in $\mathbb{H}^3_{\theta_0}$ such that the isometry sends $S \cap U$ to a surface orthogonal to the singular line in $\mathbb{H}^3_{\theta_0}$. We say that S is orthogonal to the singular locus if S is orthogonal to the singular curve of M at each intersection with the singular locus.

Definition 2.5. Let M be a hyperbolic manifold with particles and let Ω be a subset of the metric completion $\bar{M}$ of M. We say Ω is *concave* if there is no geodesic segment in the interior of Ω with endpoints in $\partial\Omega$.

Let M be a hyperbolic manifold with particles which is homeomorphic to $\Sigma \times \mathbb{R}_{>0}$ and has a metric completion $\bar{M}$ homeomorphic to $\Sigma \times \mathbb{R}_{\geq 0}$. We will write that a closed, oriented surface $S \subset \bar{M}$ (which is an incompressible surface in M and homeomorphic to Σ) is *concave* if the connected component of $\bar{M} \setminus S$ on the positive side is concave. We also assume that the surfaces are orthogonal to the singular locus.

It follows from the definition that if S is a concave surface and $x \in S$, there is at least one "local support plane" of S at x in the neighborhood of x, that is, a totally geodesic disk centered at x and not intersecting the negative side of S. In particular, if x is a singular point, then the totally geodesic support disk is orthogonal to the singular curve at x.

2.3. Hyperbolic ends with particles

In this section we consider a hyperbolic manifold with particles M which is homeomorphic to $\Sigma \times \mathbb{R}_{>0}$ and has a metric completion $\bar{M}$ homeomorphic to $\Sigma \times \mathbb{R}_{\geq 0}$. For convenience, we denote by $\partial_\infty M$ the *boundary at infinity* of M, and by $\partial_0 M$ the *metric boundary* $\bar{M} \setminus M$, which therefore corresponds to the surface $\Sigma \times \{0\}$ in the identification of $\bar{M}$ with $\Sigma \times \mathbb{R}_{\geq 0}$. We will suppose that $\partial_0 M$ is concave, in the sense of Definition 2.5, orthogonal to the particles, and that the particles start on $\partial_0 M$ and end on the boundary at infinity of M.

Let $x \in \partial_0 M$, and let $n \in T_x M$ be a non-zero vector. We will say that n is *normal* to $\partial_0 M$ if there is a half-ball centered at x in $\bar{M}$ such that n is normal to the totally geodesic part of the boundary of the half-ball. We denote by $N\partial_0 M$ the space of vectors normals to $\partial_0 M$, so that the fiber of $N\partial_0 M$ over a point where $\partial_0 M$ is totally geodesic is a line, while it is an angular sector over a point of a pleating line of $\partial_0 M$. Given $v = (x, n) \in N\partial_0 M$, we denote by $\exp(v) \in M$ the point $\gamma(1)$, where $\gamma : [0, 1] \to M$ is the geodesic such that $\gamma(0) = x$ and $\gamma'(0) = n$, if it exists. This defines a map, called exp, from a subset of $N\partial_0 M$ to M.

Lemma 2.6. exp *is a homeomorphism from $N\partial_0 M$ to M.*

Proof. Note first that since $\partial_0 M$ is concave and M is hyperbolic, exp is a local diffeomorphism from $N\partial_0 M$ to M, sending the fibers of $N\partial_0 M$ over the singular points to the cone singularities of M.

We will prove that exp is globally injective. Note that exp is injective in the neighborhood of the zero section, that is, there exists $r > 0$ such that if we set

$$N_r\partial_0 M = \{(x, n) \in N\partial_0 M \mid \|n\| < r\} ,$$

then the restriction $\exp_{|N_r\partial_0 M}$ is injective. We call r_0 the supremum of all $r > 0$ such that the restriction of exp to $N_r\partial_0 M$ is injective, and we will prove that $r_0 = \infty$.

Suppose by contradiction that r_0 is finite. It follows from the compactness of $\partial_0 M$ that there exist $(x, v), (y, w) \in N\partial_0 M$ such that $\|v\| = r_0$, $\|w\| \geq r_0$ and that $\exp(x, v) = \exp(y, w)$. Moreover, $\|w\| = r_0$, since otherwise the local injectivity of exp at (x, v) and (y, w) would imply that $\exp_{|N_r\partial_0 M}$ stops being injective for $r < r_0$.

We now consider three cases, depending on whether x and y are regular or singular points of $\partial_0 M$.

- If both x and y are singular points of $\partial_0 M$, then either the cone singularities along the singular curves starting from x and y intersect — this would contradict our definition of a hyperbolic manifold with particles, since the particles must be disjoint — or those cone singularities are in fact on the same singular line. In this second case, there is a singular segment of length $2r_0$ starting from x and ending at y. This would again contradict our definition, since the particles are requested to start on $\partial_0 M$ and end at infinity.
- If both x and y are regular points, then the locally concave surfaces $\exp(\partial(N_{r_0}\partial_0 M)$ must have point of self-tangency at $\exp(x, v) = \exp(y, w)$, again by definition of r_0. It then follows that $\exp(\{x\} \times [0, v]) \cup \exp(\{y\} \times [0, w])$ is a geodesic segment connecting x to y, contradicting the concavity of $\partial_0 M$.
- If x is a singular point and y is regular point of $\partial_0 M$. Then $\exp(\partial(N_{r_0}\partial_0 M)$ intersects the singular curve starting from x at $\exp(x, v) = \exp(y, w)$, and there is no such intersection for $r < r_0$. An elementary geometric argument

shows that this is impossible when the cone angles are less than π, since otherwise $\exp(N_r \partial_0 M)$ would already have self-intersections for $r < r_0$ close enough to r_0.

So we can conclude that $\exp : N\partial_0 M \to M$ is globally injective. It is also proper, and since it is a local homeomorphism in the neighborhood of the zero section, we can conclude that it is a homeomorphism. □

Definition 2.7. A non-degenerate hyperbolic end with particles is a non-complete hyperbolic manifold M with particles which is homeomorphic to $\Sigma \times \mathbb{R}_{>0}$, where Σ is a prescribed closed surface with marked points $\mathfrak{p}$, such that

- It has a metric completion $\bar{M}$ homeomorphic to $\Sigma \times \mathbb{R}_{\geq 0}$, which is complete on the $+\infty$ side.
- The metric boundary $\Sigma \times \{0\}$, which we will denote by $\partial_0 M$, is pleated (i.e. for each $x \in \partial_0 M \setminus \bar{M}_s$, x is contained in the interior of either a geodesic segment or a geodesic disk of $\bar{M}$ which is contained in $\partial_0 M$).
- The singular locus in $\bar{M}$ intersects $\partial_0 M$ orthogonally in totally geodesic regions.

The boundary at infinity $\partial_\infty M$ inherits a complex projective structure with cone singularities (see Proposition 3.4). The extended singular curves in $\bar{M}$ remain disjoint from each other.

Denote by $\mathfrak{Diff}_0(\Sigma \times \mathbb{R}_{>0})$ the space of diffeomorphisms on $\Sigma \times \mathbb{R}_{>0}$ isotopic to the identity among maps fixing each singular curve. Two hyperbolic ends with particles (M_1, g_1) and (M_2, g_2) are *isotopic* if there exists a map $f \in \mathfrak{Diff}_0(\Sigma \times \mathbb{R}_{>0})$ such that g_1 is the pull back by f of g_2. Let $\mathcal{HE}_\theta$ be the space of non-degenerate hyperbolic ends with particles up to isotopy. For the sake of simplicity, we shall call the elements (as isotopy classes or their representatives) in $\mathcal{HE}_\theta$ *hyperbolic ends with particles* henceforth.

Let L be the *bending locus* of $\partial_0 M$, which is the complement of those points x that admit a local support plane P such that $P \cap \partial_0 M$ is a neighborhood of x in $\partial_0 M$.

Remark 2.8. If $L = \varnothing$, $\partial_0 M$ is totally geodesic (orthogonal to the singular locus) and we say that M is Fuchsian. If $L \neq \varnothing$, it follows from the definition that L is foliated by mutually disjoint complete geodesics of $\bar{M}$. Moreover, L is the support of a measured lamination λ on $\partial_0 M$, called the bending lamination, with the transverse measure recording the bending of $\partial_0 M$ along L (see e.g. [10, Proposition 5.4], [26, Lemma A.15]).

Let (M, g) be a hyperbolic end with particles. The shape operator $B : TS \to TS$ of an embedded surface $S \subset M$ with induced metric I is defined as

$$B(u) = \nabla_u n,$$

where n is the positive-directed unit normal vector field on S and ∇ is the Levi-Civita connection of (M, g). The second and third fundamental forms on S are defined respectively as

$$II(u, v) = I(Bu, v), \qquad III(u, v) = I(Bu, Bv).$$

If S is smooth outside the intersection with singular locus in M, it is equivalent to say that S is *concave* (resp. *strictly concave*) if the principal curvatures at each regular point of S are both non-negative (resp. positive).

2.4. Convex GHM de Sitter spacetimes with particles

In order to define convex GHM de Sitter spacetimes with particles, we recall the related definitions.

2.4.1. The de Sitter 3-space

Consider the same ambient space $\mathbb{R}^{3,1}$, similarly as for $\mathbb{H}^3$. The *de Sitter 3-space* is defined as the quadric:

$$dS_3 = \{x \in \mathbb{R}^{3,1} : q(x) = 1\}.$$

It is a 3-dimensional Lorentzian symmetric space of constant curvature $+1$, diffeomorphic to $\mathbb{S}^2 \times \mathbb{R}$, where $\mathbb{S}^2$ is a 2-sphere. It is time-orientable and we choose the time orientation for which the curve $t \mapsto (\cosh t, 0, 0, \sinh t)$ is future-oriented. The group $\mathrm{Isom}_0(dS_3)$ of time-orientation and orientation preserving isometries of dS_3 is $SO^+(3,1) \cong PSL_2(\mathbb{C})$.

Consider the map $\pi : \mathbb{R}^{3,1}\setminus\{0\} \to \mathbb{S}^3$, where $\mathbb{S}^3$ is the double cover of $\mathbb{R}P^3$ and π sends a point x to the half-line from 0 passing through x. We define the *Klein model* $\mathbb{DS}_3$ of de Sitter 3-space as the image $\mathbb{DS}_3 = \pi(dS_3)$ (note that some authors define the Klein model as the projection of dS_3 to $\mathbb{R}P^3$ [2, Section 2.3], here we use $\mathbb{S}^3$ instead of $\mathbb{R}P^3$ in order to keep it time-orientable, see e.g. [4, Section 5.2.1]). The projection $\pi : dS_3 \to \mathbb{DS}_3$ is a diffeomorphism. The boundary $\partial\mathbb{DS}_3$ is the image of the quadratic $Q = \{x \in \mathbb{R}^{3,1} : q(x) = 0\}$ under π, which is a disjoint union of two 2-spheres: $\mathbb{S}^2_+ = \pi(\{x \in Q : x_4 > 0\})$ and $\mathbb{S}^2_- = \pi(\{x \in Q : x_4 < 0\})$.

A complete geodesic line in $\mathbb{DS}_3$ is *spacelike* (resp. *lightlike*, *timelike*) if it is contained in $\mathbb{DS}_3$ (resp. if it is tangent to $\mathbb{S}^2_+$ and $\mathbb{S}^2_-$, if it has endpoints lying on $\mathbb{S}^2_+$ and $\mathbb{S}^2_-$ respectively).

2.4.2. The singular de Sitter 3-space

Let $\theta_0 > 0$. We define the *singular de Sitter 3-space with cone singularities of angle* θ_0 as the space

$$dS^3_{\theta_0} := \{(t, \varphi, \alpha) \in \mathbb{R} \times [0, \pi] \times \mathbb{R}/\theta_0\mathbb{Z}\}$$

with the metric

$$-dt^2 + \cosh^2(t)(d\varphi^2 + \sin^2(\varphi)d\alpha^2).$$

The set $\mathbb{R} \times \{0, \pi\} \times \mathbb{R}/\theta_0\mathbb{Z}$ is called the *singular line* in $dS^3_{\theta_0}$ and θ_0 is called the *total angle* around this singular line. One can check that $dS^3_{\theta_0}$ is a Lorentzian manifold of constant curvature $+1$ outside the singular line. Indeed, it is obtained from the spherical surface with two cone singularities of angle θ_0 by taking a warped product with $\mathbb{R}$.

An embedded surface in $dS^3_{\theta_0}$ is *spacelike* if it intersects the singular line at exactly one point and it is spacelike outside the intersection with the singular locus.

2.4.3. De Sitter spacetimes with particles

A *de Sitter spacetime with particles* is a (singular) Lorentzian 3-manifold in which any point x has a neighborhood isometric to a subset of $dS^3_{\theta_0}$ for some $\theta_0 \in (0, \pi)$.

Let M^d be a de Sitter spacetime with particles which is homeomorphic to $\Sigma \times \mathbb{R}$. A closed embedded surface S in M^d is *spacelike* if it is locally modeled on a spacelike surface in $dS^3_{\theta_0}$ for some $\theta_0 \in (0, \pi)$. Similarly as the hyperbolic case, we can define the orthogonality of spacelike surfaces with respect to the singular locus in a de Sitter spacetime with particles.

Definition 2.9. Let $S \subset dS^3_{\theta_0}$ be a spacelike surface which intersects the singular line at a point x. S is orthogonal to the singular line at x if the causal distance from a point y of S to the totally geodesic plane P orthogonal to the singular line at x satisfies:

$$\lim_{y \in S, y \to x} \frac{d(y, P)}{d_S(x, y)} = 0,$$

where $d_S(x, y)$ is the distance between x and y with respect to the induced metric on S.

If now S is a spacelike surface in a de Sitter spacetime M^d with particles which intersects a singular line l at a point x'. S is said to be orthogonal to l at x' if there exists a neighborhood $U \subset M^d$ of x' which is isometric to a neighborhood of a singular point in $dS^3_{\theta_0}$ such that the isometry sends $S \cap U$ to a surface orthogonal to the singular line in $dS^3_{\theta_0}$. We say that S is orthogonal to the singular locus if S is orthogonal to the singular line of M^d at each intersection with the singular locus.

Definition 2.10. Let S be a spacelike surface orthogonal to the singular curves in a de Sitter spacetime with particles. We say that S is future-convex if its future $I^+(S)$ is geodesically convex. We say that S is strictly future-convex if $I^+(S)$ is strictly geodesically convex.

Definition 2.11. A de Sitter spacetime M^d with particles is convex GHM if

- M^d is convex GH: it contains a strictly future-convex spacelike surface S orthogonal to the singular curves, which intersects every inextensible timelike curve exactly once.

- M^d is maximal: if any isometric embedding of M^d into a convex GH de Sitter spacetime is an isometry.

Note that by Definition 2.11 a convex GHM de Sitter spacetime with particles is naturally future complete. Denote by $\mathfrak{Diff}_0(\Sigma \times \mathbb{R})$ the space of diffeomorphisms on $\Sigma \times \mathbb{R}$ isotopic to the identity among maps fixing each singular line. Denote by $\mathcal{DS}_\theta$ the space of isotopy classes of (future-complete) convex GHM de Sitter metrics on $\Sigma \times \mathbb{R}$ with cone singularities of angles θ_i along the singular lines $\{p_i\} \times \mathbb{R}$. Here two metrics g_1, g_2 are *isotopic* if there exists a map $f \in \mathfrak{Diff}_0(\Sigma \times \mathbb{R})$ such that g_1 is the pull back by f of g_2. For the sake of simplicity, we shall call the elements (as isotopy classes or their representatives) in $\mathcal{DS}_\theta$ *(future-complete) convex GHM de Sitter spacetime with particles* henceforth.

Let (M^d, g) be a future-complete convex GHM de Sitter spacetime with particles. Let $S \subset M^d$ be a spacelike surface which is orthogonal to the singular locus with the induced metric I. The shape operator $B : TS \to TS$ of S is defined as

$$B(u) = \nabla_u n,$$

where n is the future-directed unit normal vector field on S and ∇ is the Levi-Civita connection of (M^d, g). The second and third fundamental forms of S are defined respectively as

$$II(u, v) = I(Bu, v), \qquad III(u, v) = I(Bu, Bv).$$

If S is smooth outside the intersection with singular locus in M^d, it is equivalent to say that S is *future-convex* (resp. *strictly future-convex*) if the principal curvatures at each regular point of S are both non-negative (resp. positive).

2.5. *Minimal Lagrangian maps between hyperbolic surfaces with cone singularities*

The construction of the parametrization of $\mathcal{HE}_\theta$ by $\mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$ here depends strongly on minimal Lagrangian maps between hyperbolic surfaces with cone singularities.

Definition 2.12. Given two hyperbolic metrics $h, h' \in \mathfrak{M}^\theta_{-1}$ with cone singularities, a *minimal Lagrangian map* $m : (\Sigma, h) \to (\Sigma, h')$ is an area-preserving and orientation-preserving diffeomorphism, sending cone singularities to cone singularities, such that its graph is a minimal surface in $(\Sigma \times \Sigma, h \oplus h')$.

We introduce the following result (see [30, Theorem 1.3]).

Theorem 2.13 *(Toulisse). Let $h, h' \in \mathfrak{M}^\theta_{-1}$. Then there exists a unique minimal Lagrangian diffeomorphism $m : (\Sigma, h) \to (\Sigma, h')$ isotopic to the identity among maps sending each cone singularity of h to the corresponding cone singularity of h'.*

Minimal Lagrangian maps between hyperbolic surfaces with metrics in $\mathfrak{M}^\theta_{-1}$ have an equivalent description in terms of morphisms between tangent bundles (see e.g. [30, Proposition 6.3], [12, Proposition 2.12]).

