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Ann. I. H. Poincaré – AN 34 (2017) 293–334

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

The obstacle problem with singular coefficients near Dirichlet data

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Received 8 May 2015; received in revised form 18 November 2015; accepted 4 December 2015

Available online 11 December 2015

Abstract

In this paper we study the behaviour of the free boundary close to its contact points with the fixed boundary $B \cap \{x_1 = 0\}$ in the obstacle type problem

$$\begin{cases} \operatorname{div}(x_1^a \nabla u) = \chi_{\{u>0\}} \text{ in } B^+, \\ u = 0 \text{ on } B \cap \{x_1 = 0\} \end{cases}$$

where $a < 1$, $B^+ = B \cap \{x_1 > 0\}$, B is the unit ball in $\mathbb{R}^n$ and $n \geq 2$ is an integer.

Let $\Gamma = B^+ \cap \partial\{u > 0\}$ be the free boundary and assume that the origin is a contact point, i.e. $0 \in \overline{\Gamma}$. We prove that the free boundary touches the fixed boundary uniformly tangentially at the origin, near to the origin it is the graph of a C^1 function and there is a uniform modulus of continuity for the derivatives of this function.

MSC: 35R35; 35J60

Keywords: Free boundary; Obstacle problem; Singular coefficient; Regularity of free boundaries

1. Introduction

This paper concerns the study of the obstacle problem, where singularity or degeneracy in the operator gives rise to interesting behaviour of the solutions close to such singular or degenerate points.

The classical setting of the obstacle problem asks for the smallest supersolution u over a given obstacle ψ, in a domain D, with prescribed boundary values.

The solution to this problem then (formally) satisfies

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[1] H. Shahgholian's research was partially supported by Swedish Research Council.
[2] K. Yeressian was a long-term visitor to KTH Royal Institute of Technology and would like to thank this institute for their support and hospitality. Also support by the Institut Mittag-Leffler (Djursholm, Sweden) is gratefully acknowledged.

http://dx.doi.org/10.1016/j.anihpc.2015.12.003
0294-1449/

$$\Delta u = \Delta\psi\, \chi_{\{u=\psi\}} \text{ in } D$$

which amounts to that $v = u - \psi$ satisfies

$$\Delta v = f\, \chi_{\{v>0\}} \text{ in } D$$

where $f = -\Delta\psi$.

An important ingredient in the study of such problems is the behaviour of the solution close to a free boundary point $x^0 \in \Gamma = D \cap \partial\{v > 0\}$. Indeed, any local analysis of this problem involves the standard scaling and blowup technique, i.e., the consideration of

$$v_r(x) = \frac{v(rx + x^0)}{r^\alpha}$$

where α is the (unknown) growth rate for the function v, or the rate by which the original solution u detaches from the obstacle ψ.

Classical theory has been concerned with the case when $\alpha = 2$, which is a consequence of the assumption $\lambda_0 \le f \le \Lambda_0$, for fixed $0 < \lambda_0 \le \Lambda_0$, close to the free boundary point x^0. See [1] or [6].

For those points x^0 with $f(x^0) = 0$, until very recently, no theory had been developed. For example if f is a first order homogenous function and $x^0 = 0 \in \Gamma$, then one expects v to have a cubic growth at the origin in the noncoincidence set $\{v > 0\}$, i.e. $\alpha = 3$ (see [11]).

One expects a similar phenomena in the problem

$$\operatorname{div}\big(c(x)\nabla u\big) = \chi_{\{u>0\}} \text{ in } D.$$

The classical rate $\alpha = 2$ is a consequence of the assumption $c_0 \le c(x) \le C_0$, for fixed $0 < c_0 \le C_0$, close to the free boundary point x^0. When the coefficient $c(x)$ is degenerate or singular at the free boundary point x^0 then the corresponding rate α might be different from 2.

In this paper we consider coefficients of the form $c(x) = x_1^a$ for $a < 1$. As the singularity ($a < 0$) or degeneracy ($0 < a < 1$) of the coefficient is on the set $\{x_1 = 0\}$ we are interested to study the free boundary near to a point in this set.

One may notice that when $a \le -1$, if the corresponding energy of the solution is finite then u is constant on the set $\overline{D} \cap \{x_1 = 0\}$. But if this is a positive constant then the free boundary does not come close to the set $\{x_1 = 0\}$. Thus in the case $a \le -1$, to study the free boundary close to the set $\{x_1 = 0\}$ we should assume that $u = 0$ on $\overline{D} \cap \{x_1 = 0\}$. Because of this reason we consider a problem where $u = 0$ on $\overline{D} \cap \{x_1 = 0\}$ for all $a < 1$.

Let us now present in more details the problem we are studying in this paper. For $n \ge 2$ let $D \subset \mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_1 > 0\}$ be a bounded domain, $a < 1$, the function $g \ge 0$ satisfy

$$\int_D x_1^a\big(|\nabla g|^2 + g^2\big)dx < \infty$$

and $g = 0$ on $\partial D \cap \{x_1 = 0\}$. Let u be the unique minimiser of the energy

$$\int_D \big(x_1^a|\nabla u|^2 + 2u^+\big)dx \tag{1.1}$$

satisfying $u = g$ on ∂D.

Then we have $u \in C^{1,\alpha}_{loc}(D) \cap W^{2,p}_{loc}(D)$ for all $0 < \alpha < 1$ and $1 < p < \infty$. The equation

$$\operatorname{div}(x_1^a \nabla u) = \chi_{\{u>0\}} \text{ in } D \tag{1.2}$$

is satisfied in the sense of distributions and pointwise almost everywhere.

We define the noncoincidence set Ω and the free boundary Γ as follows

$$\Omega = \{u > 0\} \text{ and } \Gamma = D \cap \partial\Omega.$$

Let us also define the contact set

$$\Gamma' = \{x_1 = 0\} \cap \overline{\Gamma}.$$

Then we are interested in the free boundary Γ near to a contact point which has a positive distance from $\partial D \cap \mathbb{R}^n_+$.

In [8] the authors have studied the case $a = 0$, but for a more general no-sign obstacle problem, i.e. $\Delta u = \chi_{\{u \neq 0\}}$.

In the case $D = B_r^+ = B_r \cap \mathbb{R}^n_+$, $B_r = \{x \in \mathbb{R}^n \mid |x| < r\}$ for some $r > 0$, by considering $v = x_1^{a-1} u$ we obtain that v solves the problem

$$\operatorname{div}(x_1^b \nabla v) = x_1^{b-1} \chi_{\{v>0\}} \tag{1.3}$$

in B_r^+, where $b = 2 - a$ and there is no Dirichlet boundary condition on $B_r \cap \{x_1 = 0\}$.

In the case of a general domain D, close to a contact point $x^0 \in \Gamma'$ which is away from $\partial D \cap \mathbb{R}^n_+$ we investigate the free boundary by considering the problem for v.

In [3] the regularity of the solution to the obstacle problem with an elliptic Heston operator has been investigated. This problem corresponds to the case $0 < a$ when no boundary condition is assumed on $\{x_1 = 0\}$.

For the range of values $-1 < a < 1$ the operator in (1.2) is rather well understood. In [4] many properties of elliptic equations for operators with coefficients in Muckenhoupt classes have been established. In the case $-1 < a < 1$ the coefficient x_1^a belongs to the Muckenhoupt class A_2. Specially to deal with the case $a \leq -1$ we consider the function v instead of u. In the appendix we establish relevant properties of solutions of the homogenous equation with $b > 1$ by first deriving a Poisson formula for the solution in the half ball $B^+ = B_1^+$.

Also for the range $-1 < a < 1$ there is a connection between the operators considered here and an extension problem associated with fractional Laplacian. In [2] it has been established that if u solves the equation

$$\begin{cases} \operatorname{div}(x_1^a \nabla u) = 0 \text{ in } \mathbb{R}^n_+, \\ u = f \text{ on } \{x_1 = 0\} \end{cases}$$

then

$$C(-\Delta)^s f = \lim_{x_1 \to +0} x_1^a \partial_{x_1} u$$

where $C > 0$ depends only on a and n, and $s = \frac{1}{2}(1 - a)$ $(0 < s < 1)$. Although we do not use this result, we use ideas developed in [2] to derive the Poisson formula for the half ball which we have computed in the appendix.

For a generalisation of this extension technique one may refer to [10].

This paper is structured as follows. In Section 2, the main notations used in this paper have been enlisted. In Section 3, the main results of this paper are presented. In Section 4, the spaces in which the existence of the solutions is established are defined, the existence and uniqueness of the solutions to the main obstacle problem is established, similarly the existence and uniqueness of the solutions to the auxiliary obstacle problem (1.3) is established and the local reduction of the main problem to the auxiliary problem is proved. In Section 5, the optimal regularity of the solution to the auxiliary problem is proved. In Section 6, the optimal nondegeneracy of the solution to the auxiliary problem is proved. In Section 7, we consider a variation of Weiss balanced energy formula and establish its main properties. In Section 8, we classify all possible global solutions. In Section 9, by a compactness argument we prove that close to a contact point away from $\partial D \cap \mathbb{R}^n_+$ the free boundary touches the fixed boundary uniformly tangentially. In Section 10, using a compactness argument together with directional monotonicity and known results about regularity of the free boundary for the classical obstacle problem at regular points we establish that the free boundary close the a contact point away from $\partial D \cap \mathbb{R}^n_+$ might be given by a C^1 graph with a uniform modulus of continuity for the derivatives. In the appendix we have gathered some technical results including key properties of the solutions of the homogenous equations $\operatorname{div}(x_1^a \nabla u) = 0$ in D with parameter $a > 1$ and no boundary condition on $\partial D \cap \{x_1 = 0\}$.

2. Notation

c, c_1, c_2, C, C_1, C_2	generic constants;
χ_D	characteristic function of the set D ($D \subset \mathbb{R}^n$);
$\overline{D}$	the closure of D;
∂D	boundary of D;
D°	interior of D;
$\lvert \cdot \rvert$	absolute value, length of a vector, norm of a matrix, Lebesgue measure or surface measure;
$\lVert \cdot \rVert$	norm of functions;
$[\cdot]$	seminorm of functions;
x, x'	$x = (x_1, \cdots, x_n)$, $x' = (x_2, \cdots, x_n)$;
$\mathbb{R}^n_+$	$\{x \in \mathbb{R}^n \mid x_1 > 0\}$;
$B^m_r(x)$	$\{y \in \mathbb{R}^m \mid \lvert y - x\rvert < r\}$;
$B_r(x)$, $B(x)$, B	$B^n_r(x)$, $B^n_1(x)$, $B^n_1(0)$;
$B^+_r(x)$	$B_r(x) \cap \mathbb{R}^n_+$;
$\{x_1 \geq a\}$, $\{x_1 = a\}$	$\{x \in \mathbb{R}^n \mid x_1 \geq a\}$, $\{x \in \mathbb{R}^n \mid x_1 = a\}$;
$e_1, \cdots, e_n$	standard basis of $\mathbb{R}^n$;
e, υ	arbitrary unit vectors, outward normals are denoted by υ;
$e \perp e_1$	e is orthogonal to e_1;
$\partial_e f$	directional derivative of f (in the direction e);
f^+	$\max(0, f)$;
$C_b(D)$	functions in $C(D)$ with finite $\lVert \cdot \rVert_{C(D)}$ norm;
$C^1_b(D)$	functions in $C^1(D)$ with finite $\lVert \cdot \rVert_{C^1(D)}$ norm;
$\subset\subset$	compactly contained.

3. Main results

In the rest of this paper (except in the appendix) we have $a < 1$ and $b = 2 - a > 1$. Let us introduce an auxiliary obstacle problem. Let the function $g \geq 0$ satisfy

$$\int_D x_1^b \big(\lvert\nabla g\rvert^2 + g^2\big) dx < \infty.$$

Let us note that here we do not demand that $g = 0$ on $\partial D \cap \{x_1 = 0\}$.

Let v be the unique minimiser of the energy

$$\int_D \big(x_1^b \lvert\nabla v\rvert^2 + 2x_1^{b-1} v^+\big) dx \tag{3.1}$$

satisfying $v = g$ on $\partial D \cap \mathbb{R}^n_+$. Let us note that here we do not demand that $v = g$ on $\partial D \cap \{x_1 = 0\}$.

Then we have $v \in C^{1,\alpha}_{loc}(D) \cap W^{2,p}_{loc}(D)$ for all $0 < \alpha < 1$ and $1 < p < \infty$. The equation (1.3) is satisfied in the sense of distributions and pointwise almost everywhere in D.

For the solution v to the auxiliary obstacle problem we define the noncoincidence set Ω, the free boundary Γ and the contact set Γ' similarly as we defined them for the main obstacle problem.

In the following lemma, for domains $D = B^+_r$, we reduce the main obstacle problem (1.2) to the auxiliary obstacle problem (1.3).

Lemma 1. *Let $r > 0$. If u is a solution of the obstacle problem (1.2) in B^+_r then defining $v = x_1^{a-1} u$ we have that v is a solution of the obstacle problem (1.3) in B^+_r.*

By Lemma 1 to study the structure of the free boundary away from $\partial D \cap \mathbb{R}^n_+$ we might study the free boundary arising in the obstacle problem (1.3) where there is no fixed boundary condition on $\partial D \cap \{x_1 = 0\}$.

In the following theorem we prove the optimal growth of solutions. This optimal growth is the basis of optimal regularity estimates for the solution which results in uniform estimates for different blowup sequences.

Theorem 1 *(Optimal growth). There exists $C > 0$ such that if v is a solution of the obstacle problem (1.3) in D and $B^+_r(x^0) \subset D$ then we have*

$$v(x) \le C\Big(\inf_{B^+_{\frac{r}{8}}(x^0)} v + \frac{r^2}{r + x^0_1}\Big) \text{ for } x \in B^+_{\frac{r}{8}}(x^0). \tag{3.2}$$

In the following theorem we prove the optimal nondegeneracy of the solutions. Using the optimal nondegeneracy we are able to rule out trivial blowup limits.

Theorem 2 *(Optimal nondegeneracy). There exists a $c > 0$ such that if v is a solution of the obstacle problem (1.3) in D then for $x^0 \in \Omega$ and $B^+_r(x^0) \subset\subset D$ we have*

$$\sup_{\Omega \cap \partial B_r(x^0)} v \ge v(x^0) + \frac{cr^2}{r + x^0_1}. \tag{3.3}$$

Definition 1. We call v a global solution if it is a function defined on $\mathbb{R}^n_+$ such that it solves the obstacle problem (1.3) in each B^+_R for $R > 0$, $v \in C^1(\mathbb{R}^n_+) \cap C^{1,1}_{loc}(\mathbb{R}^n_+)$ and there exists $C \ge 0$ such that

$$\|v\|_{C(B^+_R)} \le C(1 + R), \; [v]_{C^1(B^+_R)} \le C \text{ for } R > 0 \tag{3.4}$$

and

$$[v]_{C^{1,1}(B_R \cap \{x_1 > \delta\})} \le \frac{C}{\delta} \text{ for } R > 0 \text{ and } \delta > 0. \tag{3.5}$$

We denote by P_∞ the set of all global solutions.

For $t_0 > 0$ and $t > 0$ let us define

$$w_{t_0}(t) = \frac{t}{b} - \frac{t_0}{b-1} + \frac{t_0^b}{b(b-1)} \frac{1}{t^{b-1}}. \tag{3.6}$$

In the following theorem by a novel method based on shrinkdowns we are able to classify all the global solutions.

Theorem 3 *(Classification of global solutions). We have*

$$P_\infty = \{0\} \cup \Big\{\frac{x_1}{b} + c \;\Big|\; c \ge 0\Big\} \cup \Big\{w_{t_0}(x_1)\chi_{\{x_1 > t_0\}} \;\Big|\; t_0 > 0\Big\}.$$

In the following theorem we prove the uniform C^1 regularity of the free boundary near contact points away from $\partial D \cap \mathbb{R}^n_+$. The proof of this result is based on the theorems mentioned above, compactness arguments, directional monotonicity and known regularity of the free boundary in the classical obstacle problem near regular points.

Theorem 4 *(C^1 regularity of the free boundary). There exists $0 < r < \frac{1}{2}$ and a modulus of continuity σ such that for v solution of the obstacle problem (1.3) in B^+ such that $0 \in \Gamma'$ there exists $g \in C^1(B^{n-1}_r)$ such that $0 \le g < r$, $g(0) = 0$, $0 \in \partial\{g > 0\}$,*

$$\Omega \cap \big((0, r) \times B^{n-1}_r\big) = \Big\{(x_1, x') \;\Big|\; g(x') < x_1 < r, \, x' \in B^{n-1}_r\Big\}$$

and σ is a modulus of continuity for $\nabla_{x'} g$ in B^{n-1}_r.

4. Preliminary analysis

4.1. Spaces

Let $n \geq 2$ be an integer and $D \subset \mathbb{R}^n_+$ be a bounded domain.

Let us define for $a \in \mathbb{R}$ and $u \in H^1_{loc}(D)$

$$\|u\|^2_{H^1(D;x_1^a)} = \int_D x_1^a \big(|\nabla u|^2 + u^2\big) dx.$$

We define $H^1_0(D; x_1^a, \{x_1 = 0\})$ and $H^1_0(D; x_1^a)$ respectively as the completion of $C^\infty_c(\mathbb{R}^n_+)$ and $C^\infty_c(D)$ with respect to the norm $\|\cdot\|_{H^1(D;x_1^a)}$.

For $a > -1$ we also define $H^1(D; x_1^a)$ and $H^1_0(D; x_1^a, \partial D \cap \mathbb{R}^n_+)$ as the completion of $C^\infty(\mathbb{R}^n)$ and $C^\infty_c\big((\overline{D} \cup \{x_1 \leq 0\})^\circ\big)$ with respect to the norm $\|\cdot\|_{H^1(D;x_1^a)}$.

One may check that for $a \in \mathbb{R}$, $H^1_0(D; x_1^a, \{x_1 = 0\})$ and $H^1_0(D; x_1^a)$ are separable Hilbert spaces and $H^1_0(D; x_1^a)$ is a closed linear subspace of $H^1_0(D; x_1^a, \{x_1 = 0\})$.

Similarly for $a > -1$, $H^1(D; x_1^a)$ and $H^1_0(D; x_1^a, \partial D \cap \mathbb{R}^n_+)$ are separable Hilbert spaces and $H^1_0(D; x_1^a; \{x_1 = 0\})$ and $H^1_0(D; x_1^a, \partial D \cap \mathbb{R}^n_+)$ are closed linear subspaces of $H^1(D; x_1^a)$.

By the boundedness of D there exists a $C > 0$ such that if $a \in \mathbb{R}$ and $u \in C^\infty_c\big((\overline{D} \cup \{x_1 \leq 0\})^\circ\big)$ then the following Poincaré type inequality holds

$$\int_D x_1^a u^2 dx \leq C \int_D x_1^a |\nabla u|^2 dx. \tag{4.1}$$

Let us note that for $a \leq -1$ the integral on the right hand side might diverge. By the definition of $H^1_0(D; x_1^a)$, $C^\infty_c(D)$ is dense in $H^1_0(D; x_1^a)$ with respect to the norm $\|\cdot\|_{H^1(D;x_1^a)}$, thus from (4.1) by a density argument we obtain that (4.1) holds for $u \in H^1_0(D; x_1^a)$. Similarly (4.1) holds for $a > -1$ and $u \in H^1_0(D; x_1^a, \partial D \cap \mathbb{R}^n_+)$.

By Lemma 12 for $a > 1$, $C^\infty_c(\mathbb{R}^n_+)$ is dense in $H^1(D; x_1^a)$ with respect to the norm $\|\cdot\|_{H^1(D;x_1^a)}$ and it follows that $H^1_0(D; x_1^a, \{x_1 = 0\}) = H^1(D; x_1^a)$. Also similarly we have the for $a > 1$, $C^\infty_c(D)$ is dense in $H^1_0(D; x_1^a; \partial D \cap \mathbb{R}^n_+)$ with respect to the norm $\|\cdot\|_{H^1(D;x_1^a)}$ and it follows that $H^1_0(D; x_1^a) = H^1_0(D; x_1^a, \partial D \cap \mathbb{R}^n_+)$.

Let $a \in \mathbb{R}$ and $r > 0$ then one may see there exists $C > 0$ such that for $u \in C^\infty_c(\mathbb{R}^n)$ we have

$$\int_{\partial B_r \cap \mathbb{R}^n_+} x_1^a u^2 s(dx) \leq C \int_{B_r^+} x_1^a \big(u^2 + |\nabla u|^2\big) dx. \tag{4.2}$$

Let us note that for $a \leq -1$ the integral on the right hand side might diverge. By a density argument for $a \in \mathbb{R}$ from (4.2) we obtain that there exists a bounded trace operator from $H^1_0(B_r^+; x_1^a, \{x_1 = 0\})$ to $L^2(\partial B_r \cap \mathbb{R}^n_+; x_1^a)$. Similarly for $a > -1$ there exists a bounded trace operator from $H^1(B_r^+; x_1^a)$ to $L^2(\partial B_r \cap \mathbb{R}^n_+; x_1^a)$.

Let $a < 1$ and $b = 2 - a$. Let $r > 0$ then for $u \in H^1_0(B_r^+; x_1^a; \{x_1 = 0\})$ defining $v = x_1^{a-1} u$ we have that $v \in H^1(B_r^+; x_1^b)$, the map from u to v is bijective and bounded (together with its inverse). Also this map from u to v, maps $H^1_0(B_r^+; x_1^a)$ to $H^1_0(B_r^+; x_1^b; \partial B_r \cap \mathbb{R}^n_+)$ and this restricted map is again bijective and bounded (together with its inverse).

