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1 Introduction

In [1] C. Dellacherie, P.A. Meyer and M. Yor proved that L^∞ is neither closed nor dense in BMO, except in trivial cases (i.e. if the underlying filtration is constant). The same is true for H^∞ (c.f. [3] section 2.6 and [5]). So one may ask, whether it is possible to find a martingale $X \in BMO$, which has a best approximation in L^∞ resp. in H^∞, i.e.

$$\inf_{Z\in L^\infty} ||X - Z||_{BMO} = ||X - \bar{Z}||_{BMO} \text{ for some } \bar{Z} \in L^\infty$$

resp.

$$\inf_{Z\in H^\infty} ||X - Z||_{BMO} = ||X - \bar{Z}||_{BMO} \text{ for some } \bar{Z} \in H^\infty.$$

It is easy to see that this is equivalent to the question: does there exist a martingale $X \in BMO$ s.t.

$$\inf_{Z\in L^\infty} ||X - Z||_{BMO} = ||X||_{BMO}$$

resp.

$$\inf_{Z\in H^\infty} ||X - Z||_{BMO} = ||X||_{BMO}.$$

holds? R. Durrett poses this problem for L^∞ in [2], p. 214, and he conjectures a solution for X. We show in this paper that a discrete time analogue of Durrett's example works, but in continuous time it does not. In the case of H^∞ we provide a class of processes (including Durrett's example), for which $\bar{Z} = 0$ is indeed the best approximation in H^∞. Note that for the negative result in L^∞ we work with the norm $||\cdot||_{BMO_1}$, as the problem was posed by Durrett in this way. For the positive result in H^∞ we use $||\cdot||_{BMO_2}$, which seems to be more natural in this case.

2 Notations and Preliminaries

We denote by BMO the space of continuous martingales X on a given stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^\infty, P)$, satisfying the "usual conditions" of completeness and right continuity, for which the the following equivalent norms are finite

$$||X||_{BMO_1} = \sup_T \{||E[|X_\infty - X_T||\mathcal{F}_T]||_\infty\} = \sup_T \left\{ \left(\frac{E[|X_\infty - X_T|]}{P[T < \infty]} \right) \right\}$$

$$||X||_{BMO_2} = \sup_T\{||E[(X_\infty - X_T)^2|\mathcal{F}_T]^{\frac{1}{2}}||_\infty\} = \sup_T\left\{\left(\frac{E[(X_\infty - X_T)^2]}{P[T<\infty]}\right)^{\frac{1}{2}}\right\} =$$
$$\sup_T\{||E[\langle X\rangle_\infty - \langle X\rangle_T|\mathcal{F}_T]^{\frac{1}{2}}||_\infty\}.$$

Here T runs through all stopping times. In the present context H^∞ denotes the space of continuous martingales M on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^\infty, P)$ s.t.

$$||M||_{H^\infty} = \text{ess sup } \langle M\rangle_\infty^{\frac{1}{2}} < \infty$$

holds. We also use the following standard notation. If M is a martingale and T a stopping time, we denote by M^T the martingale stopped at time T, i.e.

$$M_t^T = M_{t\wedge T}$$

and by ${}^TM$ the martingale started at time T, i.e.

$${}^TM = M - M^T.$$

The next easy lemma is maybe folklore, but for the convenience of the reader we provide a proof.

Lemma 2.1 *Let X be in BMO and R an arbitrary stopping time. Then we have*

$$||{}^RX||_{BMO_2} \leq ||X||_{BMO_2}.$$

Proof: We prove that $||\int HdX||_{BMO_2} \leq ||X||_{BMO_2}$, if H is previsible with $|H| \leq 1$, which immediately implies the assertion of the lemma.

$$||\int HdX||_{BMO_2}^2 = \sup_T\{||E[\langle\int HdX\rangle_\infty - \langle\int HdX\rangle_T|\mathcal{F}_T]||_\infty\} =$$
$$\sup_T\{||E[\int_T^\infty H^2 d\langle X\rangle|\mathcal{F}_T]||_\infty\} \leq \sup_T\{||E[\int_T^\infty d\langle X\rangle|\mathcal{F}_T]||_\infty\} =$$
$$\sup_T\{||E[\langle X\rangle_\infty - \langle X\rangle_T|\mathcal{F}_T]||_\infty\} = ||X||_{BMO_2}^2 \qquad \square$$

