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Fujita blow up phenomena and hair trigger effect: The role of dispersal tails

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Abstract

We consider the nonlocal diffusion equation $\partial_t u = J * u - u + u^{1+p}$ in the whole of $\mathbb{R}^N$. We prove that the Fujita exponent dramatically depends on the behavior of the Fourier transform of the kernel J near the origin, which is linked to the tails of J. In particular, for compactly supported or exponentially bounded kernels, the Fujita exponent is the same as that of the nonlinear Heat equation $\partial_t u = \Delta u + u^{1+p}$. On the other hand, for kernels with algebraic tails, the Fujita exponent is either of the Heat type or of some related Fractional type, depending on the finiteness of the second moment of J. As an application of the result in population dynamics models, we discuss the hair trigger effect for $\partial_t u = J * u - u + u^{1+p}(1-u)$.

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1. Introduction

In this work we consider solutions $u(t,x)$ to the nonlinear ($p > 0$) partial integro-differential equation

$$\partial_t u = J * u - u + u^{1+p} \quad \text{in } (0,\infty) \times \mathbb{R}^N, \tag{1}$$

in any dimension $N \geq 1$. Equation (1) is supplemented with a nonnegative and nontrivial initial data, and we aim at determining the so-called *Fujita exponent* p_F, that is the value of p that separates "systematic blow up solutions" from "blow up solutions vs global and extincting solutions" (see below for details). We shall prove that the Fujita exponent dramatically depends on the behavior of the Fourier transform of the kernel J near the origin, which is linked to the tails of J. Depending on these tails, it turns out that the Fujita phenomenon in (1) can be similar to that of the nonlinear Heat equation, or to that of a related nonlinear Fractional equation.

As an application of our main result, we consider

$$\partial_t u = J * u - u + u^{1+p}(1-u) \quad \text{in } (0,\infty) \times \mathbb{R}^N, \tag{2}$$

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which serves as a population dynamics model where both long range dispersal (via the kernel J) and a weak Allee effect (via the degeneracy of the steady state $u \equiv 0$, due to $p > 0$) are taken into account. Depending on the balance between the tails of J and the strength of the Allee effect, we discuss the so-called *hair trigger effect* — meaning that any small perturbation from $u \equiv 0$ drives the solution to $u \equiv 1$ — or the possibility of extincting solutions.

In his seminal work [10], Fujita considered solutions $u(t,x)$ to the nonlinear Heat equation

$$\partial_t u = \Delta u + u^{1+p} \quad \text{in } (0,\infty) \times \mathbb{R}^N, \tag{3}$$

supplemented with a nonnegative and nontrivial initial data. For such a problem, the Fujita exponent is $p_F = \frac{2}{N}$. Precisely, if $0 < p \leq p_F$ then any solution blows up in finite time; if $p > p_F$ then solutions with large initial data blow up in finite time whereas solutions with small initial data are global in time and go extinct as $t \to \infty$. For a precise statement we refer to [10] for the cases $0 < p < p_F$ and $p > p_F$. The critical case $p = p_F$ is studied in [15] when $N = 1, 2$, in [17] when $N \geq 3$, and in [24] via a direct and simpler approach. Let us observe that, as is well known, solutions to the Heat equation $\partial_t u = \Delta u$ tend to zero as $t \to \infty$ like $\mathcal{O}\left(t^{-\frac{N}{2}}\right)$, which is a formal argument to guess $p_F = \frac{2}{N}$.

Since then, the Fujita phenomenon has attracted much interest and the literature on refinements of the results or on various local variants of equation (3) is rather large. Let us mention for instance the works [24,19,7,21], or [22] for an overview, and the references therein.

When the Laplacian diffusion operator is replaced by the Fractional Laplacian, the situation is also well understood: the Fujita exponent for

$$\partial_t u = -(-\Delta)^{s/2} u + u^{1+p} \quad \text{in } (0,\infty) \times \mathbb{R}^N, \quad 0 < s \leq 2, \tag{4}$$

is $p_F = \frac{s}{N}$. We refer to the work of Sugitani [23]. See also, among others, [5] for a probabilistic approach, and [13] for a variant of (4). Let us observe that, as is well known, solutions to the Fractional diffusion equation $\partial_t u = -(-\Delta)^{s/2} u$ tend to zero as $t \to \infty$ like $\mathcal{O}\left(t^{-\frac{N}{s}}\right)$, which is again a formal argument to guess $p_F = \frac{s}{N}$.

As far as we know, much less is known for the nonlocal equation (1). Let us mention the work of García-Melián and Quirós [11] (and [26] for a variant) who treat the case of compactly supported dispersal kernel J. In such a situation, the Fujita exponent for (1) is the same as that of (3), namely $p_F = \frac{2}{N}$. In order to take into account rare long-distance dispersal events, which are relevant in many population dynamics models (seeds dispersal for instance), we allow in this work kernels J which have nontrivial tails. Two typical situations are when J has (light) exponential tails or (heavy) algebraic tails, the latter case meaning

$$J(x) \sim \frac{C}{|x|^\alpha} \quad \text{as } |x| \to \infty, \qquad \text{with } \alpha > N. \tag{5}$$

Owing to the decay of solutions to $\partial_t u = J * u - u$ proved by Chasseigne, Chaves and Rossi [6], we guess that $p_F = \frac{2}{N}$ in the exponential case, whereas

$$p_F = \begin{cases} \frac{\alpha}{N} - 1 & \text{if } N < \alpha \leq N+2 \\ \frac{2}{N} & \text{if } \alpha > N+2, \end{cases} \tag{6}$$

in the algebraic case (5). In other words, in the algebraic case $\alpha > N+2$ the Fujita exponent is of the Heat type (3) (and so in the exponential case), but in the algebraic case $N < \alpha \leq N+2$ the Fujita exponent becomes of the Fractional type (4) with $s = \alpha - N \in (0, 2]$. This is the role of the present paper to prove, among others, these results.

Let us comment on some technical difficulties arising from (1). Notice first that, as far as (3) and (4) are concerned, some self-similarity properties of both the Heat kernel and the fundamental solution associated to the Fractional Laplacian may be quite helpful, as seen in [23] or [24]. Those self-similarity properties are not shared by the fundamental solution of $\partial_t u = J * u - u$. Secondly, notice that, when J is compactly supported, the underlying nonlocal eigenvalue problem to (1) is rather well understood [12] and the authors in [11] took advantage of its rescaling properties. As far as we know, such informations are not available for more general dispersal kernels, as those we consider. We therefore have to adapt some techniques, in particular when dealing with blow up phenomena.

We now discuss the hair trigger effect in some population dynamics models. Let us start with the Fisher-KPP equation

$$\partial_t u = \Delta u + u(1-u),$$

which was introduced in [9,18], to model the spreading of advantageous genetic features in a population. From the linear instability of the steady state $u \equiv 0$, it is well known that any solution $u(t,x)$ to the Fisher-KPP equation, with a nonnegative and nontrivial initial data, tends to 1 as $t \to \infty$, locally uniformly in $x \in \mathbb{R}^N$. This is referred to as the hair trigger effect.

In order to take into account a weak Allee effect, meaning that the growth per capita is no longer maximal at small densities, one may consider

$$\partial_t u = \Delta u + u^{1+p}(1-u), \tag{7}$$

where $p > 0$. Then the hair trigger effect for (7) is naturally linked with the Fujita blow up phenomena for (3). Hence, in their seminal work, Aronson and Weinberger [3] showed that the hair trigger effect remains valid for (7) as long as $0 < p \leq p_F = \frac{2}{N}$, whereas some (small enough) initial data may lead to extinction, or quenching, when $p > p_F = \frac{2}{N}$. See also [25,4,28].

Based on our Fujita type results for (1), we shall discuss the hair trigger effect for (2), thus making more precise the balance between the effect of the dispersal tails and the strength of the Allee effect which allows or not the hair trigger effect. Let us mention that, rather recently, various new results studying the interplay between some heavy tails and an Allee effect have been proved. Let us mention [27,20,14,1,2], and the references therein. In those works, the issue is, in different situations permitting propagation, to determine whether invasion is performed at a constant speed or by accelerating.

2. Assumptions and results

Let us first present and discuss the assumptions on the dispersal kernel J. As observed and proved in [6], expansion (8) plays a crucial role in the behavior of the linear equation $\partial_t u = J * u - u$, and so will for the nonlinear problem (1).

Assumption 2.1 *(Dispersal kernel).* $J : \mathbb{R}^N \to \mathbb{R}$ is nonnegative, bounded, radial and satisfies $\int_{\mathbb{R}^N} J = 1$. Its Fourier transform has an expansion

$$\widehat{J}(\xi) = 1 - A|\xi|^\beta + o(|\xi|^\beta), \qquad \text{as } \xi \to 0, \tag{8}$$

for some $0 < \beta \leq 2$ and $A > 0$.