Proposition 2.14. *Let $h, h' \in \mathfrak{M}^\theta_{-1}$, and let $m : (\Sigma, h) \to (\Sigma, h')$ be a diffeomorphism fixing each singular point. Then m is a minimal Lagrangian map if and only if there exists a bundle morphism $b : T\Sigma \to T\Sigma$ defined outside the singular locus which satisfies the following properties:*

- *b is self-adjoint for h with positive eigenvalues.*
- $\det(b) = 1$.
- *b satisfies the Codazzi equation: $d^\nabla b = 0$, where ∇ is the Levi-Civita connection of h.*
- $h(b\bullet, b\bullet) = m^* h'$.
- *Both eigenvalues of b tend to 1 at the cone singularities.*

Corollary 2.15. *Let $h, h' \in \mathfrak{M}^\theta_{-1}$. Then there exists a unique bundle morphism $b : T\Sigma \to T\Sigma$ defined outside the singular locus, which is self-adjoint for h with positive eigenvalues, has determinant 1 and satisfies the Codazzi equation: $d^\nabla b = 0$, where ∇ is the Levi-Civita connection of h, such that $h(b\bullet, b\bullet)$ is isotopic to h' and both eigenvalues of b tend to 1 at the cone singularities.*

Definition 2.16. We say that a pair of hyperbolic metrics (h, h') is normalized if there exists a bundle morphism $b : T\Sigma \to T\Sigma$ defined outside the singular locus, which is self-adjoint for h, has determinant 1, and satisfies the Codazzi equation, such that $h' = h(b\bullet, b\bullet)$ and both eigenvalues of b tend to 1 at singularities, or equivalently, if the identity from (Σ, h) to (Σ, h') is a minimal Lagrangian diffeomorphism.

Remark 2.17. By Corollary 2.15, for any $(\tau, \tau') \in \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$, we can realize (τ, τ') as a normalized representative (h, h'). Note that the normalized representative of (τ, τ') is unique up to isotopies acting diagonally on both h and h'.

We also introduce the following proposition (see e.g. [18, Proposition 3.12], [23]), which provides a convenient formula to compute the (sectional) curvatures of certain metrics.

Proposition 2.18. *Let Σ be a surface with a Riemannian metric g. Let $A : T\Sigma \to T\Sigma$ be a bundle morphism such that A is everywhere invertible and $d^\nabla A = 0$, where ∇ is the Levi-Civita connection of g. Let h be the symmetric $(0, 2)$-tensor defined by $h = g(A\bullet, A\bullet)$. Then the Levi-Civita connection of h is given by*

$$\nabla^h_u(v) = A^{-1}\nabla_u(Av),$$

and its curvature is given by

$$K_h = \frac{K_g}{\det(A)}.$$

3. Hyperbolic ends with particles and complex projective structures with cone singularities

3.1. Complex projective structure on Σ with cone singularities

Let Σ be the prescribed surface with the marked points $\mathfrak{p} = (p_1, ..., p_{n_0})$ and let $\theta = (\theta_1, ..., \theta_{n_0}) \in (0, \pi)^{n_0}$. We first give a definition of a complex projective structure on Σ with cone singularities of fixed angles.

Definition 3.1. Let $\theta_0 > 0$. We call complex cone of angle θ_0, and denote by $\mathbb{C}_{\theta_0}$, the quotient of the universal covering of $\mathbb{C} \setminus \{0\}$ by a rotation of angle θ_0 centered at 0.

Definition 3.2. A complex projective structure σ on Σ with cone singularities of angle θ at $\mathfrak{p}$ is a maximal atlas of charts from $\Sigma_\mathfrak{p}$ to $\mathbb{C}P^1$ such that all transition maps are restrictions of Möbius transformations, and for each marked point p_i, there exists a neighborhood Ω_i of p_i in Σ and a complex projective map $u_i : \Omega_i \to \mathbb{C}_{\theta_i} \cup \{0\}$ sending p_i to 0, which is a diffeomorphism from $\Omega_i \setminus \{p_i\}$ to its image.

Note that in the above definition u_i is uniquely determined by the complex projective structure σ up to composition on $\mathbb{C}_{\theta_i}$ with a rotation and a homothety.

Two complex projective structures σ_1, σ_2 with prescribed cone singularities are *isotopic* if there is an orientation-preserving diffeomorphism from $\Sigma_\mathfrak{p}$ to $\Sigma_\mathfrak{p}$ isotopic to the identity that pulls back the projective charts of σ_2 to projective charts of σ_1. We denote by $\mathcal{CP}_\theta$ the set of isotopy classes of complex projective structures on Σ with cone singularities of angle θ at $\mathfrak{p}$.

Each complex projective structure σ on Σ with prescribed cone singularities defines a local diffeomorphism from the universal covering $\widetilde{\Sigma_\mathfrak{p}}$ to $\mathbb{C}P^1$, which is a complex projective map with respect to the complex projective structures on $\widetilde{\Sigma_\mathfrak{p}}$ and $\mathbb{C}P^1$. We call this map $f : \widetilde{\Sigma_\mathfrak{p}} \to \mathbb{C}P^1$ a *developing map* of σ. There is a homomorphism $\rho : \pi_1(\Sigma_\mathfrak{p}) \to PSL_2(\mathbb{C})$, called a *holonomy representation* of σ, such that f is ρ-equivariant. In particular, the image of the small loop around each marked point p_i under the holonomy ρ is an elliptic element of $PSL_2(\mathbb{C})$ of angle θ_i. We call (f, ρ) a *development-holonomy pair* and it is uniquely determined by σ up to the $PSL_2(\mathbb{C})$*-action* by $(f, \rho) \mapsto (A \circ f,\ \rho^A)$, where $\rho^A(\gamma) = A\,\rho(\gamma)\,A^{-1}$.

3.2. The cotangent bundle of $\mathcal{T}_{\Sigma,\theta}$

Note that each conformal class of a metric on $\Sigma_{\mathfrak{p}}$ with marked points admits a unique hyperbolic metric with cone singularities of angle θ_i at p_i (see [33, Theorem A] and [25]), $\mathcal{T}_{\Sigma,\theta}$ is also identified with the space of equivalence classes of conformal structures on $\Sigma_{\mathfrak{p}}$ with marked points. Two conformal structures c_1 and c_2 on $\Sigma_{\mathfrak{p}}$ are *equivalent* if there is an orientation-preserving self-homeomorphism of $\Sigma_{\mathfrak{p}}$ isotopic to the identity that pulls back the conformal charts of c_2 to conformal charts of c_1. For the sake of simplicity, we shall denote a conformal structure c and its equivalence class $[c]$ by c.

It is known that (see [31, Proposition 2.14]) for each $c \in \mathcal{T}_{\Sigma,\theta}$, the cotangent space $T^*_c\mathcal{T}_{\Sigma,\theta}$ of $\mathcal{T}_{\Sigma,\theta}$ at c is the space of meromorphic quadratic differentials (with respect to the conformal structure c) on Σ with at worst simple poles at the marked points.

We denote by $T^*\mathcal{T}_{\Sigma,\theta}$ the cotangent bundle of $\mathcal{T}_{\Sigma,\theta}$, which is a complex $6\mathfrak{g} - 6 + 2n_0$-dimensional vector space of meromorphic quadratic differentials with respect to a conformal structure in $\mathcal{T}_{\Sigma,\theta}$, with at worst simple poles at the marked points.

3.3. The complex projective structure at infinity of a hyperbolic end $M \in \mathcal{HE}_\theta$

We show that the boundary at infinity $\partial_\infty M$ of a hyperbolic end $M \in \mathcal{HE}_\theta$ admits a complex projective structure with prescribed cone singularities.

The model space V_α. Let $\alpha > 0$ and let Δ_0 be a fixed, oriented complete hyperbolic geodesic in $\mathbb{H}^3$. Denote by U the universal cover of the complement of Δ_0 in $\mathbb{H}^3$ and denote by V the metric completion of U, such that $V \setminus U$ is canonically identified with Δ_0, which is called the *singular set of* V. We define V_α (see e.g. [26, Section 3.1]) as the quotient of V by the rotation of angle α around Δ_0. The image of the singular set of V under this quotient is called the *singular set of* V_α.

Let M be a hyperbolic end with particles. It is clear that each singular point x of M has a neighborhood isometric to a subset of V_α with α equal to the total angle around the singular curve through x. Now we describe the geometry property of M near the endpoints at infinity of the singular curves in M by using the model V_α, as in the following lemma. With Lemma 2.6, the argument is similar to that in [26, Lemma 3.1, Lemma A.10] as the particular case of non-interacting particles.

Lemma 3.3. *For each point $p_i \in \partial_\infty M$ which is the endpoint at infinity of a singular curve in M, p_i has a neighborhood Ω_i isometric to a neighborhood of one of the endpoints at infinity of Δ_0 in V_{θ_i}, where θ_i is the total angle around that singular curve.*

As an analog of the complex projective structure (resp. complex projective structure with cone singularities) induced on the boundary at infinity of a hyperbolic end (resp. a quasi-Fuchsian manifold with particles), a hyperbolic end with particles also induces a complex projective structure with cone singularities on the boundary at infinity (see e.g. [26, Section 3.2]).

Proposition 3.4. *Let $M \in \mathcal{HE}_\theta$ be a hyperbolic end with particles. Then the boundary at infinity $\partial_\infty M$ is equipped with a complex projective structure with cone singularities of angle θ_i at the p_i.*

Proof. Consider the regular set M_r of M and denote its universal cover by $\widetilde{M}_r$. Let $\partial_\infty\mathbb{H}^3$ be the boundary at infinity of $\mathbb{H}^3$. Note that M_r admits a developing map $dev : \widetilde{M}_r \to \mathbb{H}^3$, which is locally isometric projection (unique up to composition on the left by an isometry of $\mathbb{H}^3$).

We define $\partial_\infty\widetilde{M}_r$ as the space of equivalence classes of geodesic rays in $\widetilde{M}_r$, where two geodesic rays are equivalent if and only if they are asymptotic. Then dev has a natural extension $dev : \widetilde{M}_r \cup \partial_\infty\widetilde{M}_r \to \mathbb{H}^3 \cup \partial_\infty\mathbb{H}^3$, which is a local homeomorphism. Note that $\partial_\infty\mathbb{H}^3$ can be identified to $\mathbb{C}P^1$ and the fundamental group of M_r acts on $\widetilde{M}_r$ by hyperbolic isometries which extend to $\partial_\infty\widetilde{M}_r$ as Möbius transformation. We can define the boundary at infinity of M_r, called $\partial_\infty M_r$, as the quotient of $\partial_\infty\widetilde{M}_r$ by the fundamental group of M_r. Then $\partial_\infty M_r$ carries a canonical $\mathbb{C}P^1$-structure.

It remains to consider the behavior of the $\mathbb{C}P^1$-structure on $\partial_\infty M$ near the endpoints of the singular locus in M. By Lemma 3.3, there exists a complex projective map $u_i : \Omega_i \to \mathbb{C}_{\theta_i} \cup \{0\}$ sending p_i to 0, which is a diffeomorphism from $\Omega_i \setminus \{p_i\}$ to its image. By Definition 3.2, $\partial_\infty M$ has a $\mathbb{C}P^1$-structure with cone singularities (at the endpoints at infinity of the singular curves) of angle equal to the total angle around the corresponding singular curve. □

3.4. The meromorphic quadratic differential induced by a complex projective structure in $\mathcal{CP}_\theta$

As the non-singular case, we can relate $\mathcal{CP}_\theta$ to the space $T^*\mathcal{T}_{\Sigma,\theta}$ by using Schwarzian derivatives with a special analysis near the cone singularities.

Note that Möbius transformations are biholomorphic on $\mathbb{C}P^1$ and $\mathbb{C}P^1$ admits a unique complex structure, a complex projective structure on Σ with cone singularities also determines a complex (or conformal) structure with marked points. Note also that a hyperbolic metric on Σ with cone singularities is a special complex projective structure on Σ with cone singularities (the Möbius transformations as transition functions preserve the unit circle). There is also a natural forgetful map

$$\pi : \mathcal{CP}_\theta \to \mathcal{T}_{\Sigma,\theta},$$

which is continuous and surjective. If $\sigma \in \mathcal{CP}_\theta$ satisfies that $\pi(\sigma) = c$, we say that σ is a complex projective structure with the *underlying conformal structure* c.

Let σ be a complex projective structure on Σ with prescribed cone singularities with the underlying conformal structure c. McOwen [25] and Troyanov [33] proved that there is a unique hyperbolic metric conformal to c, with cone singularities of the same angles as σ at the same points. Let σ_F be the complex projective structure underlying this hyperbolic metric h. We call σ_F the *Fuchsian* complex projective structure associated to σ. Note that the union of the $\mathbb{C}P^1$-atlas of σ and the $\mathbb{C}P^1$-atlas of σ_F induces a complex atlas, the identity map $id : (\Sigma_\mathfrak{p}, \sigma_F) \to (\Sigma_\mathfrak{p}, \sigma)$ is a conformal map, but not necessary a complex projective map. For convenience, we call this identity map the *natural conformal map* from σ_F to σ. Similarly, we can consider a natural conformal map from σ to σ_F.

In the non-singular case, the Schwarzian derivative measures the "difference" between a pair of complex projective structures on a Riemann surface. For the singular case, we can also use this tool to measure the difference between two complex projective structures in $\mathcal{CP}_\theta$ with the same underlying conformal structure, but one needs to analyze the behavior of the Schwarzian derivative at the cone singularities.

Let Ω is a connected open subset of $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}P^1$ be a locally injective holomorphic map. Recall that the *Schwarzian derivative* of f is the holomorphic quadratic differential on Ω.

$$\mathcal{S}(f) = \left\{ \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 \right\} dz^2$$

Recall that the Schwarzian derivative has two important properties:

(1) The Schwarzian derivative of a Möbius transformation is zero.
(2) The cocycle property: $\mathcal{S}(g \circ f) = \mathcal{S}(f) + f^*\mathcal{S}(g)$, where $f^*\mathcal{S}(g)$ is the pull back of the holomorphic quadratic differential $\mathcal{S}(g)$ under the map f.

Lemma 3.5. *Let $\sigma \in \mathcal{CP}_\theta$ be a complex projective structure. Then the Schwarzian derivative of the conformal map $id : (\Sigma_\mathfrak{p}, \sigma) \to (\Sigma_\mathfrak{p}, \sigma_F)$ is a meromorphic quadratic differential in $T_c^*\mathcal{T}_{\Sigma,\theta}$, where c is the common underlying conformal structure of σ and σ_F.*

Proof. Let φ be a local expression (which is a family of locally injective holomorphic functions with respect to the $\mathbb{C}P^1$-charts of σ and σ_F) of the map $id : (\Sigma_\mathfrak{p}, \sigma) \to (\Sigma_\mathfrak{p}, \sigma_F)$. Thanks to properties (1) and (2) above, the Schwarzian derivative of φ remains compatible with the transition functions in the overlaps of two $\mathbb{C}P^1$-charts associated to σ or σ_F, respectively. Thus $\mathcal{S}(\varphi)$ is a holomorphic quadratic differential on $\Sigma_\mathfrak{p}$.

It remains to consider the behavior of $\mathcal{S}(\varphi)$ near the cone singularities. By Definition 3.2, for each marked point p_i on the complex projective surface (Σ, σ) (resp. (Σ, σ_F)), there is a neighborhood Ω_i (resp. Ω_i^F) of p_i and a complex projective map $u : \Omega_i \to \mathbb{C}_{\theta_i} \cup \{0\}$ (resp. $u_F : \Omega_i^F \to \mathbb{C}_{\theta_i} \cup \{0\}$) sending p_i to 0, which is a diffeomorphism from

$\Omega_i \setminus \{p_i\}$ (resp. $\Omega_i^F \setminus \{p_i\}$) to its image. Note that there is a natural holomorphic local diffeomorphism from $\mathbb{C}_{\theta_i}$ to $\mathbb{C}$, defined by sending a point $u \in \mathbb{C}_{\theta_i}$ to $u^{2\pi/\theta_i}$. We denote by z, z_F the complex coordinates on Ω_i, Ω_i^F, respectively. Let f be the expression of φ near p_i under these coordinates with $f(0) = 0$. It is clear that f is a conformal map in a small punctured neighborhood of 0 with the puncture at 0 and it can be continuously extended to the point 0. Hence f is conformal in a small neighborhood of 0 and has the expansion:

$$f(z) = a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots,$$

where $a_1 \neq 0$, $a_i \in \mathbb{C}$ for $i = 1, 2, \ldots$.

Then the map φ near p_i has the following expression with respect to the complex projective coordinate u via the complex coordinates z and z_F:

$$\varphi(u) = (f((u)^{\frac{2\pi}{\theta_i}}))^{\frac{\theta_i}{2\pi}}.$$

A direct computation shows that the Schwarzian derivative $\mathcal{S}(\varphi)(u)$ has the following expansion near $u(p_i) \in \mathbb{C}_{\theta_i}$:

$$\mathcal{S}(\varphi)(u) = u^{\frac{2\pi}{\theta_i}-2}(b_1 + b_2\, u^{\frac{2\pi}{\theta_i}} + \ldots + b_n\, u^{\frac{2\pi}{\theta_i}(n-1)} + \ldots)\, du^2,$$

where $b_i \in \mathbb{C}$ for $i \geq 1$.

In the complex coordinate $z = u^{\frac{2\pi}{\theta_i}}$, the Schwarzian derivative $\mathcal{S}(\varphi)(u)$ is expressed as

$$\begin{aligned}\mathcal{S}(\varphi) \circ z^{\frac{\theta_i}{2\pi}}(z) &= z^{1-\frac{\theta_i}{\pi}}(b_1 + b_2\, z + \cdots + b_n\, z^{n-1} + \ldots)\left(dz^{\frac{\theta_i}{2\pi}}\right)^2 \\ &= \left(\frac{\theta_i}{2\pi}\right)^2 \frac{1}{z}(b_1 + b_2\, z + \cdots + b_n\, z^{n-1} + \cdots)\, dz^2.\end{aligned}$$

This implies that $\mathcal{S}(\varphi)$ is a meromorphic quadratic differential on Σ with at worst simple poles at the cone singularities, with respect to the common underlying conformal structure of σ and σ_F. The lemma follows. □

3.5. Maximal concave extension of a hyperbolic structure near infinity

To construct a hyperbolic end with particles from a complex projective structure with cone singularities, we first prove a proposition which ensures the existence and the uniqueness (up to isometry) of the maximal extension of a hyperbolic manifold with particles which has a concave metric boundary. Moreover, we show that this maximal extension is a hyperbolic end with particles, in the sense of Definition 2.7.

We first introduce two definitions.