4.2. The main obstacle problem

Let $a < 1$. For $u \in C^\infty_c(\mathbb{R}^n_+)$ we have

$$\Big|\int_D u dx\Big| = \Big|\int_D x_1^{-\frac{a}{2}} x_1^{\frac{a}{2}} u dx\Big| \leq \Big(\int_D x_1^{-a} dx\Big)^{\frac{1}{2}} \Big(\int_D x_1^a u^2 dx\Big)^{\frac{1}{2}}$$

and by the boundedness of D and $a<1$ we have $\int_D x_1^{-a}dx<\infty$. It follows that the integral $\int_D u dx$ for $u\in H_0^1(D;x_1^a)$ is a bounded linear functional on $H_0^1(D;x_1^a)$.

Using the boundedness of the linear functional mentioned above and the Poincaré inequality (4.1) for each $g\in H_0^1(D;x_1^a,\{x_1=0\})$ such that $g\geq 0$ a.e. in D there exists a unique minimiser of (1.1) among the admissible set of functions

$$\Big\{u\in H_0^1(D;x_1^a,\{x_1=0\})\ \Big|\ u=g \text{ on } \partial D\Big\}.$$

By a similar reasoning as in [6] and using the L^p (cf. [5]) estimates for elliptic equations with variable coefficients we obtain that $u\in W^{2,p}_{loc}(D)$ for all $1<p<\infty$ and (1.2) holds in the sense of distributions and pointwise a.e. in D. Also from Sobolev imbeddings it follows that $u\in C^{1,\alpha}_{loc}(D)$ for all $0<\alpha<1$.

Conversely if $u\in H_0^1(D;x_1^a,\{x_1=0\})$ and (1.2) holds in the sense of distributions then we have

$$\int_D \big(x_1^a\nabla u\cdot\nabla\varphi+\chi_{\{u>0\}}\varphi\big)dx=0 \ \text{ for }\ \varphi\in C_c^\infty(D). \tag{4.3}$$

Now let $v\in H_0^1(D;x_1^a,\{x_1=0\})$ such that $u=v$ on ∂D, i.e. $v-u\in H_0^1(D;x_1^a)$, and $v\geq 0$ a.e. in D. Then by a density argument from (4.3) we obtain

$$\int_D \big(x_1^a\nabla u\cdot\nabla(v-u)+\chi_{\{u>0\}}(v-u)\big)dx=0. \tag{4.4}$$

From (4.4) because $v\geq 0$ a.e. in D we obtain that

$$\int_D \big(x_1^a\nabla u\cdot\nabla(v-u)+(v-u)\big)dx\geq 0. \tag{4.5}$$

Because (4.5) holds for all v in the admissible set, we obtain that u satisfies the variational inequality formulation of the obstacle problem (1.2) and thus is the unique solution with its values on ∂D as boundary condition.

4.3. The auxiliary obstacle problem

Let $b>1$ then reasoning similarly as for the main obstacle problem discussed above for $g\in H^1(D;x_1^b)$ such that $g\geq 0$ a.e. in D there exists a unique minimiser of (3.1) among the admissible set of functions

$$\Big\{v\in H^1(D;x_1^b)\ \Big|\ v=g \text{ on } \partial D\cap\mathbb{R}^n_+\Big\}.$$

Also we have $v\in W^{2,p}_{loc}(D)$ for all $1<p<\infty$ and (1.3) holds in the sense of distributions and pointwise a.e. in D. From Sobolev imbeddings it follows that $v\in C^{1,\alpha}_{loc}(D)$ for all $0<\alpha<1$.

If $v\in H^1(D;x_1^b)$ and (1.3) holds in the sense of distributions then v is the unique minimiser of (3.1) with its values on $\partial D\cap\mathbb{R}^n_+$ as boundary condition.

4.4. Locally reducing the main problem to the auxiliary problem

When we say that u is a solution of the obstacle problem (1.2) we mean that $u\in H_0^1(D;x_1^a,\{x_1=0\})$, $u\geq 0$ a.e. in D and (1.2) holds in the sense of distributions. Similarly when we say that v is solution of the obstacle problem (1.3) we mean that $v\in H^1(D;x_1^b)$, $v\geq 0$ a.e. in D and (1.3) holds in the sense of distributions.

Proof of Lemma 1. As mentioned in the subsection 4.1, because $u\in H_0^1(B_r^+;x_1^a,\{x_1=0\})$ we have that $v\in H^1(B_r^+;x_1^b)$. For $\varphi\in C_c^\infty(D)$ we compute

$$\begin{aligned}
\int_D x_1^b \nabla v \cdot \nabla \varphi dx &= \int_D x_1^b \nabla (x_1^{a-1} u) \cdot \nabla \varphi dx \\
&= \int_D x_1^b \big((a-1) x_1^{a-2} u e_1 + x_1^{a-1} \nabla u\big) \cdot \nabla \varphi dx \\
&= \int_D (1-b) u \partial_{x_1} \varphi dx + \int_D x_1 \nabla u \cdot \nabla \varphi dx \\
&= \int_D (b-1) \partial_{x_1} u \varphi dx + \int_D x_1 \nabla u \cdot \nabla \varphi dx \\
&= \int_D x_1^a \nabla u \cdot \nabla (x_1^{b-1} \varphi) dx = -\int_D \chi_{\{u>0\}} x_1^{b-1} \varphi dx \\
&= -\int_D \chi_{\{v>0\}} x_1^{b-1} \varphi dx
\end{aligned}$$

which proves that (1.3) holds in the sense of distributions and this completes the proof of the lemma. □

In the rest of this paper we will study the obstacle problem (1.3).

5. Optimal regularity and proof of Theorem 1

The function $w_{t_0}(t)$ was defined in (3.6).

Lemma 2. *Let $b > 1$ and $t_0 > 0$ then*

$$w_{t_0}(t_0) = w'_{t_0}(t_0) = 0, \tag{5.1}$$

$$w_{t_0}(t) > 0 \ \text{ for } \ t \in (0, t_0) \cup (t_0, \infty) \tag{5.2}$$

and there exists $C > 0$ (depending only on b) such that for $t > \frac{1}{4} t_0$ we have

$$w_{t_0}(t) \le \frac{C}{t_0} (t - t_0)^2 \ \text{ and } \ w''_{t_0}(t) \le \frac{C}{t_0}. \tag{5.3}$$

Proof. We have

$$w_{t_0}(t) = t_0 w_1 (\frac{t}{t_0}). \tag{5.4}$$

Let $\lambda = \frac{t}{t_0}$ then we have

$$w_1(\lambda) = \frac{\lambda}{b} - \frac{1}{b-1} + \frac{1}{b(b-1)} \frac{1}{\lambda^{b-1}}$$

and computing we obtain

$$w_1(1) = w'_1(1) = 0 \tag{5.5}$$

and

$$w''_1(\lambda) = \frac{1}{\lambda^{b+1}}. \tag{5.6}$$

By (5.4) and (5.5) we obtain (5.1). By (5.4), (5.5) and (5.6) we obtain (5.2).
Now assume $t > \frac{1}{4} t_0$. We have $\lambda > \frac{1}{4}$ thus

$$w''_1(\lambda) = \frac{1}{\lambda^{b+1}} < 4^{b+1}. \tag{5.7}$$

By (5.4) and (5.7) the second inequality in (5.3) follows.

By Taylor remainder formula, (5.5) and (5.7) we obtain

$$w_1(\lambda) \le \frac{1}{2}4^{b+1}(\lambda - 1)^2. \tag{5.8}$$

By (5.4) and (5.8) the first inequality in (5.3) is also proved and this finishes the proof of the lemma. □

Proof of Theorem 1. We consider two cases, $x_1^0 \le \frac{r}{4}$ and $x_1^0 > \frac{r}{4}$.

Case $x_1^0 \le \frac{r}{4}$. Let us define $x^1 = x^0 - x_1^0 e_1$, then we have

$$B^+_{\frac{3}{4}r}(x^1) \subset B^+_{\frac{3}{4}r+|x^1-x^0|}(x^0) = B^+_{\frac{3}{4}r+x_1^0}(x^0) \subset B^+_{\frac{3}{4}r+\frac{r}{4}}(x^0) = B^+_r(x^0).$$

Let us define

$$\phi(x) = \frac{1}{b}(x_1 - \frac{3}{4}r)$$

then we have

$$\operatorname{div}(x_1^b \nabla\phi) = x_1^{b-1} \ \text{ for } \ x_1 > 0$$

and

$$\phi \le 0 \ \text{ in } \ B^+_{\frac{3}{4}r}(x^1).$$

Let us decompose

$$v = v_1 + v_2 \ \text{ in } \ B^+_{\frac{3}{4}r}(x^1)$$

where

$$\begin{cases} \operatorname{div}(x_1^b \nabla v_1) = x_1^{b-1}\chi_{\{v>0\}} \ \text{ in } \ B^+_{\frac{3}{4}r}(x^1), \\ v_1 = 0 \ \text{ on } \ \partial B_{\frac{3}{4}r}(x^1) \cap \mathbb{R}^n_+ \end{cases}$$

and

$$\begin{cases} \operatorname{div}(x_1^b \nabla v_2) = 0 \ \text{ in } \ B^+_{\frac{3}{4}r}(x^1), \\ v_2 = v \ \text{ on } \ \partial B_{\frac{3}{4}r}(x^1) \cap \mathbb{R}^n_+. \end{cases} \tag{5.9}$$

Because of

$$-\operatorname{div}(x_1^b \nabla\phi) = -x_1^{b-1} \le -x_1^{b-1}\chi_{\{v>0\}} = -\operatorname{div}(x_1^b \nabla v_1) \le 0 \ \text{ in } \ B^+_{\frac{3}{4}r}(x^1)$$

and

$$\phi \le 0 = v_1 \ \text{ on } \ \partial B_{\frac{3}{4}r}(x^1) \cap \mathbb{R}^n_+$$

by maximum principle we obtain

$$\phi \le v_1 \le 0 \ \text{ in } \ B^+_{\frac{3}{4}r}(x^1). \tag{5.10}$$

Because $v \ge 0$ by maximum principle and (5.9) we obtain that $v_2 \ge 0$. Also this follows from $v \ge 0$ and the second inequality in (5.10).

We compute

$$B^+_{\frac{r}{8}}(x^0) \subset B^+_{\frac{r}{8}+|x^0-x^1|}(x^1) = B^+_{\frac{r}{8}+x_1^0}(x^1) \subset B^+_{\frac{r}{8}+\frac{r}{4}}(x^1) = B^+_{\frac{3}{8}r}(x^1).$$

Now by Harnack inequality Lemma 19 in $B^+_{\frac{3}{8}r}(x^1)$ we have

$$\begin{aligned}\sup_{B^+_{\frac{r}{8}}(x^0)} v &= \sup_{B^+_{\frac{r}{8}}(x^0)} (v_1+v_2) \le \sup_{B^+_{\frac{r}{8}}(x^0)} v_2 \le \sup_{B^+_{\frac{3}{8}r}(x^1)} v_2 \le C \inf_{B^+_{\frac{3}{8}r}(x^1)} v_2\\ &\le C \inf_{B^+_{\frac{r}{8}}(x^0)} v_2 = C \inf_{B^+_{\frac{r}{8}}(x^0)} (v-v_1) \le C \inf_{B^+_{\frac{r}{8}}(x^0)} (v-\phi)\\ &= C \inf_{B^+_{\frac{r}{8}}(x^0)} \big(v - \frac{1}{b}(x_1 - \frac{3}{4}r)\big) \le C \inf_{B^+_{\frac{r}{8}}(x^0)} \big(v + \frac{3}{4}\frac{r}{b}\big)\\ &= C\big(\inf_{B^+_{\frac{r}{8}}(x^0)} v + \frac{3}{4}\frac{r}{b}\big) \le C_1\big(\inf_{B^+_{\frac{r}{8}}(x^0)} v + r\big).\end{aligned}$$

Thus we have proved that there exists $C>0$ (independent of v, r and x^0) such that in the case $x^0_1 \le \frac{r}{4}$ we have

$$v(x) \le C\big(\inf_{B^+_{\frac{r}{8}}(x^0)} v + r\big) \ \text{ for } \ x \in B^+_{\frac{r}{8}}(x^0). \tag{5.11}$$

Case $x^0_1 > \frac{r}{4}$. We define

$$\phi(x) = w_{x^0_1}(x_1) - \frac{C}{x^0_1}(\frac{3}{16}r)^2$$

here $C>0$ is as in (5.3). Then we have

$$\operatorname{div}(x_1^b \nabla\phi) = x_1^{b-1} \ \text{ for } \ x_1 > 0.$$

For $x \in B_{\frac{3}{16}r}(x^0)$ we have

$$x_1 > x^0_1 - \frac{3}{16}r = x^0_1 - \frac{3}{4}\frac{r}{4} > x^0_1 - \frac{3}{4}x^0_1 = \frac{1}{4}x^0_1$$

thus by (5.3) we have

$$\phi(x) = w_{x^0_1}(x_1) - \frac{C}{x^0_1}(\frac{3}{16}r)^2 \le \frac{C}{x^0_1}(x_1 - x^0_1)^2 - \frac{C}{x^0_1}(\frac{3}{16}r)^2 \le 0$$

i.e.

$$\phi \le 0 \ \text{ in } \ B_{\frac{3}{16}r}(x^0).$$

Similarly as in the previous case we decompose $v = v_1 + v_2$, but in the current case we consider the domain $B_{\frac{3}{16}r}(x^0)$.

Let us define

$$w(y) = v_2\big(x^0_1 y + (x^0)'\big) \ \text{ for } \ y \in B_{\frac{3}{16}\frac{r}{x^0_1}}(e_1).$$

We have

$$\operatorname{div}(x_1^b \nabla w) = 0 \ \text{ in } \ B_{\frac{3}{16}\frac{r}{x^0_1}}(e_1) \ \text{ and } \ \frac{1}{8}\frac{r}{x^0_1} < \frac{3}{16}\frac{r}{x^0_1} < \frac{3}{16}\frac{r}{\frac{r}{4}} = \frac{3}{4}.$$

Our operator is uniformly elliptic with variable coefficients in the domain $B_{\frac{3}{4}}(e_1)$. By Harnack inequality we obtain

$$\sup_{B_{\frac{1}{8}\frac{r}{x^0_1}}(e_1)} w \le C_1 \inf_{B_{\frac{1}{8}\frac{r}{x^0_1}}(e_1)} w.$$

By the definition of w it follows that

$$\sup_{B_{\frac{r}{8}}(x^0)} v_2 \le C_1 \inf_{B_{\frac{r}{8}}(x^0)} v_2.$$

Now we can compute

$$\begin{aligned}
\sup_{B_{\frac{r}{8}}(x^0)} v &= \sup_{B_{\frac{r}{8}}(x^0)} (v_1+v_2) \le \sup_{B_{\frac{r}{8}}(x^0)} v_2 \le C_1 \inf_{B_{\frac{r}{8}}(x^0)} v_2 \\
&= C_1 \inf_{B_{\frac{r}{8}}(x^0)} (v-v_1) \le C_1 \inf_{B_{\frac{r}{8}}(x^0)} (v-\phi) \\
&= C_1 \inf_{B_{\frac{r}{8}}(x^0)} \Big(v - \big(w_{x_1^0}(x_1) - \frac{C}{x_1^0}(\frac{3}{16}r)^2\big)\Big) \\
&\le C_1 \inf_{B_{\frac{r}{8}}(x^0)} \Big(v + \frac{C}{x_1^0}(\frac{3}{16}r)^2\Big) \\
&\le C_2 \inf_{B_{\frac{r}{8}}(x^0)} \big(v + \frac{r^2}{x_1^0}\big) \le C_2\big(\inf_{B_{\frac{r}{8}}(x^0)} v + \frac{r^2}{x_1^0}\big).
\end{aligned}$$

Thus we have proved that there exists $C>0$ (independent of v, r and x^0) such that in the case $x_1^0 > \frac{r}{4}$ we have

$$v(x) \le C\big(\inf_{B_{\frac{r}{8}}(x^0)} v + \frac{r^2}{x_1^0}\big) \quad \text{for } x \in B_{\frac{r}{8}}(x^0). \tag{5.12}$$

One may see that there exists $C_3>0$ such that

$$r\chi_{\{x_1^0 \le \frac{1}{4}r\}} + \frac{r^2}{x_1^0}\chi_{\{x_1^0 > \frac{1}{4}r\}} \le \frac{C_3 r^2}{r+x_1^0}. \tag{5.13}$$

From (5.11), (5.12) and (5.13) we obtain (3.2). □

Remark 1. Let us note that from Theorem 1 it follows that the solution has a linear bound on its growth away from a contact point (which are by definition in $\{x_1=0\}$) and that it has a usual quadratic bound on its growth away from free boundary points (which are by definition in $\mathbb{R}^n_+$ because $\Gamma \subset D \subset \mathbb{R}^n_+$).

Corollary 1. *There exists a $C>0$ such that if $R \ge 16$ and v is a solution of the obstacle problem (1.3) in B_R^+ with $\Gamma \cap B_2^+ \neq \emptyset$ then we have*

$$v(x) \le C(1+|x|) \ \textit{for} \ x \in B^+_{\frac{R}{8}}.$$

Proof. By the previous theorem choosing $16 \le r \le R$ we have

$$v(x) \le C\big(\inf_{B^+_{\frac{r}{8}}} v + r\big) \le C\big(\inf_{B_2^+} v + r\big) = Cr \quad \text{for } x \in B^+_{\frac{r}{8}}. \tag{5.14}$$

Taking $r=16$ in (5.14) we obtain

$$v(x) \le 16C \quad \text{for } x \in B_2^+. \tag{5.15}$$

For $x \in B^+_{\frac{R}{8}} \backslash B_2^+$ taking $r = 8|x|$ in (5.14) we obtain

$$v(x) \le 8C|x| \quad \text{for } x \in B^+_{\frac{R}{8}} \backslash B_2^+. \tag{5.16}$$

Combining (5.15) and (5.16) the corollary is proved. □

Lemma 3. *There exists a $C>0$ such that if v is a solution of the obstacle problem (1.3) in D, $x^0 \in D$, $d=$ $\mathrm{dist}(x^0, D\backslash\Omega)$, $B^+_{8d}(x^0) \subset D$ then*

$$|\nabla v(x^0)| \le \frac{Cd}{d+x_1^0}. \tag{5.17}$$

Proof. If $x^0 \in D\backslash\Omega$ then we have $\nabla v(x^0) = 0$ and clearly (5.17) holds.

Now assume $x^0 \in \Omega$ then we have $d = \mathrm{dist}(x^0, D\backslash\Omega) = \mathrm{dist}(x^0, \Gamma)$.