Notice that expansion (8) contains the information on the tails of J. Indeed, for kernels which have a finite second momentum, namely $m_2 := \int_{\mathbb{R}^N} |x|^2 J(x)dx < +\infty$, expansion (8) holds true with $\beta = 2$, as can be seen in [8, Chapter 2, subsection 2.3.c, (3.8) Theorem] among others. In particular, this is the case for kernels which are compactly supported, exponentially bounded, or which decrease like $\mathcal{O}\left(\frac{1}{|x|^{N+2+\varepsilon}}\right)$ with $\varepsilon > 0$.

On the other hand, when $m_2 = +\infty$ then more general expansions are possible. For example, for algebraic tails satisfying

$$J(x) \sim \frac{C}{|x|^\alpha} \quad \text{as } |x| \to \infty, \quad \text{with } N < \alpha < N+2, \tag{9}$$

then (8) holds true with $\beta = \alpha - N \in (0,2)$. This fact is related to the stable laws of index $\beta \in (0,2)$ in probability theory, and a proof can be found in [8, Chapter 2, subsection 2.7]. In particular it contains the case of the Cauchy law $J(x) = \frac{1/\pi}{1+x^2}$ (when $N = 1$), for which

$$\widehat{J}(\xi) = e^{-|\xi|} = 1 - |\xi| + o(|\xi|), \qquad \text{as } \xi \to 0,$$

and $\beta = 1$, despite the nonexistence of the first momentum $m_1 := \int |x| J(x)dx$.

Remark 2.2 *(Critical algebraic tails).* For algebraic tails

$$J(x) \sim \frac{C}{|x|^{N+2}}, \quad \text{as } |x| \to \infty,$$

which are critical for the nonexistence of the second momentum m_2, expansion (8) is replaced by

$$\widehat{J}(\xi) = 1 + A|\xi|^2 \ln|\xi| + o(|\xi|^2 \ln|\xi|), \quad \text{as } \xi \to 0.$$

Nevertheless, as proved in [6, Theorem 5.1], the solutions to $\partial_t u = J * u - u$ still tend to that of the Heat equation, but with a different time velocity. In other words, for such tails, we do believe that $p_F = \frac{2}{N}$ and that this can be proved by additional technicalities and by using [6, Theorem 5.1] rather than [6, Theorem 1] to derive an analog of Lemma 5.1.

Before going further, we need to say a word on the notion of solutions. A function $u \in C^1((0,T), L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)) \cup C^0([0,T), L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N))$ for some $T > 0$, which satisfies the equation a.e. in $(0,T) \times \mathbb{R}^N$ is a local solution to (1) with $u(0,\cdot)$ as initial data. For such solutions, the comparison principle is available. Furthermore, for a nonnegative $u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, the associated Cauchy problem (1) admits a unique solution defined on some maximal interval $[0,T)$. Moreover either $T = \infty$ and the solution is global, or $T < \infty$ and then $\|u(t,\cdot)\|_{L^\infty}$ tend to ∞ as $t \nearrow T$, which is called blow up in finite time. These facts are rather well-known, and parts of them can be found in [11] for instance.

As explained in the introduction, our main result is the identification of the Fujita exponent for the nonlocal equation (1) for a large class of dispersion kernels, namely those admitting an expansion (8). In this context, $p_F := \frac{\beta}{N}$ is the Fujita exponent. More precisely, the following holds.

Theorem 2.3 *(Systematic blow up). Let Assumption 2.1 hold. Assume $0 < p \le p_F = \frac{\beta}{N}$. Assume $u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ is nonnegative and satisfies — for some $\varepsilon > 0$, $x_0 \in \mathbb{R}^N$, $r > 0$ — $u_0(x) \ge \varepsilon$ for all $x \in B(x_0, r)$. Then the solution to the Cauchy problem (1) with u_0 as initial data blows up in finite time.*

Theorem 2.4 *(Blow up vs extinction). Let Assumption 2.1 hold. Assume $p > p_F = \frac{\beta}{N}$. Then the following holds.*

(i) There is $\delta > 0$ such that, for any nonnegative $u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ with

$$\|u_0\|_{L^1} + \|\widehat{u_0}\|_{L^1} < \delta,$$

the solution to the Cauchy problem (1) is global in time and satisfies, for some $C > 0$,

$$\|u(t,\cdot)\|_{L^\infty} \le \frac{C}{(1+t)^{N/\beta}}, \quad \textit{for any } t \ge 0. \tag{10}$$

(ii) On the other hand, assume $\lambda > 0$ and $R > 0$ are such that

$$\lambda > \left(1 - C_N \int_{|z| \le R} J(z)\, dz\right)^{1/p}, \tag{11}$$

where $0 < C_N < 1$ is a constant that depends only on the dimension N (see subsection 5.2 for the exact value of this constant). Then, the solution to the Cauchy problem (1) with $\lambda \mathbf{1}_{\{|x| \le R\}}$ as initial data blows up in finite time.

As regards condition (11), let us notice that if $\lambda > 1$ then it is satisfied for any $R > 0$, indicating that large L^∞ data always lead to blow up; if $(1 - C_N)^{1/p} < \lambda \le 1$ then (11) is satisfied by taking $R > 0$ sufficiently large, indicating that intermediate L^∞ data require large initial mass to blow up (at least in our result); if $\lambda \le (1 - C_N)^{1/p}$ then (11) is never satisfied, indicating that small L^∞ data are bad candidates for blowing up.

Let us recall that, when J is compactly supported, the fact that the Fujita exponent $p_F = \frac{2}{N}$ is the same as that of the nonlinear diffusion equation (3) was already proved in [11]. Nevertheless, our results assert further that this remains true for kernels J which have a finite second momentum. On the other hand, when $\beta < 2$ in expansion (8) the Fujita exponent becomes that of the Fractional equation (3) with $s = \beta \in (0, 2)$. Hence, depending on the tails of the dispersal kernel, the nonlocal equation (1) behaves with respect to blow up either like the local Heat equation (3), or like a Fractional equation (4). This sheds light on the richness of (1).

We now turn to the hair trigger effect for (2), whose analysis makes use of the Fujita type results for (1). Notice that if $0 \le u_0 \le \|u_0\|_\infty < +\infty$ then, from the comparison principle, we get that the solution to (2) satisfies

$$0 < u(t,x) \le \max(1, \|u_0\|_\infty), \quad \forall (t,x) \in (0,\infty) \times \mathbb{R}^N,$$

so that the solution is always global.

Corollary 2.5 *(Hair trigger effect along a subsequence vs quenching solutions). Let Assumption 2.1 hold.*

(i) Assume $0 < p \le p_F = \frac{\beta}{N}$. Assume $u_0 : \mathbb{R}^N \to [0,1]$ is continuous and nontrivial. Then the solution to the Cauchy problem (2) with u_0 as initial data satisfies

$$\limsup_{t\to\infty} \inf_{|x|\le R} u(t,x) = 1, \quad \text{for any } R \ge 0. \tag{12}$$

(ii) Assume $p > p_F = \frac{\beta}{N}$. Then there is $\delta > 0$ such that, for any nonnegative, continuous, bounded and integrable u_0 with $\|u_0\|_{L^1} + \|\widehat{u_0}\|_{L^1} < \delta$, the solution to the Cauchy problem (2) satisfies, for some $C > 0$,

$$\|u(t,\cdot)\|_{L^\infty} \le \frac{C}{(1+t)^{N/\beta}}, \quad \text{for any } t \ge 0. \tag{13}$$

Observe that Corollary 2.5 *(ii)* directly follows from Theorem 2.4 *(i)* and the comparison principle. Corollary *(i)*, whose proof is rather classical, is the hair trigger effect, but only along a subsequence of time. Under a more restrictive assumption on the exponent p, we can actually prove the following hair trigger effect.

Theorem 2.6 *(Hair trigger effect). Let Assumption 2.1 hold. Assume $0 < p < \frac{1}{2}p_F = \frac{1}{2}\frac{\beta}{N}$. Assume $u_0 : \mathbb{R}^N \to [0,1]$ is continuous and nontrivial. Then the solution to the Cauchy problem (2) with u_0 as initial data satisfies*

$$\lim_{t\to\infty} \inf_{|x|\le R} u(t,x) = 1, \quad \text{for any } R \ge 0. \tag{14}$$

The proof of the above result requires the combination of an elaborate subsolution and careful asymptotics of the solution to the linear nonlocal diffusion equation $\partial_t u = J * u - u$. Using such a strategy, it seems very difficult, if possible, to remove the restriction $0 < p < \frac{1}{2}p_F$. Hence, different approaches should be used for the range $\frac{1}{2}p_F \le p \le p_F$, where more complex scenarios may exist. We hope to address this issue in a future work.