Definition 3.6. Let M be a hyperbolic manifold with particles. Let S be a surface in $\bar{M}$. We say that a regular (resp. singular) point $x \in S$ is extremal if there exists a half-ball B in $\mathbb{H}^3$ (resp. $\mathbb{H}^3_{\theta_0}$ for some $\theta_0 \in (0, \pi)$), and an isometric embedding $\varphi : B \to \bar{M}$ sending the center of B to x, such that $\varphi(\bar{B}) \cap S = \{x\}$.

For example, all the points of a strictly concave surface in a hyperbolic manifold with particles are extremal points. The metric boundary of a hyperbolic end with particles contains no extremal points, since it is pleated (see Definition 2.7). Conversely, if the metric boundary of a hyperbolic manifold with particles is concave and contains no extremal points, then it is pleated.

Definition 3.7. Let M be a hyperbolic manifold with particles which has a concave metric boundary. We say M' is a concave extension of M if M' is a hyperbolic manifold with particles such that $\partial_0 M'$ is concave and orthogonal to the singular locus, and M can be isometrically embedded in M'. We say M' is a maximal concave extension of M if M' is a concave extension of M and any concave extension of M' is isometric to M'.

Proposition 3.8. *Let M_0 be a hyperbolic manifold with particles which has a concave metric boundary. Then there exists a unique (up to isometry) maximal concave extension of M_0, called M, in which M_0 can be isometrically embedded. Moreover, M is a hyperbolic end with particles.*

Proof. We show the first statement in the following three steps. The argument is an adaption of those for the corresponding results in globally hyperbolic spacetimes (see e.g. [13, Theorem 3], [10, Proposition 2.6]). The point is to use the concavity of the metric boundary of a hyperbolic manifold instead of the globally hyperbolicity of a spacetime.

Step 1: Let $\mathcal{E}$ be the set of all concave extensions of M_0. It is clear that $\mathcal{E}$ is non-empty since M_0 is a concave extension of itself. Given $M_1, M_2 \in \mathcal{E}$, we consider the ordered pairs (N_1, N_2) such that

- N_i is a subset of M_i in which M_0 can be isometrically embedded, for $i = 1, 2$.
- There is an isometric embedding from M_0 to M_2 which extends to an isometric embedding from N_1 to M_2 sending N_1 to N_2.

Denote by $\mathcal{C}(M_1, M_2)$ the set consisting of all such pairs for $M_1, M_2 \in \mathcal{E}$. It is clear that $\mathcal{C}(M_1, M_2)$ is partially ordered by inclusion of the first and second item of the pairs, respectively. Moreover, each totally ordered subset of $\mathcal{C}(M_1, M_2)$ has an upper bound. By Zorn's Lemma, there exists a maximal element of $\mathcal{C}(M_1, M_2)$.

Step 2: Now we give a partial order "$\leq$" for the set $\mathcal{E}$ by defining $M_1 \leq M_2$ if the isometric embedding from M_0 to M_2 extends to an isometric embedding from M_1 to M_2, here $M_1, M_2 \in \mathcal{E}$. We claim that $\mathcal{E}$ has a maximal element.

Indeed, let $(M_\alpha)_{\alpha\in\mathcal{A}}$ be a totally ordered subset of $\mathcal{E}$ and let $K = \sqcup_\alpha M_\alpha$ be the disjoint union of M_α over $\alpha \in \mathcal{A}$. We define an equivalence relation for the set K. We relate $p_\alpha \in M_\alpha$ to $p_\beta \in M_\beta$ if there exists $(N_\alpha, N_\beta) \in \mathcal{C}(M_\alpha, M_\beta)$ and an isometric embedding from N_α to M_β which sends p_α to p_β, where $\alpha, \beta \in \mathcal{A}$. One can check that this relation is an equivalence relation on K. Denote by $\bar{K}$ the quotient space of K under this equivalence relation. Then $\bar{K}$ is a manifold endowed with a natural differentiable structure and metric. Note that $M_\alpha \in \mathcal{E}$ and $M_\alpha \subset \bar{K}$ for all α, then $\bar{K}$ is a hyperbolic manifold with particles in which M_0 can be isometrically embedded.

We claim that $\bar{K}$ has a concave metric boundary orthogonal to the singular locus. This implies that $\bar{K} \in \mathcal{E}$ and $\bar{K}$ is an upper bound of (M_α). Applying Zorn's Lemma again, there exists a maximal element of $\mathcal{E}$, say M.

Now we show that $\bar{K}$ has a concave metric boundary. Note that any concave surface in a hyperbolic manifold with particles has sectional curvature at least -1. By the assumption (1) and the Gauss–Bonnet formula (see [33, Proposition 1]), the area of any incompressible concave surface (homeomorphic to Σ) has a positive lower bound. Note also that the area of a concave surface decreases exponentially with respect to the distance r along the normal flow pointing to the non-concave side of S. Combined with the fact that M_α has a concave metric boundary for all $\alpha \in \mathcal{A}$, then the metric completion of $\bar{K}$ is homeomorphic to $\Sigma \times \mathbb{R}_{\geq 0}$. Therefore, $\bar{K}$ has a metric boundary and it is naturally concave and orthogonal to the singular locus. The claim follows.

Step 3: We show that M is a concave extension of each element of $\mathcal{E}$. Let $M' \in \mathcal{E}$. We denote by $\hat{M}$ the quotient space of the disjoint union of M' and M under the equivalence relation defined above. It suffices to show $\hat{M} \in \mathcal{E}$, since this implies that $\hat{M}$ is a concave extension of both M and M'. Note that M is a maximal element of $\mathcal{E}$, then M is isometric to $\hat{M}$ and thus a concave extension of M'. This shows the uniqueness of M (up to isometry).

We now show that $\hat{M} \in \mathcal{E}$. Let (N', N) be a maximal element of $\mathcal{C}(M', M)$ (this is ensured by Step 1) and let ψ be an isometric embedding from M_0 to M which extends to an isometric embedding from N' to M sending N' to N. Denote by $\partial N'$ the boundary of N' in $\bar{M}'$ and denote by ∂N the boundary of N in $\bar{M}$. We claim that for each point $x \in \partial N'$, either $x \in \partial_0 M'$ or $\psi(x) \in \partial_0 M$. Otherwise, there exists a point $x \in \partial N'$ which is in the interior of M' with the image $\psi(x)$ in the interior of M. Note that M' and M are both locally modeled on $\mathbb{H}^3_{\theta_i}$ for some $\theta_i \in (0, \pi)$. Whatever x is a regular point or a singular point, we can choose a small neighborhood U' of x in M' and a small neighborhood U of $\psi(x)$ in M such that they are isometric to each other. It is clear that $(N' \cup U', N \cup U) \in \mathcal{C}(M', M)$. Note also that N' is a proper subset of $N' \cup U'$ in M'. This contradicts that (N', N) is the maximal element of $\mathcal{C}(M', M)$. The claim follows.

Note that $\hat{M} = (N' \cong N) \sqcup (M' \setminus N') \sqcup (M \setminus N)$. Combined with the above claim, ψ can extend to an isometric embedding from $\bar{N}'$ to $\bar{M}$ sending $\partial N'$ to ∂N, then $\hat{M}$ is Hausdorff. Note that the projection from $M \sqcup M'$ to $\hat{M}$ is open, every point of $\hat{M}$ has a neighborhood homeomorphic to $\mathbb{R}^3$. This implies that $\hat{M}$ is a manifold. Similarly as Step 2, $\hat{M}$ inherits a natural hyperbolic structure with particles. Furthermore, $\hat{M}$ has a metric completion compatible with the metric completions of M' and M, under which the metric boundary $\partial_0 \hat{M}$ is concave (one can check by using Definition 2.5) and orthogonal to the singular locus. Moreover, $\hat{M}$ is a hyperbolic manifold with particles in which M_0 can be isometrically embedded. This implies that $\hat{M} \in \mathcal{E}$.

We now show the second statement: the unique maximal concave extension M of M_0 is a hyperbolic end with particles. By definition, M is a hyperbolic manifold with particles and has a concave metric boundary $\partial_0 M$ which is orthogonal to the singular locus. It remains to show that $\partial_0 M$ is pleated.

Suppose that $\partial_0 M$ is not pleated. Then $\partial_0 M$ contains an extremal point, say p. By Definition 3.6, there exists a half-ball B in $\mathbb{H}^3$ (resp. $\mathbb{H}^3_{\theta_0}$ for some $\theta_0 \in (0,\pi)$) if p is a regular (resp. singular) point, and an isometric embedding $\varphi : B \to M$ sending the center O of B to p, such that $\varphi(\bar{B}) \cap \partial_0 M = \{p\}$. Let P be the totally geodesic disk contained in the boundary ∂B of B and let n be the unit vector orthogonal to P at O and pointing outward of B. Denote by B' the open ball in $\mathbb{H}^3$ (resp. $\mathbb{H}^3_{\theta_0}$) containing B with the same radius. We consider a geodesic plane in B', called P_ε, which is orthogonal to the geodesic $\gamma(t) = \exp_O tn$ at $\gamma(\varepsilon)$ for sufficiently small $\varepsilon > 0$. Let U be a neighborhood of O in B' with boundary containing P_ε. Let $\mathcal{C}(U)$ be the set consisting of the subsets of U which can be isometrically embedded in M. Then $\mathcal{C}(U)$ has a unique maximal element, say W. Denote by $\bar{M}$ the quotient of the disjoint union of M and U by identifying W to a subset of M under an isometric embedding from W to M. Note that we can choose ε above small enough such that the metric boundary of $\bar{M}$ is concave. By construction, $\bar{M}$ is a hyperbolic manifold with particles and has a concave metric boundary which is orthogonal to the singular locus. It is a non-trivial concave extension of M. This contradicts that M is a maximal concave extension. □

3.6. The construction of hyperbolic ends in $\mathcal{HE}_\theta$ from meromorphic quadratic differentials in $T^*\mathcal{T}_{\Sigma,\theta}$

Proposition 3.9. *Let $q \in T^*_c\mathcal{T}_{\Sigma,\theta}$ with $c \in \mathcal{T}_{\Sigma,\theta}$. Then there exists a unique hyperbolic end with particles $M \in \mathcal{HE}_\theta$ which admits a complex projective structure σ on $\partial_\infty M$ in the conformal class c, such that the Schwarzian derivative $\mathcal{S}(\phi)$ of the natural conformal map $\phi : (\partial_\infty M, \sigma) \to (\partial_\infty M, \sigma_F)$ is q.*

Proof. We construct a hyperbolic end with particles M from the given quadratic differential q on Σ (with respect to the conformal structure c) in the following two steps.

Step 1: First we construct a hyperbolic manifold with the prescribed particles M_0 which is homeomorphic to $\Sigma \times \mathbb{R}_{\geq 0}$ with a concave metric boundary $\partial_0 M_0$, by using the given data q and c.

Let I^* be a hyperbolic metric with the prescribed cone singularities in the conformal class c. Let $II^*_0 = \mathrm{Re}\, q$ be the real part of q and $II^* = \frac{1}{2}I^* + II^*_0$. Let $B^* = (I^*)^{-1}II^*$ and $III^* = I^*(B^*\bullet, B^*\bullet)$.

Let M_0 be the set $\Sigma \times [r_0, +\infty)$ with the metric

$$g_0 = dr^2 + \frac{1}{2}(e^{2r}I^* + 2II^* + e^{-2r}III^*),$$

where r_0 is to be determined. We claim that M_0 is a hyperbolic manifold with particles if we choose r_0 large enough. Denote $I_r = \frac{1}{2}(e^{2r}I^* + 2II^* + e^{-2r}III^*)$. Then we have

$$I_r = \frac{1}{2}I^*((e^r E + e^{-r}B^*)\bullet, (e^r E + e^{-r}B^*)\bullet),$$
$$II_r = \frac{1}{2}\frac{dI_r}{dr} = \frac{1}{2}I^*((e^r E + e^{-r}B^*)\bullet, (e^r E - e^{-r}B^*)\bullet)$$

Denote $B_r = (e^r E + e^{-r}B^*)^{-1}(e^r E - e^{-r}B^*)$. We show that (I_r, B_r) satisfies the following conditions:

- B_r is self-adjoint for I_r: $I_r(B_r\bullet, \bullet) = I_r(\bullet, B_r\bullet)$. This follows directly from the fact that B^* is self-adjoint for I^* (since II^*_0 is the real part of the quadratic differential q).
- (I_r, B_r) satisfies the Gauss equation for surfaces embedded in $\mathbb{H}^3$: $K_{I_r} = -1 + \det B_r$, where K_{I_r} is the sectional curvature of I_r. Indeed, by the definition of I_r and Proposition 2.18,

$$K_{I_r} = \frac{K_{I^*}}{\det(\frac{1}{\sqrt{2}}(e^r E + e^{-r}B^*))} = \frac{-2}{e^{2r} + \mathrm{tr}\, B^* + e^{-2r}\det B^*}.$$

Note that $B^* = (I^*)^{-1}II^* = \frac{1}{2}E + (I^*)^{-1}II^*_0$ and $(I^*)^{-1}II^*_0$ is traceless. We have $\mathrm{tr}\, B^* = 1$ and

$$-1 + \det B_r = \frac{-2\,\mathrm{tr}\, B^*}{e^{2r} + \mathrm{tr}\, B^* + e^{-2r}\det B^*} = \frac{-2}{e^{2r} + \mathrm{tr}\, B^* + e^{-2r}\det B^*} = K_{I_r}.$$

- (I_r, B_r) satisfies the Codazzi equation: $d^{\nabla^{I_r}} B_r = 0$, where ∇^{I_r} is the Levi-Civita connection of I_r. Denote by ∇^{I^*} the Levi-Civita connection of I^*. By Proposition 2.18,

$$\nabla^{I_r} = (e^r E + e^{-r} B^*)^{-1} \nabla^{I^*} (e^r E + e^{-r} B^*) \ .$$

  It suffices to show that $d^{\nabla^{I^*}} B^* = 0$. By the definition of ∇^{I^*}, it can be checked that $d^{\nabla^{I^*}} I^* = 0$. Note that $II_0^* = \mathrm{Re}\, q$ with q a holomorphic quadratic differential outside the marked points. Then $d^{\nabla^{I^*}} II_0^* = 0$. Therefore, $d^{\nabla^{I^*}} B^* = d^{\nabla^{I^*}} \left(\frac{1}{2} E + (I^*)^{-1} II_0^*\right) = (I^*)^{-1} d^{\nabla^{I^*}} II_0{}^* = 0$.
- (I_r, B_r) satisfies the following equality:

$$I_{r+s} = I_r((\cosh(s)E + \sinh(s)B_r)\bullet, (\cosh(s)E + \sinh(s)B_r)\bullet),$$

  for all $r, s > 0$. This follows from a direct computation.

Denote by λ^*, μ^* (resp. λ_r, μ_r) the eigenvalues of B^* (resp. B_r). By computation,

$$\lambda_r = \frac{e^r - e^{-r}\lambda^*}{e^r + e^{-r}\lambda^*}, \qquad \mu_r = \frac{e^r - e^{-r}\mu^*}{e^r + e^{-r}\mu^*}.$$

If r_0 is large enough, the eigenvalues λ_{r_0}, μ_{r_0} of (Σ, I_{r_0}) are both positive. Combined with the above properties of (I_r, B_r), this shows that M_0 is a hyperbolic manifold with particles which has a concave metric boundary.

We now show that the total angle around the singular curve $\{p_i\} \times [r_0, +\infty)$ of M_0 is θ_i. It suffices to check that $(\Sigma \times \{r\}, I_r)$ has cone singularities of angle θ_i at the intersection with the singular line through p_i. Note that $I_r = \frac{1}{2} I^*((e^r E + e^{-r} B^*)\bullet, (e^r E + e^{-r} B^*)\bullet)$. We claim that B^* tends to $\frac{1}{2}E$ at the cone singularities. Indeed $I^* = \rho(z)|dz|^2$ with $\rho(z) = e^{2u}|z|^{2(\frac{\theta_i}{2\pi}-1)}$ near the cone singularity p_i, while the quadratic differential $q = f(z)dz^2$ has at most simple pole at p_i (that is, $|f(z)| \le O(1/|z|)$ near $z(p_i) = 0$). A direct computation shows that

$$(I^*)^{-1} II_0^* = \frac{1}{2}\rho^{-1}(z) \begin{pmatrix} \mathrm{Re}\, f & -\mathrm{Im}\, f \\ -\mathrm{Im}\, f & -\mathrm{Re}\, f \end{pmatrix}.$$

Combined with the observation that $\theta_i \in (0, \pi)$ and $|\mathrm{Re}\, f|, |\mathrm{Im}\, f| \le |f| \le O(1/|z|)$ near $z(p_i) = 0$, we have that $(I^*)^{-1} II_0^*$ tends to the zero matrix at p_i. This implies that B^* tends to $\frac{1}{2}E$ at p_i. Hence, I_r tends to $\frac{1}{2}(e^r + \frac{1}{2}e^{-r})^2 I^*$ at p_i, which implies that I_r has the cone singularities of the same angle θ_i at p_i as those associated to I^*.

Step 2: We construct the desired hyperbolic end M with particles via M_0.

Indeed, by Proposition 3.8, M_0 admits a unique maximal concave extension which is a hyperbolic end with particles, say M. We will show that the induced complex projective structure σ on $\partial_\infty M$ satisfies the required condition.

A direct computation shows that $I^* = \frac{1}{2} e^{-2r} G_{r*}(I_r + 2II_r + III_r)$ (see e.g. [20, Lemma 5.1]), where G_r is the Gauss map from $(\Sigma \times \{r\}, I_r)$ to $\partial_\infty M$. This implies that the conformal structure induced on $\partial_\infty M$ by the hyperbolic metric on M is c. By [20, Lemma 8.3], the real part of the Schwarzian derivative of the natural map $\phi : (\partial_\infty M, \sigma) \to (\partial_\infty M, \sigma_F)$ is II_0^* (note that the proof of this lemma is purely local, and therefore extends to the singular setting), where σ is the complex projective structure induced on $\partial_\infty M$ and σ_F is the Fuchsian complex projective structure of σ. Hence, $\mathrm{Re}\, \mathcal{S}(\phi) = II_0^* = \mathrm{Re}\, q$. This implies that $\mathcal{S}(\phi) = q$. □

Proof of Theorem 1.1. Note that the hyperbolic end with particles in Proposition 3.9 is unique from the construction. Combined with Proposition 3.4 and Lemma 3.5, Theorem 1.1 follows. □

3.7. Hyperbolic ends with particles in terms of the bending data on the metric boundary

Now we consider the relation between $\mathcal{HE}_\theta$ and $\mathcal{T}_{\Sigma,\theta} \times \mathcal{ML}_{\mathfrak{p}}$.