Because $B^+_{8d}(x^0) \subset D$ and $\inf_{B^+_d(x^0)} v = 0$ by Theorem 1 we have

$$\sup_{B^+_d(x^0)} v \leq \frac{Cd^2}{d + x^0_1}. \tag{5.18}$$

Let us define $\phi_{x^0}(x) = b^{-1}(x_1 - x^0_1)$. Now using $B^+_d(x^0) \subset \Omega$, (5.18) and Lemma 20 we compute

$$\begin{aligned}
|\nabla v(x^0)| &= |\nabla(v - \phi_{x^0})(x^0) + \nabla\phi_{x^0}(x^0)| \\
&\leq |\nabla(v - \phi_{x^0})(x^0)| + |\nabla\phi_{x^0}(x^0)| \\
&\leq \frac{C_1}{d} \sup_{B^+_d(x^0)} |v - \phi_{x^0}| + \frac{1}{b} \\
&\leq \frac{C_1}{d} \sup_{B^+_d(x^0)} v + \frac{C_1}{d} \sup_{B^+_d(x^0)} |\phi_{x^0}| + \frac{1}{b} \\
&\leq \frac{C_1}{d} \frac{Cd^2}{d + x^0_1} + \frac{C_1}{d}\frac{d}{b} + \frac{1}{b} = \frac{C_2 d}{d + x^0_1} + C_3.
\end{aligned} \tag{5.19}$$

If $d < \frac{3}{4}x^0_1$ then for $x \in B_d(x^0)$ we have $\frac{1}{4}x^0_1 < x_1$ thus by Lemma 2 we have

$$0 \leq w_{x^0_1}(x_1) \leq \frac{C_4}{x^0_1} d^2. \tag{5.20}$$

In the case $d < \frac{3}{4}x^0_1$ using $B_d(x^0) \subset \Omega$, (5.18), (5.20) and Lemma 20 we compute

$$\begin{aligned}
|\nabla v(x^0)| &= |\nabla(v - w_{x^0_1})(x^0)| \leq \frac{C_1}{d} \sup_{B_d(x^0)} |v - w_{x^0_1}| \\
&\leq \frac{C_1}{d} \sup_{B_d(x^0)} v + \frac{C_1}{d} \sup_{B_d(x^0)} w_{x^0_1} \\
&\leq \frac{C_1}{d} \frac{Cd^2}{d + x^0_1} + \frac{C_1}{d}\frac{C_4}{x^0_1} d^2 = \frac{C_2 d}{d + x^0_1} + C_5 \frac{d}{x^0_1}.
\end{aligned} \tag{5.21}$$

By (5.19) and (5.21) we have

$$|\nabla v(x^0)| \leq \Big(\frac{C_2 d}{d + x^0_1} + C_3\Big)\chi_{\{d \geq \frac{3}{4}x^0_1\}} + \Big(\frac{C_2 d}{d + x^0_1} + C_5 \frac{d}{x^0_1}\Big)\chi_{\{d < \frac{3}{4}x^0_1\}} \leq \frac{C_6 d}{d + x^0_1}$$

and this proves the lemma. $\square$

Corollary 2. *There exists a $C > 0$ such that if $R \geq 32$, v is a solution of the obstacle problem (1.3) in B^+_R and $B^+_2\backslash\Omega \neq \emptyset$ then we have*

$$|\nabla v(x)| \leq C \text{ for } x \in B^+_{\frac{R}{18}}.$$

Proof. Let $x^0 \in B^+_{\frac{R}{18}}$. Because $B^+_2\backslash\Omega \neq \emptyset$ we have

$$d = \mathrm{dist}(x^0, B^+_R\backslash\Omega) \leq \mathrm{dist}(x^0, B^+_2\backslash\Omega) \leq |x^0| + 2. \tag{5.22}$$

We compute

$$B^+_{8d}(x^0) \subset B^+_{8d+|x^0|} \subset B^+_{8(|x^0|+2)+|x^0|} = B^+_{16+9|x^0|} \subset B^+_{16+9\frac{R}{18}} = B^+_{16+\frac{R}{2}} \subset B^+_R.$$

Thus by Lemma 3 we have

$$|\nabla v(x^0)| \le \frac{Cd}{d + x_1^0} \le C$$

which proves the corollary. □

Corollary 3. *There exists a $C > 0$ such that if $0 < r < 1$, v is a solution of the obstacle problem (1.3) in $B_r(e_1)$ and $v(e_1) = 0$ then we have*

$$|\nabla v(x)| \le C|x - e_1| \text{ for } x \in B_{\frac{r}{9}}(e_1).$$

Proof. Let $x^0 \in B_{\frac{r}{9}}(e_1)$ and $d = \operatorname{dist}(x^0, B_r(e_1)\backslash\Omega)$. We compute

$$B^+_{8d}(x^0) \subset B^+_{8d+|x^0-e_1|}(e_1) \subset B^+_{8|x^0-e_1|+|x^0-e_1|}(e_1) = B^+_{9|x^0-e_1|}(e_1) \subset B^+_r(e_1).$$

Thus by Lemma 3 we have

$$|\nabla v(x^0)| \le \frac{Cd}{d + x_1^0} \le \frac{Cd}{x_1^0} \le \frac{Cd}{1 - \frac{r}{9}} \le \frac{Cd}{1 - \frac{1}{9}} = C_1 d \le C_1|x^0 - e_1|$$

and this proves the corollary. □

Lemma 4. *There exists a $C > 0$ such that if v is a solution of the obstacle problem (1.3) in D, $y \in D$, $d = \operatorname{dist}(y, D\backslash\Omega)$, $4r < d + y_1$ and $B^+_{8d+9r}(y) \subset D$ then*

$$[v]_{C^{1,1}(B_r^+(y))} \le \frac{C}{d + y_1}.$$

Proof. For $z \in D$ we denote $d(z) = \operatorname{dist}(z, D\backslash\Omega)$. Because $4r < d(y) + y_1$ we have

$$\frac{1}{d(y) + y_1 - 2r} < \frac{2}{d(y) + y_1}.$$

For $x \in B_r^+(y)$ we have

$$d(x) + x_1 > d(y) + y_1 - 2r.$$

Thus we obtain that for $x \in B_r^+(y)$

$$\frac{1}{d(x) + x_1} < \frac{1}{d(y) + y_1 - 2r} < \frac{2}{d(y) + y_1}. \tag{5.23}$$

Also for $x \in B_r^+(y)$ we have

$$B^+_{8d(x)}(x) \subset B^+_{8d(x)+|x-y|}(y) \subset B^+_{8d(y)+9|x-y|}(y) \subset B^+_{8d(y)+9r}(y) \subset D. \tag{5.24}$$

We should show that there exists $C > 0$ such that

$$|\nabla v(x^2) - \nabla v(x^1)| \le \frac{C}{d(y) + y_1}|x^2 - x^1| \text{ for } x^1, x^2 \in B_r^+(y).$$

Fix $x^1, x^2 \in B_r^+(y)$. We consider the two cases $B^+_{|x^2-x^1|}(\frac{x^1+x^2}{2}) \subset \Omega$ and $B^+_{|x^2-x^1|}(\frac{x^1+x^2}{2}) \cap \Omega^c \ne \emptyset$ separately.

Case $B^+_{|x^2-x^1|}(\frac{x^1+x^2}{2}) \subset \Omega$. Let us denote $x^0 = \frac{x^1+x^2}{2}$. We have $d(x^0) = \mathrm{dist}(x^0, D\backslash\Omega) \geq |x^2 - x^1|$. Let us define $\phi_{x_1^0}(x) = b^{-1}(x_1 - x_1^0)$ and using (5.24), Theorem 1 and Lemma 20 we estimate

$$
\begin{aligned}
\frac{|\nabla v(x^2) - \nabla v(x^1)|}{|x^2 - x^1|} &\leq [\nabla v]_{C^{0,1}(B^+_{\frac{|x^2-x^1|}{2}}(x^0))} \\
&\leq [\nabla v]_{C^{0,1}(B^+_{\frac{d(x^0)}{2}}(x^0))} = [\nabla(v - \phi_{x_1^0})]_{C^{0,1}(B^+_{\frac{d(x^0)}{2}}(x^0))} \\
&\leq \frac{C_1}{(d(x^0))^2} \sup_{B^+_{d(x^0)}(x^0)} |v - \phi_{x_1^0}| \\
&\leq \frac{C_1}{(d(x^0))^2} \sup_{B^+_{d(x^0)}(x^0)} v + \frac{C_1}{(d(x^0))^2} \sup_{B^+_{d(x^0)}(x^0)} |\phi_{x_1^0}| \\
&\leq \frac{C_1}{(d(x^0))^2} \frac{C(d(x^0))^2}{d(x^0) + x_1^0} + \frac{C_1}{(d(x^0))^2} \frac{d(x^0)}{b} \\
&= \frac{C_2}{d(x^0) + x_1^0} + \frac{C_3}{d(x^0)}.
\end{aligned}
$$

In the case $d(x^0) < \frac{3}{4}x_1^0$ for $x \in B_{d(x^0)}(x^0)$ we have $x_1 > \frac{1}{4}x_1^0$ and using (5.24), Theorem 1, Lemmas 2 and 20 we estimate

$$
\begin{aligned}
\frac{|\nabla v(x^2) - \nabla v(x^1)|}{|x^2 - x^1|} &\leq [\nabla v]_{C^{0,1}(B^+_{\frac{|x^2-x^1|}{2}}(x^0))} \leq [\nabla v]_{C^{0,1}(B^+_{\frac{d(x^0)}{2}}(x^0))} \\
&\leq [\nabla(v - w_{x_1^0})]_{C^{0,1}(B^+_{\frac{d(x^0)}{2}}(x^0))} + [\nabla w_{x_1^0}]_{C^{0,1}(B^+_{\frac{d(x^0)}{2}}(x^0))} \\
&\leq \frac{C_1}{(d(x^0))^2} \sup_{B^+_{d(x^0)}(x^0)} |v - w_{x_1^0}| + [\nabla w_{x_1^0}]_{C^{0,1}(B^+_{\frac{d(x^0)}{2}}(x^0))} \\
&\leq \frac{C_1}{(d(x^0))^2} \sup_{B^+_{d(x^0)}(x^0)} v + \frac{C_1}{(d(x^0))^2} \sup_{B^+_{d(x^0)}(x^0)} w_{x_1^0} + [\nabla w_{x_1^0}]_{C^{0,1}(B^+_{\frac{d(x^0)}{2}}(x^0))} \\
&\leq \frac{C_1}{(d(x^0))^2} \frac{C(d(x^0))^2}{d(x^0) + x_1^0} + \frac{C_1}{(d(x^0))^2} \frac{C}{x_1^0}(d(x^0))^2 + \frac{C}{x_1^0} \\
&= \frac{CC_1}{d(x^0) + x_1^0} + (CC_1 + C)\frac{1}{x_1^0} = \frac{C_4}{d(x^0) + x_1^0} + \frac{C_5}{x_1^0}.
\end{aligned}
$$

Thus we have in the case $B^+_{|x^2-x^1|}(\frac{x^1+x^2}{2}) \subset \Omega$ the estimate

$$
\begin{aligned}
\frac{|\nabla v(x^2) - \nabla v(x^1)|}{|x^2 - x^1|} &\leq \Big(\frac{C_2}{d(x^0) + x_1^0} + \frac{C_3}{d(x^0)}\Big)\chi_{\{d(x^0) \geq \frac{3}{4}x_1^0\}} + \Big(\frac{C_4}{d(x^0) + x_1^0} + \frac{C_5}{x_1^0}\Big)\chi_{\{d(x^0) < \frac{3}{4}x_1^0\}} \\
&\leq \frac{C_6}{d(x^0) + x_1^0} \leq \frac{2C_6}{d(y) + y_1}
\end{aligned}
$$

where we used (5.23) for the last inequality.

Case $B^+_{|x^2-x^1|}(\frac{x^1+x^2}{2}) \cap \Omega^c \neq \emptyset$. We compute

$$
\begin{aligned}
&B^+_{|x^2-x^1|}(x^0) \subset B^+_{|x^2-x^1|+|x^0-y|}(y) \subset B^+_{|x^2-x^1|+\frac{1}{2}|x^1-y|+\frac{1}{2}|x^2-y|}(y) \\
&\quad \subset B^+_{\frac{3}{2}|x^1-y|+\frac{3}{2}|x^2-y|}(y) \subset B^+_{3r}(y) \subset B^+_{8d+9r}(y) \subset D
\end{aligned}
$$

thus $B^+_{|x^2-x^1|}(x^0)\cap(D\backslash\Omega)\neq\emptyset$ and by (5.24), (5.17) and (5.23) we compute

$$\begin{aligned}|\nabla v(x^2)-\nabla v(x^1)| &\leq |\nabla v(x^2)|+|\nabla v(x^1)|\\ &\leq \frac{Cd(x^2)}{d(x^2)+x_1^2}+\frac{Cd(x^1)}{d(x^1)+x_1^1}\\ &\leq \frac{C}{d(x^2)+x_1^2}\frac{3}{2}|x^2-x^1|+\frac{C}{d(x^1)+x_1^1}\frac{3}{2}|x^2-x^1|\\ &\leq \frac{3}{2}C\Big(\frac{1}{d(x^2)+x_1^2}+\frac{1}{d(x^1)+x_1^1}\Big)|x^2-x^1|\\ &\leq \frac{C_8}{d(y)+y_1}|x^2-x^1|\end{aligned}$$

and this completes the proof of the lemma. □

Corollary 4. *There exists a $C>0$ such that if $R\geq 32$, v is a solution of the obstacle problem (1.3) in B_R^+ and $B_2^+\backslash\Omega\neq\emptyset$ then*

$$[v]_{C^{1,1}(B_{\frac{R}{20}}\cap\{x_1>\delta\})}\leq\frac{C}{\delta}\ \ \textit{for}\ \ \delta>0.$$

Proof. Let $0<\delta$, $y\in B_{\frac{R}{20}}\cap\{y_1>\delta\}$, $r=\frac{y_1}{8}$ and $d=\operatorname{dist}(y,D\backslash\Omega)$. We compute

$$4r=\frac{y_1}{2}<y_1\leq d+y_1. \tag{5.25}$$

By an estimate like (5.22) we have $d\leq|y|+2$ and we compute

$$\begin{aligned}B^+_{8d+9r}(y)&\subset B^+_{8d+9r+|y|}\subset B^+_{8(|y|+2)+9r+|y|}\\ &=B^+_{9|y|+9r+16}=B^+_{9|y|+\frac{9}{8}y_1+16}\subset B^+_{9|y|+\frac{9}{8}|y|+16}\\ &\subset B^+_{10|y|+16}\subset B^+_{\frac{R}{2}+16}\subset B^+_R.\end{aligned}\tag{5.26}$$

Thus by (5.25), (5.26) and Lemma 4 we have

$$[v]_{C^{1,1}(B^+_{\frac{y_1}{8}}(y))}=[v]_{C^{1,1}(B^+_r(y))}\leq\frac{C_1}{d+y_1}\leq\frac{C_1}{y_1}<\frac{C_1}{\delta}\ \ \text{for}\ \ y\in B_{\frac{R}{20}}\cap\{y_1>\delta\}.\tag{5.27}$$

Also by Corollary 2 we have

$$|\nabla v(x)|\leq C_2\ \ \text{for}\ \ x\in B^+_{\frac{R}{18}}.\tag{5.28}$$

Using (5.27) and (5.28) we compute

$$\begin{aligned}[v]_{C^{1,1}(B_{\frac{R}{20}}\cap\{x_1>\delta\})}&=\sup_{x^1,x^2\in B_{\frac{R}{20}}\cap\{x_1>\delta\}}\frac{|\nabla v(x^2)-\nabla v(x^1)|}{|x^2-x^1|}\\ &=\max\Big(\sup_{x^1,x^2\in B_{\frac{R}{20}}\cap\{x_1>\delta\},\ |x^2-x^1|<\frac{1}{8}x_1^1}\frac{|\nabla v(x^2)-\nabla v(x^1)|}{|x^2-x^1|},\\ &\qquad\sup_{x^1,x^2\in B_{\frac{R}{20}}\cap\{x_1>\delta\},\ |x^2-x^1|\geq\frac{1}{8}x_1^1}\frac{|\nabla v(x^2)-\nabla v(x^1)|}{|x^2-x^1|}\Big)\\ &\leq\max\Big(\sup_{x^1\in B_{\frac{R}{20}}\cap\{x_1>\delta\}}\ \sup_{x^2\in B_{\frac{1}{8}x_1^1}(x^1)}\frac{|\nabla v(x^2)-\nabla v(x^1)|}{|x^2-x^1|},\end{aligned}$$

$$\begin{aligned}&\sup_{x^1,x^2\in B_{\frac{R}{20}}\cap\{x_1>\delta\}}\frac{8}{x_1^1}\big(|\nabla v(x^2)|+|\nabla v(x^1)|\big)\Big)\\ &\le \max\Big(\frac{C_1}{\delta},\frac{8}{\delta}2C_2\Big)=\frac{C_3}{\delta}\end{aligned}$$

and this proves the corollary. □

Corollary 5. *There exists a $C>0$ such that if $0<r<1$, v is a solution of the obstacle problem (1.3) in $B_r(e_1)$ and $v(e_1)=0$ then we have*

$$[v]_{C^{1,1}(B_{\frac{r}{9}}(e_1))}\le C.$$

Proof. Because $v(e_1)=0$ we have $\operatorname{dist}\big(e_1,B_r(e_1)\backslash\Omega\big)=0$. We compute

$$4\frac{r}{9}<(e_1)_1=\operatorname{dist}\big(e_1,B_r(e_1)\backslash\Omega\big)+(e_1)_1$$

and

$$B^+_{8\operatorname{dist}(e_1,B_r(e_1)\backslash\Omega)+9\frac{r}{9}}(e_1)=B_r^+(e_1).$$

Thus by Lemma 4 we have

$$[v]_{C^{1,1}(B_{\frac{r}{9}}(e_1))}\le\frac{C}{\operatorname{dist}\big(e_1,B_r(e_1)\backslash\Omega\big)+(e_1)_1}=C$$

and this proves the corollary. □

6. Optimal nondegeneracy and proof of Theorem 2

Let us define

$$p_1(y)=\frac{1}{2(n-1)}|y'|^2-\frac{1}{b+1}\frac{1}{2}y_1^2-\frac{1}{b^2-1}\frac{1}{y_1^{b-1}}+\frac{1}{2(b-1)}$$

and

$$p_{x^0}(x)=(x_1^0)^2p_1\Big(\frac{x-(x^0)'}{x_1^0}\Big).$$

Lemma 5. *There exist $c>0$ and $0<\epsilon_0<1$ such that for $0<\epsilon<\epsilon_0$ and $x,x^0\in\mathbb{R}^n_+$ we have*

$$w_{x_1^0}(x_1)+\frac{\epsilon}{x_1^0}p_{x^0}(x)\ge\frac{c}{x_1^0}\epsilon|x-x^0|^2\ \ \textit{for}\ \ x_1\le\frac{c}{\epsilon}x_1^0.$$

Proof. We compute

$$\begin{aligned}w_{x_1^0}(x_1)+\frac{\epsilon}{x_1^0}p_{x^0}(x)&=x_1^0w_1\Big(\frac{x_1}{x_1^0}\Big)+\epsilon x_1^0p_1\Big(\frac{x-(x^0)'}{x_1^0}\Big)\\ &=x_1^0\Big(\frac{x_1}{bx_1^0}-\frac{1}{b-1}+\frac{1}{b(b-1)}\frac{(x_1^0)^{b-1}}{x_1^{b-1}}\Big)\\ &\quad+\epsilon x_1^0\Big(\frac{1}{2(n-1)}\Big|\frac{x'-(x^0)'}{x_1^0}\Big|^2-\frac{1}{b+1}\frac{1}{2}\Big(\frac{x_1}{x_1^0}\Big)^2-\frac{1}{b^2-1}\Big(\frac{x_1^0}{x_1}\Big)^{b-1}+\frac{1}{2(b-1)}\Big)\\ &=\frac{\epsilon}{2(n-1)x_1^0}|x'-(x^0)'|^2+x_1^0\Big\{\frac{x_1}{bx_1^0}-\frac{1}{b-1}+\frac{1}{b(b-1)}\frac{(x_1^0)^{b-1}}{x_1^{b-1}}\\ &\quad+\epsilon\Big(-\frac{1}{b+1}\frac{1}{2}\Big(\frac{x_1}{x_1^0}\Big)^2-\frac{1}{b^2-1}\Big(\frac{x_1^0}{x_1}\Big)^{b-1}+\frac{1}{2(b-1)}\Big)\Big\}.\end{aligned}\tag{6.1}$$

Let us define

$$f(t)=\frac{t}{b}-\frac{1}{b-1}+\frac{1}{b(b-1)}\frac{1}{t^{b-1}}+\epsilon\Big(-\frac{1}{b+1}\frac{1}{2}t^2-\frac{1}{b^2-1}\frac{1}{t^{b-1}}+\frac{1}{2(b-1)}\Big) \tag{6.2}$$

then we have

$$f(1)=f'(1)=0. \tag{6.3}$$

We claim that there exists $\epsilon_0>0$ such that for $0<\epsilon<\epsilon_0$ we have

$$f(t)\geq\frac{1}{4}\epsilon(t-1)^2 \quad\text{for}\quad 0<t\leq\frac{1}{3b\epsilon}. \tag{6.4}$$

By direct computations one may see that there exists $c_1>0$ such that for $0<\epsilon<1$ we have

$$f''(t)\geq\frac{1}{2}\epsilon \quad\text{for}\quad 0<t<c_1\epsilon^{-\frac{1}{b+1}}. \tag{6.5}$$

For $0<\epsilon<c_1^{b+1}$ we have $1<c_1\epsilon^{-\frac{1}{b+1}}$. Thus for

$$0<\epsilon<\min(1,c_1^{b+1}) \tag{6.6}$$

by (6.3) and (6.5) we have

$$f(t)\geq\frac{1}{4}\epsilon(t-1)^2 \quad\text{for}\quad 0<t\leq c_1\epsilon^{-\frac{1}{b+1}}. \tag{6.7}$$

We estimate

$$\begin{aligned} f(t)&=\frac{t}{b}-\frac{\epsilon}{2(b+1)}t^2+\big(\frac{\epsilon}{2}-1\big)\frac{1}{b-1}+\big(\frac{1}{b}-\frac{\epsilon}{b+1}\big)\frac{1}{b-1}\frac{1}{t^{b-1}}\\ &\geq\frac{t}{b}-\frac{\epsilon}{2(b+1)}t^2-\frac{1}{b-1}=\Big(\frac{1}{b}-\frac{\epsilon}{2(b+1)}t\Big)t-\frac{1}{b-1}. \end{aligned} \tag{6.8}$$

In addition to (6.6) assume

$$\epsilon<\Big(c_1(b-1)\frac{1}{2b}\Big)^{1+b}. \tag{6.9}$$

Now for t such that

$$c_1\epsilon^{-\frac{1}{1+b}}\leq t\leq\frac{1}{3b\epsilon}$$

using (6.8) we estimate

$$\begin{aligned} f(t)-\epsilon t^2&\geq\big(\frac{1}{b}-\frac{\epsilon}{2(b+1)}t\big)t-\frac{1}{b-1}-\epsilon t^2\\ &=\Big(\frac{1}{b}-\big(\frac{1}{2(b+1)}+1\big)\epsilon t\Big)t-\frac{1}{b-1}\\ &\geq\Big(\frac{1}{b}-\big(\frac{1}{2(b+1)}+1\big)\frac{1}{3b}\Big)t-\frac{1}{b-1}\\ &\geq\frac{t}{2b}-\frac{1}{b-1}\geq\frac{1}{2b}\frac{c_1}{\epsilon^{\frac{1}{1+b}}}-\frac{1}{b-1}\\ &=\Big(c_1(b-1)\frac{1}{2b}\frac{1}{\epsilon^{\frac{1}{1+b}}}-1\Big)\frac{1}{b-1}\geq 0 \end{aligned} \tag{6.10}$$

for the last inequality we used (6.9).

Also from (6.9) we have

$$1<\frac{2b}{b-1}<\frac{c_1}{\epsilon^{\frac{1}{1+b}}}$$

thus for $c_1\epsilon^{-\frac{1}{1+b}}\leq t$ we have $1<t$ and it follows $(t-1)^2\leq t^2$.