Notice also that, these results remain valid for equation

$$\partial_t u = J * u - u + f(u),$$

as long as f satisfies, for instance, $f(u) \sim ru^{1+p}$ as $u \to 0$ (for some $r > 0$), $f > 0$ on $(0,1)$, $f(1) = 0$, $f'(1) < 0$, $f < 0$ on $(1,\infty)$. Indeed, in such a case, we can sandwich $mu^{1+p}(1-u) \le f(u) \le Mu^{1+p}(1-u)$ for some $m > 0$, $M > 0$, and then combine some comparison and rescaling arguments.

The paper is organized as follows. We recall basic facts in Section 3. In Section 4, we prove the systematic blow up of any solution when $0 < p \le p_F$, that is Theorem 2.3. We study the case $p > p_F$ in Section 5, proving blow up or extinction depending on the size of the initial data, as stated in Theorem 2.4. Last, in Section 6, we prove the hair trigger effect, as stated in Corollary 2.5 *(i)* and Theorem 2.6.

3. Notations and basic facts

Before proving our results, let us now introduce some notations and recall briefly some basic facts.

For any integrable function J, we define

$$K(t,\cdot) := e^{-t}\delta_0 + e^{-t}\sum_{k=1}^{+\infty} \frac{t^k}{k!} J^{*(k)} =: e^{-t}\delta_0 + \psi(t,\cdot),$$

where δ_0 is the Dirac mass at 0 and $J^{*(k)} := J * \cdots * J$ is the convolution of J with itself $k-1$ times.

Then, the (unique) bounded solution to $\partial_t u = J * u - u$ with initial condition $u_0 \in L^\infty(\mathbb{R}^N)$ is given by

$$u(t,x) = K(t,\cdot) * u_0(x) = e^{-t}u_0(x) + \psi(t,\cdot) * u_0(x).$$

Obviously, though the convolution of a Dirac mass by an L^∞ function is not pointwise well defined, we let $\delta_0 * u_0 = u_0$. Also, from the normal convergence, in $C([0,T]; L^1(\mathbb{R}^N))$, of the series $\sum_{k=1}^{+\infty} \frac{t^k}{k!} J^{*(k)}$ and $\sum_{k=1}^{+\infty} \frac{t^{k-1}}{(k-1)!} J^{*(k)}$ we deduce that the function $t \in [0,\infty) \mapsto \psi(t,\cdot) \in L^1(\mathbb{R}^N)$ is of class C^1 and that

$$\partial_t \psi(t,x) = J * \psi(t,\cdot)(x) - \psi(t,x) + e^{-t} J(x). \tag{15}$$

Notice also that

$$\int_{\mathbb{R}^N} \psi(t,x)dx = 1 - e^{-t}. \tag{16}$$

For the sake of clarity, let us state our conventions on the Fourier transform. If $f \in L^1(\mathbb{R}^N)$, we define its Fourier transform $\mathcal{F}(f) = \widehat{f}$ and its inverse Fourier transform $\mathcal{F}^{-1}(f)$ by

$$\widehat{f}(\xi) := \int_{\mathbb{R}^N} e^{-i\xi\cdot x} f(x)dx, \quad \mathcal{F}^{-1}(f)(x) := \int_{\mathbb{R}^N} e^{ix\cdot\xi} f(\xi)d\xi.$$

With this definition, we have, for $f, g \in L^1(\mathbb{R}^N)$,

$$\widehat{f * g} = \widehat{f}\,\widehat{g},$$

and $f = \frac{1}{(2\pi)^N}\mathcal{F}^{-1}(\mathcal{F}(f))$ if $f, \mathcal{F}(f) \in L^1(\mathbb{R}^N)$. Also, after defining the Fourier transform on $L^2(\mathbb{R}^N)$ we get the Plancherel formula

$$\int_{\mathbb{R}^N} f(x)g(x)dx = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \widehat{f}(\xi)\widehat{g}(\xi)d\xi,$$

for $f, g \in L^2(\mathbb{R}^N)$.

4. Systematic blow up

In this section, we first provide a priori estimates on a crucial quantity related to possible global solutions of (1). They will then enable us to prove the blow up of any solution when $0 < p \le p_F$, as stated in Theorem 2.3.

4.1. Some a priori estimates

In this subsection, we assume that u_0 is nonnegative, nontrivial, radial, continuous, bounded, and that both u_0 and $\widehat{u_0}$ are in $L^1(\mathbb{R}^N)$. We also assume that we are equipped with a global solution $u(t,x)$ of the associated Cauchy problem (1). We then define, for any $t \ge 0$, the quantity

$$f(t) := \int_{\mathbb{R}^N} e^{t(\widehat{J}(\xi)-1)}\widehat{u_0}(\xi)d\xi. \tag{17}$$

In the spirit of an original idea of Kaplan [16], also used in [10], we are going to estimate $f(t)$ from below and above as $t \to \infty$. As clear in the following, another key ingredient is the Fourier duality which enables to recast (17) as (21).

Lemma 4.1 *(Estimate from below). There is a constant $G > 0$ depending only on the dimension N and the kernel J, and a constant $t_0 > 0$ (that is allowed to depend on the initial data), such that*

$$f(t) \ge \frac{G\|u_0\|_{L^1}}{t^{N/\beta}} \quad \textit{for any } t \ge t_0. \tag{18}$$

Proof. From (8), we can select $\xi_0 > 0$ small enough so that

$$|\xi| \le \xi_0 \Longrightarrow \widehat{J}(\xi) - 1 \ge -2A|\xi|^\beta. \tag{19}$$

Since $\widehat{u_0}(0) = \int_{\mathbb{R}^N} u_0 > 0$ and $\widehat{u_0}$ is a real-valued continuous function, up to reducing $\xi_0 > 0$ if necessary, we can assume that

$$|\xi| \le \xi_0 \Longrightarrow \widehat{u_0}(\xi) \ge 0.$$

On the other hand, $\widehat{J}$ is continuous, $\widehat{J}(\xi) - 1 < 0$ for all $\xi \neq 0$, $\widehat{J}(\xi) - 1 \to -1$ as $|\xi| \to +\infty$, hence there is $\delta > 0$ such that

$$|\xi| \ge \xi_0 \Longrightarrow \widehat{J}(\xi) - 1 \le -\delta. \tag{20}$$

Now, by cutting into two pieces, we get $t^{N/\beta} f(t) = g_1(t) + g_2(t)$, where

$$|g_2(t)| \le \left| \int_{|\xi|\ge\xi_0} t^{N/\beta} e^{t(\widehat{J}(\xi)-1)} \widehat{u_0}(\xi) d\xi \right| \le t^{N/\beta} e^{-\delta t} \|\widehat{u_0}\|_{L^1} \to 0 \quad \text{as } t \to \infty,$$

and

$$\begin{aligned} g_1(t) &= t^{N/\beta} \int_{|\xi|\le\xi_0} e^{t(\widehat{J}(\xi)-1)} \widehat{u_0}(\xi) d\xi \\ &\ge t^{N/\beta} \int_{|\xi|\le\xi_0} e^{-2At|\xi|^\beta} \widehat{u_0}(\xi) d\xi \\ &= \int_{\mathbb{R}^N} e^{-2A|z|^\beta} \widehat{u_0}\left(\frac{z}{t^{1/\beta}}\right) \mathbf{1}_{(0,t^{1/\beta}\xi_0)}(|z|) dz. \end{aligned}$$

By the dominated convergence theorem, the last integral above tends, as $t \to \infty$, to the constant

$$\widehat{u_0}(0) \int_{\mathbb{R}^N} e^{-2A|z|^\beta} dz = \|u_0\|_{L^1} \int_{\mathbb{R}^N} e^{-2A|z|^\beta} dz =: \|u_0\|_{L^1} 2G,$$

where $G > 0$ depends only on the dimension N and the kernel J (via A and β). As a result, we can select $t_0 > 0$ large enough so that (18) holds true. The lemma is proved. $\square$

In order to derive an estimate from above, it is more convenient to use the dual expression (see below for a proof)

$$f(t) = (2\pi)^N \int_{\mathbb{R}^N} e^{-t} \left(\delta_0 + \sum_{k=1}^{+\infty} \frac{t^k}{k!} J^{*(k)}(x) \right) u_0(x) dx = (2\pi)^N \int_{\mathbb{R}^N} K(t,x) u_0(x) dx, \tag{21}$$

where we recall that $K(t,x)$ was defined in Section 3. Notice that, formally, the fundamental solution of $\partial_t u = J * u - u$ is $\mathcal{F}^{-1}(e^{t(\widehat{J}(\xi)-1)}) = e^{-t}\left(\delta_0 + \sum_{k=1}^{+\infty} \frac{t^k}{k!} J^{*(k)}(x)\right)$ so that expression (21) is, again formally, derived from (17) by the Plancherel formula.

