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AN INTRODUCTION TO THE COMPLETENESS OF COMPACT SEMI-RIEMANNIAN MANIFOLDS

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ABSTRACT. — The aim of this paper is twofold. First, it introduces some heuristic reasonings and examples to show that the problem of completeness of compact indefinite manifolds arises in a natural way; so, it is discussed the absence of analogous conclusions to the well-known ones from Hopf-Rinow's theorem, the existence of incomplete closed geodesics, and a heuristic way to yield incomplete Lorentzian tori, including Clifton-Pohl's torus. Second, a brief summary of known results and open questions is carried out. In this summary, the Riemannian, indefinite non compact and indefinite compact cases are compared, and some of the underlying ideas are outlined.

1. Heuristic considerations

1.0. Notation.

Throughout this article, (M, g) will denote a n-dimensional ($n \geq 2$) semi-Riemannian manifold, *i.e.* a n-manifold M endowed with a non degenerated metric g; when the index of the metric is 0 (resp. 1, strictly between 0 and n, 0 or n) it is called Riemannian (resp. Lorentzian, indefinite, definite). The results for Riemannian metrics can be extended to definite ones. If $p \in M$ then X denotes a vector of the tangent space at p, T_pM, and the causal character of X_p is defined as usual in General Relativity, that is, X_p is *timelike* if $g(X_p, X_p) < 0$, *null* if $g(X_p, X_p) = 0$ and $X_p \neq 0$, and *spacelike* if $g(X_p, X_p) > 0$ or $X_p = 0$.

If $\gamma : I \to M$, (I interval) is a geodesic then $g(\gamma', \gamma')$ is a constant, and we will say that γ is timelike, null, or spacelike according to the causal character of its velocity at any

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point. An inextendible geodesic is said to be *complete* if its domain is all R, and *incomplete* otherwise; we will say that the metric g is *timelike, null or spacelike (geodesically) complete*, according to the causal character of its complete geodesics.

1.1. Hopf-Rinow's technique does not work.

Let (M, g) be a Riemannian manifold, it is well known, by Hopf-Rinow's theorem, the equivalence among: (A) the metric completeness for the distance d_g canonically associated to g, (B) the closed and d_g-bounded subsets of M are compact and (C) the geodesic completeness of g. Obviously, if M is compact then (A) (or (B)) holds, and geodesic completeness is obtained. On the other hand, if g were indefinite, the distance d_g cannot be defined canonically, and neither (A) nor (B) can be stated. Nevertheless, if M were compact we would have: (a) any distance compatible with the topology of M is complete, and (b) any closed subset is compact. Thus, "the best possible" results analogous to (A) and (B) hold, and one could wonder if geodesic completeness could be derived.

It is well-known that the answer to this question is no; let us see now the exact step where the corresponding proof fails. Consider the following two cases: (i) g is Riemannian and satisfies (B), (ii) g is indefinite and M compact (thus, satisfying (b)). Take any geodesic $\gamma : [0, b[\to M$, $0 < b < \infty$, and try to extend it beyond b.

(1) For any sequence $\{t_n\} \to b$ the closure of the set $\{\gamma(t_n), n \in \mathrm{N}\}$ is compact in both cases (i) and (ii). Thus, there exists a converging subsequence $\{\gamma(t_{\sigma(n)})\} \to p \in M$.

(2) In the case (i), γ is continuously extendible beyond b. (This is a consequence of the inequality for the associated distance,

$$d_g(\gamma(t), p) \leq C(b - t), \qquad C = g(\gamma', \gamma')^{1/2}$$

which implies $\lim_{t \to b} \gamma(t) = p$.) Observe that in the case (ii) this step may be not true.

(3) By a standard argument (see for instance, [On], Lemma 1.56) if the geodesic γ is continuously extendible to b then it is extendible as a geodesic beyond b.

Next, we are going to see examples in which step (2) in (ii) may not hold, first by showing incomplete geodesics with image contained in a compact subset, and after constructing heuristically incomplete Lorentzian tori.

1.2. Incomplete closed geodesics.

DEFINITION 1.1. — *Let $\gamma : I \to M$ be an inextendible non-constant geodesic, γ is* closed *if there are $s_1, s_2 \in I$, $s_1 < s_2$, and $\lambda \in \mathrm{R}^+$ such that $\gamma'(s_1) = \lambda \cdot \gamma'(s_2)$.*

Observe that if $\lambda \neq 1$ then γ must be null, because of the constancy of $g(\gamma', \gamma')$. Moreover, γ gives (at least) a round not only in $[s_1, s_2]$ but also in $[s_2, s_2 + \lambda(s_2 - s_1)]$, in $[s_1 - \lambda^{-1}(s_2 - s_1), s_1]$, and in the successive intervals of length $\lambda^i(s_2 - s_1)$, $i \in \mathbf{Z}$. So, if $\lambda = 1$ γ is periodic, but if $\lambda \neq 1$ then γ is incomplete, because one of the sums $\sum_{i=1}^{\infty} \lambda^i(s_2 - s_1)$, $\sum_{i=1}^{\infty} \lambda^{-i}(s_2 - s_1)$ is finite. Summing up, we have

LEMMA 1.2. — *If γ is a closed geodesic, its corresponding λ is different to 1 if and only if γ is incomplete (and, thus, null).*

Now, we are going to construct Misner's cylinder, which provides a first example of incomplete closed geodesic (another two ways to see such cylinder can be seen in [HaEl] and [RoSa5]). Note that the step (2) in § 1.1 cannot be carried out for this kind of geodesics.