By (6.7) and (6.10) for $0 < t < \frac{1}{3b\epsilon}$ we have

$$f(t) \geq \frac{1}{4}\epsilon(t-1)^2 \chi_{\{0<t\leq c_1\epsilon^{-\frac{1}{b+1}}\}} + \epsilon t^2 \chi_{\{c_1\epsilon^{-\frac{1}{b+1}}<t\leq \frac{1}{3b}\epsilon^{-1}\}} \geq \frac{1}{4}\epsilon(t-1)^2$$

and this proves (6.4).

If $x_1 \leq \frac{1}{3b}\epsilon^{-1}x_1^0$ then by (6.4) we have

$$\begin{aligned} w_{x_1^0}(x_1) + \frac{\epsilon}{x_1^0} p_{x^0}(x) &= \frac{\epsilon}{2(n-1)x_1^0}|x' - (x^0)'|^2 + x_1^0 f(\frac{x_1}{x_1^0}) \\ &\geq \frac{\epsilon}{2(n-1)x_1^0}|x' - (x^0)'|^2 + x_1^0 \frac{1}{4}\epsilon(\frac{x_1}{x_1^0} - 1)^2 \\ &= \frac{\epsilon}{2(n-1)x_1^0}|x' - (x^0)'|^2 + \frac{1}{x_1^0}\frac{1}{4}\epsilon(x_1 - x_1^0)^2 \\ &\geq \frac{\epsilon}{x_1^0}\min\big(\frac{1}{2(n-1)}, \frac{1}{4}\big)|x - x^0|^2 \geq \frac{\epsilon}{x_1^0}\frac{1}{2n}|x - x^0|^2 \end{aligned}$$

and by taking $c = \min(\frac{1}{2n}, \frac{1}{3b})$ the lemma is proved. □

Proof of Theorem 2. Assume v is a solution of the obstacle problem (1.3) in D, $x^0 \in \Omega$ and $B_r^+(x^0) \subset\subset D$. Let $c > 0$ and $0 < \epsilon_0 < 1$ be as in Lemma 5.

We claim that if $0 < \epsilon < \epsilon_0$ and $r < (\frac{c}{\epsilon} - 1)x_1^0$ then

$$\sup_{\Omega\cap\partial B_r(x^0)} v \geq v(x^0) + \frac{c}{x_1^0}\epsilon r^2. \tag{6.11}$$

To prove this claim let us define

$$h(x) = v(x) - v(x^0) - \big(w_{x_1^0}(x_1) + \frac{\epsilon}{x_1^0} p_{x^0}(x)\big).$$

Then we have

$$\operatorname{div}(x_1^b \nabla h) = 0 \ \text{ in } \ \Omega, \ h(x^0) = 0$$

and

$$h(x) \leq -v(x^0) < 0 \ \text{ for } \ x \in \Gamma. \tag{6.12}$$

We have the inclusion

$$\big(\partial(B_r(x^0) \cap \Omega)\big) \cap \mathbb{R}^n_+ \subset \big((\overline{B}_r(x^0) \cap \mathbb{R}^n_+) \cap \Gamma\big) \cup \big(\Omega \cap \partial B_r(x^0)\big). \tag{6.13}$$

Applying the maximum principle Lemma 17 in the domain $B_r(x^0) \cap \Omega$ and using (6.13) we have

$$0 = h(x^0) \leq \sup_{(\partial(B_r(x^0)\cap\Omega))\cap\mathbb{R}^n_+} h \leq \max\Big(\sup_{(\overline{B}_r(x^0)\cap\mathbb{R}^n_+)\cap\Gamma} h, \sup_{\Omega\cap\partial B_r(x^0)} h\Big). \tag{6.14}$$

From (6.12) and (6.14) we obtain

$$0 \leq \sup_{\Omega\cap\partial B_r(x^0)} h \leq \sup_{\Omega\cap\partial B_r(x^0)} v - v(x^0) - \inf_{\Omega\cap\partial B_r(x^0)} \big(w_{x_1^0}(x_1) + \frac{\epsilon}{x_1^0} p_{x^0}(x)\big). \tag{6.15}$$

If $x \in \partial B_r(x^0)$ because $r < (\frac{c}{\epsilon} - 1)x_1^0$ we have

$$x_1 \leq x_1^0 + r < x_1^0 + (\frac{c}{\epsilon} - 1)x_1^0 = \frac{c}{\epsilon}x_1^0$$

thus by Lemma 5 we have

$$w_{x_1^0}(x_1) + \frac{\epsilon}{x_1^0} p_{x^0}(x) \geq \frac{c}{x_1^0}\epsilon r^2 \text{ for } x \in \partial B_r(x^0) \cap \mathbb{R}^n_+. \tag{6.16}$$

By (6.15) and (6.16) we prove the claim and obtain (6.11).

Now let us choose

$$\epsilon = \frac{x_1^0}{2r + x_1^0} \min(c, \epsilon_0)$$

then we have $0 < \epsilon < \epsilon_0$ and

$$\begin{aligned}
\left(\frac{c}{\epsilon} - 1\right)x_1^0 &= \Big(\frac{c}{\frac{x_1^0}{2r+x_1^0}\min(c,\epsilon_0)} - 1\Big)x_1^0 \\
&= \Big(\big(\frac{2r}{x_1^0} + 1\big)\max\big(1, \frac{c}{\epsilon_0}\big) - 1\Big)x_1^0 \\
&\geq \frac{2r}{x_1^0}\max\big(1, \frac{c}{\epsilon_0}\big)x_1^0 = 2r\max\big(1, \frac{c}{\epsilon_0}\big) \geq 2r > r.
\end{aligned}$$

We compute

$$\frac{1}{x_1^0}\epsilon r^2 = \frac{1}{x_1^0}\frac{x_1^0}{2r + x_1^0}\min(c, \epsilon_0)r^2 = \frac{1}{2r + x_1^0}\min(c, \epsilon_0)r^2 \geq \frac{1}{2}\min(c, \epsilon_0)\frac{r^2}{r + x_1^0}$$

and this proves the theorem. □

Corollary 6. *There exists a $c > 0$ such that if v is a solution of the obstacle problem (1.3) in D, $x^0 \in \partial\Omega$ and $B_r^+(x^0) \subset\subset D$ then we have*

$$\sup_{B_r^+(x^0)} v \geq \frac{cr^2}{r + x_1^0}. \tag{6.17}$$

Proof. Because $x^0 \in \partial\Omega$ there exists $x^1 \in \Omega$ such that $|x^1 - x^0| < \frac{r}{2}$. We have

$$B^+_{\frac{r}{2}}(x^1) \subset B^+_{\frac{r}{2}+|x^1-x^0|}(x^0) \subset B^+_{\frac{r}{2}+\frac{r}{2}}(x^0) = B_r^+(x^0)$$

and applying the previous lemma to the point x^1 we estimate

$$\begin{aligned}
\sup_{B_r^+(x^0)} v &\geq \sup_{B^+_{\frac{r}{2}}(x^1)} v \geq \sup_{\Omega\cap\partial B_{\frac{r}{4}}(x^1)} v \geq v(x^1) + \frac{c(\frac{r}{4})^2}{\frac{r}{4} + x_1^1} \\
&\geq \frac{c(\frac{r}{4})^2}{\frac{r}{4} + x_1^1} \geq \frac{c_1 r^2}{r + x_1^1} \geq \frac{c_1 r^2}{r + x_1^0 + \frac{r}{2}} \geq \frac{c_2 r^2}{r + x_1^0}
\end{aligned}$$

and this proves the corollary. □

7. Weiss monotonicity formula

Weiss balanced energy was introduced in [9] to study obstacle problems. Let us define for $b > 1$ and $v \in H^1(B_r^+; x_1^b)$ the Weiss balanced energy

$$W(r, v) = \frac{1}{r^{n+b}}\int_{B_r^+}\big(x_1^b|\nabla v|^2 + 2x_1^{b-1}v\big)dx - \frac{1}{r^{n+b+1}}\int_{\partial B_r\cap\mathbb{R}^n_+} x_1^b v^2 s(dx). \tag{7.1}$$

By the trace inequality (4.2), $W(r,v)$ is well defined for $v \in H^1(B_r^+; x_1^b)$.

Let v be a function defined in B^+ and $0 < r < 1$. We define the linear blowup

$$v_r(x) = \frac{v(rx)}{r} \quad \text{for } x \in B^+.$$

Lemma 6. *For $r > 0$ and $v \in H^1(B_r^+; x_1^b)$ we have $W(r,v) = W(1, v_r)$.*

For $v \in H^1(B_{r_0}^+; x_1^b)$, $W(r,v)$ as a function of $0 < r < r_0$ is locally bounded and absolutely continuous.

Let v be a solution of the obstacle problem (1.3) in $B_{r_0}^+$ such that $v \in C_b^1(B_{r_0}^+)$ and there exists a $C > 0$ such that

$$[v]_{C^{1,1}(B_{r_0} \cap \{x_1 > \delta\})} \le \frac{C}{\delta} \quad \textit{for} \ \ \delta > 0. \tag{7.2}$$

Then for $0 < r < r_0$ we have

$$\frac{d}{dr} W(r,v) = 2r \int\limits_{\partial B \cap \mathbb{R}^n_+} x_1^b (\partial_r v_r)^2 s(dx) \tag{7.3}$$

and if $W(r,v)$ is independent of $r \in (0, r_0)$ then v is first order homogenous in $B_{r_0}^+$.

Proof. Let $r > 0$ and $v \in H^1(B_r^+; x_1^b)$. We compute

$$\begin{aligned} W(r,v) &= \frac{1}{r^{n+b}} \int\limits_{B_r^+} \left(x_1^b |\nabla v|^2 + 2 x_1^{b-1} v\right) dx - \frac{1}{r^{n+b+1}} \int\limits_{\partial B_r \cap \mathbb{R}^n_+} x_1^b v^2 s(dx) \\ &= \int\limits_{B^+} \left(x_1^b |\nabla v(rx)|^2 + 2 x_1^{b-1} \frac{1}{r} v(rx)\right) dx - \frac{1}{r^2} \int\limits_{\partial B \cap \mathbb{R}^n_+} x_1^b v^2(rx) s(dx) \\ &= \int\limits_{B^+} \left(x_1^b |\nabla v_r(x)|^2 + 2 x_1^{b-1} v_r(x)\right) dx - \int\limits_{\partial B \cap \mathbb{R}^n_+} x_1^b v_r^2(x) s(dx) = W(1, v_r) \end{aligned}$$

and this proves the first claim.

Let $v \in H^1(B_{r_0}^+; x_1^b)$ then for $0 < r < r_0$ by direct computation using polar coordinates we have

$$\begin{aligned} \int\limits_{\partial B_r \cap \mathbb{R}^n_+} x_1^b v^2 s(dx) = -2 r^{b+n-1} &\int\limits_{B_{r_0}^+ \setminus B_r^+} \frac{1}{|x|^{n+b}} x_1^b v(x) \nabla v(x) \cdot x dx \\ &+ \left(\frac{r}{r_0}\right)^{b+n-1} \int\limits_{\partial B_{r_0} \cap \mathbb{R}^n_+} x_1^b v^2(x) s(dx). \end{aligned} \tag{7.4}$$

The equation (7.4) together with the fact that for $f \in L^1_{loc}(\mathbb{R}^n)$, $\int_{B_r^+} f dx$ as a function of r is locally bounded and absolutely continuous function proves the second claim.

Let v be a solution of the obstacle problem (1.3) in $B_{r_0}^+$ such that $v \in C_b^1(B_{r_0}^+)$ and there exists $C > 0$ such that (7.2) holds.

Then by the Theorem of Rademacher we have that the distributional second derivatives $\partial_{x_i x_j} v$ satisfy

$$|\partial_{x_i x_j} v(x)| \le \frac{C}{\delta} \quad \text{for a.e. } \ x \in B_{r_0} \cap \{x_1 > \delta\}.$$

Let $\delta_k = \frac{r_0}{2^k}$ for $k \in \{0\} \cup \mathbb{N}$. We may decompose

$$B_{r_0}^+ = \cup_{k \in \mathbb{N}} \left(B_{r_0} \cap \left\{\delta_k < x_1 \le \delta_{k-1}\right\}\right).$$

Then for $k \in \mathbb{N}$ there exists $E_k \subset B^+_{r_0}$ such that $|E_k| = 0$ (here $|\cdot|$ denotes the n-dimensional Lebesgue measure) and for $x \in B_{r_0} \cap \{\delta_k < x_1 \le \delta_{k-1}\} \setminus E_k$ we have

$$|\partial_{x_i x_j} v(x)| \le \frac{C}{\delta_k} = \frac{2C}{\delta_{k-1}} \le \frac{2C}{x_1}. \tag{7.5}$$

Now defining $E = \cup_{k\in\mathbb{N}} E_k$ we have that $|E| = 0$ and for $x \in B^+_{r_0} \setminus E$ the inequality (7.5) holds. Thus

$$x_1|\partial_{x_i x_j} v(x)| \le 2C \quad \text{for a.e. } \; x \in B^+_{r_0}.$$

Now we have enough regularity and for $0 < r < r_0$ we compute

$$\begin{aligned}
\frac{1}{2}\frac{d}{dr}W(r,v) &= \frac{1}{2}\frac{d}{dr}W(1,v_r) \\
&= \frac{1}{2}\int_{B^+} \big(x_1^b 2\nabla v_r \cdot \nabla \partial_r v_r + 2x_1^{b-1}\partial_r v_r\big)dx \\
&\quad - \frac{1}{2}\int_{\partial B\cap\mathbb{R}^n_+} x_1^b 2 v_r \partial_r v_r s(dx) \\
&= \int_{B^+} \big(\operatorname{div}(x_1^b \partial_r v_r \nabla v_r) - \operatorname{div}(x_1^b \nabla v_r)\partial_r v_r + x_1^{b-1}\partial_r v_r\big)dx \\
&\quad - \int_{\partial B\cap\mathbb{R}^n_+} x_1^b v_r \partial_r v_r s(dx) \\
&= \int_{B^+} \big(-\operatorname{div}(x_1^b \nabla v_r)\partial_r v_r + x_1^{b-1}\partial_r v_r\big)dx \\
&\quad + \int_{\partial B^+} x_1^b \partial_r v_r \nabla v_r \cdot \upsilon s(dx) - \int_{\partial B\cap\mathbb{R}^n_+} x_1^b v_r \partial_r v_r s(dx) \\
&= \int_{\partial B^+} x_1^b \partial_r v_r \nabla v_r \cdot \upsilon s(dx) - \int_{\partial B\cap\mathbb{R}^n_+} x_1^b v_r \partial_r v_r s(dx) \\
&= \int_{\partial B\cap\mathbb{R}^n_+} x_1^b \partial_r v_r \nabla v_r \cdot \upsilon s(dx) - \int_{\partial B\cap\mathbb{R}^n_+} x_1^b v_r \partial_r v_r s(dx) \\
&= \int_{\partial B\cap\mathbb{R}^n_+} x_1^b \big(\partial_\upsilon v_r - v_r\big)\partial_r v_r s(dx).
\end{aligned} \tag{7.6}$$

It is easy to see that on $\partial B \cap \mathbb{R}^n_+$ we have

$$\partial_\upsilon v_r - v_r = r\partial_r v_r. \tag{7.7}$$

By (7.6) and (7.7) we obtain (7.3).
The last claim follows from (7.3) and (7.7). $\square$

8. Global solutions and proof of Theorem 3

A global solution was defined in Definition 1 and the set P_∞ was defined as the set of all global solutions. We denote by $P_{\infty,hom.}$ the set of first oder homogenous global solutions.

Lemma 7 *(Classification of homogenous global solutions). We have*

$$P_{\infty,hom.} = \left\{0, \frac{x_1}{b}\right\}.$$

Proof. Let us consider the two cases when $\Omega = \mathbb{R}^n_+$ and $\Omega \neq \mathbb{R}^n_+$ separately.

Case $\Omega = \mathbb{R}^n_+$**.** We write $v = \frac{x_1}{b} + h$ where $h \in H^1(B_r^+; x_1^b)$ for each $r > 0$, $\operatorname{div}(x_1^b \nabla h) = 0$ in $\mathbb{R}^n_+$ and h is first order homogenous. We have that ∇h is 0-th order homogenous. Thus for $r > 0$ by the Poisson kernel formula proved in Lemma 16 we have

$$\nabla h(x) = \nabla h(rx) = \int\limits_{\partial B \cap \mathbb{R}^n_+} \nabla K(rx, y; b) h(y) s(dy).$$

Sending $r \to 0$ we obtain

$$\nabla h(x) = \int\limits_{\partial B \cap \mathbb{R}^n_+} \nabla K(0, y; b) h(y) s(dy).$$

Therefore ∇h is a constant vector. So we have

$$h(x) = c + p \cdot x$$

for some $c \in \mathbb{R}$ and $p \in \mathbb{R}^n$, and by first order homogeneity of h we obtain that $c = 0$.

We compute

$$0 = \operatorname{div}(x_1^b \nabla h) = \operatorname{div}\big(x_1^b \nabla(p \cdot x)\big) = b p_1 x_1^{b-1}$$

from which it follows that $p_1 = 0$. So we have

$$h(x) = p' \cdot x'$$

for some $p' \in \mathbb{R}^{n-1}$.

Because $v = \frac{x_1}{b} + h$ should be nonnegative we obtain that $p' = 0$ and thus $h = 0$ and $v = \frac{x_1}{b}$.

Case $\Omega \neq \mathbb{R}^n_+$**.** Let $e \perp e_1$ and $w = \partial_e v$. Similarly as in the proof of Lemma 6 we have $x_1|\nabla w| \leq x_1|\nabla^2 v| \leq C$ a.e. in $\mathbb{R}^n_+$. It follows that $w \in H^1(B_r^+; x_1^b)$ for all $r > 0$. Also the following equation holds

$$\operatorname{div}(x_1^b \nabla w) = 0 \quad \text{in} \quad \Omega \tag{8.1}$$

and w is 0-th order homogenous.

We claim that $w \geq 0$ in Ω. Otherwise we denote $m = \inf_{\partial B \cap \mathbb{R}^n_+} w < 0$. Then there exists $x^0 \in \partial B \cap \overline{\mathbb{R}^n_+}$ such that $m = w(x^0)$. Let us define $\Omega_{rel.}$ to be the interior of $\overline{\Omega}$ in the relative topology of $\overline{\mathbb{R}^n_+}$. If $v(x^0) = 0$ then because $m = w(x^0) = \partial_e v(x^0) < 0$ we would have $v(x^0 + te) < 0$ for small enough $t > 0$ which is in contradiction with $v \geq 0$, thus $v(x^0) > 0$. So in particular $x^0 \in \Omega_{rel.}$. Let $\Omega^0_{rel.}$ be the open component of $\Omega_{rel.}$ in the relative topology of $\overline{\mathbb{R}^n_+}$ such that $x^0 \in \Omega^0_{rel.}$. By 0-th order homogeneity we have that

$$w(x^0) = \inf_{\partial B \cap \mathbb{R}^n_+} w = \inf_{\Omega^0_{rel.}} w.$$

Now because w solves (8.1) in the (usual) interior of $\Omega^0_{rel.}$ and attains its infimum in $\Omega^0_{rel.}$, by strong maximum principle Lemma 18 it is a constant function in $\Omega^0_{rel.}$. Because $\partial \Omega^0_{rel.} \cap \partial \Gamma \neq \emptyset$ and $w = 0$ on Γ we obtain that $w = 0$ on $\Omega^0_{rel.}$, a contradiction with $w(x^0) < 0$.

Hence for arbitrary $e \perp e_1$ we have proved that $\partial_e v \geq 0$ in Ω.

Therefore $\partial_e v = 0$ in Ω for all $e \perp e_1$. It follows that v does not depend on x' and depends only on x_1.

If $\Omega \neq \emptyset$ then because also we consider the case $\Omega \neq \mathbb{R}^n_+$, there exists $\eta > 0$ such that $v(x) = 0$ for $x \in \{x_1 = \eta\}$. Now by first order homogeneity of v we obtain that $v = 0$ in $\mathbb{R}^n_+$ which contradicts with $\Omega \neq \emptyset$.

So we should have $\Omega = \emptyset$ and $v = 0$ in $\mathbb{R}^n_+$. $\square$

Definition 2. For $0 \le \delta < 1$ let us define the open cone

$$\mathcal{C}_\delta = \{x \in \mathbb{R}^n \mid x_1 > \delta|x|\}. \tag{8.2}$$

Proof of Theorem 3. Assume $v \in P_\infty$ and $v \neq 0$.

Step 1. In this step we show that $v_R = \frac{1}{R}v(Rx) \to \frac{x_1}{b}$ in $C_b(B^+)$ as $R \to \infty$.

By the assumptions (3.4) and (3.5) we have

$$\|v_R\|_{C^1(B^+)} \le C \text{ and } [v_R]_{C^{1,1}(B\cap\{x_1>\delta\})} \le \frac{C}{\delta} \text{ for } R > 1 \text{ and } \delta > 0. \tag{8.3}$$

Thus v_R is uniformly bounded in $C_b^1(B^+)$ and in $C^{1,1}(B \cap \{x_1 > \delta\})$ for any $0 < \delta < 1$. By compact embeddings and diagonal selection argument there exists a sequence $R_j \to \infty$ and $v_\infty \in C^{0,1}(B^+)$ such that $v_\infty \in C^{1,1}(B \cap \{x_1 > \delta\})$ for $0 < \delta < 1$, $v_{R_j} \to v_\infty$ in $C^{0,\alpha}(B^+)$ for any $0 < \alpha < 1$, $v_{R_j} \to v_\infty$ in $C^{1,\alpha}(B \cap \{x_1 > \delta\})$ for any $0 < \alpha < 1$ and $0 < \delta < 1$ and $\nabla v_{R_j} \to \nabla v_\infty$ weakly in $L^2(B^+)$.