Let M be a hyperbolic end with particles. It follows from Remark 2.8 that $\partial_0 M$ has a bending lamination, say λ.

Note that the singular lines are orthogonal to $\partial_0 M$ and the total angles around the singular curves are less than π. The distance from the singular points in $\bar{M}$ to the support L of the bending lamination is bounded away from 0. In particular, if $x \in \partial_0 M$ is a singular point, then $\partial_0 M$ has a local support plane at x in $\bar{M}$, say P, such that $P \cap \partial_0 M$ contains a neighborhood of x in P.

It follows from those facts that $\partial_0 M$ can be locally isometrically embedded into a complete pleated surface in $\mathbb{H}^3$ (resp. a totally geodesic plane orthogonal to the singular line in $\mathbb{H}^3_{\theta_i}$ for some θ_i) away from the singular points (resp. near each singular point). Therefore, $\partial_0 M$ carries an intrinsic hyperbolic metric, say h, with cone singularities (at the intersections with singular locus) of angle equal to the total angle around the corresponding singular curve. Thus we obtain (up to isotopy) a pair $(h, \lambda) \in \mathcal{T}_{\Sigma,\theta} \times \mathcal{ML}_{\mathfrak{p}}$.

Proposition 3.10. *The map sending a hyperbolic end with particles to (h, λ), the induced metric and measured bending lamination on its metric boundary is a bijection between $\mathcal{HE}_\theta$ and $\mathcal{T}_{\Sigma,\theta} \times \mathcal{ML}_{\mathfrak{p}}$.*

Proof. The construction above shows that the pair (h, λ) is uniquely determined by the choice of a hyperbolic end.

Conversely, we will show that given a hyperbolic metric $h \in \mathcal{T}_{\Sigma,\theta}$ and a measured lamination $\lambda \in \mathcal{ML}_{\mathfrak{p}}$, there is a unique hyperbolic end with particles, say M, such that h and λ are the induced metric and bending lamination on $\partial_0 M$. The argument is similar to that in [10, Proposition 5.8] which considers the case of AdS manifolds with particles.

Denote by $\widetilde{\Sigma_{\mathfrak{p}}}$ the universal cover of $\Sigma_{\mathfrak{p}}$. We claim that h and λ determine a local isometric embedding $dev_\lambda : \widetilde{\Sigma_{\mathfrak{p}}} \to \mathbb{H}^3$, which is equivariant under a homomorphism $\rho_\lambda : \pi_1(\Sigma_{\mathfrak{p}}) \to PSL_2(\mathbb{C})$. Indeed, associated to λ we can define a bending cocycle $\beta_\lambda : \widetilde{\Sigma_{\mathfrak{p}}} \times \widetilde{\Sigma_{\mathfrak{p}}} \to PSL_2(\mathbb{C})$ (see [2, Chapter 4.1] and [11, Definition II 3.5.2]), which satisfies the following two equalities:

$$\beta_\lambda(x, y) \circ \beta_\lambda(y, z) = \beta_\lambda(x, z),$$
$$\beta_\lambda(\gamma x, \gamma y) = \rho(\gamma)\beta_\lambda(x, y)\rho(\gamma)^{-1},$$

where $\rho : \pi_1(\Sigma_{\mathfrak{p}}) \to PSL_2(\mathbb{R}) \le PSL_2(\mathbb{C})$ is the holonomy representation of h.

In particular, the map dev_λ can be expressed in terms of β_λ, that is,

$$dev_\lambda(x) = \beta_\lambda(x_0, x) I(dev(x)),$$

where $x_0 \in \widetilde{\Sigma_{\mathfrak{p}}}$ is a fixed point, dev is the developing map of h, and I is the isometric embedding of $\mathbb{H}^2$ into $\mathbb{H}^3$. We define $\rho_\lambda : \pi_1(\Sigma_{\mathfrak{p}}) \to PSL_2(\mathbb{C})$ as

$$\rho_\lambda(\gamma) = \beta_\lambda(x_0, \gamma x_0) \circ \rho(\gamma),$$

for all $\gamma \in \pi_1(\Sigma_{\mathfrak{p}})$.

One can check that dev_λ is locally injective and it is ρ_λ-equivariant. Note that as the singular locus of h on Σ stay away from λ, the cocycle $\beta_\lambda(x_0, x)$ is trivial in $\pi^{-1}(U_i)$ for a neighborhood U_i of a marked point $p_i \in \mathfrak{p}$, where $\pi : \widetilde{\Sigma_{\mathfrak{p}}} \to \Sigma_{\mathfrak{p}}$ is the universal cover. This implies that the map dev_λ is conjugated to dev in $\pi^{-1}(U_i)$. Let S be the surface equipped with the developing map dev_λ and the holonomy representation ρ_λ. Then S admits a hyperbolic metric on $\Sigma_{\mathfrak{p}}$ with cone singularities of the same angle as h at $\mathfrak{p}$, and bending along λ (in terms of the local chart in $\mathbb{H}^3$ given by $(dev_\lambda, \rho_\lambda)$-data) with the bending angle equal to the corresponding transverse measure. Let us denote by S_r the regular set of S and by $\widetilde{S_r}$ the universal cover of S_r. Then $dev_\lambda : \widetilde{S_r} \to \mathbb{H}^3$ is a ρ_λ-equivariant developing map of S_r. Now we consider the normal exponential map, called exp, of $dev_\lambda(\widetilde{S_r}) \subset \mathbb{H}^3$.

$$\exp : N(dev_\lambda(\widetilde{S_r})) \to \mathbb{H}^3,$$

where $N(dev_\lambda(\widetilde{S_r}))$ is the set of the pairs (x, v) with $x \in dev_\lambda(\widetilde{S_r})$ and v a locally concave-directed vector at x which is orthogonal to a totally geodesic disk centered at x and supporting on $dev_\lambda(\widetilde{U_{\tilde{x}}}) \subset \mathbb{H}^3$, here $\widetilde{U_{\tilde{x}}} \subset \widetilde{S_r}$ is a neighborhood of a point $\tilde{x} \in dev_\lambda^{-1}(x)$ such that $dev_\lambda|_{\widetilde{U_{\tilde{x}}}}$ is homeomorphic. Define $\exp(x, v) = \exp_x(v)$. Note that $dev_\lambda(\widetilde{S_r})$ is locally concave in $\mathbb{H}^3$ and then exp is well-defined and indeed a local homeomorphism by construction. Hence $dev_\lambda(\widetilde{S_r})$ inherits a natural metric from the hyperbolic metric on $\mathbb{H}^3$.

Note also that the holonomy representation ρ_λ for S_r induces a natural action on $N(dev_\lambda(\widetilde{S_r}))$: for any $(x, v) \in N(dev_\lambda(\widetilde{S_r}))$ and $\gamma \in \pi_1(S_r)$, we define $\rho_\lambda(\gamma)(x, v) = (\rho_\lambda(\gamma)(x), \rho_\lambda(\gamma)_*(v))$, where $\rho_\lambda(\gamma)_*(v)$ is the put-forward vector at $\rho_\lambda(\gamma)(x)$ by $\rho_\lambda(\gamma)$ of the vector v at x. Now we define an identification on $\exp(N(dev_\lambda(\widetilde{S_r})))$ by identifying $\exp(x, v)$ with $\exp(x', v')$ if (x, v) is related to (x', v') by an action induced by $\rho_\lambda(\gamma)$ for some $\gamma \in \pi_1(S_r)$. One can check that the quotient of $\exp(N(dev_\lambda(\widetilde{S_r})))$ by this identification is a hyperbolic manifold homeomorphic to $S_r \times (0, +\infty)$ (since dev_λ is locally homeomorphic and ρ_λ-equivariant, exp is locally homeomorphic, and the induced metric on $\exp(N(dev_\lambda(\widetilde{S_r})))$ is invariant under this identification).

Let M be the metric completion of this quotient manifold. Observe that for each small loop $\gamma_i \in \pi_1(S_r)$ around the marked point p_i, $\rho_\lambda(\gamma_i)$ is an elliptic element in $PSL_2(\mathbb{C})$ of angle θ_i up to conjugation. Note also that the distance from the support of λ to the cone singularities of S is bounded away from 0. Then the small neighborhood of the line $l_i = \{p_i\} \times (0, +\infty)$ in M is locally modeled on $\mathbb{H}^3_{\theta_i}$, thus l_i is a singular curve in M with cone singularities of angle θ_i at each point. Therefore, M is a hyperbolic end with particles in $\mathcal{HE}_\theta$, which has a concave pleated boundary (identified to S) with the induce metric h and the bending lamination λ.

Let $f : \mathcal{T}_{\Sigma,\theta} \times \mathcal{ML}_\mathfrak{p} \to \mathcal{HE}_\theta$ be the map constructed above. It follows from the construction that f is well-defined, with the inverse as exactly the induced hyperbolic metric and bending lamination on $\partial_0 M$. This completes the proof. □

3.8. Comparing parameterizations of $\mathcal{HE}_\theta$

We now sum up the various parameterizations of the space of hyperbolic ends with particles, and the relations among them.

Proposition 3.11. *The following maps are homeomorphisms.*

- *The map $f : \mathcal{T}_{\Sigma,\theta} \times \mathcal{ML}_\mathfrak{p} \to \mathcal{HE}_\theta$ sending (m, l) to the unique hyperbolic end with particles such that the induced metric and measured bending lamination on the metric boundary are m and l, see Proposition 3.10,*
- *the map $f_1 : \mathcal{HE}_\theta \to \mathcal{CP}_\theta$ sending a hyperbolic end with particles to the complex projective structure at infinity, see Proposition 3.4,*
- *the map $f_2 : \mathcal{CP}_\theta \to T^*\mathcal{T}_{\Sigma,\theta}$ sending a complex projective structure to the Schwarzian derivative of its map to the Fuchsian complex projective structure with the same underlying complex structure, see Lemma 3.5,*
- *the map $f_3 : T^*\mathcal{T}_{\Sigma,\theta} \to \mathcal{HE}_\theta$ reconstructing a hyperbolic end with particles from the data of a hyperbolic metric and a traceless Codazzi tensor on the boundary at infinity, see Proposition 3.9.*

Moreover, the triangle on the right-hand side of Fig. 2 commutes.

Proof. It is sufficient to show the continuity of the maps f, f^{-1}, f_1, f_2, f_3 in the following diagram.

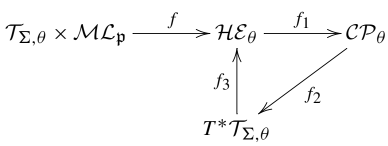

Fig. 2. A diagram showing the relations among several spaces related to $\mathcal{HE}_\theta$.

Note that the induced metric and the bending lamination on $\partial_0 M$ of a hyperbolic end M with particles are completely determined by the intrinsic geometry of M. Conversely, a hyperbolic end with particles is obtained as the image under the exponential map exp (defined before Proposition 3.10) of the normal bundle NS, which depends continuously on the $(dev_\lambda, \rho_\lambda)$-data determined by the bending data $(h, \lambda) \in \mathcal{T}_{\Sigma,\theta} \times \mathcal{ML}_\mathfrak{p}$. Therefore, f and f^{-1} are naturally continuous.

As for the map f_1, observe that the complex projective structure induced on $\partial_\infty M$ is determined by the canonical complex projective structure on $\partial_\infty M_r$ (considered as an extended $\big(PSL_2(\mathbb{C}), \partial_\infty \mathbb{H}^3\big)$-structure on $\partial_\infty M_r$, which depends continuously on the $\big(PSL_2(\mathbb{C}), \mathbb{H}^3\big)$-structure on M_r) and the asymptotic geometry near the endpoints at infinity of the singular curves in M (see Lemma 3.3, which ensures that the complex projective structure at infinity has cone singularities of angle θ_i at the endpoint at infinity of the singular curve $\{p_i\} \times (0, +\infty)$). Hence, f_1 is naturally continuous.

A well-known fact in complex analysis says that uniformly convergent holomorphic maps have uniformly convergent derivatives of arbitrary order (on compact subsets). Note also that the natural maps from a complex projective structure with cone singularities to the corresponding Fuchsian complex projective structure extend conformally to the marked points (with respect to the complex charts) and there is a natural holomorphic local diffeomorphism from the

$\mathbb{C}P^1$-chart in $\mathbb{C}_{\theta_i}$ to the complex chart in $\mathbb{C}$ at the singular point p_i (see e.g. Lemma 3.5). Therefore, the Schwarzian derivative induces a continuous map on the space of the natural conformal maps from a complex projective structure σ with cone singularities to the corresponding Fuchsian complex projective structure σ_F. Moreover, the sequence of natural conformal maps $\varphi_n : (\Sigma, \sigma_n) \to (\Sigma, (\sigma_n)_F)$ converges to the natural conformal map $\varphi : (\Sigma, \sigma) \to (\Sigma, \sigma_F)$ (with respect to the $\mathbb{C}P^1$-charts) as σ_n converges to σ in $\mathcal{CP}_\theta$ (under the topology defined using development-holonomy pairs). It follows that f_2 is continuous.

Recall the proof of Proposition 3.9 that the geometry of the obtained hyperbolic end M with particles from a given quadratic differential $q \in T^*\mathcal{T}_{\Sigma,\theta}$ is completely determined by the first and second fundamental form I^*, II^* (defined by q) on $\partial_\infty M$. More precisely, I^* is the hyperbolic metric with cone singularities of fixed angles in the conformal class of the underlying conformal structure of q and $II^* = \frac{1}{2}I^* + \mathrm{Re}\, q$. This implies that I^* and II^* depend continuously on $q \in T^*\mathcal{T}_{\Sigma,\theta}$. As a result, we obtain the continuity of f_3.

Combining the above results, any two spaces in Fig. 2 are homeomorphic. □

3.9. The grafting map on hyperbolic surfaces with prescribed cone singularities

In non-singular case, it was proved by Thurston that the grafting map $Gr : \mathcal{T} \times \mathcal{ML} \to \mathcal{CP}$ is a homeomorphism (see e.g. [15,21,29]), where $\mathcal{T}$ denotes the Teichmüller space of a closed oriented surface S of genus at least 2, $\mathcal{ML}$ denotes the space of measured laminations on S and $\mathcal{CP}$ is the space of complex projective structures on S, up to isotopy. Here we generalize this result to hyperbolic surfaces with cone singularities of angles less than π by showing that the grafting map is indeed the composition of the maps f and f_1 in Proposition 3.11.

Recall that for a hyperbolic surface with cone singularities p_i of angles $\theta_i \in (0, \pi)$, each p_i has a neighborhood of a radius $r_i = r(\theta_i) > 0$ (depending only on θ_i) which is disjoint from any simple closed geodesic (see [14, Theorem 3]). Note also that the weighted multicurves are dense in $\mathcal{ML}_\mathfrak{p}$. Then the distance from the support of any measured laminations in $\mathcal{ML}_\mathfrak{p}$ to $\{p_1, ..., p_{n_0}\}$ has a uniformly positive lower bound. Therefore, the grafting operation can be naturally generalized to the case with cone singularities.

Let S be a hyperbolic surface with the metric $h \in \mathcal{T}_{\Sigma,\theta}$ and let $t\gamma$ be a t-weighted simple closed geodesic on S. We perform a *grafting operation*: cut S open along γ and glue a cylinder $\gamma \times [0, t]$ along the cutting on both side. For a disjoint union $\cup_i t_i \gamma_i$ of weighted simple closed geodesics, we can also perform this operation for each weighted geodesic $t_i \gamma_i$. Note that this operation is done outside the union of the neighborhood U_{r_i} of each singular point p_i on S with a radius r_i. As the non-singular case (see e.g. [15, Section 4.1]), we can consider the corresponding operation in the universal cover of the regular set of S. It is not hard to see that the obtained surface admits a complex projective structure with prescribed cone singularities.

For non-singular case, Thurston has shown that grafting along weighted simple closed curves extends continuously to arbitrary measured laminations. Note again that the distance from the support of any measured lamination to the cone points is bounded away from 0. Under a limit process, we can also consider the grafting along a measured lamination $\lambda \in \mathcal{ML}_\mathfrak{p}$ as the limit of the obtained complex projective structure under the grafting operation along $\cup_i t_i \gamma_i$ with $\sum_i t_i \gamma_i \to \lambda$ in $\mathcal{ML}_\mathfrak{p}$ (note that this is independent of the choice of $\cup_i t_i \gamma_i$).

Definition 3.12. Let $Gr_\theta : \mathcal{T}_{\Sigma,\theta} \times \mathcal{ML}_\mathfrak{p} \to \mathcal{CP}_\theta$ be the map associates to (h, λ) the complex projective structure obtained by the above grafting operation on a hyperbolic surface (Σ, h) along λ. We call it the grafting map.

Lemma 3.13. $Gr_\theta = f_1 \circ f$.

Proof. It suffices to show that for each hyperbolic end $M \in \mathcal{HE}_\mathfrak{p}$, the complex projective structure induced on $\partial_\infty M$ can be obtained as the image of the pair (h, λ) under the grafting map Gr_θ, where h and λ are the induced hyperbolic metric and the bending lamination on $\partial_0 M$, respectively. Indeed, we only need to prove this for the case that λ is a simple closed geodesic γ with the weight $\alpha > 0$ which records the bending angle at γ.

Let $S = \partial_0 M$ and consider the normal exponential map $\exp : N^1 S \times (0, +\infty) \to M$ defined in Lemma 2.6. For each $r > 0$, the subset $\exp(N^1(S \setminus \gamma) \times \{r\})$ of the equidistant surface S_r at distance r from S has induced metric $I_r = \cosh^2(r)h$ for all $x \in S \setminus \lambda$. Moreover, the image $\exp(N^1(\gamma) \times \{r\})$ is an annulus A_r embedded in S_r. By computation, A_r has two boundary components of length $a_r = \cosh(r)\ell_\gamma(h)$ and the shortest distance between these two boundary components is $b_r = \sinh(r)\alpha$.