Consider the two-dimensional Lorentz-Minkowski spacetime in usual null coordinates (u, v), $\mathbf{L}^2 = (\mathbf{R}^2, g = du \otimes dv + dv \otimes du \equiv 2dudv)$. Choose $\lambda > 1$ and define the isometry ψ^λ of $\mathbf{L}^2$ by $\psi^\lambda(u, v) = (\lambda u, \lambda^{-1}v)$, $\forall (u, v) \in \mathbf{R}^2$. The corresponding action of the group of isometries generated by ψ^λ on $\mathbf{L}^2$ satisfies:

(1) The origin is a fixed point, and so we have a non-discontinuous $\mathbf{Z}$-action on $\mathbf{L}^2$; the corresponding quotient is not a manifold.

(2) If we remove the origin, a discontinuous but not properly discontinuous action of $\mathbf{Z}$ on $\mathbf{L}^2 - \{(0, 0)\}$ is naturally obtained. The quotient can be regarded as a non-Hausdorff Lorentzian manifold (note that the projections of a point of each axis cannot be separated by two disjoint open subsets). Recall that all discontinuous action by isometries of a Riemannian manifold is properly discontinuous ([KoNo] Proposition 4.4, Chapter I); thus, we have obtained a counterexample for the corresponding indefinite case.

(3) If we consider just the right open semiplane, $\mathbf{R}^+ \times \mathbf{R}$, the corresponding induced action on it is properly discontinuous, and it is easy to check that the quotient is topologically a cylinder, *Misner's cylinder*. The reparametrization of the u-axis which makes it a (null) geodesic $\bar{\rho}$, projects onto an incomplete closed geodesic ρ of the cylinder. Moreover, every non constant geodesic $\bar{\eta}$ on $\mathbf{R}^+ \times \mathbf{R}$ which is not a reparametrization of a vertical straight line projects onto an incomplete geodesic in the cylinder; note that the incomplete side of η has as accumulation points the image of ρ.

1.3. Incomplete Lorentzian tori.

We have constructed incomplete geodesics in non-compact manifolds with image contained in a compact subset. Now, we are going to construct a family of incomplete semi-Riemannian metrics on a torus. The properties of this family is widely studied in

[Sa2] (see also [Sa1]), and the procedure is also discussed in [RoSa2], even though here some new heuristic arguments are introduced.

Consider again Misner's cylinder fixing $\lambda = e$, $S \equiv \psi^e$. Call (C, g') to the corresponding cylinder generated from $(\mathbf{R}^+ \times \mathbf{R}, g = 2dudv)$, and compare this cylinder with the one (C_0, g_0') generated from $(\mathbf{R}^2, g_0 = 2dxdy)$ (x, y usual coordinates) by the action induced from the translation $T(x,y) = (x, y+1), \forall (x,y) \in \mathbf{R}^2$. The complete metric g_0' can be induced naturally on a torus by (i) fixing two different circles, S_η with fixed coordinate $x = \eta$, and $S_{\eta'}$, with $x = \eta'$, (ii) cutting C_0 by these circles and (iii) glueing (by using a x-translation) each point of S_η with the corresponding one on $S_{\eta'}$. But the incomplete metric g' cannot be induced on a torus by this (or by a different) method, because it is flat and, by a result of Carrière [Ca], no compact flat Lorentzian manifold is incomplete. Now, our purpose is to construct a new metric h' on a cylinder with a behaviour: (1) as the incomplete metric g' near an incomplete closed geodesic ρ (so h' will be incomplete) and (2) as the complete metric g_0' out a compact subset (so h' will be inducible in a torus by cutting and glueing suitably chosen circles as before).

We must have the next caution to construct the metric h'. Two Riemannian metrics defined on subsets of a cylinder as g' and g_0' above can be easily extended to a unique metric on all the cylinder by using a partition of unity and standard arguments. But these arguments cannot be directly extended to the Lorentzian case because Lorentzian metrics on a vector space V are not a convex subset of the set of all the metrics on V. That is, given two Lorentzian metrics g_1, g_2 on V, the metrics,

$$L_{g_1,g_2}(t) = tg_1 + (1-t)\cdot g_2$$

are not necessarily Lorentzian for all $t \in [0,1]$. To skip this obstacle, we need the next concept.