By the second bound in (8.3) and the pointwise convergence $\nabla v_{R_j} \to \nabla v_\infty$ in $B \cap \{x_1 > \delta\}$ we have

$$[v_\infty]_{C^{1,1}(B\cap\{x_1>\delta\})} \le \frac{C}{\delta} \text{ for } \delta > 0. \tag{8.4}$$

We have that v_{R_j} is a minimiser in B^+, i.e.

$$\int_{B^+} \left(x_1^b|\nabla v_{R_j}|^2 + 2x_1^{b-1}v_{R_j}^+\right)dx \le \int_{B^+} \left(x_1^b|\nabla \psi|^2 + 2x_1^{b-1}\psi^+\right)dx \tag{8.5}$$

for all $\psi \in H^1(B^+; x_1^b)$ such that $\psi = v_{R_j}$ on $\partial B^+ \cap \mathbb{R}^n_+$.

Let $\varphi \in H^1(B^+; x_1^b)$ and $\varphi = v_\infty$ on $\partial B^+ \cap \mathbb{R}^n_+$. Let $\psi = \varphi + v_{R_j} - v_\infty$ in (8.5), then we obtain

$$\begin{aligned}&\int_{B^+} \left(x_1^b|\nabla v_{R_j}|^2 + 2x_1^{b-1}v_{R_j}^+\right)dx \\ &\quad \le \int_{B^+} \left(x_1^b|\nabla(\varphi + v_{R_j} - v_\infty)|^2 + 2x_1^{b-1}(\varphi + v_{R_j} - v_\infty)^+\right)dx.\end{aligned} \tag{8.6}$$

Using the convergence $v_{R_j} \to v_\infty$ in $C_b(\overline{B^+})$ we have

$$\int_{B^+} 2x_1^{b-1}\left(v_{R_j}^+ - (\varphi + v_{R_j} - v_\infty)^+\right)dx \to \int_{B^+} 2x_1^{b-1}\left(v_\infty^+ - \varphi^+\right)dx. \tag{8.7}$$

Using the convergence $\nabla v_{R_j} \to \nabla v_\infty$ weakly in $L^2(B^+)$ we compute

$$\begin{aligned}&\int_{B^+} x_1^b\left(|\nabla v_{R_j}|^2 - |\nabla(\varphi + v_{R_j} - v_\infty)|^2\right)dx \\ &\quad = \int_{B^+} x_1^b\left(-|\nabla\varphi|^2 - 2\nabla\varphi\cdot\nabla(v_{R_j} - v_\infty) + 2\nabla v_{R_j}\cdot\nabla v_\infty - |\nabla v_\infty|^2\right)dx \\ &\quad \to \int_{B^+} x_1^b\left(|\nabla v_\infty|^2 - |\nabla\varphi|^2\right)dx.\end{aligned} \tag{8.8}$$

From (8.6), (8.7) and (8.8) we obtain

$$\int_{B^+} \left(x_1^b|\nabla v_\infty|^2 + 2x_1^{b-1}v_\infty^+\right)dx \le \int_{B^+} \left(x_1^b|\nabla\varphi|^2 + 2x_1^{b-1}\varphi^+\right)dx$$

which proves that v_∞ is a minimiser in B^+.

Let us consider $\varphi \in C_c^\infty(B)$. By the equation satisfied by v_{R_j} we have

$$-\int_{B^+} x_1^b \nabla v_{R_j} \cdot \nabla(v_{R_j}\varphi)dx = \int_{B^+} x_1^{b-1}\chi_{\{v_{R_j}>0\}} v_{R_j}\varphi dx.$$

Also we have

$$\int_{B^+} x_1^b \nabla v_{R_j} \cdot \nabla(v_{R_j}\varphi)dx = \int_{B^+} x_1^b |\nabla v_{R_j}|^2 \varphi dx + \int_{B^+} x_1^b \nabla v_{R_j} \cdot \nabla\varphi v_{R_j} dx$$

thus

$$\int_{B^+} x_1^b |\nabla v_{R_j}|^2 \varphi dx = -\int_{B^+} x_1^{b-1}\chi_{\{v_{R_j}>0\}} v_{R_j}\varphi dx - \int_{B^+} x_1^b \nabla v_{R_j} \cdot \nabla\varphi v_{R_j} dx. \tag{8.9}$$

Similarly we have

$$\int_{B^+} x_1^b |\nabla v_\infty|^2 \varphi dx = -\int_{B^+} x_1^{b-1}\chi_{\{v_\infty>0\}} v_\infty\varphi dx - \int_{B^+} x_1^b \nabla v_\infty \cdot \nabla\varphi v_\infty dx. \tag{8.10}$$

Now because we might pass to the limit on the right hand side of (8.9) and obtain the terms on the right hand side of (8.10) we have

$$\int_{B^+} x_1^b |\nabla v_\infty|^2 \varphi dx = \lim_{j\to\infty} \int_{B^+} x_1^b |\nabla v_{R_j}|^2 \varphi dx. \tag{8.11}$$

Fix $0 < r < 1$. Let $r < r_1 < 1$ and $\varphi \in C_c^\infty(B)$ such that $0 \le \varphi \le 1$ in B, $\varphi = 1$ in B_r and $\varphi = 0$ in $B \backslash B_{r_1}$. Then from (8.11) it follows that

$$\int_{B_{r_1}^+} x_1^b |\nabla v_\infty|^2 dx \ge \lim_{j\to\infty} \int_{B_r^+} x_1^b |\nabla v_{R_j}|^2 dx.$$

Sending $r_1 \to r$ we obtain

$$\int_{B_r^+} x_1^b |\nabla v_\infty|^2 dx \ge \lim_{j\to\infty} \int_{B_r^+} x_1^b |\nabla v_{R_j}|^2 dx.$$

Therefore for all $0 < r < 1$ we have $v_{R_j} \to v_\infty$ strongly in $H^1(B_r^+; x_1^b)$.

Also we have the convergence $v_{R_j} \to v_\infty$ in $C_b(\overline{B^+})$.

Now for $0 < r < 1$ by the convergence $v_{R_j} \to v_\infty$ strongly in $H^1(B_r^+; x_1^b)$ and $v_{R_j} \to v_\infty$ in $C_b(\overline{B^+})$ we may pass to the limit in the balanced energy $W(r, v_{R_j})$ and obtain that $W(r, v_{R_j}) \to W(r, v_\infty)$.

For any $R > 1$ by (8.3) we have

$$\begin{aligned} W(R, v) = W(1, v_R) &\le \int_{B^+} \left(x_1^b |\nabla v_R|^2 + 2x_1^{b-1} v_R\right)dx \\ &\le \int_{B^+} \left(x_1^b C^2 + 2x_1^{b-1} C\right)dx = C_1 \end{aligned}$$

thus as a function of $R > 1$, $W(R, v)$ is uniformly bounded.

By (3.4), v has enough regularity and thus the equation (7.3) holds and $W(R, v)$ is monotonically nondecreasing for $R > 1$.

Therefore a finite limit $\lim_{R\to\infty} W(R,v) = W(+\infty, v)$ exists.

We compute for $0<r<1$

$$W(r,v_\infty) = \lim_{j\to\infty} W(r,v_{R_j}) = \lim_{j\to\infty} W(rR_j, v) = W(+\infty, v)$$

thus $W(r,v_\infty)$ is independent of $0<r<1$. Now by Lemma 6 we obtain that v_∞ is first order homogenous.

We have $\Omega \neq \emptyset$ thus there exists $x^0 \in \Omega$. Let $R > 2|x^0|$ then

$$B^+_{\frac{R}{2}}(x^0) \subset B^+_{\frac{R}{2}+|x^0|} \subset B^+_{\frac{R}{2}+\frac{R}{2}} = B^+_R$$

and by Theorem 2 we have

$$\sup_{B^+} v_R \ge \frac{1}{R} \sup_{\Omega\cap\partial B_{\frac{R}{2}}(x^0)} v \ge \frac{1}{R}\Big(v(x^0) + \frac{c(\frac{R}{2})^2}{\frac{R}{2}+x^0_1}\Big) \ge \frac{c}{4}\frac{R}{\frac{R}{2}+x^0_1} \ge \frac{c}{4}\frac{R}{\frac{R}{2}+|x^0|} \ge \frac{c}{4}.$$

Passing to the limit $R = R_j \to \infty$ we obtain

$$\sup_{B^+} v_\infty \ge \frac{c}{4} \tag{8.12}$$

thus $v_\infty \neq 0$ in B^+.

Because v_∞ is a first order homogenous function in B^+ we have $v_\infty(0)=0$ and $v_\infty(x) = v(\frac{x}{2|x|})(2|x|)$ for all $x \in B^+\backslash\{0\}$. Then we might extend v_∞ as a first order homogenous function in $\mathbb{R}^n_+$ as follows, $v_\infty(x) = v_\infty(\frac{x}{2|x|})(2|x|)$ for all $x \in \mathbb{R}^n_+\backslash B^+$.

Because $v_\infty \in C^{0,1}(B^+)$ and (8.4) holds we obtain that (3.4) holds for (the extended) v_∞.

One may see that because v_∞ is solution in B^+, its extension in $\mathbb{R}^n_+$ as above is a solution in each B^+_r for $r>0$.

Thus v_∞ is a homogenous global solution, i.e. $v_\infty \in P_{\infty,hom.}$. Because $v_\infty \neq 0$ in B^+ from Lemma 7 we obtain that $v_\infty = \frac{x_1}{b}$. By the uniqueness of the limit v_∞ we conclude that $v_R \to v_\infty = \frac{x_1}{b}$ in $C_b(B^+)$ as $R\to\infty$, as desired.

Step 2. In this step we show that v depends only on x_1.

Let us denote for $R>0$

$$\epsilon(R) = \sup_{r\ge R}\sup_{B^+} |v_r - \frac{x_1}{b}|$$

and

$$h = v - \frac{x_1}{b}.$$

By the definition of $\epsilon(R)$ we know that it is a nonnegative and nonincreasing function of $R>0$ and by the first step we have that $\epsilon(R)\to 0$ as $R\to\infty$.

Next

$$\begin{aligned}\sup_{y\in B^+_R} |h(y)| &= \sup_{y\in B^+_R} |v(y) - \frac{y_1}{b}| = R \sup_{y\in B^+_R} |\frac{1}{R}v(y) - \frac{1}{R}\frac{y_1}{b}| \\ &= R\sup_{x\in B^+} |\frac{1}{R}v(Rx) - \frac{x_1}{b}| = R\sup_{x\in B^+}|v_R(x) - \frac{x_1}{b}| \le R\epsilon(R)\end{aligned}$$

which implies

$$v(y) = \frac{y_1}{b} + h(y) \ge \frac{y_1}{b} - |h(y)| \ge \frac{y_1}{b} - |y|\epsilon(|y|)$$

and consequently

$$\Big\{y\in\mathbb{R}^n_+ \;\Big|\; y_1 > b|y|\epsilon(|y|)\Big\} \subset \Omega.$$

Let $0<\delta<1$ and $R_\delta>0$ be large enough such that $\epsilon(R_\delta) < \frac{1}{b}\delta$.

It follows that

$$\mathcal{C}_\delta \cap B^c_{R_\delta} \subset \left\{ y \in \mathbb{R}^n_+ \;\middle|\; y_1 > b|y|\epsilon(|y|) \right\} \subset \Omega$$

where $\mathcal{C}_\delta$ is defined in (8.2).

Assume $\delta < \frac{1}{3}$ and

$$x^0 \in \mathcal{C}_{3\delta} \cap B^c_{2R_\delta}$$

then one may see that

$$B_{\delta|x^0|}(x^0) \subset \mathcal{C}_\delta \cap B^c_{R_\delta} \subset \Omega.$$

We have that h solves the linear equation $\operatorname{div}(x_1^b \nabla h) = 0$ in $B_{\delta|x^0|}(x^0) \subset \Omega$, therefore by Lemma 20 for $e \perp e_1$ we have

$$\begin{aligned}
|\partial_e h(x^0)| &\le \frac{C}{\delta|x^0|} \sup_{B_{\delta|x^0|}(x^0)} |h| \le \frac{C}{\delta|x^0|} \sup_{B^+_{\delta|x^0|+|x^0|}} |h| \\
&\le \frac{C}{\delta|x^0|}(1+\delta)|x^0|\epsilon\big((1+\delta)|x^0|\big) \\
&= \frac{C}{\delta}(1+\delta)\epsilon\big((1+\delta)|x^0|\big) \le \frac{C_1}{\delta}\epsilon(|x^0|).
\end{aligned}$$

We have that $\partial_e h = \partial_e v - \partial_e(\frac{x_1}{b}) = \partial_e v = 0$ on Γ and by (3.4) we have $|\partial_e h| = |\partial_e v| \le |\nabla v| \le C_2$.

Let $R > 2R_\delta$. For $x \in \partial(\Omega \cap B_R) \cap \mathbb{R}^n_+$ we have

$$|\partial_e h(x)| \le \begin{cases} 0, & x \in \partial\Omega \cap B^+_R, \\ C_2, & x \in \Omega \cap \partial B_R \cap \mathcal{C}^c_{3\delta}, \\ \dfrac{C_1}{\delta}\epsilon(R), & x \in \Omega \cap \partial B_R \cap \mathcal{C}_{3\delta}. \end{cases} \tag{8.13}$$

Let us define for $x \in \partial B_R \cap \mathbb{R}^n_+$ the function

$$g(x) = \begin{cases} C_2, & x \in \partial B_R \cap \mathcal{C}^c_{3\delta} \cap \mathbb{R}^n_+, \\ \dfrac{C_1}{\delta}\epsilon(R), & x \in \partial B_R \cap \mathcal{C}_{3\delta}. \end{cases} \tag{8.14}$$

And let q be solution of $\operatorname{div}(x_1^b \nabla q) = 0$ in B^+_R with boundary data $q = g$ on $\partial B_R \cap \mathbb{R}^n_+$. The shrinkdown q_R is a solution in B^+ thus for $x \in B^+_{\frac{R}{2}}$ we have

$$\begin{aligned}
q(x) = Rq_R(\frac{x}{R}) &= R \int_{\partial B \cap \mathbb{R}^n_+} K(\frac{x}{R}, y; b) q_R(y) s(dy) \\
&= \int_{\partial B \cap \mathbb{R}^n_+} K(\frac{x}{R}, y; b) q(Ry) s(dy) \\
&= \int_{\partial B \cap \mathbb{R}^n_+} K(\frac{x}{R}, y; b) g(Ry) s(dy) \\
&= \int_{\partial B \cap \{0 < y_1 \le 3\delta\}} K(\frac{x}{R}, y; b) g(Ry) s(dy) \\
&\quad + \int_{\partial B \cap \{y_1 > 3\delta\}} K(\frac{x}{R}, y; b) g(Ry) s(dy)
\end{aligned}$$

$$
\begin{aligned}
&\le C_2 \int\limits_{\partial B\cap\{0<y_1\le 3\delta\}} K(\frac{x}{R}, y; b)s(dy) \\
&\quad + \frac{C_1}{\delta}\epsilon(R) \int\limits_{\partial B\cap\{y_1>3\delta\}} K(\frac{x}{R}, y; b)s(dy) \\
&\le C_2 \int\limits_{\partial B\cap\{0<y_1\le 3\delta\}} C_4 y_1^b s(dy) \\
&\quad + \frac{C_1}{\delta}\epsilon(R) \int\limits_{\partial B\cap\{y_1>3\delta\}} C_4 y_1^b s(dy) \\
&\le C_5\delta^{b+1} + \frac{C_6}{\delta}\epsilon(R).
\end{aligned} \tag{8.15}
$$

By (8.13), (8.14) and the maximum principle we have that

$$-q(x) \le \partial_e h \le q(x) \ \text{ for } \ x \in \Omega \cap B_R^+$$

but then from (8.15) it follows that

$$|\partial_e h| \le C_5\delta^{b+1} + \frac{C_6}{\delta}\epsilon(R) \ \text{ in } \ \Omega \cap B_{\frac{R}{2}}^+.$$

For a fixed $x \in \Omega$ and $0 < \delta < \frac{1}{3}$ by sending $R \to \infty$ we obtain

$$|\partial_e h(x)| \le C_5\delta^{b+1}.$$

Because this holds for all $x \in \Omega$ and $0 < \delta < \frac{1}{3}$ we obtain $\partial_e h(x) = 0$. Thus $\partial_e h = 0$ in $\mathbb{R}^n_+$ for all $e \perp e_1$ and this proves that h depends only on x_1. Therefore $v = \frac{x_1}{b} + h$ depends only on x_1.

Step 3. In this step we finish the proof of the theorem.

Let Ω^0 be a component of Ω. Because h depends only on x_1 and $\operatorname{div}(x_1^b \nabla h) = 0$ in Ω^0 we have that there exist C_1 and C_2 such that

$$v = \frac{x_1}{b} + C_1 + \frac{C_2}{x_1^{b-1}} \ \text{ in } \ \Omega^0. \tag{8.16}$$

If $\Omega = \mathbb{R}^n_+$ then $\Omega^0 = \mathbb{R}^n_+$. By (3.4), v has finite energy in B_R^+ for each $R > 0$. Because x_1^{1-b} has infinite energy we should have $C_2 = 0$. Also because we should have $v(0) \ge 0$ it follows that $C_1 \ge 0$.

If $\Omega \ne \mathbb{R}^n_+$ then Ω^0 is a proper subset of $\mathbb{R}^n_+$. Thus there exists $t_0 > 0$ such that $t_0 e_1 \in \partial\Omega^0 \cap \mathbb{R}^n_+ \subset \Gamma$. Therefore we should have

$$v(t_0 e_1) = \partial_{x_1} v(t_0 e_1) = 0. \tag{8.17}$$

Now by finding the correct values of the coefficients C_1 and C_2 in (8.16) such that the two equations in (8.17) hold we obtain that $v(x_1) = w_{t_0}(x_1)$. Now from (5.2) it follows that either Ω^0 is equal to $(0, t_0) \times \mathbb{R}^{n-1}$ or $(t_0, \infty) \times \mathbb{R}^{n-1}$. But in the case $\Omega^0 = (0, t_0) \times \mathbb{R}^{n-1}$ we will have infinite energy thus the only possibility is when $\Omega^0 = (t_0, \infty) \times \mathbb{R}^{n-1}$ and $v = w_{t_0}(x_1)\chi_{\{x_1>t_0\}}$ in Ω^0.

Because all connected open sets (t_0, ∞) for $t_0 > 0$ intersect. We obtain that Ω has only one component and $v = w_{t_0}(x_1)\chi_{\{x_1>t_0\}}$. This finishes the proof of the theorem. □

9. Tangential touch of the free boundary

Definition 3. For $0 < \eta$ we call σ a modulus of continuity defined on $[0, \eta)$ if $\sigma : [0, \eta) \to [0, \infty)$, $\sigma(0) = 0$, σ is nondecreasing and $0 = \sigma(+0) = \lim_{\tau\to 0,\ \tau>0} \sigma(\tau)$.

Lemma 8. *There exists $0 < \eta_1 < 1$ (depending only on n and b) and a modulus of continuity σ_1 (defined on $[0, \eta_1)$ and depending only on n and b) such that if v is a solution of the obstacle problem (1.3) in B^+ and $0 \in \Gamma'$ then*

$$\sup_{B^+} |v_r - \frac{x_1}{b}| \leq \sigma_1(r) \ \text{ for } \ 0 < r < \eta_1. \tag{9.1}$$

Proof. We argue by contradiction. If the claim of the lemma does not hold then there exists $\epsilon > 0$, v^j solutions of the obstacle problem (1.3) in B^+, $0 \in \Gamma'_{v^j}$ and $0 < r^j \to 0$ such that

$$\sup_{B^+} |v^j_{r^j} - \frac{x_1}{b}| \geq \epsilon. \tag{9.2}$$

We have that $v^j_{r^j}$ is solution of the obstacle problem (1.3) in $B^+_{(r^j)^{-1}}$ and $0 \in \Gamma'(v^j_{r_j})$ (here $\Gamma'(v^j_{r_j})$ denotes the contact set associated with $v^j_{r_j}$) thus by the Corollaries 1, 2 and 4 there exists $C > 0$ such that for any $R > 0$ and large enough j we have the uniform bounds

$$\|v^j_{r^j}\|_{C(B^+_R)} \leq C(1 + R), \ [v^j_{r^j}]_{C^1(B^+_R)} \leq C$$

and

$$[v^j_{r^j}]_{C^{1,1}(B_R \cap \{x_1 > \delta\})} \leq \frac{C}{\delta} \ \text{ for } \ \delta > 0.$$

Arguing as in the first step of the proof of Theorem 3 we obtain that there exists a subsequence j_k and a global solution v_0 such that $v^{j_k}_{r_{j_k}} \to v_0$ in $C^{0,\alpha}(B^+_R)$ for any $R > 0$ and $0 < \alpha < 1$ and in $C^{1,\alpha}(B_R \cap \{x_1 > \delta\})$ for any $R > 0$, $0 < \alpha < 1$ and $\delta > 0$.