Proof of (21). From (17) we get

$$e^t f(t) - \int_{\mathbb{R}^N} \widehat{u_0}(\xi) d\xi = \int_{\mathbb{R}^N} \sum_{k=1}^{+\infty} \frac{t^k}{k!} \widehat{J}^k(\xi) \widehat{u_0}(\xi) d\xi = \sum_{k=1}^{+\infty} \int_{\mathbb{R}^N} \frac{t^k}{k!} \widehat{J}^k(\xi) \widehat{u_0}(\xi) d\xi,$$

since $\sum_k \int |\frac{t^k}{k!} \widehat{J}^k(\xi) \widehat{u_0}(\xi)| d\xi \le \sum_k \frac{t^k}{k!} \|\widehat{u_0}\|_{L^1} < +\infty$ (recall that $|\widehat{J}(\xi)| \le 1$). Next $J \in L^1(\mathbb{R}^N)$ implies $\widehat{J}^k(\xi) = \widehat{J^{*(k)}}(\xi)$, so that

$$e^t f(t) - \int_{\mathbb{R}^N} \widehat{u_0}(\xi)d\xi = \sum_{k=1}^{+\infty} \int_{\mathbb{R}^N} \frac{t^k}{k!} \widehat{J^{*(k)}}(\xi)\widehat{u_0}(\xi)d\xi.$$

Next, both $J^{*(k)}$, $u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N) \subset L^2(\mathbb{R}^N)$ so that we can apply the Plancherel formula to get

$$\begin{aligned} e^t f(t) - \int_{\mathbb{R}^N} \widehat{u_0}(\xi)d\xi &= (2\pi)^N \sum_{k=1}^{+\infty} \int_{\mathbb{R}^N} \frac{t^k}{k!} J^{*(k)}(x)u_0(x)dx \\ &= (2\pi)^N \int_{\mathbb{R}^N} \sum_{k=1}^{+\infty} \frac{t^k}{k!} J^{*(k)}(x)u_0(x)dx, \end{aligned} \tag{22}$$

since $\sum_k \int |\frac{t^k}{k!} J^{*(k)}(x)u_0(x)|dx \le \sum_k \frac{t^k}{k!} \|J\|_{L^\infty} \|u_0\|_{L^1} < +\infty$. Last, notice that $\int_{\mathbb{R}^N} \widehat{u_0} = \widehat{\widehat{u_0}}(0)$. Since u_0 is real and radial, so is $\widehat{u_0}$, which in turn implies $\widehat{\widehat{u_0}} = \mathcal{F}^{-1}(\widehat{u_0}) = (2\pi)^N u_0$ so that $\int_{\mathbb{R}^N} \widehat{u_0} = (2\pi)^N u_0(0) = (2\pi)^N \int_{\mathbb{R}^N} \delta_0 u_0$, which we plug into (22) to conclude the proof of (21). □

Equipped with the dual formula (21), we can now prove the following.

Lemma 4.2 *(Estimate from above). We have*

$$f(t) \le (2\pi)^N \left(\left(\frac{p+1}{p} \right)^{1/p} \frac{1}{t^{1/p}} + e^{-t} u_0(0) \right) \quad \text{for any } t > 0. \tag{23}$$

Proof. Let $T > 0$ be given. Denote $C_T := \max_{0 \le \tau \le T+1} \|u(\tau, \cdot)\|_{L^\infty} + \|u(\tau, \cdot)\|_{L^1} < +\infty$. Fix some $0 < t \le T$.

First observe that (21) is recast

$$h(t) := \frac{f(t)}{(2\pi)^N} = \int_{\mathbb{R}^N} \left(e^{-t}\delta_0 + \psi(t,x)\right) u_0(x)dx = e^{-t}u_0(0) + \int_{\mathbb{R}^N} \psi(t,x)u_0(x)dx, \tag{24}$$

where $\psi(t,x) = e^{-t} \sum_{k=1}^{+\infty} \frac{t^k}{k!} J^{*(k)}(x)$ is as in Section 3.

For $0 < \varepsilon \le 1$, let us define

$$g_\varepsilon(s) := \int_{\mathbb{R}^N} \psi(t-s+\varepsilon, x)u(s,x)dx, \quad 0 \le s \le t. \tag{25}$$

Notice that, if k is sufficiently large, the support of $J^{*(k)}$ meets that of $u(s,\cdot)$, and therefore $\int_{\mathbb{R}^N} J^{*(k)}(x)u(s,x)dx > 0$, which in turn implies $g_\varepsilon(s) > 0$. Notice also that, using the dominated convergence theorem, we see that, as $\varepsilon \to 0$,

$$g_\varepsilon(0) = \int_{\mathbb{R}^N} \psi(t+\varepsilon, x)u_0(x)dx \to \int_{\mathbb{R}^N} \psi(t,x)u_0(x)dx = \frac{f(t)}{(2\pi)^N} - e^{-t}u_0(0). \tag{26}$$

Using the constant C_T defined above one can dominate the partial derivative with respect to s of the integrand in (25), and therefore prove that g_ε is differentiable. Using equations (15) and (1), we then compute

$$\begin{aligned} g'_\varepsilon(s) &= \int_{\mathbb{R}^N} (-J * \psi(t-s+\varepsilon, \cdot) + \psi(t-s+\varepsilon, \cdot) - e^{-(t-s+\varepsilon)} J)u(s,\cdot) \\ &\quad + \int_{\mathbb{R}^N} \psi(t-s+\varepsilon, \cdot)(J * u(s,\cdot) - u(s,\cdot) + u^{1+p}(s,\cdot)) \\ &= -\int_{\mathbb{R}^N} e^{-(t-s+\varepsilon)} Ju(s,\cdot) + \int_{\mathbb{R}^N} \psi(t-s+\varepsilon, \cdot)u^{1+p}(s,\cdot), \end{aligned} \tag{27}$$

by Fubini theorem. From the expression of ψ, we see that $e^{-\tau}J(x) \le \frac{\psi(\tau,x)}{\tau}$ for $\tau > 0$, so that

$$\int_{\mathbb{R}^N} e^{-(t-s+\varepsilon)} Ju(s,\cdot) \le \frac{1}{t-s+\varepsilon} g_\varepsilon(s). \tag{28}$$

Next, we write

$$\begin{aligned}\int_{\mathbb{R}^N} \psi(t-s+\varepsilon,\cdot)u^{1+p}(s,\cdot) &= (1-e^{-(t-s+\varepsilon)}) \int_{\mathbb{R}^N} \frac{\psi(t-s+\varepsilon,\cdot)}{1-e^{-(t-s+\varepsilon)}} u^{1+p}(s,\cdot)\\ &\ge \frac{1}{(1-e^{-(t-s+\varepsilon)})^p} g_\varepsilon^{1+p}(s)\\ &\ge g_\varepsilon^{1+p}(s),\end{aligned} \tag{29}$$

where we have used the Jensen inequality (notice that $\int_{\mathbb{R}^N} \frac{\psi(t-s+\varepsilon,\cdot)}{1-e^{-(t-s+\varepsilon)}} = 1$ in view of (16)). Plugging (28) and (29) into (27) and multiplying by the integrating factor $(t-s+\varepsilon)^p$ we arrive at

$$\left(\frac{g_\varepsilon'(s)}{g_\varepsilon^{1+p}(s)} + \frac{1}{t-s+\varepsilon}\frac{1}{g_\varepsilon^p(s)}\right)(t-s+\varepsilon)^p \ge (t-s+\varepsilon)^p.$$

The left hand side member is nothing else that $\frac{d}{ds}\left(\frac{(t-s+\varepsilon)^p}{-pg_\varepsilon^p(s)}\right)$ so that integrating from 0 to t, we get

$$-\frac{1}{p}\frac{1}{g_\varepsilon^p(t)}\varepsilon^p + \frac{1}{p}\frac{1}{g_\varepsilon^p(0)}(t+\varepsilon)^p \ge -\frac{\varepsilon^{p+1}}{p+1} + \frac{(t+\varepsilon)^{p+1}}{p+1},$$

which in turn implies

$$\frac{1}{g_\varepsilon^p(0)} \ge \frac{p}{p+1}(t+\varepsilon) - \frac{p}{p+1}\frac{\varepsilon^{p+1}}{(t+\varepsilon)^p}.$$

Letting $\varepsilon \to 0$ and using (26), we get estimate (23), which concludes the proof of Lemma 4.2. □

4.2. *Proof of systematic blow up*

Proof of Theorem 2.3. Let $u_0 \in L^\infty(\mathbb{R}^N)$ be nonnegative and satisfying — for some $\varepsilon > 0$, $x_0 \in \mathbb{R}^N$, $r > 0$ — $u_0(x) \ge \varepsilon$ for all $x \in B(x_0, r)$. Thus there exists a nonnegative, nontrivial, radial and $C_c^\infty(\mathbb{R}^N)$ function that is smaller than u_0. By the comparison principle, it is enough to prove blow up for such an initial data. Hence we can assume without loss of generality that u_0 is nonnegative, nontrivial, bounded, that $u_0 \in C_c^\infty(\mathbb{R}^N)$, and thus $\widehat{u_0} \in \mathcal{S}(\mathbb{R}^N)$. Hence, assuming by contradiction existence of a global solution, all the results of subsection 4.1 are available.