Definition 1.3. — *Let g_1 and g_2 be two Lorentzian metrics on a vector space V. We will say that the timelike cone of g_1 is greater or equal to the timelike cone of g_2 if $g_2(v,v) < 0$ implies $g_1(v,v) < 0$, $v \in V$. In this case, we write $g_1^{\mathrm{tcon}} \geq g_2^{\mathrm{tcon}}$.*

Then, it is straightforward to prove:

Lemma 1.4. — *If $g_1^{\mathrm{tcon}} \geq g_2^{\mathrm{tcon}}$ then $L_{g_1,g_2}(t)$ is a Lorentzian metric for all $t \in [0,1]$.*

On the other hand, if V is two dimensional, and g_1, g_2 have a common null vector then necessarily one of the next inequalities hold:

$$g_1^{\mathrm{tcon}} \geq g_2^{\mathrm{tcon}}, \quad g_2^{\mathrm{tcon}} \geq g_1^{\mathrm{tcon}}, \quad g_1^{\mathrm{tcon}} \geq (-g_2)^{\mathrm{tcon}}, \quad (-g_2)^{\mathrm{tcon}} \geq g_1^{\mathrm{tcon}}.$$

So, we have:

LEMMA 1.5. — *If two Lorentzian metrics g_1, g_2 on a two-dimensional vector space V have a common null vector then the metrics $L_{g_1,g_2}(t)$ or the metrics $L_{g_1,-g_2}(t)$ are Lorentzian for all $t \in [0,1]$.*

Now, come back to our problem of finding h'. Consider a diffeomorphism Φ : $\mathbf{R}^2 \to \mathbf{R}^+ \times \mathbf{R}$, $\Phi_\varepsilon(x,y) = (u(x,y), \varepsilon \cdot v(x,y))$, where $\varepsilon \in \{\pm 1\}$. If $u(x,y)$ does not depend on the first variable, $u(x,y) \equiv u(y)$, then the coordinate vector field $\partial/\partial x$ is null not only for g_0 but also for $\Phi_\varepsilon^* g$. So, Lemma 1.5 can be applied to g_0 and $\Phi_\varepsilon^* g$ on each tangent vector space of $\mathbf{R}^2$, and we will also have $\Phi_{+1}^* g = -\Phi_{-1}^* g$. If we want that Φ_ε can be induced on the cylinders C_0 and C, then we impose $\Phi_\varepsilon \circ T = S \circ \Phi_\varepsilon$, that is, $u(y+1) = e \cdot u(y), v(x, y+1) = e^{-1} \cdot v(x,y), \forall (x,y) \in \mathbf{R}^2$.

Thus, choose the diffeomorphism $\Phi : \mathbf{R}^2 \to \mathbf{R}^+ \times \mathbf{R}$, $\Phi(x,y) = (\exp(y), x \cdot \exp(-y))$. Clearly Φ induces a diffeomorphism $\Phi' : C_0 \to C$, and setting $g^* = \Phi^* g$ we have $g^*_{(x,y)} = 2dxdy - 2x(dy)^2$, $\forall (x,y) \in \mathbf{R}^2$. Let μ be a smooth function on $\mathbf{R}$, $0 \leq \mu \leq 1$, with value 1 on $]-\varepsilon, \varepsilon[$, $0 < \varepsilon < 1$, and value 0 on $\mathbf{R} -]-1, 1[$. Consider the following Lorentzian metric h on $\mathbf{R}^2$:

$$h_{(x,y)} = \mu(x) g^*_{(x,y)} + (1 - \mu(x))(g_0)_{(x,y)} = 2dxdy - 2\mu(x)x(dy)^2.$$

This metric is incomplete and induces the required metric h' on the cylinder C_0. Even more, as a first generalization, we can consider, for any smooth function τ on $\mathbf{R}$, the Lorentzian metric on $\mathbf{R}^2$

$$h^\tau_{(x,y)} = 2dxdy - 2\tau(x)dy^2. \tag{1.1}$$

If there exists $a \in \mathbf{R}$ such that $\tau(a) = 0$ and $\tau'(a) \neq 0$, the null geodesic $\gamma(t) = (a, b \log(t + (1/b)))$, $b = \tau'(a)$, is incomplete. Clearly, h^τ is always inducible on a cylinder and, if τ is assumed to be periodic, then h^τ yields an incomplete Lorentzian metric on a torus.

As a bigger generalization, we can consider the metric on $\mathbf{R}^2$,

$$g_{(x,y)} = \alpha(x)dx^2 + \beta(x)2dxdy - \delta(x)dy^2 \tag{1.2}$$

where α, β, δ are smooth periodic functions on $\mathbf{R}$ with the same period (g inducible on a torus) and satisfying $\alpha\delta + \beta^2 > 0$ (g Lorentzian). It is not difficult to check that if δ vanishes but it is not identically 0, then g is timelike, null and spacelike incomplete, [Sa2]. It is worth pointing out that Clifton-Pohl's torus (as the incomplete examples in [RoSa1] and [Ga]), is included in this incomplete family of metrics. To see this, note that this torus is constructed taking the quotient of $\mathbf{R}^2 - \{(0,0)\}$, $g = 2dudv/(u^2 + v^2))$ by the group of isometries generated by $(u,v) \to (2u, 2v)$. Putting $\alpha(x) = \delta(x) = -\pi^2 \sin 2\pi x$, $\beta(x) = \pi^2 \cos 2\pi x$, $x \in \mathbf{R}$, we can construct an isometry from the corresponding torus and Clifton Pohl's one by taking into account the locally isometric covering map $\mathbf{R}^2 \to \mathbf{R}^2 - \{(0,0)\}$, $(x,y) \to (u(x,y), v(x,y))$, $u(x,y) = (1/\pi)\exp(\pi y)\sin \pi x$, $v = (1/\pi)\exp(\pi y)\cos \pi x$.