We have $0 \in \Gamma'_{v^j} \subset \partial\Omega_{v^j}$ thus by Corollary 6 for $\tau > 0$ and large enough j we have

$$\sup_{B^+_\tau} v^j_{r_j} = \frac{1}{r_j} \sup_{B^+_{r_j\tau}} v^j \geq c\tau$$

and passing to the limit $r_{j_k} \to 0$ we obtain

$$\sup_{B^+_\tau} v_0 \geq c\tau \ \text{ for } \ \tau > 0.$$

So we have $0 \in \partial\{v_0 > 0\}$.

Because $v^j_{r_j}$ is uniformly continuous in B^+ and $0 \in \Gamma'(v^j_{r_j})$ we have $v^j_{r_j}(0) = 0$. Because $v^{j_k}_{r_{j_k}} \to v_0$ uniformly in $\overline{B^+}$ we have $v_0(0) = 0$.

Thus we have $v_0 \in P_\infty$, $v_0(0) = 0$ and $0 \in \partial\{v_0 > 0\}$ and consequently from Theorem 3 we obtain $v_0 = \frac{x_1}{b}$ and this contradicts (9.2). □

Theorem 5. *Let $b > 1$. There exists $0 < \eta_2 < 1$ (depending only on n and b) and a modulus of continuity σ_2 (defined on $[0, \eta_2)$ and depending only on n and b) such that if v is a solution of the obstacle problem (1.3) in B^+ (corresponding to the parameter b) and $0 \in \Gamma'$ then we have*

$$\left\{x \in (0, \eta_2) \times B^{n-1}_{\eta_2} \ \middle| \ x_1 > |x'|\sigma_2(|x'|)\right\} \subset \Omega.$$

Proof. Let $x \in B^+_{\eta_1}$, $r = |x|$ and $y = \frac{x}{|x|}$ then using (9.1) we have

$$\begin{aligned} v(x) = r v_r(y) &= r\frac{y_1}{b} + r\big(v_r(y) - \frac{y_1}{b}\big) \geq \frac{x_1}{b} - r|v_r(y) - \frac{y_1}{b}| \\ &\geq \frac{x_1}{b} - r\sup_{B^+}|v_r - \frac{x_1}{b}| \geq \frac{x_1}{b} - r\sigma_1(r) = \frac{x_1}{b} - |x|\sigma_1(|x|) \end{aligned}$$

thus

$$\left\{x \in B^+_{\eta_1} \;\middle|\; x_1 > b|x|\sigma_1(|x|)\right\} \subset \Omega. \tag{9.3}$$

Let $0 < \eta_2 < \frac{1}{2}\eta_1$ be small enough such that $2b\sigma_1(2\eta_2) < 1$. Let us define $\sigma_2(\eta) = 2b\sigma_1(2\eta)$ for $\eta \in [0, \eta_2)$. We claim that

$$\left\{x \in (0,\eta_2) \times B^{n-1}_{\eta_2} \;\middle|\; x_1 > |x'|\sigma_2(|x'|)\right\} \subset \left\{x \in B^+_{\eta_1} \;\middle|\; x_1 > b|x|\sigma_1(|x|)\right\}. \tag{9.4}$$

Assume $x \in (0,\eta_2) \times B^{n-1}_{\eta_2}$ such that $|x'|\sigma_2(|x'|) < x_1$.

We compute

$$|x| \le x_1 + |x'| \le 2\eta_2 < \eta_1$$

thus $x \in B^+_{\eta_1}$.

In the case $|x'| \le x_1$ we estimate

$$b|x|\sigma_1(|x|) \le b(x_1 + |x'|)\sigma_1(x_1 + |x'|) \le 2bx_1\sigma_1(2x_1) \le 2b\sigma_1(2\eta_2)x_1 < x_1$$

and in the case $x_1 < |x'|$ we estimate

$$\begin{aligned} b|x|\sigma_1(|x|) &\le b(x_1 + |x'|)\sigma_1(x_1 + |x'|) \\ &\le bx_1\sigma_1(x_1 + |x'|) + b|x'|\sigma_1(x_1 + |x'|) \\ &\le bx_1\sigma_1(2\eta_2) + b|x'|\sigma_1(2|x'|) \\ &\le \frac{1}{2}x_1 + \frac{1}{2}|x'|\sigma_2(|x'|) < \frac{1}{2}x_1 + \frac{1}{2}x_1 = x_1. \end{aligned}$$

thus $b|x|\sigma_1(|x|) < x_1$ holds in both cases of $|x'| \le x_1$ and $x_1 < |x'|$. Therefore the inclusion (9.4) holds.

By (9.3) and (9.4) the theorem is proved. □

10. C^1 regularity of the free boundary and proof of Theorem 4

For a function v defined in B^+ and $x^0 \in B^+$ we define

$$v_{x^0}(y) = \frac{1}{x_1^0} v\big(x_1^0(y - e_1) + x^0\big) \quad \text{for} \quad y \in B^+_{\frac{1}{x_1^0}}\big(-\frac{1}{x_1^0}(x^0)'\big). \tag{10.1}$$

For $0 < \eta < \frac{1}{2}$ and $x^0 \in (0,\eta) \times B^{n-1}_{\frac{1}{2}}$ we have $x^0 \in B^+$ and

$$\begin{aligned} B^+_{\frac{1}{2\eta}} &\subset B^+_{\frac{1}{2\eta} + \frac{1}{x_1^0}|(x^0)'|}\big(-\frac{1}{x_1^0}(x^0)'\big) = B^+_{\frac{1}{x_1^0}(\frac{x_1^0}{2\eta} + |(x^0)'|)}\big(-\frac{1}{x_1^0}(x^0)'\big) \\ &\subset B^+_{\frac{1}{x_1^0}(\frac{\eta}{2\eta} + \frac{1}{2})}\big(-\frac{1}{x_1^0}(x^0)'\big) = B^+_{\frac{1}{x_1^0}}\big(-\frac{1}{x_1^0}(x^0)'\big) \end{aligned}$$

and therefore v_{x^0} is well defined in $B^+_{\frac{1}{2\eta}}$.

Lemma 9. *There exists $0 < \eta_3 < \frac{1}{4}$ (depending only on n and b) and a modulus of continuity σ_3 (defined on $[0, \eta_3)$ and depending only on n and b) such that if v is a solution of the obstacle problem (1.3) in B^+ and $x^0 \in \big((0,\eta_3) \times B^{n-1}_{\frac{1}{2}}\big) \cap \Gamma_v$ then we have*

$$\|v_{x^0} - w_1(y_1)\chi_{\{y_1 > 1\}}\|_{C^1(B_{\frac{1}{2}}(e_1))} \le \sigma_3(x_1^0)$$

where the functions v_{x^0} and w_1 are respectively defined in (10.1) and (3.6). Let us note that because $x_1^0 < \eta_3 < \frac{1}{4}$ by the remark before the lemma we have that v_{x^0} is well defined in B_2^+ and this contains $B_{\frac{1}{2}}(e_1)$.

In short, this lemma says that v_{x^0} (a scaling of v) is $\sigma_3(x_1^0)$ close to the class P_∞.

Proof. We argue by contradiction. If the claim of the lemma does not hold then there exists $\epsilon > 0$, v^j solutions of the obstacle problem (1.3) in B^+, $x^j \in \big((0, \frac{1}{4}) \times B^{n-1}_{\frac{1}{2}}\big) \cap \Gamma_{v^j}$ with $x^j_1 \to 0$ such that

$$\|v^j_{x^j} - w_1(y_1)\chi_{\{y_1>1\}}\|_{C^1(B_{\frac{1}{2}}(e_1))} \geq \epsilon. \tag{10.2}$$

For $R > 0$ because $x^j_1 \to 0$ by the remark before the lemma we have that $v^j_{x^j}$ is well defined in B^+_R for large enough j. One may check that actually it is a solution of the obstacle problem (1.3) in B^+_R.

Arguing as in Lemma 8 there exists a subsequence $j_k \to \infty$ and a global solution v_0 such that $v^{j_k}_{x^{j_k}} \to v_0$ in $C^{0,\alpha}(B^+_R)$ for any $0 < \alpha < 1$ and $R > 0$ and in $C^{1,\alpha}(B_R \cap \{x_1 > \delta\})$ for any $0 < \alpha < 1$, $R > 0$ and $0 < \delta < 1$.

We have $x^j \in \Gamma_{v^j}$ thus by Corollary 6 for $0 < \tau < 1$ and large enough j we have

$$\sup_{B_\tau(e_1)} v^j_{x^j} = \frac{1}{x^j_1} \sup_{B_{x^j_1\tau}(x^j)} v \geq \frac{1}{x^j_1} \frac{c(x^j_1\tau)^2}{x^j_1\tau + x^j_1} = \frac{c\tau^2}{\tau+1}$$

and passing to the limit $j_k \to \infty$ we obtain

$$\sup_{B_\tau(e_1)} v_0 \geq \frac{c\tau^2}{\tau+1} \quad \text{for} \quad \tau > 0$$

thus $e_1 \in \overline{\{v_0 > 0\}}$.

Because $x^j \in \Gamma_{v^j}$ we have that $v^j_{x^j}(e_1) = 0$ thus we have $v_0(e_1) = 0$. So together with $e_1 \in \overline{\{v_0 > 0\}}$ we obtain that $e_1 \in \Gamma_{v_0}$.

Hence we have $v_0 \in P_\infty$ and $e_1 \in \Gamma_{v_0}$. From Theorem 3 we obtain $v_0 = w_1(y_1)\chi_{\{y_1>1\}}$. By the $C^{1,\frac{1}{2}}(B_2 \cap \{y_1 > \frac{1}{4}\})$ convergence of $v^j_{x^j}$ to v_0 we come to contradiction with (10.2) and this proves the lemma. □

Remark 2. By considering respectively $\frac{1}{2}\eta_3$ and $\frac{\eta_3\tau}{\eta_3 - 2\tau} + \frac{1}{\tau}\int_\tau^{2\tau} \sigma_3(s)ds$ instead of η_3 and $\sigma_3(\tau)$ we might assume that $\sigma_3(+0) = 0$, σ_3 is strictly increasing, σ_3 is continuous on $(0, \eta_3)$, $\tau \leq \sigma_3(\tau)$ for $0 < \tau < \eta_3$ and $\lim_{\tau\to\eta_3, \tau<\eta_3} \sigma_3(\tau) = +\infty$.

Let us note that by Remark 2 for $0 \leq t < +\infty$ the inverse $0 \leq \sigma_3^{-1}(t) < \eta_3$ is well defined.

Lemma 10. *There exists $0 < \zeta_1 < 1$ and $\epsilon_1 > 0$ such that if v is a solution of the obstacle problem (1.3) in $B(e_1)$, $v(e_1) = 0$, $\upsilon \in \partial B \cap \{\upsilon_1 > 0\}$ and*

$$\|v - w_1(y_1)\chi_{\{y_1>1\}}\|_{C^1(B_{\frac{1}{2}}(e_1))} \leq \epsilon_1 \upsilon_1$$

then

$$\partial_\upsilon v \geq 0 \quad in \quad B_{\zeta_1}(e_1).$$

Proof. Let $c > 0$ and $\upsilon \in \partial B \cap \{\upsilon_1 > 0\}$. Near to $y_1 = 1$ we have that $\partial_\upsilon w_1(y_1)$ is linear and $w_1(y_1)$ is quadratic thus one expects to have $c\partial_\upsilon w_1(y_1) - w_1(y_1) \geq 0$ for large enough $c > 0$. We shall however give an exact estimate for $1 < y_1 < 2$

$$\begin{aligned} c\partial_\upsilon w_1(y_1) - w_1(y_1) &= c\upsilon_1 w_1'(y_1) - w_1(y_1) \\ &= c\upsilon_1\Big(\frac{1}{b}(1 - \frac{1}{y_1^b})\Big) - \Big(\frac{y_1}{b} - \frac{1}{b-1} + \frac{1}{b(b-1)}\frac{1}{y_1^{b-1}}\Big) \\ &= \frac{1}{by_1^b}(y_1^b - 1)\Big(c\upsilon_1 - y_1\frac{(b-1)y_1^b - by_1^{b-1} + 1}{(b-1)(y_1^b - 1)}\Big) \\ &\geq \frac{1}{by_1^b}(y_1^b - 1)(c\upsilon_1 - y_1) \geq \frac{1}{by_1^b}(y_1^b - 1)(c\upsilon_1 - 2). \end{aligned} \tag{10.3}$$

Thus taking

$$c = \frac{2}{\upsilon_1} \tag{10.4}$$

we have

$$c\partial_\upsilon w_1(y_1) - w_1(y_1) \geq 0 \text{ in } B(e_1) \cap \{y_1 > 1\}. \tag{10.5}$$

Let $c_1 > 0$ and $0 < \epsilon_0 < 1$ be as in Lemma 5, $C_2 > 0$ be the constant in Corollary 3 and define

$$\epsilon = \min\Big(\frac{\epsilon_0}{2}, \frac{c_1}{2}\Big), \; r_1 = \min\Big(\frac{1}{2}, \frac{9}{2bC_2}\Big) \text{ and } c_3 = \frac{\epsilon}{16}\Big(\frac{r_1}{18}\Big)^2 c_1.$$

We claim that if $\upsilon \in \partial B \cap \{\upsilon_1 > 0\}$, c as in (10.4) and

$$\big\|v - w_1(y_1)\chi_{\{y_1>0\}}\big\|_{C(B_{\frac{1}{9}r_1}(e_1))} + \sum_{j=1}^{n} \big\|\partial_{y_j} v - \partial_{y_j} w_1(y_1)\chi_{\{y_1>0\}}\big\|_{C(B_{\frac{1}{9}r_1}(e_1))} \leq \frac{1}{2}c_3\upsilon_1 \tag{10.6}$$

then $c\partial_\upsilon v - v \geq 0$ in $B_{\frac{r_1}{18}}(e_1)$.

Let us argue by contradiction. Assume there exists $y^0 \in B_{\frac{r_1}{18}}(e_1)$ such that $c\partial_\upsilon v(y^0) - v(y^0) < 0$. Let us define

$$h(y) = v(y) - c\partial_\upsilon v(y) - \frac{1}{2}\Big(w_{y_1^0}(y_1) + \frac{\epsilon}{y_1^0} p_{y^0}(y)\Big).$$

We have

$$B_{\frac{r_1}{18}}(y^0) \subset B_{\frac{r_1}{18}+\frac{r_1}{18}}(e_1) = B_{\frac{r_1}{9}}(e_1). \tag{10.7}$$

In particular we have that $B_{\frac{r_1}{18}}(y^0) \subset B_{\frac{r_1}{9}}(e_1) \subset \mathbb{R}^n_+$, where the last inclusion holds because by the definition of r_1 we have $\frac{r_1}{9} < 1$.

Because $y^0 \in B_{\frac{r_1}{18}}(e_1)$ we have $1 - \frac{r_1}{18} < y_1^0$ and because $\epsilon \leq \frac{c_1}{2}$ we have $2 \leq \frac{c_1}{\epsilon}$, therefore for $y \in B_{\frac{r_1}{18}}(y^0)$ we have

$$y_1 < y_1^0 + \frac{r_1}{18} < y_1^0 + 1 - \frac{r_1}{18} < 2y_1^0 \leq \frac{c_1}{\epsilon} y_1^0 \tag{10.8}$$

and by Lemma 5 we have

$$w_{y_1^0}(y_1) + \frac{\epsilon}{y_1^0} p_{y^0}(y) \geq \frac{c_1}{y_1^0}\epsilon|y - y^0|^2 \geq 0 \text{ for } y \in B_{\frac{r_1}{18}}(y^0). \tag{10.9}$$

In Ω we have

$$\Delta h(y) + \frac{b}{y_1}\partial_{y_1} h(y) = \frac{1}{2y_1} + \frac{c\upsilon_1}{y_1^2}\big(1 - b\partial_{y_1} v(y)\big).$$

By Corollary 3 and the definition of r_1 we have

$$|\nabla v(y)| \leq C_2|y - e_1| < \frac{1}{9}C_2 r_1 \leq \frac{1}{2b} \text{ for } y \in B_{\frac{r_1}{9}}(e_1). \tag{10.10}$$

Using (10.7) and (10.10) for $y \in \Omega \cap B_{\frac{r_1}{18}}(y^0)$ we have

$$\Delta h(y) + \frac{b}{y_1}\partial_{y_1} h(y) \geq \frac{1}{2y_1} + \frac{c\upsilon_1}{y_1^2}\big(1 - b|\nabla v(y)|\big) \geq \frac{1}{2y_1} + \frac{c\upsilon_1}{2y_1^2} \geq 0.$$

Since

$$h(y) \leq 0 \text{ for } y \in B_{\frac{r_1}{18}}(y^0) \cap \Gamma \tag{10.11}$$

and

$$\partial(B_{\frac{r_1}{18}}(x^0)\cap\Omega)\subset\big(\overline{B}_{\frac{r_1}{18}}(x^0)\cap\Gamma\big)\cup\big(\partial B_{\frac{r_1}{18}}(x^0)\cap\Omega\big)$$

after applying the maximum principle in the domain $B_{\frac{r_1}{18}}(y^0)\cap\Omega$ we arrive at

$$\begin{aligned}0<v(y^0)-c\partial_\upsilon v(y^0)=h(y^0)&\le\sup_{\partial(B_{\frac{r_1}{18}}(y^0)\cap\Omega)}h\\&=\max\Big(\sup_{\overline{B}_{\frac{r_1}{18}}(y^0)\cap\Gamma}h,\ \sup_{\partial B_{\frac{r_1}{18}}(y^0)\cap\Omega}h\Big).\end{aligned}\tag{10.12}$$

From (10.11) and (10.12) we obtain

$$\begin{aligned}0\le\sup_{\partial B_{\frac{r_1}{18}}(y^0)\cap\Omega}h\le&\sup_{\partial B_{\frac{r_1}{18}}(y^0)\cap\Omega}(v-c\partial_\upsilon v)\\&-\frac{1}{2}\inf_{y\in\partial B_{\frac{r_1}{18}}(y^0)\cap\Omega}\big(w_{y_1^0}(y_1)+\frac{\epsilon}{y_1^0}p_{y^0}(y)\big).\end{aligned}\tag{10.13}$$

We have

$$\begin{aligned}\frac{1}{2}\frac{c_1}{y_1^0}\epsilon(\frac{r_1}{18})^2&\ge\frac{1}{4c}(1+c)\frac{c_1}{y_1^0}\epsilon(\frac{r_1}{18})^2=\frac{1}{8}(1+c)\frac{c_1}{y_1^0}\epsilon(\frac{r_1}{18})^2\upsilon_1\\&\ge\frac{1}{8}(1+c)\frac{c_1}{2}\epsilon(\frac{r_1}{18})^2\upsilon_1=(1+c)c_3\upsilon_1.\end{aligned}\tag{10.14}$$

By (10.9), (10.13) and (10.14) we obtain

$$(1+c)c_3\upsilon_1\le\frac{1}{2}\frac{c_1}{y_1^0}\epsilon(\frac{r_1}{18})^2\le\sup_{\partial B_{\frac{r_1}{18}}(y^0)\cap\Omega}(v-c\partial_\upsilon v).\tag{10.15}$$

Using (10.5) and (10.7) we estimate

$$\begin{aligned}\sup_{\partial B_{\frac{r_1}{18}}(y^0)\cap\Omega}\big(v-c\partial_\upsilon v\big)\le&\sup_{y\in\partial B_{\frac{r_1}{18}}(y^0)\cap\Omega}\big(w_1(y_1)\chi_{\{y_1>0\}}-c\partial_\upsilon(w_1(y_1)\chi_{\{y_1>0\}})\big)\\&+\big\|v-c\partial_\upsilon v-\big(w_1(y_1)\chi_{\{y_1>0\}}-c\partial_\upsilon(w_1(y_1)\chi_{\{y_1>0\}})\big)\big\|_{C(B_{\frac{r_1}{18}}(y^0))}\\\le&\sup_{y\in B(e_1)}\big(w_1(y_1)\chi_{\{y_1>0\}}-c\partial_\upsilon(w_1(y_1)\chi_{\{y_1>0\}})\big)\\&+\big\|v-c\partial_\upsilon v-\big(w_1(y_1)\chi_{\{y_1>0\}}-c\partial_\upsilon(w_1(y_1)\chi_{\{y_1>0\}})\big)\big\|_{C(B_{\frac{r_1}{9}}(e_1))}\\\le&(1+c)\Big(\big\|v-w_1(y_1)\chi_{\{y_1>0\}}\big\|_{C(B_{\frac{r_1}{9}}(e_1))}\\&+\sum_{j=1}^{n}\big\|\partial_{y_j}v-\partial_{y_j}w_1(y_1)\chi_{\{y_1>0\}}\big\|_{C(B_{\frac{r_1}{9}}(e_1))}\Big).\end{aligned}\tag{10.16}$$

By (10.6), (10.15) and (10.16) we come to a contradiction and this proves our claim.
Because $c>0$ from the claim by taking $\zeta_1=\frac{r_1}{18}$ the lemma is proved. □

Corollary 7. *Let ζ_1, ϵ_1 and v be as in Lemma 10. Let $0<\delta<1$ and*

$$\|v-w_1(y_1)\chi_{\{y_1>1\}}\|_{C^1(B_{\frac{1}{2}}(e_1))}\le\epsilon_1\delta$$

then for all $y\in\Gamma\cap B_{\frac{1}{2}\zeta_1}(e_1)$ we have

$$B_{\frac{1}{2}\zeta_1}(y)\cap\big(y+\mathcal{C}_\delta\big)\subset\{v>0\}\ \ \textit{and}\ \ B_{\frac{1}{2}\zeta_1}(y)\cap\big(y-\mathcal{C}_\delta\big)\subset\{v=0\}.\tag{10.17}$$

Proof. By Lemma 10 and the definition of $\mathcal{C}_\delta$ we have that for all $\upsilon \in \mathcal{C}_\delta \cap \partial B$

$$\partial_\upsilon v \geq 0 \ \text{ in } \ B_{\zeta_1}(e_1). \tag{10.18}$$

From (10.18) because $v \geq 0$ it follows that

$$\text{if } \ y \in B_{\frac{1}{2}\zeta_1}(e_1) \ \text{ and } \ v(y) = 0 \ \text{ then } \ B_{\frac{1}{2}\zeta_1}(y) \cap (y - \mathcal{C}_\delta) \subset \{v = 0\}. \tag{10.19}$$

This in particular proves the second inclusion in (10.17).