3 The case L^∞ - a discrete time example

We give in this section an example of a discrete-time process, for which $\bar{Z} = 0$ is indeed the best approximation in L^∞, if we use the space bmo_1 (c.f. [4]) as an analogue to BMO_1 in the continuous-time setting. Let $(W_n)_{n=0}^\infty$ be a standard random walk on $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0}^\infty, P)$ with natural filtration, i.e. $P[\{\Delta W_n = 1\}] = P[\{\Delta W_n = -1\}] = \frac{1}{2}$ and $W_0 = 0$. Let T be the stopping time $T = inf\{n|\Delta W_n = -1\}$, and $B_n = W_n^T = W_{T\wedge n}$. This is a discrete-time analogue of the continuous martingale, which we consider in section 4, and which was suggested by Durrett in [2].

Denoting the bmo_1-norm by $||\cdot||_*$, an easy calculation gives

$$||B||_* = \sup_S ||E[|B_\infty - B_S||\mathcal{F}_S]||_\infty = \sup_{k\in N_0} ||E[|B_\infty - B_S||B_S = k]||_\infty =$$
$$||E[|B_\infty - B_S||B_S = 0]||_\infty = E[|B_\infty|] = \sum_{r=-1}^\infty |r|2^{-(r+2)} = 1,$$

where the supremum is taken over all stopping times S and N_0 denotes the set $\{0,1,2,...\}$. We denote by $L^\infty(\Omega,\mathcal{F},(\mathcal{F}_n)_{n=0}^\infty,P)$ the space of all bounded martingales with respect to the given filtration. Our claim is

Proposition 3.1

$$\inf_{Z\in L^\infty}\|B-Z\|_* = 1,$$

i.e. $\bar{Z}=0$ *is the best approximation in* L^∞ *of* B.

Proof: We shall show that assuming the existence of a $Z\in L^\infty$, which fulfills $\|B-Z\|_* = \alpha < 1$, leads to a contradiction.

As the definition of the bmo_1-norm is invariant with respect to an additive constant, we assume that $Z\geq 0$ holds. Furthermore the function $f(t) = \|B-tZ\|_*$ is a continuous convex function with $f(0)=1$ and $f(1)=\alpha$. Therefore we may assume w.l.o.g. that $\|Z\|_\infty < \frac{1}{4}$ holds, and we remain with

$$Z_\infty = a_k \qquad \text{on } C_k \text{ for } k=-1,0,1,...,$$

where

$$0\leq a_k \leq \frac{1}{4} \tag{1}$$

holds, and the atoms C_k are defined by $C_k=\{B_\infty = k\}$. Since the filtration is given by

$$\mathcal{F}_n = \{C_{-1},C_0,...,C_{n-2},(C_{n-1}\cup C_n\cup ...)\} \qquad n=0,1,2,...,$$

and $P[C_k]=2^{-(k+2)}$, one can easily calculate $Z_n = E[Z_\infty|\mathcal{F}_n]$ for $n=0,1,2,....$

$$Z_n = a_k \qquad \text{on } C_k \text{ for } k=-1,0,1,...,n-2$$
$$Z_n = \sum_{r=n-1}^{\infty} a_r 2^{-(r+2-n)} =: \gamma_{n-1} \qquad \text{on } (C_{n-1}\cup C_n\cup ...)$$

Hence we get for $n=0,1,2,...$

$$B_\infty - B_n = Z_\infty - Z_n = 0 \qquad \text{on } C_{-1}\cup C_0\cup ...\cup C_{n-2},$$

resp.

$$B_\infty - B_n = s-n \qquad \text{on } C_s \text{ for } s=n-1,n,n+1,...$$

and

$$Z_\infty - Z_n = a_s - \gamma_{n-1} \qquad \text{on } C_s \text{ for } s=n-1,n,n+1,....$$