Therefore, for $x \in S \setminus \lambda$, the induced metric I_r of the set $\exp(N^1(S\setminus\gamma)\times\{r\})$ satisfies that $e^{-2r}I_r \to h$ as $r \to +\infty$. On the other hand, the ration (or module) of A_r, as $r \to +\infty$, satisfies that

$$\frac{a_r}{b_r} = \frac{\cosh(r)}{\sinh(r)}\frac{\ell_\gamma(h)}{\alpha} \to \frac{\ell_\gamma(h)}{\alpha} = \mathrm{Mod}(A_\gamma),$$

where $A_\gamma = \gamma \times [0, \alpha]$ is the annulus replacing γ in the grafting operation and $\mathrm{Mod}(A_\gamma)$ is the module of A_γ. Therefore, the complex projective structure on $\partial_\infty M$ is $Gr_\theta(h, \lambda)$. □

Proof of Theorem 1.2. This follows from Proposition 3.11 and Lemma 3.13. □

4. De Sitter spacetimes with particles and complex projective structures with cone singularities

In this section, we consider the "dual" manifolds of hyperbolic ends with particles, that is, future-complete convex GHM de Sitter spacetimes with particles (see Definition 2.11). We describe this dual relation in terms of the complex projective structures induced on the boundary at infinity of either of these two mutually dual manifolds.

It is interesting to ask whether every future-complete GHM de Sitter spacetime with particles contains a strictly future-convex spacelike surface. This relates closely to a question posed in [18, Section 6] whether every future-complete GHM de Sitter spacetime with particles contains a constant mean curvature spacelike surface, and a question asked in [7] whether every future-complete GHM flat spacetime with particles contains a uniformly future-convex spacelike surface.

4.1. The complex projective structure at infinity of a de Sitter spacetime $M^d \in \mathcal{DS}_\theta$

Recall that every de Sitter spacetime in $\mathcal{DS}_\theta$ is future-complete. We denote by $\partial_\infty M^d$ the boundary at infinity of a de Sitter spacetime $M^d \in \mathcal{DS}_\theta$ and will show that $\partial_\infty M^d$ admits a complex projective structure with cone singularities of the same angles as the particles.

The model space W_α. Let $\alpha > 0$ and let Γ_0 be a fixed, future-oriented complete timelike geodesic in $\mathbb{DS}_3$. Denote by U the universal cover of the complement of Γ_0 in $\mathbb{DS}_3$ and denote by W the completion of U, such that $W \setminus U$ is canonically identified with Γ_0, which is called the *singular set of* W. We define W_α as the quotient of W by the rotation of angle α around Γ_0. The image of the singular set of W under this quotient is called the *singular set of* W_α.

Let M^d be a future-complete convex GHM de Sitter spacetime with particles. It is clear that each singular point x of M^d has a neighborhood isometric to a subset of W_α with α equal to the total angle around the singular curve through x. Now we describe the geometry property of M^d near the endpoints at infinity of the singular curves in M^d by using the model W_α, see the following lemma. Since M^d contains a strictly future-convex spacelike surface, with an alternative version of Lemma 2.6 for the de Sitter case with particles, the argument for the hyperbolic case with particles is adapted to the de Sitter case.

Lemma 4.1. *For each point $p_i \in \partial_\infty M^d$ which is the endpoint at infinity of a singular curve in M^d, p_i has a neighborhood U_i in M^d isometric to a neighborhood of the endpoint at infinity of Γ_0 in W_{θ_i} which lies on $\mathbb{S}^2_+$, where θ_i is the total angle around that singular curve.*

Proof. Now we prove the lemma in the following four steps:

Step 1: Let $S^d \subset M^d$ be a strictly future-convex spacelike surface and let NS^d be the space of future-directed vectors normal to S^d (note that at a singular point $x \in S^d$, the "normal" vector is directed along the singular curve through x). Given $v = (x, n) \in NS^d$, we denote by $\exp(v) \in M^d$ the point $\gamma(1)$, where $\gamma : [0, 1] \to M^d$ is the geodesic such that $\gamma(0) = x$ and $\gamma'(0) = n$, if it exists. This defines a map exp from a subset of NS^d to M^d.

Step 2: We claim that the map $\exp : NS^d \to M^d$ is well-defined on NS^d and it is a homeomorphism onto its image. Note that S^d is a Cauchy surface in M^d and every geodesic starting in the direction of NS^d is timelike, then there is no geodesic segment in the future of S^d connecting two points of S^d in the directions of NS^d. Applying an analogous argument used in Lemma 2.6 for hyperbolic case, we have the claim.

Step 3: The exponential map $\exp_\infty : NS^d \to \partial_\infty M^d$ is a homeomorphism, where $\exp_\infty$ is defined as the equivalence class of the geodesic ray which is the fiber of NS^d over $x \in S^d$. This follows directly from Step 2.

Step 4: By Step 3, for each point $p_i \in \partial_\infty M^d$ which is the endpoint at infinity of a singular curve in M^d, the singular curve is unique and we denote it by l_i. Assume that this singular curve l_i intersects S^d at x_i. Let F_i be the fiber of NS^d over x_i and let H_i be a small neighborhood of F_i in NS^d. Consider $U_i = \exp(H_i)$. Note that M^d is locally modeled on W_{θ_i} near the singular curve l_i (where l_i is identified as the singular set Γ_0 in W_{θ_i} and θ_i is the total angle around l_i). By the definition of de Sitter metrics with particles, U_i contains a cylinder of exponentially expanding radius around l_i along the future-direction. This implies the desired result. □

Note that the regular set M^d_r of M^d has a $(PSL_2(\mathbb{C}), \mathbb{DS}_3)$-structure and it is future-complete, we can define the boundary at infinity of M^d_r, denoted by $\partial_\infty M^d_r$, as for the hyperbolic case in Proposition 3.4. Moreover, $\partial_\infty M^d_r$ carries a canonical complex projective structure. Combined with the geometric property of M^d near the endpoints at infinity of the singular curves, as presented in Lemma 4.1, we have the following proposition.

Proposition 4.2. *Let $M^d \in \mathcal{DS}_\theta$ be a future-complete convex GHM de Sitter spacetime with particles. Then the boundary at infinity $\partial_\infty M^d$ is endowed with a complex projective structure with cone singularities of angle θ_i at the p_i.*

4.2. The construction of de Sitter spacetimes in $\mathcal{DS}_\theta$ from complex projective structures in $\mathcal{CP}_\theta$

To construct a convex GHM de Sitter spacetime with particles from a complex projective structure with cone singularities, we give the following result which ensures the existence and the uniqueness (up to isometry) of the maximal extension of a convex GH de Sitter spacetime with particles. This can be proved by adapting verbatim the argument given for the anti-de Sitter case in [5, Proposition 6.24].

Proposition 4.3. *Let M^d_0 be a convex GH de Sitter spacetime with particles. Then there exists a unique (up to isometry) maximal extension of M^d_0, called M^d, in which M^d_0 can be isometrically embedded.*

Proposition 4.4. *Let $\sigma \in \mathcal{CP}_\theta$ be a complex projective structure with cone singularities. Then there is a unique future-complete convex GHM de Sitter spacetime with particles $M^d \in \mathcal{DS}_\theta$, such that $\partial_\infty M^d$ is endowed with the complex projective structure σ.*

Proof. By Lemma 3.5, the Schwarzian derivative of the conformal map $id : (\Sigma_{\mathfrak{p}}, \sigma) \to (\Sigma_{\mathfrak{p}}, \sigma_F)$ is a meromorphic quadratic differential q in $T^*_c\mathcal{T}_{\Sigma,\theta}$, where c is the common underlying conformal structure of σ_F and σ.

Now we use q to construct a future-complete convex GHM de Sitter spacetime M^d with particles in the following two steps, as in Proposition 3.9 for the hyperbolic case.

Step 1: First we construct a future-complete GH de Sitter spacetime M^d_0 with the prescribed particles which is homeomorphic to $\Sigma \times \mathbb{R}_{\geq 0}$.

As in the hyperbolic case (see the proof of Proposition 3.9), we use the same data at infinity. Let I^* be a hyperbolic metric with the prescribed cone singularities in the conformal class c. Recall the notations that $II^*_0 = \mathrm{Re}\, q$, $II^* = \frac{1}{2}I^* + II^*_0$, $B^* = (I^*)^{-1}II^*$ and $III^* = I^*(B^*\bullet, B^*\bullet)$.

Let M^d_0 be the set $\Sigma \times [t_0, +\infty)$ with the metric

$$g^d_0 = -dt^2 + \frac{1}{2}(e^{2t}I^* - 2II^* + e^{-2t}III^*),$$

where t_0 is to be determined. We claim that M^d_0 is a convex GH de Sitter spacetime with particles if we choose t_0 large enough. Denote $I^d_t = \frac{1}{2}(e^{2t}I^* - 2II^* + e^{-2t}III^*)$. Then we have

$$I^d_t = \frac{1}{2}I^*((e^tE - e^{-t}B^*)\bullet, (e^tE - e^{-t}B^*)\bullet),$$
$$II^d_t = \frac{1}{2}\frac{dI^d_t}{dt} = \frac{1}{2}I^*((e^tE - e^{-t}B^*)\bullet, (e^tE + e^{-t}B^*)\bullet).$$

Denote $B^d_t = (e^tE - e^{-t}B^*)^{-1}(e^tE + e^{-t}B^*)$. Similarly as the hyperbolic case (see Proposition 3.9), one can check that (I^d_t, B^d_t) satisfies the following conditions:

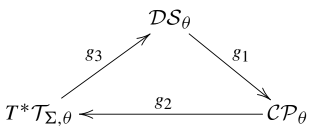

Fig. 3. A diagram showing the relations among the spaces related to $\mathcal{DS}_\theta$.

- B_t^d is self-adjoint for I_t^d: $I_t^d(B_t^d\bullet,\bullet)=I_t^d(\bullet,B_t^d\bullet)$. This follows from the fact that II^* is self-adjoint for I^* (since II_0^* is the real part of a quadratic differential q).
- (I_t^d, B_t^d) satisfies the Gauss equation for surfaces in dS_3: $K_{I_t^d}=1-\det B_t^d$, where $K_{I_t^d}$ is the sectional curvature of I_t^d.
- (I_t^d, B_t^d) satisfies the Codazzi equation: $d^{\nabla^{I_t^d}}B_t^d=0$, where $\nabla^{I_t^d}$ is the Levi-Civita connection of I_t^d.
- (I_t^d, B_t^d) satisfies the following equality:

$$I_{t+s}^d=I_t^d((\cosh(s)E+\sinh(s)B_t^d)\bullet,(\cosh(s)E+\sinh(s)B_t^d)\bullet),$$

  for all $t,s>0$. This follows from a direct computation.

Denote by λ^*, μ^* (resp. λ_t^d, μ_t^d) the eigenvalues of B^* (resp. B_t^d). By computation,

$$\lambda_t^d=\frac{e^t+e^{-t}\lambda^*}{e^t-e^{-t}\lambda^*},\qquad \mu_t^d=\frac{e^t+e^{-t}\mu^*}{e^t-e^{-t}\mu^*}.$$

If t_0 is large enough, the eigenvalues $\lambda_{t_0}^d$, $\mu_{t_0}^d$ of $(\Sigma\times\{t_0\}, I_{t_0}^d)$ are both positive. Let the positive direction of t be the future direction. Combined with the above properties of (I_t^d, B_t^d), this shows that M_0^d is a future-complete convex GH de Sitter spacetime with particles.

From the argument in Proposition 3.9, we have that B^* tends to $\frac{1}{2}E$ at each cone singularity p_i. Therefore, I_t^d tends to $\frac{1}{2}(e^t-\frac{1}{2}e^{-t})^2I^*$ at p_i. This shows that the total angle around the singular curve $\{p_i\}\times[t_0,+\infty)$ of M_0^d is θ_i.

Step 2: We construct the desired de Sitter spacetime M^d with particles via M_0^d.

Indeed, by Proposition 4.3, M_0^d admits a unique maximal extension, say M^d. We will show that the induced complex projective structure on $\partial_\infty M^d$ satisfies the required condition.

A direct computation shows that $I^*=\frac{1}{2}e^{-2t}G_{t\,*}^d(I_t^d+2II_t^d+III_t^d)$, where G_t^d is the Gauss map from $(\Sigma\times\{t\}, I_t^d)$ to $\partial_\infty M^d$. This implies that the conformal structure induced on $\partial_\infty M^d$ by the de Sitter metric on M^d is c. Note that the expressions of the first, second and third fundamental forms of the surfaces Σ_t in the foliation near the boundary at infinity of M^d can be obtained by replacing the shape operator B^* in Proposition 3.9 by $-B^*$. An adaption of the argument for the hyperbolic case (see [20, Lemma 8.3]) shows that the real part of the Schwarzian derivative of the natural map $\phi:(\partial_\infty M^d,\sigma^d)\to(\partial_\infty M^d,\sigma_F^d)$ is II_0^*, where σ^d is the complex projective structure induced on $\partial_\infty M^d$ and σ_F^d is the Fuchsian complex projective structure of σ^d. Hence, $\mathrm{Re}\,\mathcal{S}(\phi)=II_0^*=\mathrm{Re}\,q$. This implies that $\mathcal{S}(\phi)=q$. Note also that $\sigma_F^d=I^*=\sigma_F$, then $\sigma^d=\sigma$. This implies that the complex projective structure induced on $\partial_\infty M^d$ is exactly σ. □

For convenience, we give the commutative diagram in Fig. 3, which shows the relations among the spaces related to $\mathcal{DS}_\theta$ and the following maps g_1, g_2, g_3 are all homeomorphisms (see e.g. Proposition 3.11).

- the map $g_1:\mathcal{DS}_\theta\to\mathcal{CP}_\theta$ sending a de Sitter spacetime with particles to the complex projective structure at infinity, see Proposition 4.2,
- the map g_2 defined to be the map f_2 in Proposition 3.11,
- the map $g_3:T^*\mathcal{T}_{\Sigma,\theta}\to\mathcal{DS}_\theta$ reconstructing a de Sitter spacetime with particles from the data of a hyperbolic metric and a traceless Codazzi tensor on the boundary at infinity, see Proposition 4.4.

Proof of Theorem 1.4. This follows from Proposition 4.2 and Theorem 4.4. □

4.3. The duality between $\mathcal{HE}_\theta$ and $\mathcal{DS}_\theta$

Combining Theorem 1.1 and Theorem 1.4, we can define a natural map, say δ, which is a homeomorphism from $\mathcal{HE}_\theta$ to $\mathcal{DS}_\theta$ sending a hyperbolic end with particles to the unique future-complete convex GHM de Sitter spacetime with the same complex projective structure at infinity.

Let $M \in \mathcal{HE}_\theta$ be a non-degenerate hyperbolic end with particles, and let $M^d \in \mathcal{DS}_\theta$ be the dual future-complete convex GHM de Sitter spacetime with particles. We also describe a duality between closed strictly concave surfaces in M and closed strictly future-convex surfaces S^d in M^d.

Let $S \subset M$ be a closed, strictly concave surface. We define a dual surface $S^d \subset M^d$ of S, as the surface satisfying the following properties:

- S^d is a strictly future-convex spacelike surface in M^d.
- There is a unique diffeomorphism $u : S \to S^d$ such that $u^* I_d = III$ and $u^* III_d = I$, where I, III are the induced metric and third fundamental form on S, and I_d and III_d are the induced metric and third fundamental form on S^d.

Conversely, given a closed, strictly future-convex spacelike surface $S^d \subset M^d$, we can also define a dual surface $S \subset M$ of S^d, in an analogous way as above. It remains to show that the definition of the duality for surfaces is well-defined. By observation, it suffices to show the existence and uniqueness of the dual surface $S^d \subset M^d$ of a closed, strictly concave surface S in a hyperbolic end M with particles defined above. Equivalently, it suffices to show Theorem 1.5.

To show Theorem 1.5, it is convenient to clarify the relation between hyperbolic ends with particles (resp. convex GHM de Sitter spacetimes with particles) and the data at infinity. We will then see in the next subsection that the same description applies in the de Sitter case.

4.4. Hyperbolic ends with particles and the data at infinity

4.4.1. The data at infinity obtained from an equidistant foliation near the boundary at infinity of $M \in \mathcal{HE}_\theta$

Let M be a hyperbolic end with particles and let S be a strictly concave surface in M, with the induced metric I, the shape operator B, and the second fundamental form II. Consider an equidistant foliation $(S_r)_{r>0}$, with S_r the equidistant surface obtained at distance r from S along the orthogonal geodesics on the concave side of S. Define the data at infinity (I^*, II^*) as follows:

$$
\begin{aligned}
I^* &= \frac{1}{2} e^{-2r} G_{r*}(I_r + 2II_r + III_r), \\
II^* &= \frac{1}{2} e^{-2r} G_{r*}(I_r - III_r),
\end{aligned} \tag{2}
$$

where I_r, II_r, III_r are respectively the induced metric, second and third fundamental forms on S_r in M, while G_r is the Gauss map from S_r to the boundary at infinity $\partial_\infty M$ of M. One can check by direct computation that the data (I^*, II^*) defined above is independent of r.

It is not hard to check that (see e.g. [20, Remark 5.4 and Remark 5.5]) the data (I^*, II^*) satisfies the Codazzi equation and a modified version of the Gauss equation for surfaces embedded in $\mathbb{H}^3$:

$$
\begin{aligned}
d^{\nabla^{I^*}} II^* &= 0, \\
\mathrm{tr}_{I^*} II^* &= -K_{I^*}\ ,
\end{aligned} \tag{3}
$$

where ∇^{I^*} is the Levi-Civita connection of I^* and K_{I^*} is the Gauss curvature of I^*.

Conversely, I_r, II_r, and the shape operator B_r of S_r can be rewritten by using the data at infinity (I^*, II^*) in the following way (see [20, Lemma 5.6]).

$$
\begin{aligned}
I_r &= \frac{1}{2} I^*((e^r E + e^{-r} B^*)\bullet, (e^r E + e^{-r} B^*)\bullet), \\
II_r &= \frac{1}{2} I^*((e^r E + e^{-r} B^*)\bullet, (e^r E - e^{-r} B^*)\bullet), \\
B_r &= (e^r E + e^{-r} B^*)^{-1}(e^r E - e^{-r} B^*),
\end{aligned} \tag{4}
$$

where $B^* = (I^*)^{-1}II^*$.