- When $0 < p < \frac{\beta}{N}$, letting $t \to \infty$ in (18) and (23) immediately gives a contradiction.
- In the critical case $p = \frac{\beta}{N}$, letting $t \to \infty$ in (18) and (23) only provides $\|u_0\|_{L^1} \le C$, where the constant $C > 0$ depends on the dimension N and the kernel J but not on the size of the initial data. Thus, by regarding $u(t,\cdot)$ as an initial value, we derive that

$$m(t) := \|u(t,\cdot)\|_{L^1} \le C, \qquad \text{for any } t \ge 0. \tag{30}$$

Integrating equation (1) over $x \in \mathbb{R}^N$ and using Fubini theorem, we get

$$\frac{d}{dt}m(t) = \int_{\mathbb{R}^N} u^{1+p}(t,x)\,dx,$$

so that $\int_0^t \int_{\mathbb{R}^N} u^{1+p}(t,x)\,dxdt = m(t) - m(0) \le C$, for all $t \ge 0$. As a result we know that

$$\int_0^\infty \int_{\mathbb{R}^N} u^{1+p}(t,x)\,dxdt < +\infty. \tag{31}$$

We are going to derive below a contradiction, using a modification of an original technique of [21] for a local equation. Notice that our kernel J may not have finite second nor first moment, so we need to derive further estimates. Also, we shall again take advantage of the Fourier duality.

Consider $\rho \in C_c^\infty(\mathbb{R})$ such that $\rho \equiv 1$ on $(-1, 1)$, $0 \le \rho \le 1$ and $\operatorname{Supp} \rho = [-2, 2]$. Let $T > 0$ be given. Let $\varepsilon > 0$ be given. For $R > 0$, we define

$$\psi_R(t) := \rho\left(\frac{t-T}{R^{Np}}\right) = \rho\left(\frac{t-T}{R^\beta}\right), \quad \theta_R(x) := \rho\left(\varepsilon\frac{|x|}{R}\right).$$

We multiply equation (1) by $\theta_R(x)\psi_R(t)$ and integrate over $(t, x) \in (T, \infty) \times \mathbb{R}^N$ to get

$$\begin{aligned}
\int_T^\infty \int_{\mathbb{R}^N} u^{1+p}(t,x)\theta_R(x)\psi_R(t) &= -\int_T^\infty \int_{\mathbb{R}^N} (J * u - u)(t,x)\theta_R(x)\psi_R(t) \\
&\quad + \int_T^\infty \int_{\mathbb{R}^N} \partial_t u(t,x)\theta_R(x)\psi_R(t) \\
&\le -\int_T^\infty \int_{\mathbb{R}^N} (J * \theta_R - \theta_R)(x)u(t,x)\psi_R(t) \\
&\quad - \int_T^\infty \int_{\mathbb{R}^N} u(t,x)\theta_R(x)\psi_R'(t) =: -I_1 - I_2, \qquad (32)
\end{aligned}$$

where we have used Fubini theorem, integration by part in time, in the first, respectively the second, integral of the right hand side member. In the sequel, we denote by C a positive constant that may change from place to place but that is always independent of $\varepsilon > 0$ and $R > 0$.

Let us deal with $I_2 = I_2(R)$. Observe that

$$|\psi_R'(t)| = |\frac{1}{R^\beta}\rho'\left(\frac{t-T}{R^\beta}\right)| \le \frac{C}{R^\beta}\mathbf{1}_{(T+R^\beta, T+2R^\beta)}(t),$$

so that

$$\begin{aligned}
|I_2| &\le \frac{C}{R^\beta}\int_{T+R^\beta}^{T+2R^\beta} \int_{|x|\le 2R/\varepsilon} u(t,x) \qquad (33)\\
&\le \frac{C}{R^\beta}\left(\int_{T+R^\beta}^{T+2R^\beta} \int_{|x|\le 2R/\varepsilon} 1\right)^{\frac{p}{p+1}} \left(\int_{T+R^\beta}^{T+2R^\beta} \int_{|x|\le 2R/\varepsilon} u^{1+p}(t,x)\right)^{\frac{1}{p+1}} \\
&= \frac{C}{R^\beta}\left(R^\beta\left(\frac{2R}{\varepsilon}\right)^N\right)^{\frac{p}{p+1}} \left(\int_{T+R^\beta}^{T+2R^\beta} \int_{|x|\le 2R/\varepsilon} u^{1+p}(t,x)\right)^{\frac{1}{p+1}} \\
&= \frac{C}{\varepsilon^{\frac{Np}{p+1}}} \left(\int_{T+R^\beta}^{T+2R^\beta} \int_{|x|\le 2R/\varepsilon} u^{1+p}(t,x)\right)^{\frac{1}{p+1}},
\end{aligned}$$

where we have used the Hölder inequality and equality $\beta = Np$. In view of (31), the last integral above tends to zero as $R \to \infty$, and so does I_2.

Let us deal with $I_1 = I_1(R) = \int_T^{T+2R^\beta} \int_{\mathbb{R}^N} B(x)u(t,x)$, where

$$B(x) := (J * \theta_R - \theta_R)(x) = \int_{\mathbb{R}^N} (\theta_R(z-x) - \theta_R(x))\, J(z)dz.$$

First observe that if $|x| \geq 2R/\varepsilon$ then $\theta_R(x) = 0$ so that $B(x) \geq 0$. As a result

$$I_1 \geq I_1' := \int_T^{T+2R^\beta} \int_{|x|<2R/\varepsilon} B(x)u(t,x). \tag{34}$$

In order to estimate $B(x)$, we use the Plancherel formula and get

$$\begin{aligned}
(2\pi)^N B(x) &= (2\pi)^N \int_{\mathbb{R}^N} \theta_R(z-x)J(z)dz - (2\pi)^N \theta_R(x) \\
&= \int_{\mathbb{R}^N} \widehat{J}(\xi) e^{-ix\cdot\xi} \widehat{\theta_R}(\xi)\, d\xi - (2\pi)^N \theta_R(x) \\
&= \int_{\mathbb{R}^N} (1 - \mathcal{A}(\xi)|\xi|^\beta) e^{-ix\cdot\xi} \widehat{\theta_R}(\xi)\, d\xi - (2\pi)^N \theta_R(x),
\end{aligned}$$

where function $\mathcal{A}$ is bounded in view of (8). Since $\int_{\mathbb{R}^N} e^{-ix\cdot\xi} \widehat{\theta_R}(\xi)\, d\xi = \mathcal{F}(\mathcal{F}(\theta_R))(x) = (2\pi)^N \theta_R(x)$ and $\widehat{\theta_R}(\xi) = \left(\frac{R}{\varepsilon}\right)^N \widehat{\rho_N}(\frac{R}{\varepsilon}\xi)$ — where $\rho_N : \mathbb{R}^N \to \mathbb{R}$ is defined by $\rho_N(x) := \rho(|x|)$ — we get

$$(2\pi)^N B(x) = -\left(\frac{\varepsilon}{R}\right)^\beta \int_{\mathbb{R}^N} \mathcal{A}\left(\frac{\varepsilon}{R}\xi'\right) |\xi'|^\beta e^{-i\frac{\varepsilon}{R}x\cdot\xi'} \widehat{\rho_N}(\xi')\, d\xi',$$

so that

$$|B(x)| \leq \frac{1}{(2\pi)^N} \left(\frac{\varepsilon}{R}\right)^\beta \|\mathcal{A}\|_\infty \int_{\mathbb{R}^N} |\xi'|^\beta |\widehat{\rho_N}(\xi')|\, d\xi' = C\frac{\varepsilon^\beta}{R^\beta},$$

since $\widehat{\rho_N} \in \mathcal{S}(\mathbb{R}^N)$. As a result

$$|I_1'| \leq C\frac{\varepsilon^\beta}{R^\beta} \int_T^{T+2R^\beta} \int_{|x|<2R/\varepsilon} u(t,x).$$

We are now in the footsteps of (33) so that — notice the presence of the crucial multiplicative factor ε^β — similar arguments (Hölder inequality and $\beta = Np$) yield

$$|I_1'| \leq C\varepsilon^{\frac{\beta p}{p+1}} \left(\int_T^{T+2R^\beta} \int_{|x|\leq 2R/\varepsilon} u^{1+p}(t,x) \right)^{\frac{1}{p+1}}. \tag{35}$$

To conclude, plugging (34) and (35) into (32), we get

$$\int_T^\infty \int_{\mathbb{R}^N} u^{1+p}(t,x)\theta_R(x)\psi_R(t) \leq |I_2| + C\varepsilon^{\frac{\beta p}{p+1}} \left(\int_T^{T+2R^\beta} \int_{|x|\leq 2R/\varepsilon} u^{1+p}(t,x) \right)^{\frac{1}{p+1}}.$$

Letting $R \to \infty$ yields

$$\int_T^\infty \int_{\mathbb{R}^N} u^{1+p} \le C\varepsilon^{\frac{\beta p}{p+1}} \left(\int_T^\infty \int_{\mathbb{R}^N} u^{1+p} \right)^{\frac{1}{p+1}}.$$

From the arbitrariness of $\varepsilon > 0$ and $T > 0$ we deduce that $u \equiv 0$ on $(0, \infty) \times \mathbb{R}^N$, which is a contradiction. This concludes the proof of Theorem 2.3. $\square$

5. Blow up vs extinction

In this section, we prove that when $p > p_F = \frac{\beta}{N}$, depending on the size of the initial data, the solution to the Cauchy problem (1) can be global and extincting, or blowing up in finite time, as stated in Theorem 2.4.