As the supremum over all stopping times in the definition of the bmo_1-norm can be replaced by a supremum over all fixed times n, we calculate $E[|B_\infty - B_n - Z_\infty + Z_n||\mathcal{F}_n]$, which is 0 on $C_{-1},C_0,...,C_{n-2}$ and

$$\sum_{s=-1}^{\infty} |s - a_{n+s} + \gamma_{n-1}| 2^{-(s+2)}$$

on $C_{n-1}\cup C_n \cup ,....$

Using eq. (1) and our assumption $\|B-Z\|_* = \alpha < 1$, we conclude that

$$(a_{n-1}-\gamma_{n-1})\frac{1}{2}+|a_n-\gamma_{n-1}|(\frac{1}{2})^2+\sum_{s=1}^{\infty}(-a_{n+s}+\gamma_{n-1})2^{-(s+2)} \leq -\rho := \alpha - 1$$

has to hold for $n = 0, 1, 2, \ldots$. We now distinguish two cases.
Case 1: $-a_n + \gamma_{n-1} \geq 0$
A simple calculation gives

$$a_{n-1} \leq \gamma_{n-1} - \rho.$$

Case 2: $-a_n + \gamma_{n-1} < 0$
In this case we get $-\gamma_{n-1} + \frac{2}{3}a_{n-1} + \frac{1}{3}a_n \leq -\frac{2}{3}\rho$. This inequality and our assumption in case 2 allow us to conclude that $a_{n-1} < a_n$ has to hold, and we finally get

$$a_{n-1} < \gamma_{n-1} - \frac{2}{3}\rho.$$

Denoting now $\sigma = \frac{2}{3}\rho > 0$, we can combine case 1 and case 2, which yields

$$-a_{n-1} + \gamma_{n-1} > \sigma \qquad n = 0, 1, 2, \ldots$$

or

$$-\frac{1}{2}a_{n-1} + \sum_{s=n}^{\infty} a_s 2^{-(s-n+2)} > \sigma \qquad n = 0, 1, 2, \ldots. \tag{2}$$

Defining $A = \sup_{s=-1,0,..} a_s$, implies the existence of an M, s.t. $a_M > A - \sigma$, and we infer that

$$-\frac{1}{2}a_M + \sum_{s=M+1}^{\infty} a_s 2^{-(s-M+1)} < \frac{\sigma}{2}$$

holds, which is a contradiction to eq. (2). □

4 The case L^∞ - a continuous time example

In contrast to the discrete case it seems to be not so easy to find a martingale in BMO in continuous time, which has a best approximation $\bar{Z} = 0$ in L^∞. It is shown in this section that the - in some sense - natural guess of Durrett [2] of a martingale, which is quasi-stationary, in a sense to be defined later, does not work. However, it will be shown in section 5 that this quasistationarity is sufficient to guarantee a best approximation $\bar{Z} = 0$ in H^∞.

Let $(W_t)_{t=0}^{\infty}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^{\infty}, P)$. As in [2] we define $R_0 = 0$, $R_n = \inf\{t > R_{n-1} : |W_t - W_{R_{n-1}}| > 1\}$, $N = \inf\{n : W_{R_n} - W_{R_{n-1}} = -1\}$ and finally $X_t = W_{t \wedge R_N}$. The following formula is valid for $a \in (-1, 1)$ (c.f. [2],p. 208)

$$\|X\|_{BMO_1} = \sup_T \|E[|X_\infty - X_T||\mathcal{F}_T]\|_\infty = \sup_{a\in(-1,1)} E[|X_\infty - X_T||X_T = a] =$$
$$\sup_{a\in(-1,1)} 1_{(-1,0]}(a)(1-a^2) + 1_{(0,1]}(a)\frac{(a+1)(2-a)}{2} = \frac{9}{8}.$$

Our claim is now

Proposition 4.1

$$\inf_{Z \in L^\infty} \|X - Z\|_{BMO_1} < \frac{9}{8},$$

where $L^\infty = L^\infty(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ *is the space of continuous bounded martingales.*