2. Several known results and open questions

2.0. Some previous technical results.

Now, several results on completeness in semi-Riemannian manifolds are summarized, comparing what occurs in the cases: (a) (compact or not) Riemannian, (b) non-compact indefinite and (c) compact indefinite; this summary extends and updates the one in [RoSa5]. But previously, the next results are outlined for later reference.

PROPOSITION 2.1. — *Let (M, g) be a semi-Riemannian manifold, and let $\gamma : [0, b[\to M$ be a geodesic, $0 < b < \infty$. The following assertions are equivalent: (i) γ is extendible as a geodesic beyond b. (ii) for any complete Riemannian metric g_R on M, we have that $g_R(\gamma', \gamma')$ is bounded. (iii) there exists a sequence, $\{t_n\} \to b$, such that $\{\gamma'(t_n)\}$ converges in the tangent bundle TM.*

Proof.

(i) $\Rightarrow$ (ii) is obvious, and for (ii) $\Rightarrow$ (iii) note that γ' lies in a compact subset of TM. The implication (iii) $\Rightarrow$ (ii) is a consequence of the two following results: (A) The velocities of geodesics of (M, g) can be seen as integral curves of the geodesic vector field G on TM (if $X_p \in TM$, the value of G at X_p is the initial velocity of the curve in $TM : t \to (\exp(tX_p))'$, where exp is the exponential map for g). (B) If an integral curve $\rho : [0, b[\to N, b < \infty$, of a vector field X on a manifold N admits a sequence $\{t_n\} \to b$ such that $\{\rho(t_n)\}$ converges in N, then ρ is extendible as an integral curve of X beyond b ([On], Lemma 1.56). ∎

Remark. — As a consequence if γ is incomplete its velocity is not contained in any compact subset of TM, even if M is compact. The converse is not true, that is, there are complete geodesics on compact indefinite manifolds with velocity (defined in all $\mathbb{R}$) not contained in any compact subset of TM. Counterexamples can be found in the family of metrics defined in (1.2), (see [Sa2]), and also in the (locally) warped products (of dimension greater or equal to 2) constructed in [RoSa3], Remark 3.18. In the former counterexamples there are also incomplete geodesics, but in the later all the geodesics are complete.

For the next result, divide the punctured tangent bundle $TM^* = TM$-{zero-section} of (M, g) in two subsets, C and J, consisting of all vectors tangent to inextendible geodesics with domain upper unbounded, and all vectors tangent to (incomplete) inextendible geodesics with domain upper bounded, respectively.

PROPOSITION 2.2. — *Let (M, g) be a semi-Riemannian manifold, if M is com-*

pact and g incomplete then the closure of the set of incomplete vectors J in TM^* has null vectors.

Proof. — Let $\gamma : [0, b[\to M$ be a geodesic inextendible to b, and consider an arbitrary Riemannian metric g on M. If we take a sequence $\{t_n\} \to b$, the sequence $\{Z_n\}$ in the g_R-unitary bundle, given by

$$Z_n := g_R(\gamma'(t_n), \gamma'(t_n))^{-1/2} \cdot \gamma'(t_n),\ n \in \mathbb{N},$$

has a converging subsequence $\{Z_{\sigma(n)}\} \to Z \in TM^*$. From Proposition 2.1 we know that $\{g_R(\gamma'(t_n), \gamma'(t_n))\} \to \infty$. So, if $C = g(\gamma'(t), \gamma'(t))$ we conclude

$$g(Z,Z) = \lim_{n\to\infty} g(Z_{\sigma(n)}, Z_{\sigma(n)}) = \lim_{n\to\infty} \left[C/g_R(\gamma'(t_{\sigma(n)}), \gamma'(t_{\sigma(n)}))\right] = 0. \quad \blacksquare$$

Remark. — Given a convergent sequence in TM^*, $\{Z_n\} \to Z$, we can construct the sequence of inextendible geodesics $\{\sigma_n\}$, $\sigma_n(t) := \exp(tZ_n)$, $n \in \mathbb{N}$. We can think of the inextendible geodesic $\sigma(t) = \exp(tZ)$, as a limit in a rather strong sense of the sequence $\{\sigma_n\}$ ([RoSa1], Proposition 2.1). So, Proposition 2.2 says that, in a certain sense, incomplete geodesics on compact manifolds converge to null geodesics. This convergence if closely related to the concept of *tangential convergence* systematically studied with other topological properties of the space of the geodesics in [BePa1] (see also the previous topology for null geodesics in [Lo], and [Sa] Capítulo V).