Let $y \in B_{\frac{1}{2}\zeta_1}(e_1) \cap \Gamma$. If there exists $y^1 \in B_{\frac{1}{2}\zeta_1}(y) \cap (y + \mathcal{C}_\delta)$ such that $v(y^1) = 0$, then by (10.19) we have that $v = 0$ in $B_{\frac{1}{2}\zeta_1}(y^1) \cap (y^1 - \mathcal{C}_\delta)$. From $y^1 \in y + \mathcal{C}_\delta$ it follows that $y \in y^1 - \mathcal{C}_\delta$, thus y is in the interior of $B_{\frac{1}{2}\zeta_1}(y^1) \cap (y^1 - \mathcal{C}_\delta)$ where we have $v = 0$ and this contradicts with $y \in \Gamma$. This proves the first inclusion in (10.17). □

Lemma 11. *There exists $0 < \zeta_2 < 1$, $c_1 > 0$ and $C_2 > 0$ such that if v is a solution of the obstacle problem (1.3) in $B(e_1)$ and $v(e_1) = 0$ then v solves the obstacle problem*

$$\Delta v = f \chi_{\{v>0\}} \ \text{ in } \ B_{\zeta_2}(e_1)$$

with $f \geq c_1$ in $B_{\zeta_2}(e_1)$ and $\|f\|_{C^{0,1}(B_{\zeta_2}(e_1))} \leq C_2$.

Proof. Because v solves the obstacle problem (1.3) in $B(e_1)$ we have

$$\Delta v = f \chi_{\{v>0\}} \ \text{ in } \ B(e_1)$$

where

$$f(y) = (1 - b\partial_{y_1} v) y_1^{-1} \ \text{ for } \ y \in B(e_1).$$

By Corollary 3, choosing ζ_2 small enough there exists $c_1 > 0$ and $C_2 > 0$ such that $c_1 \leq f \leq C_2$ in $B_{\zeta_2}(e_1)$.

By Corollary 5 and Theorem of Rademacher we have

$$|\nabla^2 v(y)| \leq C_3 \ \text{ for a.e. } \ y \in B_{\zeta_2}(e_1).$$

It follows that $|\nabla_{y'} f| \leq C_4$ a.e. in $B_{\zeta_2}(e_1)$. Together with Corollary 3 it follows that $|\partial_{y_1} f| \leq C_5$ a.e. in $B_{\zeta_2}(e_1)$ and this completes the proof of the lemma. □

Proof of Theorem 4. **Step 1.** Let $0 < \zeta_1 < 1$ and $\epsilon_1 > 0$ as in Lemma 10.

By Lemma 9 and Corollary 7 for all $0 < \delta < 1$ if v is a solution of the obstacle problem (1.3) in B^+ with $0 \in \Gamma'$ and $x^0 \in \big((0, \sigma_3^{-1}(\epsilon_1\delta)) \times B^{n-1}_{\frac{1}{2}}\big) \cap \Gamma_v$ then for all $y \in \Gamma_{v_{x^0}} \cap B_{\frac{1}{2}\zeta_1}(e_1)$ we have

$$B_{\frac{1}{2}\zeta_1}(y) \cap \big(y + \mathcal{C}_\delta\big) \subset \{v_{x^0} > 0\} \ \text{ and } \ B_{\frac{1}{2}\zeta_1}(y) \cap \big(y - \mathcal{C}_\delta\big) \subset \{v_{x^0} = 0\}.$$

Step 2. In this step we show that there exists $0 < r_1 < \frac{1}{2}$ such that if v is a solution of the obstacle problem (1.3) in B^+ with $0 \in \Gamma'$ then

$$\mathcal{C}_{\frac{1}{2}\sqrt{2}} \cap \big((0, r_1] \times B^{n-1}_{r_1}\big) \subset \Omega. \tag{10.20}$$

Let η_2 and σ_2 as in Theorem 5. By choosing $0 < r_1 < \eta_2$ small enough such that $\sigma_2(r_1) < 1$ from the definition of $\mathcal{C}_{\frac{1}{2}\sqrt{2}}$ it follows that

$$\mathcal{C}_{\frac{1}{2}\sqrt{2}} \cap \big((0, r_1] \times B^{n-1}_{r_1}\big) \subset \Big\{x \in (0, \eta_2) \times B^{n-1}_{\eta_2} \ \Big| \ x_1 > |x'|\sigma_2(|x'|)\Big\}. \tag{10.21}$$

Now because $0 \in \Gamma'$ from Theorem 5 and (10.21) we obtain (10.20).

Step 3. In this step we show that there exists $0 < r_2 < r_1$ such that for v solution of the obstacle problem (1.3) in B^+ with $0 \in \Gamma'$ there exists $g_v : B^{n-1}_{r_2} \to [0, r_2)$ such that

$$\Omega \cap \big((0, r_2) \times B^{n-1}_{r_2}\big) = \Big\{x \;\Big|\; g_v(x') < x_1 < r_2,\, x' \in B^{n-1}_{r_2}\Big\} \tag{10.22}$$

and

$$g_v(x') \le |x'| \;\text{ for }\; x' \in B^{n-1}_{r_2}. \tag{10.23}$$

Let us define

$$r_2 = \min\Big(\frac{\sqrt{2}}{2} r_1, \frac{1}{2}, \sigma_3^{-1}\big(\frac{1}{2}\epsilon_1\big)\Big).$$

Fix $x' \in B^{n-1}_{r_2}$ and define $k(t) = v(t, x')$ for $t \in (0, r_1]$. By the second step we know that $k(r_1) > 0$. Then by the continuity of k we know that in a neighbourhood of r_1, k is positive.

Let us denote

$$U = \big\{t \in (0, r_1] \mid k(t) > 0\big\}$$

then U is open in the relative topology of $(0, r_1]$.

Let us show that U is connected. We assume that U has a component I not containing r_1. Then we have $I = (t_1, t_2)$ with $t_2 < r_1$. Let us define $x^0 = (t_2, x')$ then we have $x^0 \in \Gamma_v$. For $x^0 \in (0, r_2) \times B^{n-1}_{r_2}$ we have $x^0 \in \big(0, \sigma_3^{-1}(\frac{1}{2}\epsilon_1)\big) \times B^{n-1}_{\frac{1}{2}}$. Thus by the first step we have

$$B_{\frac{1}{2}\zeta_1}(e_1) \cap \big(e_1 - \mathcal{C}_{\frac{1}{2}}\big) \subset \{v_{x^0} = 0\}.$$

But this contradicts with $k > 0$ in (t_1, t_2). This contradiction proves that U is connected.

Let $U = (t_0, r_1]$ and let us define $g_v(x') = t_0$.

By the definition of U the equation (10.22) follows. By the second step the inequality (10.23) follows.

Step 4. In this step we show that there exists $0 < r_3 < r_2$, $0 < r_4 < 1$ and $0 < \delta_1 < 1$ such that if $0 < \delta < \delta_1$, v is solution of the obstacle problem (1.3) in B^+ with $0 \in \Gamma'$ and

$$x^0 \in \Gamma_v \cap \big((0, \sigma_3^{-1}(\epsilon_1 \delta)) \times B^{n-1}_{r_3}\big)$$

then we have that $(0, 2) \times B^{n-1}_1$ is in the domain of v_{x^0} and

$$\Omega_{v_{x^0}} \cap \big((0, 2) \times B^{n-1}_1\big) = \Big\{y \;\Big|\; g_{v,x^0}(y') < y_1 < 2,\, y' \in B^{n-1}_1\Big\}$$

where

$$g_{v,x^0}(y') = \frac{1}{x^0_1} g_v\big(x^0_1 y' + (x^0)'\big) \;\text{ for }\; y' \in B^{n-1}_1,$$

$g_{v,x^0}(0) = 1$ and g_{v,x^0} is Lipschitz continuous in $B^{n-1}_{r_4}$ with Lipschitz constant not exceeding 2δ.

Let

$$x = x^0_1(y - e_1) + x^0 = x^0_1 y + (x^0)' \;\text{ and }\; y = \frac{1}{x^0_1}\big(x - (x^0)'\big),$$

$r_3 = \frac{1}{2} r_2$ and take $0 < \delta_1 \le \frac{1}{2}$ small enough such that $\sigma_3^{-1}(\epsilon_1 \delta_1) \le r_3$.

Because $2r_3 = r_2$, by the third step we have

$$\Omega \cap \big((0, 2r_3) \times B^{n-1}_{2r_3}\big) = \Big\{x \;\Big|\; g_v(x') < x_1 < 2r_3,\, x' \in B^{n-1}_{2r_3}\Big\}.$$

From here by translating by $-(x^0)'$ and scaling by $\frac{1}{x^0_1}$ we obtain

$$\begin{aligned}&\Omega_{v_{x^0}} \cap \Big((0, \frac{1}{x^0_1} 2r_3) \times B^{n-1}_{\frac{1}{x^0_1} 2r_3}\big(-\frac{1}{x^0_1}(x^0)'\big)\Big)\\ &\quad = \Big\{y \;\Big|\; g_{v,x^0}(y') < y_1 < \frac{1}{x^0_1} 2r_3,\, y' \in B^{n-1}_{\frac{1}{x^0_1} 2r_3}\big(-\frac{1}{x^0_1}(x^0)'\big)\Big\}.\end{aligned} \tag{10.24}$$

One may see that because $x^0 \in (0, r_3) \times B^{n-1}_{r_3}$ we have

$$(0,2) \times B^{n-1}_1 \subset (0, \frac{1}{x^0_1} 2r_3) \times B^{n-1}_{\frac{1}{x^0_1}2r_3}\big(-\frac{1}{x^0_1}(x^0)'\big).$$

Thus from (10.24) we obtain

$$\Omega_{v_{x^0}} \cap \big((0,2) \times B^{n-1}_1\big) = \Big\{y \;\Big|\; g_{v,x^0}(y') < y_1 < 2,\, y' \in B^{n-1}_1\Big\}. \tag{10.25}$$

By the first step we have that for $0 < \delta < \delta_1$ because $x^0 \in \big((0, \sigma_3^{-1}(\epsilon_1\delta)) \times B^{n-1}_{r_3}\big) \cap \Gamma_v$ for $y \in \Gamma_{v_{x^0}} \cap B_{\frac{1}{2}\zeta_1}(e_1)$ we have

$$B_{\frac{1}{2}\zeta_1}(y) \cap (y + \mathcal{C}_\delta) \subset \{v_{x^0} > 0\} \text{ and } B_{\frac{1}{2}\zeta_1}(y) \cap (y - \mathcal{C}_\delta) \subset \{v_{x^0} = 0\}. \tag{10.26}$$

Because $0 < \delta < \delta_1 \le \frac{1}{2}$ we have that $\mathcal{C}_{\frac{1}{2}} \subset \mathcal{C}_\delta$ thus taking $y = e_1$ we have

$$B_{\frac{1}{2}\zeta_1}(e_1) \cap (e_1 + \mathcal{C}_{\frac{1}{2}}) \subset \{v_{x^0} > 0\} \text{ and } B_{\frac{1}{2}\zeta_1}(e_1) \cap (e_1 - \mathcal{C}_{\frac{1}{2}}) \subset \{v_{x^0} = 0\}. \tag{10.27}$$

From (10.27) it follows that by defining $r_4 = \frac{1}{4}\zeta_1$ for $y' \in B^{n-1}_{r_4}$ we have $y = (g_{v,x^0}(y'), y') \in \Gamma_{v_{x^0}} \cap B_{\frac{1}{2}\zeta_1}(e_1)$. Now from the inclusions in (10.26) it follows that g_{v,x^0} is Lipschitz continuous in $B^{n-1}_{r_4}$ with Lipschitz constant not exceeding $(1-\delta^2)^{-\frac{1}{2}}\delta$.

Because $0 < \delta < \frac{1}{2}$ we have that $(1-\delta^2)^{-\frac{1}{2}} < 2$ and this finishes the proof of the claim of this step.

Step 5. Let r_3, r_4 and δ_1 as in the forth step. In this step we show that there exists $0 < r_5 < r_4$, $0 < \delta_2 < \delta_1$ and a modulus of continuity σ_4 such that for $0 < \delta < \delta_2$, v a solution of the obstacle problem (1.3) in B^+ with $0 \in \Gamma'$ and

$$x^0 \in \Gamma_v \cap \big((0, \sigma_3^{-1}(\epsilon_1\delta)) \times B^{n-1}_{r_3}\big)$$

we have $g_{v,x^0} \in C^1(B^{n-1}_{r_5})$, σ_4 is a modulus of continuity for $\nabla_{y'} g_{v,x^0}$ in $B^{n-1}_{r_5}$ and $|\nabla_{y'} g_{v,x^0}| \le 2\delta$ in $B^{n-1}_{r_5}$.

We have that v_{x^0} is a solution in $B(e_1)$ with $v_{x^0}(e_1) = 0$. By Lemma 11 we have that v_{x^0} is the solution of the obstacle problem

$$\Delta v_{x^0} = f\chi_{\{v_{x^0}>0\}} \text{ in } B_{\zeta_2}(e_1)$$

with $f \ge c_1 > 0$ in $B_{\zeta_2}(e_1)$ and $\|f\|_{C^{0,1}(B_{\zeta_2}(e_1))} \le C_2$.

By the forth step we have that g_{v,x^0} is Lipschitz continuous in $B^{n-1}_{r_4}$ with Lipschitz constant not exceeding 2δ.

Let $\tilde{r}_5 = \min(\frac{2\sqrt{5}}{5}\zeta_2, r_4)$ and $\tilde{\delta}_2 = \min(\delta_1, \frac{1}{4})$.

We have

$$(1 - \frac{1}{2}\tilde{r}_5, 1 + \frac{1}{2}\tilde{r}_5) \times B^{n-1}_{\tilde{r}_5} \subset B_{\zeta_2}(e_1).$$

For $\delta < \tilde{\delta}_2 \le \frac{1}{4}$ we have that g_{v,x^0} is Lipschitz continuous in $B^{n-1}_{\tilde{r}_5} \subset B^{n-1}_{r_4}$ with Lipschitz constant not exceeding $\frac{1}{2}$.

Thus by Lemma 21 there exists $0 < r_5 < \tilde{r}_5$, $d > 0$ and a modulus of continuity σ_4 such that if

$$\{x_1 < 1 - d\} \cap \big((1 - \frac{1}{2}\tilde{r}_5, 1 + \frac{1}{2}\tilde{r}_5) \times B^{n-1}_{\tilde{r}_5}\big) \subset \{v_{x^0} = 0\}$$

then $g_{v,x^0} \in C^1(B^{n-1}_{r_5})$ and σ_4 is a modulus of continuity for $\nabla_{x'} g_{v,x^0}$ in $B^{n-1}_{r_5}$.

Let us define $\delta_2 = \min(\tilde{\delta}_2, \frac{1}{2}\frac{d}{\tilde{r}_5})$. Then for $0 < \delta < \delta_2$ we have that

$$1 - d < 1 - 2\delta\tilde{r}_5 \le 1 - 2\delta|y'| \le g_{v,x^0}(y') \text{ for } y' \in B^{n-1}_{\tilde{r}_5}$$

and this proves the claim of this step.

Step 6. Let r_3, r_5, δ_2 and σ_4 as in the fifth step. In this step we show that there exists $0 < r_6 < r_3$ such that for v a solution of the obstacle problem (1.3) in B^+ with $0 \in \Gamma'$, $z \in B^{n-1}_{r_6}$ and $g_v(z) > 0$ we have

$$g_v \in C^1\big(B^{n-1}_{r_5 g_v(z)}(z)\big),$$

$\sigma_4(\frac{\cdot}{g_v(z)})$ is a modulus of continuity for $\nabla_{x'} g_v$ in $B^{n-1}_{r_5 g_v(z)}(z)$ and $|\nabla_{x'} g_v| \le \frac{1}{\epsilon_1}\sigma_3(g_v(z))$ in $B^{n-1}_{r_5 g_v(z)}(z)$.

It follows directly from the fifth step and the definition of g_{v,x^0} that for $0<\delta<\delta_2$ and

$$x^0 \in \Gamma_v \cap \big((0, \sigma_3^{-1}(\epsilon_1\delta)) \times B^{n-1}_{r_3}\big) \tag{10.28}$$

we have

$$g_v \in C^1\big(B^{n-1}_{r_5 x_1^0}((x^0)')\big),$$

$\sigma_4(\frac{\cdot}{x_1^0})$ is a modulus of continuity for $\nabla_{x'} g_v$ in $B^{n-1}_{r_5 x_1^0}((x^0)')$ and $|\nabla_{x'} g_v| \le 2\delta$ in $B^{n-1}_{r_5 x_1^0}((x^0)')$.

Let us define $r_6 = \min\big(\sigma_3^{-1}(\epsilon_1\delta_2), r_3\big)$. Let $z \in B^{n-1}_{r_6}$ such that $g_v(z) > 0$. By the third step we have that $g_v(z) \le r_6 \le \sigma_3^{-1}(\epsilon_1\delta_2)$ thus $\sigma_3(g_v(z)) \le \epsilon_1\delta_2$ and defining $\delta = \frac{1}{2\epsilon_1}\sigma_3(g_v(z))$ we have $0<\delta<\delta_2$. Let us define $x^0 = (g_v(z), z)$. Then (10.28) holds, so we have

$$g_v \in C^1\big(B^{n-1}_{r_5 g_v(z)}(z)\big),$$

$\sigma_4(\frac{\cdot}{g_v(z)})$ is a modulus of continuity for $\nabla_{x'} g_v$ in $B^{n-1}_{r_5 g_v(z)}(z)$ and $|\nabla_{x'} g_v| \le 2\delta = \frac{1}{\epsilon_1}\sigma_3(g_v(z))$ in $B^{n-1}_{r_5 g_v(z)}(z)$.

Step 7. Let r_6 as in step 6. In this step we show that for v a solution of the obstacle problem (1.3) in B^+ with $0 \in \Gamma'$ we have $g_v \in C(B^{n-1}_{r_6})$ and for $z \in B^{n-1}_{r_6}$ the gradient $\nabla_{x'} g_v(z)$ exists and if $g_v(z) = 0$ then $\nabla_{x'} g_v(z) = 0$.

For those $z \in B^{n-1}_{r_6}$ such that $g_v(z) > 0$ the continuity and differentiability of g_v at z follows from the sixth step.

Now let us consider a $z \in B^{n-1}_{r_6}$ such that $g_v(z) = 0$. Either $g_v = 0$ in a neighbourhood of z, in which case clearly g_v is continuous and differentiability at z and $\nabla_{x'} g_v(z) = 0$, or $(0,z)$ is a contact point. In the case $(0,z) \in \Gamma'$, let us translate the origin of $\mathbb{R}^n$ to $(0,z)$ and consider $\tilde{v}(x) = 2v(\frac{1}{2}x + (0,z))$. Then $\tilde{v}$ is a solution in B^+ with $0 \in \Gamma'_{\tilde{v}}$ Now applying a similar reasoning as in step 3 to $\tilde{v}$ we obtain that

$$\Omega_{\tilde{v}} \cap \big((0, r_2) \times B^{n-1}_{r_2}\big) = \Big\{x \;\Big|\; g_{\tilde{v}}(x') < x_1 < r_2,\, x' \in B^{n-1}_{r_2}\Big\}$$

and

$$g_{\tilde{v}}(x') \le |x'| \;\text{ for }\; x \in B^{n-1}_{r_2}.$$

From which by the definition of $\tilde{v}$ it follows that

$$\Omega_v \cap \Big((0, \frac{r_2}{2}) \times B^{n-1}_{\frac{r_2}{2}}(z)\Big) = \Big\{x \;\Big|\; g_v(x') < x_1 < \frac{r_2}{2},\, x' \in B^{n-1}_{\frac{r_2}{2}}(z)\Big\} \tag{10.29}$$

and

$$g_v(x') \le |x'-z| \;\text{ for }\; x' \in B^{n-1}_{\frac{r_2}{2}}(z). \tag{10.30}$$

From (10.30) it immediately follows that g_v is continuous at z.