5.1. *Extinction for small initial data*

Proof of Theorem 2.4 *(i)*. The proof, as that in [11], relies strongly on [6] which provides the rate of decrease of the L^∞ norm of the solution of the nonlocal linear equation $\partial_t v = J * v - v$.

Lemma 5.1 *(See Theorem 1 in [6] and Theorem 5 in [11]). There is $C > 0$ such that, for any initial data $v_0 \in L^1(\mathbb{R}^N)$ such that $\widehat{v_0} \in L^1(\mathbb{R}^N)$, the solution of the Cauchy problem $\partial_t v = J * v - v$ satisfies*

$$\|v(t,\cdot)\|_{L^\infty} \le \frac{C(\|v_0\|_{L^1} + \|\widehat{v_0}\|_{L^1})}{(1+t)^{N/\beta}}, \quad \textit{for any } t \ge 0.$$

We look after a supersolution to (1) in the form $g(t)v(t,x)$, where $g(t) > 0$ is to be determined (with $g(0) = 1$) and $v(t,x)$ is the solution of $\partial_t v = J * v - v$ with u_0 as initial data. A straightforward computation shows that it is enough to have $\frac{g'(t)}{g^{1+p}(t)} \ge \|v(t,\cdot)\|_{L^\infty}^p$. By the above lemma, it is therefore enough to have

$$\frac{g'(t)}{g^{1+p}(t)} = \frac{C^p(\|u_0\|_{L^1} + \|\widehat{u_0}\|_{L^1})^p}{(1+t)^{pN/\beta}}, \quad g(0) = 1.$$

If $\|u_0\|_{L^1} + \|\widehat{u_0}\|_{L^1} < \delta := \frac{1}{C}\left(\frac{\frac{pN}{\beta}-1}{p}\right)^{1/p}$ (notice that $\frac{pN}{\beta} - 1 > 0$) then the solution of the above Cauchy problem

$$g(t) = \frac{1}{\left(1 - \frac{pC^p(\|u_0\|_{L^1}+\|\widehat{u_0}\|_{L^1})^p}{\frac{pN}{\beta}-1}\left(1 - \frac{1}{(1+t)^{\frac{pN}{\beta}-1}}\right)\right)^{1/p}},$$

exists for all $t \ge 0$, is increasing and bounded. It therefore follows from the comparison principle that $u(t,x) \le g(t)v(t,x) \le \|g\|_\infty v(t,x)$ so that the solution $u(t,x)$ of (1) is global in time and, in view of Lemma 5.1, satisfies estimate (10). This concludes the proof of Theorem 2.4 *(i)*. $\square$

5.2. *Blow up for large initial data*

Proof of Theorem 2.4 *(ii)*. Let $\lambda > 0$ and $R > 0$ be given such that (11) holds. Now, let us consider the solution $u(t,x)$ to (1) with initial data $u_0 = \lambda \mathbf{1}_{\{|x| \le R\}}$ and prove the blow up of the "localized mass"

$$m(t) := \int_{|x|\le R} u(t,x)dx, \tag{36}$$

which is enough to prove the blow up of the solution.

Integrating equation (1), we get

$$\frac{d}{dt}m(t) = \int_{|x|\le R} J * u(t,\cdot)(x)\,dx - m(t) + \int_{|x|\le R} u^{1+p}(t,x)\,dx. \tag{37}$$

Denoting by B_N the volume of the unit ball in $\mathbb{R}^N$, we estimate the last term in the above right hand side member by

$$\int_{|x|\le R} u^{1+p}(t,\cdot) = B_N R^N \int_{|x|\le R} \frac{1}{B_N R^N} u^{1+p}(t,\cdot) \ge \frac{1}{B_N^p R^{Np}} m^{1+p}(t), \tag{38}$$

thanks to the Jensen inequality. Let us now turn to the first term in the right hand side member of (37). Using Fubini theorem yields

$$\begin{aligned}\int_{|x|\le R} J * u(t,\cdot) &= \int_{\mathbb{R}^N} u(t,y) \int_{|x|\le R} J(x-y)\,dxdy\\ &\ge \int_{|y|\le R} u(t,y) \int_{|z-y|\le R} J(z)\,dzdy.\end{aligned}$$

Now we claim that, for any y such that $0<|y|<R$,

$$\int_{|z-y|\le R} J(z)\,dz \ge C_N \int_{|z|\le R} J(z)\,dz, \tag{39}$$

where $0<C_N<1$ is a constant that depends only on the dimension N. We postpone the proof of (39) and obtain

$$\int_{|x|\le R} J * u(t,\cdot) \ge m(t) C_N \int_{|z|\le R} J(z)\,dz. \tag{40}$$

Then, plugging (38) and (40) in (37), we arrive at the differential inequality

$$\frac{d}{dt}m(t) \ge m(t)\left[\frac{m^p(t)}{B_N^p R^{Np}} - \left(1 - C_N \int_{|z|\le R} J(z)\,dz\right)\right].$$

Since $m(0) = \int_{|x|\le R} u_0 = \lambda B_N R^N > \left(1 - C_N \int_{|z|\le R} J(z)\,dz\right)^{1/p} B_N R^N$ thanks to (11), the above differential inequality enforces[1] the blow up of $m(t)$ in finite time. This concludes the proof of Theorem 2.4 *(ii)*. □

For the convenience of the reader, and also to give the exact value of the constant C_N, we prove below the rather intuitive claim (39).

Proof of claim (39). In dimension $N=1$, (39) clearly holds true with $C_1 = \frac{1}{2}$ since J is even. Let us now assume $N \ge 2$. We denote by S_{N-1} the unit hypersphere of $\mathbb{R}^N$. Since J is radial we have

$$\int_{|z|\le R} J(z)\,dz = |S_{N-1}| \int_0^R r^{N-1} J(r)\,dr, \tag{41}$$

where we recall that $|S_{N-1}| = \frac{2\pi^{N/2}}{\Gamma(N/2)}$. Let us take y such that $0 < r_0 := |y| < R$. Define $(e_1 := \frac{y}{r_0}, e_2, \cdots, e_N)$ an orthonormal basis of $\mathbb{R}^N$. For a generic point $z \in \mathbb{R}^N$ we denote by $(z_1,\cdots,z_N)$ its cartesian coordinates in $(e_1,\cdots,e_N)$ and $(r,\theta_1,\cdots,\theta_{N-1}) \in [0,\infty) \times [-\frac{\pi}{2},\frac{\pi}{2}[^{N-2} \times [-\pi,\pi)$ its polar coordinates, which are related through

[1] Indeed, the Bernoulli equation $\dot{x} = ax^{1+p} - bx$, $a>0$, $b>0$, can be solved explicitly and blows up in finite time as soon as $x(0) > \left(\frac{b}{a}\right)^{1/p}$.

$$
\begin{aligned}
z_1 &= r\cos\theta_1\cos\theta_2 \ldots \cos\theta_{N-2}\cos\theta_{N-1}\\
z_2 &= r\cos\theta_1\cos\theta_2 \ldots \cos\theta_{N-2}\sin\theta_{N-1}\\
z_3 &= r\cos\theta_1\cos\theta_2 \ldots \sin\theta_{N-2}\\
&\ldots\\
z_{N-1} &= r\cos\theta_1\sin\theta_2\\
z_N &= r\sin\theta_1.
\end{aligned}
$$

We claim that

$$
D := \left\{ z : 0 < r < R, |\theta_i| < \theta^* := \arccos\frac{1}{2^{\frac{1}{N-1}}} \right\} \subset \{z : |z-y| < R\}. \tag{42}
$$