4.4.2. The hyperbolic end with particles determined by a particular couple on Σ

Let (I'^*, II'^*) be a couple with I'^* a Riemannian metric on Σ, and II'^* a bilinear symmetric form on $T\Sigma$ (defined outside the singular locus) satisfying the following conditions, called *Condition* ($\star$) for convenience.

- (I'^*, II'^*) assumes the two equations in (3) by replacing (I^*, II^*) with (I'^*, II'^*).
- The determinant of II'^* with respect to I'^* remains bounded.

In particular, the previous data at infinity (I^*, II^*) obtained from $(S_r)_{r>0}$ in Section 4.4.1 satisfies Condition ($\star$). Denote $B'^* = (I'^*)^{-1}II'^*$. Consider the manifold $\Sigma \times [0, +\infty)$ with the following metric

$$g_0 = dr^2 + I'_r,$$

where I'_r is defined as the formula for I_r in (4) by replacing (I^*, II^*, B^*) with (I'^*, II'^*, B'^*). By Condition ($\star$), it can be checked as Step 1 in the proof of Proposition 3.9 that (I'_r, B'_r) determines a hyperbolic end with particles, denoted by M', with $(\Sigma \times \{r\})_{r>0}$ an equidistant foliation near the boundary at infinity of M'. Moreover, the data at infinity obtained from $(\Sigma \times \{r\})_{r>0}$ as in Section 4.4.1 is exactly the given couple (I'^*, II'^*). This shows that the prescribed couple (I'^*, II'^*) completely determines a hyperbolic end with particles.

To verify Theorem 1.5, we also need the following proposition, which follows from a particular case (i.e. the case of 2+1 dimensional Poincaré–Einstein manifold) of Theorem 1.9 in [28].

Proposition 4.5. *Let (I_1^*, II_1^*) and (I_2^*, II_2^*) be two couples satisfying Condition ($\star$). Then (I_1^*, II_1^*) and (I_2^*, II_2^*) characterize the same hyperbolic end with particles if and only if they satisfy the following relation:*

$$\begin{aligned} I_2^* &= e^{2u} I_1^*, \\ II_2^* &= II_1^* + \mathrm{Hess}(u) - du \otimes du + \frac{1}{2}||du||^2_{I_1^*} I_1^*, \end{aligned} \tag{5}$$

where u is a continuous function on Σ and $\mathcal{C}^2$ function on $\Sigma_{\mathfrak{p}}$. Moreover, $\mathcal{HE}_\theta$ is parameterized by the space of the couples satisfying Condition ($\star$), up to the relation (5).

4.5. De Sitter spacetimes with particles and the data at infinity

4.5.1. The data at infinity obtained from an equidistant foliation near the boundary at infinity of $M^d \in \mathcal{DS}_\theta$

Similarly, we can define the data at infinity, called (I^{d*}, II^{d*}), of a future-complete convex GHM de Sitter spacetime with particles M^d by an equidistant foliation $(S_t^d)_{t>0}$, where S_t^d is the equidistant surface obtained at distance t from S^d along the orthogonal geodesics on the convex side of a strictly future-convex spacelike surface S^d in M^d. Define the data at infinity (I^{d*}, II^{d*}) as follows:

$$\begin{aligned} I^{d*} &= \frac{1}{2} e^{-2t} G^d_{t\,*}(I_t^d + 2II_t^d + III_t^d), \\ II^{d*} &= \frac{1}{2} e^{-2t} G^d_{t\,*}(III_t^d - I_t^d), \end{aligned} \tag{6}$$

here I_t^d, II_t^d, III_t^d are respectively the induced metric, second and third fundamental forms on S_t^d in M^d, while G_t^d is the Gauss map from S_t^d to the boundary at infinity $\partial_\infty M^d$ of M^d. One can check by direct computation that the data (I^{d*}, II^{d*}) defined above is independent of t.

It is not hard to check that the data (I^{d*}, II^{d*}) satisfies the Codazzi equation and a modified version of the Gauss equation for surfaces embedded in dS_3 (indeed, these equations turn out to be the same as those in (3) for the hyperbolic case):

$$\begin{aligned} d^{\nabla^{I^{d*}}} II^{d*} &= 0, \\ \mathrm{tr}_{I^{d*}} II^{d*} &= -K_{I^{d*}}, \end{aligned} \tag{7}$$

where $\nabla^{I^{d*}}$ is the Levi-Civita connection of I^{d*} and $K_{I^{d*}}$ is the Gauss curvature of I^{d*}.

Conversely, I_t^d, II_t^d, and the shape operator B_t^d of S_t^d can be rewritten by using the data at infinity (I^{d*}, II^{d*}) in the following way (one can check this by direct computation).

$$\begin{aligned} I_t^d &= \frac{1}{2} I^{d*}((e^t E - e^{-t} B^{d*})\bullet, (e^t E - e^{-t} B^{d*})\bullet), \\ II_t^d &= \frac{1}{2} I^{d*}((e^t E - e^{-t} B^{d*})\bullet, (e^t E + e^{-t} B^{d*})\bullet), \\ B_t^d &= (e^t E - e^{-t} B^{d*})^{-1}(e^t E + e^{-t} B^{d*}), \end{aligned} \tag{8}$$

where $B^{d*} = (I^{d*})^{-1} II^{d*}$.

4.5.2. The de Sitter spacetime with particles determined by a particular couple on Σ

Let (I'^*, II'^*) be a couple satisfying Condition ($\star$). In particular, the previous data at infinity (I^{d*}, II^{d*}) obtained from $(S_t^d)_{t>0}$ in Section 4.5.1 satisfies Condition ($\star$).

Denote $B'^* = (I'^*)^{-1} II'^*$. Consider the manifold $\Sigma \times [0, +\infty)$ with the following metric

$$g_0^d = -dt^2 + I_t',$$

where I_t' is defined as the formula for I_t^d in (8) by replacing $(I^{d*}, II^{d*}, B^{d*})$ with (I'^*, II'^*, B'^*). By Condition ($\star$), it can be checked as Step 1 in the proof of Proposition 4.4 that (I_t', B_t') determines a future-complete convex GHM de Sitter spacetime with particles, denoted by M'^d, with $(\Sigma \times \{t\})_{t>0}$ an equidistant foliation near the boundary at infinity of M'^d. Moreover, the data at infinity obtained from $(\Sigma \times \{t\})_{t>0}$ as in Section 4.5.1 is exactly the given couple (I'^*, II'^*). This shows that the prescribed couple (I'^*, II'^*) completely determines a future-complete convex GHM de Sitter spacetime with particles.

As a consequence, we have a result for the de Sitter case analogous to Proposition 4.5 for the hyperbolic case.

Proposition 4.6. *Let (I_1^*, II_1^*) and (I_2^*, II_2^*) be two couples satisfying Condition ($\star$). Then (I_1^*, II_1^*) and (I_2^*, II_2^*) characterize the same future-complete convex GHM de Sitter spacetime with particles if and only if they satisfy the relation (5). Moreover, $\mathcal{DS}_\theta$ is parameterized by the space of the couples satisfying Condition ($\star$), up to the relation (5).*

Proof of Theorem 1.5. By the definition of the dual relation between M and M^d (see Section 4.3) and combining Proposition 4.5 and Proposition 4.6, M and M^d are indeed parameterized by the same data at infinity, denoted by (I'^*, II'^*), which is obtained from the same complex projective structure with cone singularities induced at infinity of M and M^d (see e.g. Proposition 3.9 and Proposition 4.4).

Note that from the given embedded strictly concave surface $S \subset M$ we can construct an equidistant foliation $(S_r)_{r>0}$ near $\partial_\infty M$. Hence, M is also characterized by the couple (I^*, II^*), which is the data at infinity obtained from the foliation $(S_r)_{r>0}$, as shown in Section 4.4.1. Proposition 4.5 implies that (I^*, II^*) and (I'^*, II'^*) satisfy the relation (5).

Now we construct a future-complete convex GHM de Sitter spacetime with particles, called M_1^d, by using an adapted embedding data, denoted by (I^d, B^d), obtained from (S, I, B), where B is the shape operator of S in M, (I^d, B^d) is defined as follows:

$$I^d := III, \quad B^d := B^{-1}.$$

It is not difficult to check that (I^d, B^d) satisfies the Codazzi–Gauss equations for surfaces embedded in dS_3 (this follows from a computation using Proposition 2.18 and the fact that (I, B) satisfies the Codazzi–Gauss equations for surfaces embedded in $\mathbb{H}^3$). Moreover, B^d is self-disjoint for I^d with positive eigenvalues. Now we consider the manifold $\Sigma \times [0, +\infty)$, called M_0^d, with the following metric:

$$g_0^d = -dt^2 + I^d((\cosh(t)E + \sinh(t)B^d)\bullet, (\cosh(t)E + \sinh(t)B^d)\bullet) \ ,$$

where E is the identity isomorphism on $T\Sigma$ and $t \in [0, +\infty)$. Combined with the above properties of (I^d, B^d), it follows that M_0^d is a future-complete convex GH dS spacetime with particles. Let M_1^d be the (unique) maximal extension of M_0^d (this is ensured by Proposition 4.3). Moreover, $\Sigma \times \{t\}$, called S_t^d, is the equidistant surface in M_1^d at a distance t on the convex side from the strictly future-convex surface $\Sigma \times \{0\}$, called S^d, with the induced metric I^d and shape operator B^d.

Therefore, M_1^d has the data at infinity, called (I^{d*}, II^{d*}), which is obtained from the foliation $(S_t^d)_{t>0}$ near $\partial_\infty M_1^d$. One can check by using the formulas (2) and (6) that $(I^{d*}, II^{d*}) = (I^*, II^*)$. Therefore, (I^{d*}, II^{d*}) and (I'^*, II'^*) satisfy the relation (5). Using Proposition 4.6 again, the manifold M_1^d characterized by (I^{d*}, II^{d*}) is the same as the manifold M^d characterized by (I'^*, II'^*).

Therefore, S^d is a strictly future-convex spacelike surface in M^d. Since the boundary at infinity $\partial_\infty M$ (resp. $\partial_\infty M^d$) of M (resp. M^d) can be identified as a complex projective surface with prescribed cone singularities. There is a natural correspondence between the points on $\partial_\infty M$ and the points on $\partial_\infty M^d$ through the Gauss normal flow starting from $S \subset M$ (resp. $S^d \subset M^d$). Let $u := (G^d)^{-1} \circ G$, where G^d (resp. G) is the Gauss map from S^d (resp. S) to $\partial_\infty M^d$ (resp. $\partial_\infty M$). Then $u : S \to S^d$ is a diffeomorphism (outside the singular locus) such that $u^* I^d = III$ and $u^* III^d = I$. Note that a closed, strictly concave surface in M (resp. strictly future-convex spacelike surface in M^d) is uniquely determined by its embedding data, the uniqueness of S^d and u follows. This completes the proof of Theorem 1.5. □

Proof of Proposition 1.6. Denote by K the Gauss curvature of a strictly concave surface $S \subset M$ in Theorem 1.5 and denote by K^d the Gauss curvature of the dual strictly future-convex surface $S^d \subset M^d$. It follows from the argument of Theorem 1.5 that K^d is equal to the Gauss curvature of the third fundamental form on S, that is, $K^d = K/(K+1)$. Conversely, K is equal to the Gauss curvature of the third fundamental form on S^d, that is, $K = K^d/(1-K^d)$. Therefore, K is a constant in $(-1, 0)$ if and only if K^d is a constant in $(-\infty, 0)$, related by an equality $K^d = K/(K+1)$. This shows Proposition 1.6. □

5. Parametrization of $\mathcal{HE}_\theta$ by $\mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$ in terms of constant curvature surfaces

In this section, we will prove Theorem 1.3 by parameterizing $\mathcal{HE}_\theta$ in terms of constant curvature surfaces. We consider hyperbolic manifolds with particles homeomorphic to $\Sigma \times \mathbb{R}_{>0}$, with a metric boundary orthogonal to the singular locus. Moreover, the surfaces we consider in a hyperbolic manifold with particles are assumed to be incompressible embedded closed surfaces (homeomorphic to Σ) and orthogonal to the singular curves. In order to define the parameterization map, we first give the following lemma.

5.1. The definition of the map ϕ_K

Lemma 5.1. *Let $K \in (-1, 0)$ and let $(h, h') \in \mathfrak{M}_{-1}^\theta \times \mathfrak{M}_{-1}^\theta$ be a pair of normalized metrics. Then there exists a unique hyperbolic end M with particles which contains a surface of constant curvature K, with the induced metric $I = (1/|K|)h$ and the third fundamental form $III = (1/|K^*|)h'$, where $K^* = K/(1+K)$.*

Proof. Let $b : T\Sigma \to T\Sigma$ be the bundle morphism associated to (h, h') by Definition 2.16, so that $h' = h(b\bullet, b\bullet)$. Let $I = (1/|K|)h$. We equip Σ with the metric I and consider a bundle morphism $B : T\Sigma \to T\Sigma$, which is defined by $B = \sqrt{1+K}b$. By the properties of h and b, it follows that

- (Σ, I) has constant curvature K.
- B is self-adjoint for I with positive eigenvalues.
- B satisfies the Codazzi equation: $d^{\nabla^I} B = 0$, where ∇^I is the Levi-Civita connection of I.
- B satisfies the Gauss equation: $K = -1 + \det(B)$.

Consider the manifold $\Sigma \times (-\varepsilon, +\infty)$ with the following metric (here $\varepsilon > 0$ is a sufficiently small number):

$$g_0 = dt^2 + I((\cosh(t)E + \sinh(t)B)\bullet, (\cosh(t)E + \sinh(t)B)\bullet),$$

where E is the identity isomorphism on $T\Sigma$ and $t \in (-\varepsilon, +\infty)$. One can check that $\Sigma \times (-\varepsilon, +\infty)$ endowed with the metric g_0 is a hyperbolic manifold with particles, denoted by M_0, which has a concave metric boundary (note that B has positive eigenvalues, then $\Sigma \times \{0\}$ with the induced metric is strictly concave and we can construct such a manifold by taking ε small enough), and each line $\{p_i\} \times (-\varepsilon, +\infty)$ corresponds to a singular curve, around which the total angle is θ_i. Furthermore, for each $t \in (-\varepsilon, +\infty)$, the surface $\Sigma \times \{t\}$ is the equidistant surface at an oriented distance t from $\Sigma \times \{0\}$, where $t > 0$ corresponds to the concave side of $\Sigma \times \{0\}$.

By Proposition 3.8, there exists a unique maximal concave extension M of M_0, which is a hyperbolic end with particles, such that the metric on M restricted to the subset $\Sigma \times (-\varepsilon, +\infty)$ is exactly g_0. In particular, M contains a concave surface of constant curvature K (which is orthogonal to the singular curves) at $\Sigma \times \{0\}$, with the induced metric $I = (1/|K|)h$ and the third fundamental form

$$III = I(B\bullet, B\bullet) = \frac{1}{|K|}h(\sqrt{1+K}b\bullet, \sqrt{1+K}b\bullet) = \frac{1}{|K^*|}h',$$

where $K^* = K/(1+K)$. This shows the existence of the required manifold M. The uniqueness follows directly from the constrain conditions of the hyperbolic end with particles. □

It can be checked as Lemma 3.3 in [12] that for any $(\tau, \tau') \in \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$, if (h, h') and (h_1, h'_1) are two normalized representatives of (τ, τ'), then the hyperbolic end with particles associated to (h, h') and (h_1, h'_1), as described in Lemma 5.1, are isotopic. Now we are ready to give the definition of the parametrization map ϕ_K.

Definition 5.2. For any $K \in (-1, 0)$, define the map $\phi_K : \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta} \to \mathcal{HE}_\theta$ by assigning to an element $(\tau, \tau') \in \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$ the isotopy class of the hyperbolic end with particles satisfying the property prescribed in Lemma 5.1.

We show that the map ϕ_K is a homeomorphism, as stated in the following proposition.

Proposition 5.3. *For any $K \in (-1, 0)$ and $\theta = (\theta_1, ..., \theta_{n_0}) \in (0, \pi)^{n_0}$, the map $\phi_K : \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta} \to \mathcal{HE}_\theta$ is a homeomorphism.*

The proof will be given below, after some preliminary lemmas and propositions.

5.2. *The injectivity of the map ϕ_K*

We prove this property by applying the Maximum Principle outside the singular locus and a specialized analysis near cone singularities. The idea is similar to that given in [12, Section 3.2] for the case of AdS manifold with particles. Indeed, this argument is applicable to two concave surfaces which behave "umbilically" (i.e. the limits of the principal curvatures tend to be the same) at singular points and satisfy the property that the supremum of the Gauss curvatures over all the points of one surface is less than the infimum of those of the other surface (see Lemma 5.7 for more details).

Let $M \in \mathcal{HE}_\theta$ be a hyperbolic end with particles. Let $S \subset M$ be a concave surface of constant curvature $K \in (-1, 0)$. Consider the minimal Lagrangian map (see Corollary 2.15) associated to two hyperbolic metrics $|K|I, |K^*|III \in \mathfrak{M}^\theta_{-1}$, where $K^* = K/(1+K)$, and I (resp. III) is the first (resp. third) fundamental form of S. By the last statement of Proposition 2.14, both principal curvatures on S tend to $k = \sqrt{1+K}$ at the intersection p_i with the singular curve l_i in M for $i = 1, ..., n_0$.

The following theorem is an alternative version of the Maximum Principle Theorem (see e.g. [3, Lemma 2.3], [12, Theorem 3.10]) for the case of hyperbolic ends with particles.

Theorem 5.4 *(Maximum Principle). Let M be a hyperbolic end with particles. Let S and S' be two concave surfaces in M. Assume that S and S' intersect at a regular point x, and assume that S' is contained on the concave side of S in M. Then the product of the principal curvatures of S' at x is smaller than or equal to that of S.*

To show the injectivity of ϕ_K, we first state the following two lemmas, which follow from a direct computation.