PROPOSITION 2.3. — *Let (M, g) be a compact semi-Riemannian manifold and let $g^* = e^{2\omega} g$ (ω smooth function on M) be a (pointwise) conformal metric to g. Then g is null complete if and only if g^* is.*

Proof. — Consider a geodesic $\alpha(t)$ of g^* and let $\bar{\alpha}(s) = \alpha(t(s))$ be a reparametrization of α such that $dt/ds = \exp(2\omega(\bar{\alpha}(s)))$. A direct computation shows,

$$(D/ds)\bar{\alpha}' = (\overline{C}/2)(\nabla e^{2\omega}) \circ \bar{\alpha} \tag{2.1}$$

where $\overline{C} = g^*(\alpha', \alpha')$ and ∇ is the Levi-Civita connection of g. Thus, if α is null then $\bar{\alpha}$ is a null geodesic respect to g, and as $0 < \operatorname{Inf}|dt/ds| \leq \operatorname{Sup}|dt/ds| < \infty$, the result easily follows (see also [RoSa2], Lemma 4.1). $\blacksquare$

Remarks.

(1) From (2.1) we have that any null geodesic for g is a pregeodesic for g^*. This is more evident in the two dimensional case, because null cones determine two one dimensional foliations which are invariant under conformal changes, and null geodesics are integral curves of these foliations. This fact has been used in [CaRo], § 1–3, to reobtain Proposition 2.3 in 2-dimensional case by other methods.

(2) Observe that (2.1) can be seen as the trajectory of a curve $\bar{\alpha}$ under the potential $V = -(\overline{C}/2)e^{2\omega}$. So, the completeness of g^* can be seen as the completeness of curves accelerated by the potential V, and the completeness of g as the particular case in which $V = 0$.

Given a semi-Riemannian manifold, (M, g), recall that a conformal-Killing vector field is a vector field K on M such that the Lie derivative L_K of g with respect to K satisfies, for a (smooth) function σ on M, $L_K g = \sigma g$.

PROPOSITION 2.4. — *Let (M, g) be a compact Lorentzian manifold. If there exists a timelike conformal-Killing vector field K on M then (i) for any geodesic $\gamma : [0, b[\to M$, $0 < b < \infty$, the closure of the image of γ' in TM is compact (ii) g is complete.*

Proof. — Note that (ii) is straightforward from (i) and Proposition 2.1. For (i) it is enough to see that the projection of γ' on the subbundle Span$\{K\}$ lies in a compact subset, because $g(\gamma', \gamma')$ is a constant C. As $\mathrm{Inf}\,|g(K, K)| > 0$, we have just to check that $g(K, \gamma')$ is bounded. But our assumption on K implies

$$(d/dt)g(K, \gamma') = \frac{1}{2}C(\sigma \circ \gamma)$$

so, $(d/dt)g(K, \gamma')$ and, as a consequence, $g(K, \gamma')$ is bounded on $[0, b[$. ∎

Remarks.

(1) The result of completeness improves the one in [Ka] Theorem A because no assumption on curvature is imposed (see also [RoSa4]).

(2) Note that, clearly, if K is a Killing vector field ($\sigma \equiv 0$) for g, then $L_K(e^{2\omega}g) = K(e^{2\omega})g$, for any conformal factor $e^{2\omega}$. Thus, K is conformal-Killing for any conformal metric to g. The next converse also holds: if K is a timelike (or spacelike without zeroes) conformal-Killing vector field on a semi-Riemannian manifold (M, g) then K is Killing for the metric $|g(K, K)|^{-1/2} \cdot g$ (compare with [Har]).

(3) The assumptions of Proposition 2.4 can be weakened. A completely analogous proof works if we are given: (a) a compact semi-Riemannian manifold (b) so many timelike (resp. spacelike) pointwise independent conformal-Killing (or affine) vector fields as the index (resp. the co-index) of the semi-Riemannian metric. Moreover, more general hypothesis have been given in [RoSa3], [Sa], where the compactness of M is dropped just assuming some regularity conditions on the timelike vector fields. A particular case of these results has alternatively proved recently in [GuLa].

(4) The timelike assumption on the conformal Killing vector field cannot be weakened in non-spacelike (and, thus, without zeroes); the counterexamples can be obtained from the family of metrics h^τ in (1.1) with, for instance, $\tau(x) = 1 + \sin(x)$, $x \in \mathbf{R}$

(note that the vector field $\partial/\partial y$ is Killing; see [Sa2], [Sa] for details). On the other hand, for an inextendible geodesic γ as in Proposition 2.4, Im γ' may have non compact closure (see the complete counterexamples quoted in the Remark to Proposition 2.1).

(5) In Physics, space-times admitting a timelike Killing vector field are called stationary, and they are well-known [SaWu]; those admitting a timelike conformal Killing vector field are also useful and have been used to model imperfect fluids, see [Har], [CoTu1], [CoTu2], [EIMM] and references therein.

2.1. Independence of the three kinds of causal completeness.

As we can speak about timelike, null or spacelike completeness (of course, no in the Riemannian case) we can wonder if these three kinds of causal completeness are independent, or if there exists any logical dependence among them. In the non-compact indefinite case, Kundt [Kun], Geroch [Ge] and Beem [Be1] gave enough examples to show the complete logical independence among the three kinds of completeness. On the other hand, in several particular cases may be relations among them; for instance, Lafuente [La] proved the complete logical equivalence among the three kinds of completeness for locally symmetric semi-Riemannian manifolds (recall that symmetric semi-Riemannian manifolds are always complete, [On] Lemma 8.20).