Indeed, for $z \in D$, we have

$$
\begin{aligned}
|z-y|^2 &= (z_1 - r_0)^2 + z_2^2 + \cdots + z_N^2 = r^2 - 2rr_0\cos\theta_1 \ldots \cos\theta_{N-1} + r_0^2\\
&\le r^2 - 2rr_0\cos^{N-1}\theta^* + r_0^2 = r^2 - rr_0 + r_0^2 \le \max(r^2, r_0^2) < R^2.
\end{aligned}
$$

It therefore follows from (42) that

$$
\begin{aligned}
\int_{|z-y|\le R} J(z)dz &\ge \int_D J(z)dz\\
&= \int_{-\theta^*}^{\theta^*} (\cos\theta_1)^{N-2} d\theta_1 \int_{-\theta^*}^{\theta^*} (\cos\theta_2)^{N-3} d\theta_2 \ldots \int_{-\theta^*}^{\theta^*} d\theta_{N-1} \int_0^R r^{N-1} J(r)dr\\
&= C_N \int_{|z|\le R} J(z)dz,
\end{aligned}
$$

in view of (41) and where

$$
\begin{aligned}
C_N &:= \frac{\int_{-\theta^*}^{\theta^*} (\cos\theta_1)^{N-2} d\theta_1 \int_{-\theta^*}^{\theta^*} (\cos\theta_2)^{N-3} d\theta_2 \ldots \int_{-\theta^*}^{\theta^*} d\theta_{N-1}}{|S_{N-1}|}\\
&= \frac{\int_{-\theta^*}^{\theta^*} (\cos\theta_1)^{N-2} d\theta_1 \int_{-\theta^*}^{\theta^*} (\cos\theta_2)^{N-3} d\theta_2 \ldots \int_{-\theta^*}^{\theta^*} d\theta_{N-1}}{\int_{-\pi/2}^{\pi/2} (\cos\theta_1)^{N-2} d\theta_1 \int_{-\pi/2}^{\pi/2} (\cos\theta_2)^{N-3} d\theta_2 \ldots \int_{-\pi}^{\pi} d\theta_{N-1}} \in (0,1),
\end{aligned}
$$

which concludes the proof of (39). □

6. Hair trigger effect

6.1. Hair trigger effect along a subsequence

Following the strategy of [22, Theorem 18.7], we prove here the hair trigger effect along a subsequence of time.

Proof of Corollary 2.5 *(i)*. First, let $v_0 \in C(\mathbb{R}^N, [0,1])$ be such that $v_0(x_0 + \cdot)$ is radial nonincreasing, for some $x_0 \in \mathbb{R}^N$. Let $v(t,x)$ be the global solution of (2) with v_0 as initial data. Then, J being radial, $v(t, x_0 + \cdot)$ remains radial nonincreasing for later times $t > 0$. Let us prove that

$$
\limsup_{t\to\infty} v(t, x_0) = 1. \tag{43}
$$

Assume by contradiction that there are $0 < \varepsilon < 1$ and $T > 0$ such that $v(t, x_0) \le 1 - \varepsilon$ for all $t \ge T$, which in turn implies $v(t,x) \le 1 - \varepsilon$, for all $(t,x) \in [T, \infty) \times \mathbb{R}^N$. As a result

$$
\partial_t v \ge J * v - v + \varepsilon v^{1+p} \quad \text{in } (T,\infty) \times \mathbb{R}^N.
$$

Hence $w := \varepsilon^{1/p} v$ satisfies $\partial_t w \geq J * w - w + w^{1+p}$ in $(T, \infty) \times \mathbb{R}^N$. Since $0 < p \leq p_F = \frac{\beta}{N}$, it follows from Theorem 2.3 and the comparison principle that w is non-global, which is a contradiction.

Now, let $u_0 : \mathbb{R}^N \to [0, 1]$ be as in Corollary 2.5 *(i)*, that is continuous and nontrivial. We need to prove

$$\limsup_{t\to\infty} \inf_{|x|\leq R} u(t, x) = 1, \qquad \text{for any } R \geq 0. \tag{44}$$

By a time shift if necessary, we can assume further that $u_0 > 0$. Therefore u_0 dominates some $\tilde{u}_0 : \mathbb{R}^N \to [0, 1]$ which is nontrivial and radial nonincreasing. Hence, by comparison, it suffices to prove (44) for $\tilde{u}(t, x)$ the solution of (2) with $\tilde{u}_0$ as initial data, for which we can take advantage of the fact that $\tilde{u}(t, \cdot)$ is radial nonincreasing for later times $t > 0$. Again, by a time shift if necessary, we can assume further that $\tilde{u}_0 > 0$. Now, for a given $x_0 \in \mathbb{R}^N$, $\tilde{u}_0$ dominates some $v_0 \in C(\mathbb{R}^N, [0, 1])$ such that $v_0(x_0 + \cdot)$ is radial nonincreasing. It follows from (43) and the comparison principle that

$$\limsup_{t\to\infty} \tilde{u}(t, x_0) = 1.$$

Since x_0 is arbitrary and since $\tilde{u}(t, \cdot)$ is radial nonincreasing, this implies

$$\limsup_{t\to\infty} \inf_{|x|\leq R} \tilde{u}(t, x) = 1, \qquad \text{for any } R \geq 0.$$

This concludes the proof of (44). □

6.2. *Actual hair trigger effect*

In this subsection, we prove the actual hair trigger effect as stated in Theorem 2.6. This requires the combination of an elaborate subsolution involving two different time scales — see [28] for a related argument in a local case — and the following asymptotics for the solution to the linear nonlocal diffusion equation.

Lemma 6.1 *(The linear equation from below). Let Assumption 2.1 hold. For a given $R > 0$, let $\varphi(t, x)$ be the solution of $\partial_t \varphi = J * \varphi - \varphi$ with initial data $\varphi_0 \equiv \mathbf{1}_{B_R}$. Then there are $\gamma > 0$ and $m > 0$ such that*

$$\varphi(t, x) \geq \frac{\gamma}{t^{N/\beta}} \mathbf{1}_{B_{mt^{1/\beta}}}(x),$$

for $t > 0$ large enough and $x \in \mathbb{R}^N$.

Proof. This is a direct consequence of [6]. Indeed in virtue of [6, Corollary 2.1], we have

$$\begin{aligned} t^{N/\beta}\varphi(t, x) &= t^{N/\beta}\varphi\left(t, t^{1/\beta}\frac{x}{t^{1/\beta}}\right) - \|\varphi_0\|_{L^1} G_A\left(\frac{x}{t^{1/\beta}}\right) + \|\varphi_0\|_{L^1} G_A\left(\frac{x}{t^{1/\beta}}\right) \\ &= o(t) + \|\varphi_0\|_{L^1} G_A\left(\frac{x}{t^{1/\beta}}\right), \end{aligned}$$

as $t \to \infty$, where

$$G_A(y) := \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{iy\cdot\xi} e^{-A|\xi|^\beta} d\xi.$$

Noticing that $G_A(y) \to \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{-A|\xi|^\beta} d\xi > 0$ as $y \to 0$ enables to select $m > 0$ small enough so that, for any $x \in B_{mt^{1/\beta}}$, we have $\|\varphi_0\|_{L^1} G_A\left(\frac{x}{t^{1/\beta}}\right) \geq 2\gamma$ for some $\gamma > 0$. This concludes the proof of the lemma. □

Proof of Theorem 2.6. Let $u_0 : \mathbb{R}^N \to [0, 1]$ be as in Theorem 2.6, that is continuous and nontrivial. Let $\varepsilon > 0$ and $R > 0$ be given. In view of the hair trigger effect along a subsequence (44) and thanks to the comparison principle, it is enough to consider the solution $u(t, x)$ to (2) with the initial datum $u_0 \equiv (1 - \varepsilon)\mathbf{1}_{B_R}$.