Lemma 5.5. *Let M be a hyperbolic end with particles and let S be a concave surface in M. Consider a map $\psi^t : S \to M$ defined by $\psi^t(x) = \exp_x(t \cdot n_x)$, where n_x is the $\partial_\infty M$-directed unit normal vector at x of S in M. Then for each regular point $x \in S$, we have*

(1) ψ^t *is an embedding in a neighborhood of* x *for all* $t > 0$.
(2) The principal curvatures of $\psi^t(S)$ *at the point* $\psi^t(x)$ *are given by*

$$\lambda^t(\psi^t(x)) = \frac{\lambda(x) + \tanh(t)}{1 + \lambda(x)\tanh(t)}, \qquad \mu^t(\psi^t(x)) = \frac{\mu(x) + \tanh(t)}{1 + \mu(x)\tanh(t)},$$

where $\lambda(x)$ *and* $\mu(x)$ *are the principal curvatures of* S *at* x.
(3) Fix $x \in S$, *if* $\lambda(x)\mu(x) \in (0, 1)$, *then* $F(t) = \lambda^t(\psi^t(x)) \cdot \mu^t(\psi^t(x))$ *is strictly increasing in* $(0, +\infty)$.

Lemma 5.6. *Let* M *be a hyperbolic end with particles. Let* S, S' *be two concave surfaces in* M. *Assume that* S *and* S' *intersect at a singular point* x *such that the limits of both principal curvatures of* S *at* x *are equal to* $k > 0$, *and the limits of both principal curvatures of* S' *at* x *are equal to* $k' > 0$. *If there exists a neighborhood* U *of* x *in* S *and a neighborhood* U' *of* x *in* S' *such that* U' *is on the concave side of* U, *then* $k' \le k$.

Let S be a concave surface in a hyperbolic end M with particles. Define the *principal curvatures at a singular point* $x \in S$ as the limit of the principal curvatures as the regular points converge to x. Now we give the following result by applying the maximum principle and the above two lemmas.

Lemma 5.7. *Let* M *be a hyperbolic end with particles. Assume that* S_1 *and* S_2 *are two strictly concave surfaces in* M *such that the supremum of the Gauss curvatures over all the points on* S_1 *is less than the infimum of the Gauss curvatures over all the points on* S_2, *and the limits of both principal curvatures at singular points on* S_1 *(resp.* S_2*) are the same. Then* S_2 *is strictly on the concave side of* S_1.

Proof. Denote by λ_i, μ_i the principal curvatures of S_i for $i = 1, 2$. Denote $C_1 = \sup_{x\in S_1} \lambda_1(x)\mu_1(x)$ and $C_2 = \inf_{x\in S_2} \lambda_2(x)\mu_2(x)$. By assumption, we have $C_1 < C_2$, and the Gauss–Bonnet formula shows that $C_2 < 1$.

Suppose that S_2 is not strictly on the concave side of S_1. Note that S_1 and S_2 are both concave, therefore there exist points of S_2 where the $\partial_\infty M$-directed orthogonal geodesic rays from S_2 intersect the part of S_1 on the concave side exactly once. Consider $\psi^t : S_2 \to M$ defined by $\psi^t(x) = \exp_x(t \cdot n_x)$, where n_x is the $\partial_\infty M$-directed unit normal vector at x of S_2 in M. Let $t_0 = \sup\{t > 0 : \psi^t(x) \in S_1 \text{ for some } x \in S_2\}$ and let $S_2^{t_0} = \psi^{t_0}(S_2)$. Since S_1 and S_2 are both compact, then t_0 is attained at a point $x_0 \in S_2$. It follows from Lemma 5.5 that $S_2^{t_0}$ is a concave surface which intersects S_1 at a point $y_0 = \psi^{t_0}(x_0)$, and it stays on the concave side of S_1. Denote by $\lambda_2^{t_0}$, $\mu_2^{t_0}$ the principal curvatures of $S_2^{t_0}$.

If y_0 is a regular point, combining Theorem 5.4 and Statement (3) of Lemma 5.5, we have

$$C_2 \le (\lambda_2\mu_2)(x_0) \le (\lambda_2^{t_0}\mu_2^{t_0})(y_0) \le (\lambda_1\mu_1)(y_0) \le C_1. \tag{9}$$

This contradicts that $C_1 < C_2$.

If y_0 is a singular point, note that S_1 and S_2 behave "umbilically" at singular points, and it follows from Statement (2) of Lemma 5.5 that $S_2^{t_0}$ has an "umbilical" point at y_0. Applying Statement (3) of Lemma 5.5 and Lemma 5.6 we have the same inequality (9). This contradicts again that $C_1 < C_2$. Therefore, S_2 is strictly on the concave side of S_1. □

Using a similar argument as Lemma 5.7, we have the following proposition.

Proposition 5.8. *Let* S_i, $i = 1, 2$ *be concave surfaces of constant curvature* $K_i \in (-1, 0)$ *in a hyperbolic end* M *with particles for* $i = 1, 2$. *Then we have the following statements:*

(1) $K_1 < K_2$ *if and only if* S_2 *is strictly on the concave side of* S_1.
(2) $K_1 = K_2$ *if and only if* S_1 *coincides with* S_2.

Proof. *Proof of Statement (1)*: First we show that $K_1 < K_2$ implies that S_2 is strictly on the concave side of S_1. Note that $K_1 < K_2$ and the constant curvature surfaces S_1, S_2 behave "umbilically" at singular points. This statement follows directly from Lemma 5.7.

Now we prove the sufficiency, that is, if S_2 is strictly on the concave side of S_1, then $K_2 > K_1$. Denote $S_1^t = \psi^t(S_1)$. Set $\delta_0 = \sup\{d(z, S_2) : z \in S_1\}$. Obviously, $\delta_0 > 0$. Assume that δ_0 is attained at a point $z_0 \in S_1$ and denote $w_0 = \psi^{\delta_0}(z_0) \in S_2 \cap S_1^{\delta_0}$. Discussing w_0 in two cases (as a regular or singular point) as Lemma 5.7 again, we have

$$\lambda_1^{\delta_0}(w_0)\mu_1^{\delta_0}(w_0) > \lambda_1(z_0)\mu_1(z_0) = 1 + K_1,$$
$$\lambda_1^{\delta_0}(w_0)\mu_1^{\delta_0}(w_0) \leq \lambda_2(w_0)\mu_2(w_0) = 1 + K_2.$$

Thus $K_2 > K_1$.

Proof of Statement (2): The sufficiency is obvious. Now we show the necessity. By assumption, $K_1 = K_2$. Set $d_1 = \sup\{d(x, S_1) : x \in S_2$ is on the concave side of S_1 or lying on $S_1\}$ and $d_2 = \sup\{d(x, S_2) : x \in S_1$ is on the concave side of S_2 or lying on $S_2\}$. Note that $S_1 = S_2$ if and only if $d_1 = d_2 = 0$.

If $d_1 > 0$, consider the surface $S_1^{d_1}$ obtained by pushing S_1 along orthogonal geodesics in a distance d_1 in the positive direction. Using the argument as above, we obtain the contradiction that $K_1 < K_2$. This implies that $d_1 = 0$.

If $d_2 > 0$, consider the surface $S_2^{d_2}$ obtained by pushing S_2 along orthogonal geodesics in a distance d_2 in the positive direction. Using the same argument as above, we obtain the contradiction that $K_1 > K_2$. This implies that $d_2 = 0$. Therefore, $S_1 = S_2$. □

Proposition 5.9. *For any $K \in (-1, 0)$, the map $\phi_K : \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta} \to \mathcal{HE}_\theta$ is injective.*

Proof. Assume that (h, h'), $(h_1, h_1') \in \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$ satisfy that $\phi_K(h, h') = \phi_K(h_1, h_1') := M$. Then M contains a concave surface S of constant curvature K, with the induced metric $I = (1/|K|)h$ and the third fundamental form $III = (1/|K^*|)h'$, and also contains a concave surface S_1 of constant curvature K, with the induced metric $I_1 = (1/|K|)h_1$ and the third fundamental form $III = (1/|K^*|)h_1'$. By Proposition 5.8, we have $S = S_1$. Then $h = h_1$ and $h' = h_1'$, which implies that $(h, h') = (h_1, h_1')$. □

5.3. The continuity of the map ϕ_K

The map ϕ_K relates deeply to the minimal Lagrangian maps between two hyperbolic surfaces with cone singularities in $\mathcal{T}_{\Sigma,\theta}$, which provides the embedding data to construct a hyperbolic end with particles. With the result in [12, Lemma 3.19] (which shows that the minimal Lagrangian maps (isotopic to the identity) between h_k and h_k' converge to the minimal Lagrangian map (isotopic to the identity) between h and h', as (h_k) converge to h and (h_k') converge to h'), we have the following proposition.

Proposition 5.10. *For any $K \in (-1, 0)$, the map $\phi_K : \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta} \to \mathcal{HE}_\theta$ is continuous.*

Proof. It suffices to prove that if the sequence $(h_k, h_k')_{k\in\mathbb{N}}$ converges to $(h, h') \in \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$, then the sequence $(\phi_K(h_k, h_k'))_{k\in\mathbb{N}}$ converges to $\phi_K(h, h') \in \mathcal{HE}_\theta$. Denote by m_k the unique minimal Lagrangian map between (Σ, h_k) and (Σ, h_k') isotopic to the identity and by m the unique minimal Lagrangian map between (Σ, h) and (Σ, h') isotopic to the identity.

By the proof in [12, Lemma 3.19], the sequence $(m_k)_{k\in\mathbb{N}}$ converges to m. Let $b_k : T\Sigma \to T\Sigma$ be the bundle morphism defined outside the singular locus which is described in Proposition 2.14 with the property $m_k^*(h_k') = h_k(b_k\bullet, b_k\bullet)$. Then b_k converges to a bundle morphism from $T\Sigma$ to $T\Sigma$, say b.

Let $I_k = (1/|K|)h_k$ and $B_k = \sqrt{1+K}b_k$. Then $(\Sigma, I_k, B_k)_{k\in\mathbb{N}}$ converges to (Σ, I, B), in the sense that I_k and B_k converge to $I = (1/|K|)h$ and $B = \sqrt{1+K}b$, respectively. This implies that $(\phi_K(h_k, h_k'))_{k\in\mathbb{N}}$ converges to $\phi_K(h, h')$ in $\mathcal{HE}_\theta$. The lemma follows. □

5.4. The properness of the map ϕ_K

To prove this property of ϕ_K, we first give a comparison between the lengths of closed geodesics in the same isotopy class on the metric boundary $\partial_0 M$ and on a strictly concave surface in a hyperbolic end M with particles.

Lemma 5.11. *Let M be a hyperbolic end with particles. Let S be a strictly concave surface in M. Then for any closed geodesic γ on $\partial_0 M$, the length of γ is smaller than the length of any closed minimizing geodesic γ' on S isotopic to γ in M.*

Proof. Let $r : M \to \partial_0 M$ be the closest point projection of M to the metric boundary $\partial_0 M$ (this is well-defined since $\partial_0 M$ is concave). Note that if $x \in M$ is a singular point, then the closet point projection is along the singular curve through x. Then r is 1-Lipschitz with respect to the hyperbolic metric on M and the induced metric on $\partial_0 M$. Therefore, the marked length spectrum of $\partial_0 M$ is bounded by the marked length spectrum of S. This completes the proof. □

Let X be a topological space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements in X. We say that $(x_n)_{n\in\mathbb{N}}$ *tends to infinity* if $(x_n)_{n\in\mathbb{N}}$ is not contained in any compact subset of X.

Now we recall a result in Teichmüller spaces of hyperbolic surfaces with cone singularities of prescribed angles less than π. This follows from an analysis on the parametrization of $\mathcal{T}_{\Sigma,\theta}$ by Fenchel–Nielsen coordinates associated to a fixed pants decomposition and the Collar lemma for hyperbolic cone-surfaces (see [14, Theorem 3]).

Lemma 5.12. *Let $(h_n)_{n\in\mathbb{N}}$ be a sequence of elements in $\mathcal{T}_{\Sigma,\theta}$. Then the following two statements are equivalent:*

(1) $(h_n)_{n\in\mathbb{N}}$ tends to infinity.
(2) For any $k \in \mathbb{N}^+$, there exists a simple closed curve γ_k on Σ and an integer $N > 0$ (depending on k and γ_k), such that $\ell_{\gamma_k}(h_N) < (1/k)\,\ell_{\gamma_k}(h_0)$.

Proposition 5.13. *For any $K \in (-1, 0)$, the map $\phi_K : \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta} \to \mathcal{HE}_\theta$ is proper.*

Proof. Denote $\phi_K(h_n, h'_n) = (M_n, g_n)$ for $n \in \mathbb{N}$. We suppose that $(M_n, g_n)_{n\in\mathbb{N}}$ converges to a limit (M, g), and will prove that $(h_n)_{n\in\mathbb{N}}$ and $(h'_n)_{n\in\mathbb{N}}$ must remain bounded.

It follows from the hypothesis that $(m_n)_{n\in\mathbb{N}}$ and $(l_n)_{n\in\mathbb{N}}$ remain bounded (see Proposition 3.11), where m_n and l_n are the induced metric and measured bending lamination on $\partial_0 M_n$ for g_n. After extracting a subsequence, we can suppose that $(m_n)_{n\in\mathbb{N}}$ converges to a limit m, and $(l_n)_{n\in\mathbb{N}}$ converges to a limit l, where m and l are the induced metric and measured bending lamination on $\partial_0 M$ for g.

Note that the concave surface $\Sigma_{K,n}$ of constant curvature K in M_n has the induced metric $I_n = (1/|K|)h_n$. It follows from Lemma 5.11 that $\ell_\gamma(m_n) < \ell_\gamma(I_n) = (1/\sqrt{|K|})\ell_\gamma(h_n)$ for all simple closed curves γ on Σ. Suppose that $(h_n)_{n\in\mathbb{N}}$ is not bounded. Combined with Lemma 5.12, this shows that, for any $k > 0$, there exists a simple closed curve γ_k on Σ and an integer $N > 0$ (depending on k and γ_k), such that $\ell_{\gamma_k}(m_N) < (k\sqrt{|K|})^{-1}\ell_{\gamma_k}(h_0)$. Applying Lemma 5.12 again, we find that $(m_n)_{n\in\mathbb{N}}$ tends to infinity, which leads to a contradiction.

We first note that there exists $r > 0$ such that for all $x \in \Sigma_{K,n}$, the distance from x to $\partial_0 M_n$ is at most r. Otherwise, there would be a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in \Sigma_{K,n}$ and $d_{g_n}(x_n, \partial_0 M_n) \to \infty$, and this would contradict the fact that I_n, m_n are converging to metrics of constant curvature, l_n is converging to l, and the area of a concave surface in M_n expands exponentially with respect to the distance r along the normal flow starting from $\partial_0 M_n$.

Let $S_{r,n}$ be the set of points at distance r from $\partial_0 M_n$ for g_n, with the induced metric $I_{r,n}$. For all n, $S_{r,n}$ is a smooth (outside the singular locus), strictly concave surface. Let $III_{r,n}$ be the third fundamental form of $I_{r,n}$. Notice that since $m_n \to m$ and $l_n \to l$, $III_{r,n}$ must also converge to a limit III_r.

We claim that the length spectrum of the third fundamental form III_n of $\Sigma_{K,n}$ is smaller than the length spectrum of $III_{r,n}$ of $S_{r,n}$. This is equivalent to proving that the length spectrum of the induced metric on the dual surface $\Sigma^d_{K,n}$ in M^d_n, the GHM de Sitter spacetime with particles dual to M_n as seen in Section 4, is smaller than the length spectrum of the induced metric on the surface $S^d_{r,n}$ dual to $S_{r,n}$. To prove this dual statement, note that the definition of the duality shows that $S^d_{r,n}$ is the set of points at distance r from the initial singularity $(\partial_0 M_n)^d$ of M^d_n. As a consequence, the open segments of length r orthogonal to $S^d_{r,n}$ in the past foliate the past of $S^d_{r,n}$, and the de Sitter metric on the past of $S^d_{r,n}$ can be written as

$$-dt^2 + I^d_{t,n},\ t \in (0, r)\ ,$$

where $I^d_{t,n}$ is the induced metric on $S^d_{t,n}$ and therefore isometric to $III_{t,n}$.

Since the $S^d_{t,n}$ are future-convex, $I^d_{t,n}$ is increasing in t, and therefore $I^d_{t,n} \leq I^d_{r,n}$ for all $t \leq r$. It follows that the induced metric on the surface $\Sigma^d_{K,n}$ can be written as

$$I^d_n = -dt^2 + I^d_{t,n} \leq I^d_{t,n} \leq I^d_{r,n} \ ,$$

where we are using the identification between $\Sigma^d_{K,n}$, $S^d_{t,n}$ and $S^d_{r,n}$ through the normal flow of the $(S^d_{t,n})_{t\in(0,r)}$. Here t is the function defined on $\Sigma^d_{K,n}$ as the distance to the initial singularity of M^d_n.

We have now established that the length spectrum of III_n is smaller than that of $III_{r,n}$, and so uniformly bounded. This shows that, after extracting a subsequence, $(III_n)_{n\in\mathbb{N}}$ converges to a limit. Recall that in Lemma 5.1 we showed that $h'_n = |K^*|III_n$, where $K^* = K/(1+K)$. Therefore, $(h'_n)_{n\in\mathbb{N}}$ also converges to a limit. □

Proof of Proposition 5.3. By Proposition 3.11, $\mathcal{HE}_\theta$ is homeomorphic to $T^*\mathcal{T}_{\Sigma,\theta}$. Therefore $\mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$ and $\mathcal{HE}_\theta$ are both simply connected. Note that $\mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$ and $\mathcal{HE}_\theta$ have the same dimension and have no boundary. Combined with Proposition 5.9, Proposition 5.10, and Proposition 5.13, it follows that ϕ_K is a homeomorphism. □

5.5. The convergence of K-surfaces

Fix a hyperbolic end M with particles. By Proposition 5.3, M contains a locally concave surface S_K of constant curvature K for all $K \in (-1, 0)$ (since ϕ_K is surjective). Furthermore, the constant curvature K-surface in M is unique (since ϕ_k is injective) and distinct constant curvature K-surfaces are disjoint from each other (this follows from Proposition 5.8).

To show that M admits a foliation by locally concave constant curvature surfaces, it suffices to prove that the union of constant curvature K-surfaces S_K over all $K \in (-1, 0)$ is exactly M. In particular, we show that the sequence $(S_{K_n})_{n\in\mathbb{N}}$ of constant curvature K_n-surfaces in M converges to S_K in the C^2-topology (outside the singular locus) if $K_n \to K \in (-1, 0)$.