Applying Theorem 5 to $\tilde{v}$ and obtain

$$\Big\{x \in (0,\eta_2) \times B^{n-1}_{\eta_2} \;\Big|\; x_1 > |x'|\sigma_2(|x'|)\Big\} \subset \Omega_{\tilde{v}}$$

and by the definition of $\tilde{v}$ it follows that

$$\Big\{x \in (0, \frac{\eta_2}{2}) \times B^{n-1}_{\frac{\eta_2}{2}}(z) \;\Big|\; x_1 > |x'-z|\sigma_2(2|x'-z|)\Big\} \subset \Omega_v. \tag{10.31}$$

Because $r_2 \le \eta_2$ by (10.29) and (10.31) we have

$$\begin{aligned}&\Big\{x \;\Big|\; |x'-z|\sigma_2(2|x'-z|) < x_1 < \frac{r_2}{2},\, x' \in B^{n-1}_{\frac{r_2}{2}}(z)\Big\}\\ &\quad\subset \Big\{x \;\Big|\; g_v(x') < x_1 < \frac{r_2}{2},\, x' \in B^{n-1}_{\frac{r_2}{2}}(z)\Big\}\end{aligned}$$

from which it follows that

$$g_v(x') \le |x'-z|\sigma_2(2|x'-z|) \ \text{ for } \ x' \in B^{n-1}_{\frac{r_2}{2}}(z). \tag{10.32}$$

Finally from (10.32) we obtain that g_v is differentiable at z and $\nabla_{x'} g_v(z) = 0$.

Step 8. In this step we finish the proof of the theorem.

Let v be a solution of the obstacle problem (1.3) in B^+ with $0 \in \Gamma'$. From the sixth and seventh steps it follows that if $z \in B^{n-1}_{r_6}$ then $|\nabla_{x'} g_v(z)| \le \frac{1}{\epsilon_1}\sigma_3(g_v(z))$.

Let $\epsilon > 0$ and take t_ϵ small enough such that $\frac{2}{\epsilon_1}\sigma_3(t_\epsilon) < \epsilon$ and γ_ϵ small enough such that $\gamma_\epsilon \le r_5 t_\epsilon$ and $\sigma_4(\frac{\gamma_\epsilon}{t_\epsilon}) < \epsilon$.

Let $z^1, z^2 \in B^{n-1}_{r_6}$ such that $|z^2 - z^1| < \gamma_\epsilon$.

In the case $g_v(z^1), g_v(z^2) < t_\epsilon$ we have

$$\begin{aligned}
\big|\nabla_{x'} g_v(z^2) - \nabla_{x'} g_v(z^1)\big| &\le \big|\nabla_{x'} g_v(z^2)\big| + \big|\nabla_{x'} g_v(z^1)\big| \\
&\le \frac{1}{\epsilon_1}\sigma_3(g_v(z^2)) + \frac{1}{\epsilon_1}\sigma_3(g_v(z^1)) \\
&\le \frac{1}{\epsilon_1}\sigma_3(t_\epsilon) + \frac{1}{\epsilon_1}\sigma_3(t_\epsilon) = \frac{2}{\epsilon_1}\sigma_3(t_\epsilon) < \epsilon
\end{aligned}$$

In the case when either $g_v(z^1) \ge t_\epsilon$ or $g_v(z^1) \ge t_\epsilon$ we might assume that $g_v(z^1) \ge t_\epsilon$.

Then we have

$$|z^2 - z^1| < \gamma_\epsilon \le r_5 t_\epsilon \le r_5 g_v(z^1)$$

thus by the sixth step we have

$$\big|\nabla_{x'} g_v(z^2) - \nabla_{x'} g_v(z^1)\big| \le \sigma_4\big(\frac{|z^2 - z^1|}{g_v(z^1)}\big) \le \sigma_4(\frac{\gamma_\epsilon}{t_\epsilon}) < \epsilon.$$

We have shown that for all $\epsilon > 0$ and $z^1, z^2 \in B^{n-1}_{r_6}$ if $|z^2 - z^1| < \gamma_\epsilon$ then $|\nabla_{x'} g_v(z^2) - \nabla_{x'} g_v(z^1)| < \epsilon$ and this completes the proof of the theorem. □

Conflict of interest statement

There is no conflict of interest.

Appendix A. Technical results

In this appendix we shall list technical results. Some are well known but not easy to find a reference to.

A.1. Spaces

Lemma 12. *For $a > 1$, $C_c^\infty(\mathbb{R}^n_+)$ is dense in $H^1(D; x_1^a)$.*

Proof. It is enough to show that for $u \in C^\infty(\mathbb{R}^n)$ there exists $u_\epsilon \in H^1(D; x_1^a)$ such that $\operatorname{supp} u_\epsilon \subset \{x_1 > \frac{1}{2}\epsilon\}$ and $u_\epsilon \to u$ in $H^1(D; x_1^a)$ as $\epsilon \to 0$.

Fix $u \in C^\infty(\mathbb{R}^n)$. Let us define for $\epsilon > 0$

$$u_\epsilon(x) = \max\big(0, \min(1, \frac{x_1}{\epsilon} - 1)\big)u(x) \ \text{ for } \ x \in D.$$

It is easy to see that $u_\epsilon \in H^1(D; x_1^a)$ and $\operatorname{supp} u_\epsilon \subset \{x_1 > \frac{1}{2}\epsilon\}$. Using $a > 1$ and particularly the regularity $u \in C^\infty(\mathbb{R}^n)$ one may see that $\|u - u_\epsilon\|_{H^1(D; x_1^a)} \to 0$ as $\epsilon \to 0$. □

A.2. Poisson kernel

Lemma 13. *Let $a \geq 2$ be an integer, $u \in H^1(B^+; x_1^a)$ and $\operatorname{div}(x_1^a \nabla u) = 0$ in B^+ (in the sense of distributions), then we have*

$$u(x) = \int\limits_{\partial B \cap \mathbb{R}^n_+} K(x, y; a) u(y) s(dy) \ \text{ for } \ x \in B^+ \tag{A.1}$$

where

$$K(x, y; a) = 2^{a-1} \frac{|\partial B^a|}{|\partial B^{n+a}|} \big(1 - |x|^2\big) y_1^a \int\limits_0^1 \frac{(t(1-t))^{\frac{a}{2}-1}}{\big(|x - \bar{y}|^2(1-t) + |x-y|^2 t\big)^{\frac{n+a}{2}}} dt$$

and $\bar{y} = (-y_1, y_2, \cdots, y_n)$.

Proof. Let a and u be as in the statement of the lemma.

For $x \in \mathbb{R}^{n+a}$ we denote $X_1 = (x_1, \cdots, x_{n-1})$ and $X_2 = (x_n, \cdots, x_{n+a})$. Let us define the function w defined on B^{n+a} by

$$w(x) = u\big(X_1, |X_2|\big) \ \text{ for } \ x \in B^{n+a}.$$

Then we have $w \in H^1(B^{n+a})$ and $\Delta w = 0$ in B^{n+a}. We might write w using the Poisson kernel for the Laplacian in unit ball. Then writing w in terms of u, after some computations we obtain the equation (A.1). □

It follows that for $a \geq 2$ an integer, $K(x, y; a)$ satisfies the equations

$$\int\limits_{\partial B \cap \mathbb{R}^n_+} K(x, y; a) s(dy) = 1 \ \text{ for } \ x \in B^+ \tag{A.2}$$

and

$$\operatorname{div}_x\big(x_1^a \nabla_x K(x, y; a)\big) = 0 \ \text{ for } \ x \in B^+ \ \text{ and } \ y \in \partial B \cap \mathbb{R}^n_+. \tag{A.3}$$

Writing surface area of unit balls using the Γ function, we have $|\partial B^m| = 2\pi^{\frac{m}{2}} (\Gamma(\frac{m}{2}))^{-1}$, thus we may write

$$K(x, y; a) = \frac{2^{a-1}}{\pi^{\frac{n}{2}}} \frac{\Gamma(\frac{n+a}{2})}{\Gamma(\frac{a}{2})} \big(1 - |x|^2\big) y_1^a \int\limits_0^1 \frac{(t(1-t))^{\frac{a}{2}-1}}{\big(|x - \bar{y}|^2(1-t) + |x-y|^2 t\big)^{\frac{n+a}{2}}} dt. \tag{A.4}$$

We consider for a positive number $c > 0$ and a complex number z, $c^z = e^{z \ln(c)}$.

By (A.4), $K(x, y; a)$ has clearly an analytic extension for $\Re a > 0$.

Lemma 14. *For $a > 0$ not necessarily an integer, $K(x, y; a)$ retains the properties (A.2) and (A.3).*

Proof. Carlson's theorem [7] states that if f is analytic for $\Re z > 0$, continuous for $\Re z \geq 0$, $f(n) = 0$ for $n = 0, 1, 2, \cdots$, there exists $c > 0$ such that $|f(z)| \leq C_1 e^{c|z|}$ and there exists $0 < c < \pi$ such that $|f(iz_2)| \leq C_2 e^{c|z_2|}$ then $f(z) = 0$ for $\Re z \geq 0$.

We consider the left hand sides of the equations (A.2) and (A.3) as analytic functions of a for $\Re a > 0$. We have that (A.2) and (A.3) hold for integer values of $a \geq 2$ and one may check that the growth conditions assumed in Carlson's theorem are also satisfied. Applying Carlson's theorem we prove the claim. □

Lemma 15. *Let $a > 1$, $g \in C_c^\infty(\mathbb{R}^n_+)$ and u be given by (A.1). Then we have $u \in C_b^1(B^+)$.*

Proof. Let $0<\epsilon<1$ such that $\operatorname{supp} g\subset\{x_1>\epsilon\}$. Let us fix $0<r<\frac{\epsilon}{16}$ to be chosen later. We decompose

$$B^+=\big(B\cap\{0<x_1\le\frac{\epsilon}{2}\}\big)\cup\big(B_{1-r}\cap\{x_1>\frac{\epsilon}{2}\}\big)\cup\big((B\backslash B_{1-r})\cap\{x_1>\frac{\epsilon}{2}\}\big).$$

There exists $c_{\epsilon,r}>0$ such that for $x\in(B\cap\{0<x_1\le\frac{\epsilon}{2}\})\cup(B_{1-r}\cap\{x_1>\frac{\epsilon}{2}\})$, $0<t<1$ and $y\in\partial B\cap\{y_1>\epsilon\}$ we have

$$|x-\bar{y}|^2(1-t)+|x-y|^2t\ge c_{\epsilon,r}$$

from which it follows that $u\in C^1_b((B\cap\{0<x_1\le\frac{\epsilon}{2}\})\cup(B_{1-r}\cap\{x_1>\frac{\epsilon}{2}\}))$.

We consider the cover

$$(B\backslash B_{1-r})\cap\{x_1>\frac{\epsilon}{2}\}\subset\cup_{x^0\in\partial B\cap\{x_1>\frac{\epsilon}{2}\}}(B\cap B_{2r}(x^0)).$$

Thus it is enough to show that $u\in C^1_b(B\cap B_{2r}(x^0))$ for each $x^0\in\partial B\cap\{x_1>\frac{\epsilon}{2}\}$.

Using (A.2) we have

$$\int\limits_{\partial B\cap\mathbb{R}^n_+}\partial_{x_i}K(x,y;a)s(dy)=0\ \text{ for }\ x\in B^+,\, i=1,\cdots,n.$$

Thus we may write

$$\partial_{x_i}u(x)=\int\limits_{\partial B\cap\mathbb{R}^n_+}\partial_{x_i}K(x,y;a)\big(g(y)-g(\frac{x}{|x|})\big)s(dy)$$

and we consider the decomposition $\partial_{x_i}u(x)=v_1(x)+v_2(x)$ where

$$v_1(x)=\int\limits_{B_{4r}(x^0)\cap\partial B\cap\{y_1>\epsilon\}}\partial_{x_i}K(x,y;a)\big(g(y)-g(\frac{x}{|x|})\big)s(dy)$$

and

$$v_2(x)=\int\limits_{B^c_{4r}(x^0)\cap\partial B\cap\{y_1>\epsilon\}}\partial_{x_i}K(x,y;a)\big(g(y)-g(\frac{x}{|x|})\big)s(dy).$$

For $x\in B_{2r}(x^0)\cap B$ and $y\in B^c_{4r}(x^0)\cap\partial B\cap\{y_1>\epsilon\}$ we have

$$|x-\bar{y}|^2(1-t)+|x-y|^2t\ge 4r^2$$

thus $v_2\in C_b(B_{2r}(x^0)\cap B)$.

Thus we should only show that v_1 is in $C_b(B_{2r}(x^0)\cap B)$. Around x^0 we might straighten the boundary part $B_{4r}(x^0)\cap\partial B$ of the domain $B_{4r}(x^0)\cap B$. Then by similar estimates as for the boundedness of the derivatives of harmonic functions in half spaces given by the half space Poisson kernel we show that v_1 is bounded in $B_{2r}(x^0)\cap B$. And this proves the lemma. □

Lemma 16. *The statement of Lemma 13 holds for $a>1$ (not necessarily an integer).*

Proof. Let $a>1$, $g\in C^\infty_c(\mathbb{R}^n_+)$ and u be given by (A.1). Then by the previous lemma we have that $u\in C^1_b(B^+)\subset H^1(B^+;x_1^a)$ and by (A.3) we have that $\operatorname{div}(x_1^a\nabla u)=0$ in B^+. By (A.2) and the property $K(x,y;a)\to 0$ for $x\to x^0\ne y$ we have that $u=g$ on $\partial B\cap\mathbb{R}^n_+$. Thus u is the solution of the desired equation with boundary condition g. Now by the boundedness of the solution operator we have that $\|u\|_{H^1(B^+;x_1^a)}\le C\|g\|_{H^1(B^+;x_1^a)}$ for some constant $C>0$ independent of g. By this inequality and Lemma 12 the lemma is proved. □

A.3. Maximum principles

Lemma 17. *Let $a > 1$, $u \in H^1(D; x_1^a)$ and $\operatorname{div}(x_1^a \nabla u) = 0$ in D (in the sense of distributions), then $u(x) \leq \sup_{\partial D \cap \mathbb{R}^n_+} u$ for a.e. $x \in D$.*

Proof. Let $M = \sup_{\partial D \cap \mathbb{R}^n_+} u$, we have $(u - M)^+ \in H_0^1(D; x_1^a, \partial D \cap \mathbb{R}^n_+)$. By testing the equation satisfied by u, by $(u - M)^+$ we obtain

$$\int_D x_1^a \nabla u \cdot \nabla (u - M)^+ dx = 0$$

from which it follows that

$$\int_D x_1^a |\nabla (u - M)^+|^2 dx = 0.$$

Now by applying the Poincaré inequality (4.1) we obtain $(u - M)^+ = 0$ a.e. in D, i.e. $u \leq M$ a.e. in D which proves the lemma. □

Lemma 18. *Let $D_{rel.}$ be the interior of $\overline{D}$ in the relative topology of $\overline{\mathbb{R}^n_+}$. Let $a > 1$, $u \in H^1(D; x_1^a)$ and $\operatorname{div}(x_1^a \nabla u) = 0$ in D (in the sense of distributions). If u attains its maximum in $D_{rel.}$ then it is constant.*

Proof. We claim that if $x^0 \in D_{rel.}$ such that $u(x) \leq u(x^0)$ for $x \in D$ and $r > 0$ such that $B_r^+(x^0) \subset D$ then $u = u(x^0)$ in $B^+_{\frac{r}{4}}(x^0)$.

If $3x_1^0 < r$ we define $x^1 = x^0 - x_1^0 e_1$ and then we have $x^0 \in B^+_{\frac{2}{3}r}(x^1)$ and

$$B^+_{\frac{2r}{3}}(x^1) \subset B^+_{\frac{2r}{3}+x_1^0}(x^0) \subset B_r^+(x^0) \subset D.$$

Using the Poisson formula (A.1) (after a translation and scaling to bring the problem to the domain B^+) we obtain that $u = u(x^0)$ in $B^+_{\frac{2r}{3}}(x^1)$. We have

$$B^+_{\frac{r}{3}}(x^0) \subset B^+_{\frac{r}{3}+x_1^0}(x^1) \subset B^+_{\frac{2r}{3}}(x^1)$$

thus $u = u(x^0)$ in $B^+_{\frac{r}{3}}(x^0)$.

If $3x_1^0 \geq r$ then we have $B_{\frac{r}{4}}(x^0) \subset \{x_1 > x_1^0 - \frac{r}{4}\} \subset \{x_1 > \frac{r}{12}\}$ and we might apply the usual strong maximum principle for uniformly elliptic with variable coefficient elliptic equations in $B_{\frac{r}{4}}(x^0)$. Thus $u = u(x^0)$ in $B_{\frac{r}{4}}(x^0)$ and this finishes the proof of the claim.

From connectedness of D it is easy to see that from the claim the proof of the lemma follows. □

A.4. Harnack inequality

Lemma 19. *Let $a > 1$. For $0 < \gamma < 1$ there exists a $C > 0$ such that if $u \in H^1(B_r^+; x_1^a)$, $\operatorname{div}(x_1^a \nabla u) = 0$ in B_r^+ (in the sense of distributions) and $u \geq 0$ in B_r^+ then $\sup_{B^+_{\gamma r}} u \leq C \inf_{B^+_{\gamma r}} u$.*

Proof. There exists $c_1, C_2 > 0$ such that

$$c_1 \leq K(x, y; a) \leq C_2 \quad \text{for } x \in B_\gamma^+ \text{ and } y \in \partial B \cap \mathbb{R}^n_+. \tag{A.5}$$

Let us define $v(z) = u(rz)$ for $z \in B^+$. We have that v is a nonnegative solution in B^+.

By the Poisson equation (A.1) and the inequalities in (A.5), for $z^1, z^2 \in B^+_\gamma$ we have

$$v(z^1) = \int\limits_{\partial B\cap\mathbb{R}^n_+} K(z^1, y; a)v(y)s(dy) \le C_2 \int\limits_{\partial B\cap\mathbb{R}^n_+} v(y)s(dy)$$
$$\le \frac{C_2}{c_1} \int\limits_{\partial B\cap\mathbb{R}^n_+} K(z^2, y; a)v(y)s(dy) = \frac{C_2}{c_1} v(z^2).$$

It follows that

$$\sup_{B^+_\gamma} v \le C_3 \inf_{B^+_\gamma} v$$

by which the lemma is proved. □

A.5. Regularity estimates

Lemma 20. *Let $a > 1$ and $k = 1, 2$. There exists a $C > 0$ such that if $u \in H^1(B^+_r(x^0); x_1^a)$ and $\operatorname{div}(x_1^a \nabla u) = 0$ in $B^+_r(x^0)$ (in the sense of distributions) then*

$$|\nabla^k u(x^0)| \le \frac{C}{r^k} \sup_{B^+_r(x^0)} |u|$$

here $|\nabla^2 u(x^0)|$ means a norm of the matrix $\nabla^2 u(x^0)$, for example the Frobenius norm which is the square root of the sum of the square power of all entries.

Proof. We might assume that $(x^0)' = 0$, i.e. $x^0 = x_1^0 e_1$.

In the case $x_1^0 \le \frac{r}{4}$ we have

$$B^+_{\frac{3}{4}r} \subset B^+_{\frac{3}{4}r+|x^0|}(x^0) = B^+_{\frac{3}{4}r+x_1^0}(x^0) \subset B^+_{\frac{3}{4}r+\frac{r}{4}}(x^0) = B^+_r(x^0).$$

Let us define $v(z) = u(\frac{3}{4}rz)$ for $z \in B^+$. We have that v is solution in B^+ thus by the Poisson equation (A.1) we have

$$|\nabla^k v(\frac{4}{3r}x^0)| \le C \sup_{B^+} |v|$$

and by the definition of v it follows that

$$|\nabla^k u(x^0)| \le \frac{C}{(\frac{3r}{4})^k} \sup_{\frac{3r}{4}B^+} |u| \le \frac{C_1}{r^k} \sup_{B^+_r(x^0)} |u|.$$

Now let us consider the case $\frac{1}{4}r < x_1^0$. We have $\frac{r}{8x_1^0} < \frac{1}{2}$. Let us define $v(z) = u(x_1^0 z)$ then v solves

$$\Delta v + \frac{b}{z_1}\partial_{z_1} v = 0 \ \text{ in } \ B_{\frac{r}{8x_1^0}}(e_1).$$

By regularity theory for uniformly elliptic with variable coefficients we have

$$|\nabla^k v(e_1)| \le \frac{C}{(\frac{r}{8x_1^0})^k} \sup_{B_{\frac{r}{8x_1^0}}(e_1)} |v|$$

and by the definition of v it follows that

$$|\nabla^k u(x^0)| \le \frac{1}{(x_1^0)^k} \frac{C}{(\frac{r}{8x_1^0})^k} \sup_{B_{\frac{r}{8x_1^0}}(e_1)} |v| \le \frac{C}{(\frac{r}{8})^k} \sup_{B_{\frac{r}{8}}(x^0)} |u| \le \frac{C_1}{r^k} \sup_{B^+_r(x^0)} |u|$$

and this completes the proof of the lemma. □

A.6. Regularity of classical obstacle problem at regular points

Lemma 21. *Let $c_1, C_2, M > 0$ and $0 < r_1 < 1$ then there exists $0 < r_2 < r_1$, $d > 0$ and a modulus of continuity σ_4 such that if w is a solution of the obstacle problem*

$$\Delta w = f\chi_{\{w>0\}} \ \text{in} \ U = (1 - Mr_1, 1 + Mr_1) \times B_{r_1}^{n-1}$$

with $f \geq c_1$ in U, $\|f\|_{C^{0,1}(U)} \leq C_2$, there exists a function g defined on $B_{r_1}^{n-1}$ such that g is Lipschitz with Lipschitz constant not exceeding M, $g(0) = 1$,

$$\Omega = \left\{x \in U \ \middle| \ g(x') < x_1\right\}$$

and

$$\{x_1 < 1 - d\} \cap U \subset \{w = 0\}$$

then $g \in C^1(B_{r_2}^{n-1})$ and σ_4 is a modulus of continuity for $\nabla_{x'} g$ in $B_{r_2}^{n-1}$.

Proof. Cf. [6, Chapter 6]. □

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