Let $\varphi(t, x)$ denote the solution to $\partial_t \varphi = J * \varphi - \varphi$ with initial datum $u_0 \equiv (1 - \varepsilon)\mathbf{1}_{B_R}$. Notice that, from the comparison principle, we get $\varphi \leq u$. From Lemma 6.1 we deduce that there is $\tau_0 > 0$ such that

$$\varphi(\tau,x)\geq(1-\varepsilon)\frac{\gamma}{\tau^{N/\beta}}\mathbf{1}_{B_{m\tau^{1/\beta}}}(x)=:\Phi_0(x),\quad \forall(\tau,x)\in(\tau_0,\infty)\times\mathbb{R}^N.$$

Let $\Phi(t,x)$ denote the solution to $\partial_t\Phi=J*\Phi-\Phi$ with initial datum Φ_0. Let $U(t,x)$ denote the solution to $\partial_t U=J*U-U+U^{1+p}(1-U)$ with initial datum Φ_0. Since $U(0,x)=\Phi_0(x)\leq\varphi(\tau,x)\leq u(\tau,x)$, the comparison principle yields

$$u(\tau+t,x)\geq U(t,x),\quad \forall(t,x)\in(0,\infty)\times\mathbb{R}^N. \tag{45}$$

Next, for $X>0$, let us define

$$w(t,X):=\frac{1}{\left(X^{-p}-\varepsilon pt\right)^{1/p}}=\frac{X}{(1-\varepsilon ptX^p)^{1/p}},\quad 0<t<\frac{1}{\varepsilon pX^p},$$

that is the solution to the Cauchy problem

$$\partial_t w(t,X)=\varepsilon w^{1+p}(t,X),\quad w(0,X)=X.$$

Notice that $\partial_X w=\frac{w^{1+p}}{X^{1+p}}$ and $\partial_{XX}w=(1+p)\frac{w^{1+p}}{X^{2+2p}}(w^p-X^p)\geq 0$ so that $X\mapsto w(t,X)$ is convex.

Lemma 6.2 *(A subsolution). Function*

$$W(t,x):=w(t,\Phi(t,x))$$

is a subsolution to (2) on $(0,T)\times\mathbb{R}^N$, where time

$$T=T(\tau):=\frac{1}{\varepsilon p}\left(\frac{\tau^{pN/\beta}}{(1-\varepsilon)^p\gamma^p}-\frac{1}{(1-\varepsilon)^p}\right) \tag{46}$$

is positive for any $\tau\geq\tau_0$, up to enlarging $\tau_0>0$ if necessary.

Proof. Since $\Phi(t,x)\leq\|\Phi_0\|_{L^\infty}=\frac{(1-\varepsilon)\gamma}{\tau^{N/\beta}}$, we see that $W(t,x)\leq 1-\varepsilon$ on $(0,T)\times\mathbb{R}^N$. As a result

$$\begin{aligned}\partial_t W-(J*W-W)-W^{1+p}(1-W)&\leq\partial_t W-(J*W-W)-\varepsilon W^{1+p}\\&=\frac{W^{1+p}}{\Phi^{1+p}}(J*\Phi-\Phi)-(J*W-W),\end{aligned}$$

by a direct computation. On the other hand

$$\begin{aligned}(J*W-W)(t,x)&=\int_{\mathbb{R}^N}J(y)\left(w(t,\Phi(t,x-y))-w(t,\Phi(t,x))\right)dy\\&\geq\int_{\mathbb{R}^N}J(y)(\Phi(t,x-y)-\Phi(t,x))\frac{w^{1+p}(t,\Phi(t,x))}{\Phi^{1+p}(t,x)}dy\\&=\frac{W^{1+p}(t,x)}{\Phi^{1+p}(t,x)}(J*\Phi-\Phi)(t,x),\end{aligned}$$

where we have used the convexity of $X\mapsto w(t,X)$. This concludes the proof of the lemma. □

Since $W(0,x)=\Phi_0(x)=U(0,x)$, the comparison principle yields

$$\begin{aligned}U(T,x)\geq W(T,x)&=\left(\Phi(T,x)^{-p}-\varepsilon pT\right)^{-1/p}\\&=\left(\Phi(T,x)^{-p}-\frac{\tau^{pN/\beta}}{(1-\varepsilon)^p\gamma^p}+\frac{1}{(1-\varepsilon)^p}\right)^{-1/p},\end{aligned}$$

by definition of time T. But, in view of Section 3, one can write

$$\begin{aligned}\Phi(T,x) &= K(T,\cdot)*\Phi_0(x) = \frac{(1-\varepsilon)\gamma}{\tau^{N/\beta}} K(T,\cdot)*\mathbf{1}_{B_{m\tau^{1/\beta}}}(x)\\ &= \frac{(1-\varepsilon)\gamma}{\tau^{N/\beta}}\left(e^{-T}\mathbf{1}_{B_{m\tau^{1/\beta}}}(x)+\psi(T,\cdot)*\mathbf{1}_{B_{m\tau^{1/\beta}}}(x)\right)\\ &= \frac{(1-\varepsilon)\gamma}{\tau^{N/\beta}}\left(1-e^{-T}(1-\mathbf{1}_{B_{m\tau^{1/\beta}}}(x)) - \int_{|y|\geq m\tau^{1/\beta}} \psi(T,x-y)dy\right),\end{aligned}$$

using (16). From now, we restrict ourselves to $x \in B_R$. Hence, up to enlarging $\tau_0 > 0$ if necessary, we can get rid of the term $1 - \mathbf{1}_{B_{m\tau^{1/\beta}}}(x)$ for any $\tau \geq \tau_0$ and get

$$\Phi(T,x)^{-p} = \frac{\tau^{pN/\beta}}{(1-\varepsilon)^p\gamma^p}\left(1-\int_{|y|\geq m\tau^{1/\beta}} \psi(T,x-y)dy\right)^{-p},$$

so that

$$U(T,x) \geq \left(\frac{1}{(1-\varepsilon)^p\gamma^p}\tau^{pN/\beta}\left[\left(1-\int_{|y|\geq m\tau^{1/\beta}} \psi(T,x-y)dy\right)^{-p}-1\right]+\frac{1}{(1-\varepsilon)^p}\right)^{-1/p}. \tag{47}$$

We now need to estimate $\int_{|y|\geq m\tau^{1/\beta}} \psi(T,x-y)dy$. Again, up to enlarging $\tau_0 > 0$, we have for any $\tau \geq \tau_0$

$$\int_{|y|\geq m\tau^{1/\beta}} \psi(T,x-y)dy \leq \int_{|z|\geq \frac{m}{2}\tau^{1/\beta}} \psi(T,z)dz = e^{-T}\sum_{k=1}^{\infty}\frac{T^k}{k!}\int_{|z|\geq \frac{m}{2}\tau^{1/\beta}} J^{*(k)}(z)dz. \tag{48}$$

We use the fact that the decay of the kernel J is associated to the behavior of $\widehat{J}$ near zero. Precisely, quoting [8, Chapter 2, subsection 2.3.c, (3.5)], we get a constant $C > 0$ such that, for all $k \geq 1$,

$$\begin{aligned}\int_{|z|\geq \frac{m}{2}\tau^{1/\beta}} J^{*(k)}(z)dz &\leq C\tau^{N/\beta}\int_{|\xi|\leq \frac{1}{C\tau^{1/\beta}}} (1-\widehat{J^{*(k)}}(\xi))d\xi\\ &= C\tau^{N/\beta}\int_{|\xi|\leq \frac{1}{C\tau^{1/\beta}}} (1-\widehat{J}^k(\xi))d\xi\\ &\leq C\tau^{N/\beta}\int_{|\xi|\leq \frac{1}{C\tau^{1/\beta}}} k(1-\widehat{J}(\xi))d\xi.\end{aligned}$$

Since $1-\widehat{J}(\xi) \sim A|\xi|^\beta$ as $\xi \to 0$, it follows that, up to enlarging $\tau_0 > 0$, we have for all $\tau \geq \tau_0$ and all $k \geq 1$

$$\int_{|z|\geq \frac{m}{2}\tau^{1/\beta}} J^{*(k)}(z)dz \leq kC2A\tau^{N/\beta}\int_{|\xi|\leq \frac{1}{C\tau^{1/\beta}}} |\xi|^\beta d\xi \leq k\frac{C'}{\tau},$$

for some $C' > 0$. Plugging this into (48) we get

$$\int_{|y|\geq m\tau^{1/\beta}} \psi(T,x-y)dy \leq e^{-T}\sum_{k=1}^{+\infty}\frac{T^k}{k!}k\frac{C'}{\tau} = C'\frac{T}{\tau}. \tag{49}$$

To conclude, the key point is that, in view of (49), (46) and assumption $0 < p < \frac{1}{2}\frac{\beta}{N}$,

$$\tau^{pN/\beta}\left[\left(1 - \int_{|y|\geq m\tau^{1/\beta}} \psi(T, x-y)dy\right)^{-p} - 1\right] \to 0, \qquad \text{as } \tau \to \infty,$$

uniformly with respect to $x \in B_R$. Hence, in view of (47) and up to enlarging $\tau_0 > 0$, we have $U(T, x) \geq 1 - 2\varepsilon$ for any $\tau \geq \tau_0$, any $x \in B_R$. Hence, we deduce from (45) that

$$u(\tau + T(\tau), x) \geq 1 - 2\varepsilon, \quad \forall \tau \geq \tau_0, \forall x \in B_R,$$

which in turn implies

$$u(t, x) \geq 1 - 2\varepsilon, \quad \forall t \geq t_0 := \tau_0 + T(\tau_0), \forall x \in B_R.$$

This concludes the proof of Theorem 2.6. □

Conflict of interest statement

None declared.

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