But compact counterexamples showing any kind of independence among the three kinds of causal completeness have not been found yet. So, this problem remains completely open in the compact case. Specially, it is open the next assertion, which we will call Dependence Null Assertion (DNA), *a compact incomplete indefinite manifold is null incomplete.* About this assertion it is known: (1) given a compact and incomplete indefinite manifold (M, g), if the set of incomplete vectors is closed, then g is incomplete (see Proposition 2.2; note that if J were open it would be obvious that the null incompleteness of the corresponding indefinite manifold, compact or not, would imply also spacelike and timelike incompleteness), (2) J may be neither closed nor open, even though it was thought to be closed (the counterexamples and discussion can be seen in [RoSa1]), (3) if we consider tori with one of the two null foliations by circles, then DNA is "generically true", but there are arguments to think that probably a counterexample could be found among these tori [CaRo].

2.2. Completeness of conformal metrics.

As it was shown in Proposition 2.3, a null geodesic for an indefinite metric is a null pregeodesic for every conformal metric, and equation (2.1) can be seen as a equation generalizing geodesic equation. Then, we can wonder if completeness can be gained (or lost) by conformal changes of metric. (From another point of view, conformal geodesics are studied in [FrSc].)

In the compact Riemannian case all the metrics are clearly complete; so this question is trivial. In the non compact Riemannian case, Nomizu and Ozeki [NoOz] proved that any Riemannian metric is conformal (1) to a complete metric and (2) to a metric of finite diameter and, thus, incomplete. Thus, this question is fully answered in the Riemannian case.

In the indefinite non compact case the question is much more difficult; in the important (but very particular) case of *globally hyperbolic* spacetimes, which are non compact Lorentzian manifolds, we have that all of them are conformally timelike and null complete [Se], [Cl], but it is not known what occurs for spacelike completeness.

Note that the equation of reparametrization for null pregeodesics $dt/ds = -\exp(2w(\bar{\alpha}(s)))$ in the proof of Proposition 2.3 determine when it is null complete an indefinite metric which is conformal to a null complete metric. In fact, this proposition shows that null completeness is a conformal invariant in the compact case. Moreover, Proposition 2.4 can also be seen as a result on completeness under conformal changes of metric, but in the general case, it is not known if the next assertion, which we will call Conformal Completeness Assertion (COCA) is true: *a compact indefinite manifold which is conformal to a complete one is complete.* Clearly, Proposition 2.3 yields DNA $\Rightarrow$ COCA.

2.3. Completeness of warped products.

Recall that a warped product $(B \times F, g^f)$ with base the semi-Riemannian manifold (B, g_B), fiber the semi-Riemannian manifold (F, g_F) and warping function $f : B \to \mathbf{R}$, is the product manifold $B \times F$ endowed with the metric g^f:

$$g^f = \pi_B^* g_B + (f \circ \pi_B)^2 \pi_F^* g_F \tag{2.2}$$

(being $\pi_B : B \times F \to B$, $\pi_F : B \times F \to F$ the natural projections). Many interesting families of semi-Riemannian manifolds, as the relativistic Robertson-Walker spacetimes, are included in the family of warped products (for spacetimes in Physics generalizing Robertson-Walker ones, see [ARS]). Even more, we can generalize warped products as follows. Let $E(B, F)$ be a (differentiable) fiber bundle with base B and fiber F. Denote by $\pi_B : E \to B$ the canonical projection and, for each fiber-chart $\Phi : \pi_B^{-1}(U) \to U \times F$ (U is an open subset of B which will be called *trivializing*), denote by π_F the projection onto F. Let g_B , g_F be semi-Riemannian metrics on B, F respectively, and let $f > 0$ be a smooth function on B. A metric g^f on E is defined to be the *local warped product* of g_B and g_F with *warping function* f if there is a covering of B consisting of trivializing open subsets such that in the corresponding open subsets of E, with the usual identifications, the metric g^f is a warped product as in (2.2). It is easy to find non trivial fiber bundles with local warped metrics which are not (global) warped ones; Moebius strip yields naturally a straightforward example (see also Example 3.12 in [RoSa3]).

It is not difficult to show that if we take complete and Riemannian manifolds as base and fiber, then any local warped product constructed from them is complete (the proof for warped products of [On] Lemma 7.40 essentially holds for locally warped products). But in the indefinite case, Beem and Buseman gave the next simple counterexample of incomplete warped product of two complete and definite metrics: base **R** with its canonical metric g_0, fiber $(\mathbf{R}, -g_0)$, warping function $f(x) = e^x$ for all x ([On], Example 7.41).

The completeness of indefinite locally warped products has extensively studied in [RoSa3], and the results can be summarized as follows:

(A) Let (B, g_B) be a semi-Riemannian manifold and let $f>0$ be a smooth function on B. The following assertions are equivalent: (i) there is a complete manifold with an indefinite metric (F_0, g_{F_0}) and a fiber bundle $E_0(B, F_0)$ such that $(E_0(B, F_0), g^f)$ is complete (resp. timelike, null or spacelike complete), and (ii) every $(E(B, F), g^f)$ is complete (resp. timelike, null or spacelike complete) for any complete manifold with indefinite metric (F, g_F). On the other hand, it is not difficult to see that if the fiber is incomplete then any locally warped product is incomplete [Sa] in all possible causal senses. So it is natural to state the next definition: let (B, g_B) be a semi-Riemannian manifold and $f > 0$ be a smooth function on B, (B, g_B, f) is said to be *warped-complete* (resp. *timelike, null* or *spacelike warped-complete*) if every $(E(B, F), g^f)$ is complete (resp. timelike, null or spacelike complete) for every complete semi-Riemannian manifold (F, g_F). It is remarkable that if g_B is incomplete then (B, g_B, f) is not warped-complete, but it can be warped-complete in one or two causal senses.