Note that the singularities on a constant curvature surface in M behave like "umbilical" points and the cone angles are less than π, the theorem given by F. Labourie [22, Theorem D] (which describes a degenerating phenomenon of a sequence of isometric embedding of a surface with the determinants of second fundamental form bounded below by $\varepsilon > 0$ in a Riemannian 3-manifold with sectional curvature less than K_0 for a real number K_0) can be generalized to the following case of hyperbolic ends with cone singularities.

Theorem 5.14. *Let M be a hyperbolic end with particles and let S_n be a sequence of surfaces in M with the determinants of second fundamental forms bounded below by $\varepsilon > 0$, with the induced metric g_n. Let f_n be an embedding of the prescribed surface Σ into M with the image $f_n(\Sigma) = S_n$. Assume that $f^*_n(g_n)$ converges to a Riemannian metric g_∞ in the C^2-topology, and f_n converges to an embedding $f_\infty : \Sigma \to M$ in the C^0-topology but not in the C^3-topology (outside the singular locus), then there exists a complete geodesic γ of (Σ, g_∞) such that $f_\infty|_\gamma$ is an isometry from γ into a geodesic of M.*

Lemma 5.15. *Let M be a hyperbolic manifold with particles which has a concave metric boundary. Assume that $\bar{M}$ contains a complete geodesic γ which stays in a bounded distance from $\partial_0 M$, then γ lies on the metric boundary $\partial_0 M$.*

Proof. Consider a function $u : \gamma \to \mathbb{R}_{\geq 0}$ defined by

$$u(x) = \sinh d(x, \partial_0 M).$$

Denote by g the metric on $\bar{M}$. It is known that u satisfies the equality $\mathrm{Hess}(u) \geq ug$ in the distributional sense (see e.g. [26, Lemma A.12]), since $\partial_0 M$ is concave and the map $\exp : N\partial_0 M \to M$ is a homeomorphism (see Lemma 2.6). Assume that γ is a geodesic parameterized by arclength, then $(u \circ \gamma)'' \geq u \circ \gamma$. Note that γ stays at bounded distance from $\partial_0 M$. Applying the maximum principle, we obtain that $u \circ \gamma = 0$ for all $t \in \mathbb{R}$. Therefore, the complete geodesic γ lies on the metric boundary $\partial_0 M$. □

Lemma 5.16. *Let (M, g) be a hyperbolic end with particles. Let $(S_{K_n})_{n\in\mathbb{N}}$ be a sequence of locally concave surfaces in M of constant curvature $K_n \in (-1, 0)$. Then the following statements hold.*

(1) If $K_n \to K \in [-1, 0)$ with $K_n \neq K$ for any $n \in \mathbb{N}$, then the sequence $(S_{K_n})_{n\in\mathbb{N}}$ converges to S_K in the compact-open topology (or C^0-topology). Moreover, if $K \in (-1, 0)$, then the sequence $(S_{K_n})_{n\in\mathbb{N}}$ converges to S_K in the $\mathcal{C}^2$-topology (outside the singular locus).

(2) If $K_n \to 0$, then the (least) distance from the surface S_{K_n} to the metric boundary $\partial_0 M$ tends to infinity as $n \to \infty$.

Proof. *Proof of Statement (1)*: Denote by Φ the $\partial_\infty M$-directed normal flow, given by the exponential map $\exp : N\partial_0 M \to M$ (see the map exp in Lemma 2.6). By the Gauss–Bonnet formula for surfaces with cone singularities (see e.g. [33, Proposition 1]), the area of S_{K_n} is equal to $(2\pi/K_n)\,\chi(\Sigma, \theta)$, where

$$\chi(\Sigma,\theta) = \chi(\Sigma) + \sum_{i=1}^{n_0}(\theta_i/2\pi - 1) < 0.$$

Therefore, $Area(S_{K_n}) \to (2\pi/K)\,\chi(\Sigma,\theta) = Area(S_K)$ as $n \to \infty$, where $K \in [-1, 0)$.

We claim that S_{K_n} converges to S_K in the compact-open topology as $n \to \infty$. Indeed, we first fix an embedding map $f_\infty : \Sigma \to M$ such that $f_\infty(\Sigma) = S_K$. Then let $f_n : \Sigma \to M$ be the embedding map compatible with the flow Φ, that is, the map $f_n \circ f_\infty^{-1} : S_K \to S_{K_n}$ coincides with the homeomorphism from S_K to S_{K_n} induced by the flow Φ for all $n \in \mathbb{N}$. Suppose that there exists a compact subset $U \subset \Sigma$, such that the sequence $(f_n(U))_{n\in\mathbb{N}}$ does not converge to $f_\infty(U)$ in M. Then there exists a neighborhood V of $f_\infty(U)$ in M such that we can find a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ with $f_{n_k}(U)$ disjoint from V for all $k \in \mathbb{N}$. By Proposition 5.8, there exists an integer $N > 0$, such that $f_n(U)$ is disjoint from V for $n \geq N$, and S_{K_n} is disjoint from S_K for all $K_n \neq K$. Combined with the construction of f_n, the distance from $f_\infty(U) \subset S_K$ to $f_n(U) \subset S_{K_n}$ along the flow Φ is bigger than a positive number r_0 for all $n \geq N$. Note that the induced metric by M is strictly increasing along the normal flow Φ. This implies that the sequence $(|Area(S_K) - Area(S_{K_n})|)_{n\in\mathbb{N}^+}$ does not converge to zero, which leads to a contradiction.

Now we show that $(S_{K_n})_{n\in\mathbb{N}}$ converges to S_K in the C^2-topology outside the singular locus for all $K \in (-1, 0)$. Denote by g_n the induced metric on S_{K_n} for all $n \in \mathbb{N}$. Note that S_{K_n} is orthogonal to the singular lines l_k (which are homeomorphic to $\{p_k\} \times \mathbb{R}$) and the angle of the singularity on S_{K_n} at the intersection with l_k is $\theta_k \in (0, \pi)$ for $k = 1, ..., n_0$. Therefore, the metrics g_n can be written as follows:

$$g_n = (1/|K_n|)\widehat{g_n},$$

where $\widehat{g_n} \in \mathfrak{M}^\theta_{-1}$ for all $n \in \mathbb{N}^+$.

For convenience, we assume that $S_{K_0} = \partial_0 M$, that is, $K_0 = -1$. By Lemma 5.11, for any simple closed curve γ on Σ, we have

$$\ell_{f_n(\gamma)}(g_n) \geq \ell_{f_0(\gamma)}(g_0),$$

for all $n \in \mathbb{N}$. Note that K_n converges to $K \in (-1, 0)$. Without loss of generality, we assume that K_n increasingly converges to K. Then

$$\ell_{f_n(\gamma)}(\widehat{g_n}) = \ell_{f_n(\gamma)}(|K_n|g_n) = \sqrt{|K_n|}\ell_{f_n(\gamma)}(g_n) \geq \sqrt{|K|}\,\ell_{f_0(\gamma)}(g_0) = \sqrt{K/K_0}\,\ell_{f_0(\gamma)}(\widehat{g_0}),$$

for all $n \in \mathbb{N}$. Here $K/K_0 < 1$.

Denote by $f_n^*(\widehat{g_n})$ the pull-back metric on Σ of $\widehat{g_n}$ under f_n and still denote by $f_n^*(\widehat{g_n})$ its isotopy class in $\mathcal{T}_{\Sigma,\theta}$ for all $n \in \mathbb{N}$. For any simple closed curve γ on Σ, we get

$$\ell_\gamma(f_n^*(\widehat{g_n})) \geq \sqrt{K/K_0}\,\ell_\gamma({f_0}^*(\widehat{g_0})).$$

By Lemma 5.12, the set $\{f_n^*(\widehat{g_n}) : n \in \mathbb{N}\}$ is compact in $\mathcal{T}_{\Sigma,\theta}$. Therefore, up to extracting a subsequence, $(f_n^*(\widehat{g_n}))_{n\in\mathbb{N}}$ converges in $\mathcal{T}_{\Sigma,\theta}$. Note that $(f_n)_{n\in\mathbb{N}}$ is compatible with the flow Φ. $(f_n^*(\widehat{g_n}))_{n\in\mathbb{N}}$ converges to $f_\infty^*(\widehat{g_K})$ in the $\mathcal{C}^2$-topology (outside the singular locus), where $\widehat{g_K} = |K|\,g_K$ and g_K is the induced metric on S_K in M. In particular, $f_n^*(g_n)$ converges to $g_\infty = f_\infty^*(g_K)$ in the $\mathcal{C}^2$-topology (outside the singular locus). Note that Σ is compact and by the above result we have f_n converges to f_∞ in the $\mathcal{C}^0$-topology.

We claim that f_n converges to f_∞ in the $\mathcal{C}^3$-topology. Otherwise, it follows from Theorem 5.14 that there exists a complete geodesic γ of (Σ, g_∞) such that $f_\infty|_\gamma$ is an isometry from γ into a geodesic of (M, g). Note that the geodesic $f_\infty(\gamma)$ lies on S_K and thus stays in a bounded distance from $\partial_0 M$. Combined with Lemma 5.15, $f_\infty(\gamma)$ is contained in $\partial_0 M$ which is disjoint from S_K. This leads to a contradiction. Therefore, Statement (1) follows.

Proof of Statement (2): We first fix the surface S_{K_1} and denote $S = S_{K_1}$. Consider a map $\psi^t : S \to M$ defined by $\psi^t(x) = \exp_x(t \cdot n_x)$, where n_x is the $\partial_\infty M$-directed unit normal vector at x of S in M. For any $T > 0$, we denote $S^T = \psi^T(S)$ and denote by λ^T, μ^T the principal curvatures of S^T. By Lemma 5.5, the principal curvatures of S^T are

$$\lambda^T(\psi^T(x)) = \frac{\lambda(x) + \tanh(T)}{1 + \lambda(x)\tanh(T)}, \qquad \mu^T(\psi^T(x)) = \frac{\mu(x) + \tanh(T)}{1 + \mu(x)\tanh(T)},$$

where $\lambda(x)$ and $\mu(x)$ are the principal curvatures of S at x.

Let $C^T = \sup_{y \in S^T} \lambda^T(y)\mu^T(y)$. Since $\lambda^t(\psi^t(x))\mu^t(\psi^t(x))$ increasingly tends to 1 as $t \to +\infty$ for all $x \in S$, and S^T is also locally concave and compact, so $C^T \in (0, 1)$.

By assumption, $K_n \to 0$ (without lost of generality, we assume that K_n increasingly converges to 0). Therefore there exists $N_T > 0$ (depending only on T) such that for all $n \geq N_T$, we have

$$-1 + C^T < K_n < 0. \tag{10}$$

Note that S^T and S_{K_n} are strictly concave surfaces and behave "umbilically" at singular points. It follows from the inequality (10) and Lemma 5.7 that S_{K_n} is on the concave side of S^T for all $n \geq N_T$. Observe that $C_T \to 1$ as $T \to +\infty$, and the distance from S^T to $\partial_0 M$ tends to infinity as $T \to +\infty$. Combined with the result above, the distance from S_{K_n} to $\partial_0 M$ tends to infinity as $n \to \infty$. □

The following corollary is a direct consequence of Proposition 5.8 and Lemma 5.16.

Corollary 5.17. *Let M be a hyperbolic end with particles. Then the union of the constant curvature K-surfaces S_K in M over all $K \in (-1, 0)$ provides a C^2-foliation of the regular part of M.*

Proof of Theorem 1.3. As discussed in the beginning of Section 5.5, it follows directly from Proposition 5.3 and Corollary 5.17. □

5.6. Applications to smooth grafting

In the non-singular case, the landslide flow is defined in [8,9] as a map $L : S^1 \times \mathcal{T} \times \mathcal{T} \to \mathcal{T} \times \mathcal{T}$, sending $(e^{i\alpha}, h, h^*)$ to the pair (h_1, h_2), where h_1 is the left metric of the unique GHM AdS spacetime containing a future-convex spacelike surface with induced metric $\cos^2(\alpha/2)h$ and third fundamental form $\sin^2(\alpha/2)h^*$ and h_2 is the left metric of the unique GHM AdS spacetime containing a future-convex spacelike surface with induced metric $\cos^2(\alpha/2)h^*$ and third fundamental form $\sin^2(\alpha/2)h$.

It is also proved there that the landslide map, composed with the canonical projection on the first factor, has a complex extension as the "smooth grafting" map $sgr : (0, 1) \times \mathcal{T} \times \mathcal{T} \to \mathcal{T}$, sending (r, h, h^*) to the conformal metric at infinity of the unique hyperbolic end containing a constant curvature surface with induced metric $\frac{(1+r)^2}{4r}h$ and third fundamental form $\frac{(1-r)^2}{4r}h^*$. This surface has constant curvature $-4r/(1+r)^2$. The map sgr is obtained from another grafting map $SGr : (0, 1) \times \mathcal{T} \times \mathcal{T} \to \mathcal{CP}$ by composition on the left with the forgetful map from $\mathcal{CP}$ to $\mathcal{T}$.

The landslide map limits in a precise sense to the earthquake map $\mathcal{T} \times \mathcal{ML} \to \mathcal{T}$, while the smooth grafting map limits in a precise sense to the grafting map $\mathcal{T} \times \mathcal{ML} \to \mathcal{T}$.

The results of [12] on constant Gauss curvature foliations in convex GHM AdS spacetimes with particles lead to an extension of the landslide flow to hyperbolic surfaces with cone singularities of angles less than π. In the same manner, the results presented here on constant curvature foliations of hyperbolic ends with particles lead directly, by extending the arguments of [8] without any serious change, to the definition of the smooth grafting maps $sgr_\theta : (0, 1) \times \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta} \to \mathcal{T}_{\Sigma,\theta}$ and $SGr_\theta : (0, 1) \times \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta} \to \mathcal{CP}_\theta$.

It can be proved, using the same arguments as in [8], that:

(1) The smooth grafting map sgr_θ provides a complex extension of the landslide map. More precisely, if $L^1 : S^1 \times \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta} \to \mathcal{T}_{\Sigma,\theta}$ is the landslide map followed by projection on the first factor, then the "complex landslide" map:

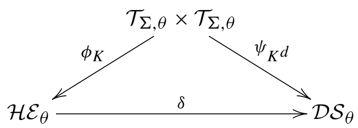

Fig. 4. A diagram showing the parametrizations of $\mathcal{HE}_\theta$ and $\mathcal{DS}_\theta$ by $\mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$, respectively.

$$D \times \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta} \to \mathcal{T}_{\Sigma,\theta}$$
$$(re^{i\alpha}, h, h^*) \mapsto sgr(r, L_{e^{i\alpha}}(h, h^*))$$

defines a holomorphic map from the unit disk D to $\mathcal{T}_{\Sigma,\theta}$ extending L^1 to the unit disk, for any fixed h and h^*.

(2) The smooth grafting maps sgr_θ and SGr_θ limit, in the same suitable sense as in [8], to the grafting maps $gr_\theta : \mathcal{T}_{\Sigma,\theta} \times \mathcal{ML}_\mathfrak{p} \to \mathcal{T}_{\Sigma,\theta}$ and $Gr_\theta : \mathcal{T}_{\Sigma,\theta} \times \mathcal{ML}_\mathfrak{p} \to \mathcal{CP}_\theta$.

6. Foliations of de Sitter spacetimes with particles by constant curvature surfaces

In this last section, we prove that convex GHM de Sitter spacetimes with particles admit a unique foliation by constant Gauss curvature surfaces orthogonal to the particles. As a consequence, for each $K^d \in (-\infty, 0)$, the space of convex GHM de Sitter spacetimes with particles can be parameterized by the product of two copies of $\mathcal{T}_{\Sigma,\theta}$ in terms of constant curvature K^d-surface.

6.1. Foliation of de Sitter spacetimes with particles by K-surfaces

As a consequence of Proposition 1.6, each foliation of a non-degenerate hyperbolic end with particles has a dual foliation of the dual future-complete convex GHM de Sitter space-time with particles.

Observe that the curvature K^d varies from $-\infty$ to 0 in Proposition 1.6, combined with Theorem 1.3, we therefore obtain Corollary 1.7, which states that every future-complete convex GHM de Sitter spacetime M^d with particles admits a unique foliation by surfaces of constant curvature K^d, with K^d varying from $-\infty$ near the initial singularity to 0 near the boundary at infinity. In particular, for each $K^d \in (-\infty, 0)$, M^d contains a unique closed surface of constant curvature K^d. Combined with Theorem 1.5 and Corollary 5.17, the union of the constant curvature K^d-surfaces in M^d over all $K^d \in (-\infty, 0)$ provides a C^2-foliation of the regular part of M^d.

6.2. A parametrization of $\mathcal{DS}_\theta$ by $\mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$

We can also give a parametrization of $\mathcal{DS}_\theta$ in terms of constant curvature surfaces.

Let $K^d \in (-\infty, 0)$ and let $(h, h') \in \mathfrak{M}^\theta_{-1} \times \mathfrak{M}^\theta_{-1}$ be a pair of normalized metrics. Using a similar argument as in Lemma 5.1, there exists a unique convex GHM de Sitter spacetime M^d with particles which contains a surface of constant curvature K^d, with the induced metric $I^d = (1/|K^d|)h'$ and the third fundamental form $III^d = (1/|K^{d*}|)h$, where $K^{d*} = K^d/(1 - K^d)$.

For any $K^d \in (-\infty, 0)$, define the map $\psi_{K^d} : \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta} \to \mathcal{DS}_\theta$ by assigning to an element $(\tau, \tau') \in \mathcal{T}_{\Sigma,\theta} \times \mathcal{T}_{\Sigma,\theta}$ the isotopy class of the de Sitter spacetime with particles satisfying the above property. Combining Proposition 5.3 and the duality between strictly concave surfaces in a hyperbolic end M with particles and strictly future-convex spacelike surfaces in the dual de Sitter spacetimes M^d with particles (see Theorem 1.5), it follows that the parametrization ψ_{K^d} is equal to the composition map $\delta \circ \phi_K$, and therefore a homeomorphism (as shown in Fig. 4).

Conflict of interest statement

There is no conflict of interest.

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