(B) If (B, g_B) were definite and complete, several conditions on the decreasing of f yield the (timelike, null or spacelike) warped-completeness of (B, g_B, f). This conditions are not satisfied by the exponential, because it decreases too fast (so, Beem-Buseman counterexample), but if $\mathrm{Inf}(f) > 0$ (in particular, if B is compact) then (B, g_B, f) is warped-complete. (On the other hand, if B is one-dimensional as in Robertson-Walker spacetimes, then the equation of the projection of the geodesics on B can be explicitly integrated; these techniques can be extended for locally warped products with more than one fiber, as Reissner-Nordsöm intermediate type, see [Sa3].)

(C) If (B, g_B) were indefinite and complete there are examples of f with $\mathrm{Inf}(f)>0$ such that (B, g_B, f) is not warped-complete, but in these examples B is not compact. So, the next assertion, or Warped Assertion on Completeness, (WAC), remains open: *any* (B, g_B, f) *with compact, complete and indefinite base is warped-complete.* WAC is especially important, because the next logical implications among the previous open questions are shown:

$$\text{DNA} \Rightarrow \text{WAC} \Rightarrow \text{COCA}.$$

2.4. Completeness of homogeneous manifolds.

It is easy to show that an homogeneous Riemannian manifold is complete and, moreover, that a symmetric semi-Riemannian manifold is too. But in the non compact Lorentzian case it is also easy to construct homogeneous manifolds which are incomplete (so, for instance, $(\mathbf{R}^+ \times \mathbf{R}, 2dudv)$; more examples are systematically constructed in [DuIh]).

Nevertheless, Marsden proved [Ma] that any compact homogeneous semi-Riemannian manifold is complete. Marsden's proof is a consequence of Proposition 2.1 and the next fact: the tangent bundle TM of a compact homogeneous manifold M can be divided into subsets which are invariant under the geodesic flow. It is worth pointing out that Marsden's theorem can be extended, by using a technique as in Proposition 2.4, to (globally) conformally homogeneous compact semi-Riemannian manifolds [RoSa3], [Sa]; nevertheless, in this case TM cannot be divided as before (see Remark to Proposition 2.1, and the complete counterexamples quoted there). Note that this result yield a new partial answer to the problem in § 2.2. On the other hand, locally homogeneous compact semi-Riemannian manifolds may be non complete [GuLa].

Further results can be studied by considering affine homogeneous manifolds. In this case, Goldman and Hirsch proved that a compact affine homogeneous manifold is complete if and only if it has parallel volume [GoHi], solving in this case the well known Markus'conjecture (a compact *affine* manifold is complete if and only if it has parallel volume). As a consequence, one can obtain, [DuIh] Theorem 2.1, that a compact flat pseudo-Riemannian homogeneous manifold is complete, which can be regarded as a particular case of Marsden's theorem. (For more results on completeness in affine – compact– manifolds, see [Ca], [GoHi], [DuIh], [Ka] and references therein.)

2.5. Relation with curvature.

We cannot find any clear relation between curvature and completeness in non compact semi-Riemannian manifolds; to remove a point of a complete semi-riemannian manifold can convince us of this assertion.

For the compact indefinite case, we can construct complete and incomplete Lorentzian metrics on a torus with the same curvature [RoSa2]. In fact, consider the h^τ metrics on $\mathbf{R}^2$ given by (1.1) with τ periodic. The curvature of these metrics is equal to $-\tau''$. Choose now τ_1 such that h^{τ_1} is incomplete and $\tau_2 = \tau_1 + \mathrm{Max}|\tau_1| + 1$. Then clearly the curvatures of h^{τ_1} and h^{τ_2} are the same, but h^{τ_2} is complete because $\partial/\partial y$ is a timelike Killing vector field and, thus, Proposition 2.4 can be claimed.

Anyway, this result does not mean that no relations between curvature and completeness can be found in the compact case. So, we have:

(a) Carrière [Ca] proved an important case of Markus' conjecture, and, as a consequence, that *a flat compact Lorentzian manifold is complete* (a proof of this result for the particular case of a torus was given previously in [FGH], [FuFe]). Anyway, the result is open for other indexes or constant curvature. (On the other hand, recall that the hypothesis of completeness is essential for the classification of compact indefinite manifolds of constant curvature; see [Ka] and references therein, [RoSa6]).

(b) Taking into account Carrière's result, we can also wonder the particular case of COCA: are all the (globally) conformally flat Lorentzian metrics on a compact manifold M complete? If M is a n-torus or a nilmanifold, the answer of this question is yes, because then there exists a timelike conformal Killing vector field, and Proposition 2.4 can be claimed. But the question is open in general.

(c) As a further generalization, we can consider locally symmetric manifolds. In this case, one has that *all compact locally symmetric 1-connected semi-Riemannian manifolds are symmetric and, thus, complete* [FuAr]. Moreover, by using Lafuente's result quoted in § 2.1 and the conformal invariance of null completeness of Proposition 2.3, the result on completeness can be extended to metrics which are just conformal to locally symmetric (which gives another partial answer to COCA). Anyway, we can wonder if the 1-connection assumption is necessary (recall that there are incomplete 1-connected compact manifolds, [GuLa]).

2.6. Other problems.

Of course, the previous list of results and questions does not cover all interesting problems on completeness of compact indefinite manifolds. Now, let us point out a couple of questions more.

(A) *Geodesic connectedness.* From Hopf-Rinow's theorem, a complete Riemannian manifold is always geodesically connected. But in the non-compact indefinite case this property does not hold, being the pseudosphere S_1^n an example of complete manifold which is not geodesically connected. Taking in mind General Relativity, this leads naturally to the problem of when two points of a Lorentzian manifold can be joined by a geodesic. This question has been widely studied in [BFM], [BFG], [GiMa], [Gi], [Sa3] and references therein. In the compact Lorentzian case, Clifton-Pohl torus was known to be non geodesically connected ([On], p. 260). More examples of such non geodesically connected tori are systematically constructed in [Sa2] (in particular, the non geodesically connected torus in [Be2] is essentially shown to be as Clifton-Pohl's one). All this tori are incomplete and, even more, incomplete tori which are geodesically connected are also shown. But, (see [Sp]) can we construct a Lorentzian torus which is complete but not geodesically connected?

(B) *Conformal moduli on surfaces.* For a 1-connected Riemannian manifold of dimension 2, (M, g), it is a well known consequence of the Uniformization Theorem that it must be conformal to a disk, a 2-sphere or the Euclidean space $\mathbf{R}^2$. Moreover, if we identify all the conformally related Riemannian metrics on a torus, the quotient is naturally C, [FaKr]. In the indefinite case, note that a topological 2-sphere cannot admit a Lorentzian metric; on the other hand Kulkarni gave some results on conformally related Lorentzian metrics on manifolds diffeomorphic to $\mathbf{R}^2$ [Kul], but the results are here much more complicated. An interesting question for the compact case would be then: how looks like the quotient of conformally related Lorentzian metrics on a torus? This problem seems to be rather complicated, and in a first approach we could consider just complete or conformally flat metrics. An introduction to this problem can be seen in [RoSa2] (there some problems that could carry the Whitney unstability of null completeness and null incompleteness are pointed out; more general results on Whitney unstability can be seen in [BeEh], [BePa2]).

2.7. A question from General Relativity.

Compact space-times have been usually neglected in Physics because they do have closed timelike curves. Nevertheless, this property does not seem to be enough to overlook them. So, wormholes also have them (probably, see [FMNEKTY], [FrNo], [Haw]) and we must bear in mind physicists usually compactify manifolds to develop field theories with good boundary conditions. Frequently Riemannian manifolds are compactified, and the results are reinterpreted in a Lorentzian way by using a standard "Wick rotation". Anyway, this trick does not seem appropriate for an arbitrarily curved manifold; thus, from a physical point of view, it seems natural to study field theory on (Lorentzian) space-times. Quite a few of reasons justifying the importance of compact space-times, from both, physical and mathematical points of view, can found in [Yu].

But now we are going to see a reason to study completeness of compact Lorentzian manifolds, independently of the fact that compact space-times are taken or not as models of physical universe. First, recall that according to the classical classification scheme of singularities by Ellis and Schmidt [ElSc], to each space-time (M, g) can be attached a boundary ∂M. The boundary points are associated to certain kinds of inextendible curves, and, even though there are different ways to attach this boundary, in all of them a point of the boundary must be assigned if there exists an incomplete timelike or null geodesic. If the space-time can be extended through a boundary point p, then p is called a *regular* point, and the singularity is considered "removable" and not relevant. Otherwise, p is called *singular*, and we can distinguish another two cases. Take a curve γ associated to p, if the components of the curvature tensor R_{abcd} with respect to any parallel frame along γ are well behaved (continuously extendible to the frontier point, differentiable) then p is a *quasi-regular singularity*, otherwise is a *curvature singularity*. Black holes and other physical objects are curvature singularities, but there is no physical interpretation for quasi-regular singularities.

So, an interesting question from a physical point of view would be: (a) to find a good physical interpretation for quasi-regular singularities, or, if not possible, (b) to find a good physical condition for space-times such that quasi-regular singularities cannot occur. Observe that for quasi-regular singularities there are neither divergences of physical quantities nor removed points of a bigger manifold, so, the behavior of incomplete geodesics in compact spacetimes seems to be representative of the behavior of quasi-regular singularities. Thus, a more manageable question with a similar importance for Physics is: (A) to find a good physical interpretation for incomplete compact spacetimes, or if not possible, (B) to find a good physical condition for space-times such that compact spacetimes satisfying it are complete.

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