This answers negatively the question posed by Durrett in Ex. 1 of sect. 7.7 in [2].
Proof: In order to prove the proposition some further notation is needed. We define

$$\begin{array}{ll} A_r = \{\omega : X_\infty = r\} & r = -1, 0, 1, \ldots \\ S_n = \inf\{t > R_{n-1} : X_t - X_{R_{n-1}} = \frac{1}{2}\} & n = 1, 2, 3, \ldots, \end{array}$$

where we use the convention $\inf \emptyset = \infty$. Furthermore we need

$$\begin{array}{l} A_r^b = A_r \cap \{S_{r+2} = \infty\} \\ A_r^g = A_r \cap \{S_{r+2} < \infty\}, \quad r = -1, 0, 1, \ldots \end{array}$$

and finally

$$M_t = \begin{cases} 0 & R_{n-1} \leq t < S_n \\ 1 & S_n \leq t < R_n, \end{cases} \qquad n = 1, 2, 3, \ldots.$$

The process M indicates, whether X has reached the value $X_{R_{n-1}} + \frac{1}{2}$ in the stochastic interval $[[R_{n-1}, R_n[[$ or not, and is essential for the calculation of the conditional expectations occurring in the sequel.

A straightforward but lengthy application of the optional stopping theorem yields the following table of conditional probabilities, which we will need later on.

$r = 0, 1, 2, \ldots$	$-1 < a \leq \frac{1}{2}$	$\frac{1}{2} < a < 1$
$P[A_{-1}^b \mid X_T = a, M_T]$	$\frac{1-2a}{3}(1 - M_T)$	0
$P[A_{-1}^g \mid X_T = a, M_T]$	$\frac{a+1}{6}(1 - M_T) + \frac{1-a}{2} M_T$	$\frac{1-a}{2}$
$P[A_r^b \mid X_T = a, M_T]$	$\frac{1+a}{6} \frac{1}{2^r}$	$\frac{1+a}{6} \frac{1}{2^r}$
$P[A_r^g \mid X_T = a, M_T]$	$\frac{1+a}{12} \frac{1}{2^r}$	$\frac{1+a}{12} \frac{1}{2^r}$

We define now a bounded continuous martingale Z, which gives a better approximation of X than the trivial approximation $Z = 0$:

$$Z_\infty = \delta 1_{\cup_{r=-1}^\infty A_r^b}$$
$$Z_t = E[Z_\infty | \mathcal{F}_t]$$

with $\delta > 0$. This yields

$$Z_T = \delta P[S_N = \infty | \mathcal{F}_T].$$

Again it suffices to consider $a \in (-1, 1)$. Since $\|Z\|_{BMO_1} \leq \delta$ holds, we only have to show that

$$E[|X_\infty - X_T - Z_\infty + Z_T||X_T = a, M_T] < E[|X_\infty - X_T||X_T = a] = \frac{(a+1)(2-a)}{2}$$

holds for $a \in [\frac{1}{4}, \frac{3}{4}]$ uniformly in M_T, and then to choose δ small enough.

Using again the optional stopping theorem, an easy calculation gives the following table.

$r = -1, 0, 1, \ldots$	X_∞	X_T	Z_∞	Z_T: $-1 < a < \frac{1}{2}$	Z_T: $\frac{1}{2} \le a < 1$
A_r^b	r	a	δ	$\delta(\frac{2-a}{3}(1 - M_T) + \frac{a+1}{3} M_T)$	$\delta \frac{a+1}{3}$
A_r^g	r	a	0	"	"

Putting things together we arrive - after a lot of algebra - at

$$E[|X_\infty - X_T - Z_\infty + Z_T||X_T = a, M_T] =$$

$$\begin{cases} \frac{(a+1)(2-a)}{2} + \frac{\delta}{6}(M_T(a^2 - 1) + (1 - M_T)(-a^2 - a)) & \frac{1}{4} \le a \le \frac{1}{2} \\ \frac{(a+1)(2-a)}{2} + \frac{\delta}{6}(a^2 - 1) & \frac{1}{2} \le a \le \frac{3}{4}, \end{cases}$$

which clearly proves our assertion. □

5 The case H^∞

In this section we introduce a class of processes for which $\bar{Z} = 0$ is indeed the best approximation in H^∞. This class includes also the example of section 4. We start with definitions.

Definition 5.1 *Let X be in BMO. Then we call a stopping time T proper for X, if $P[\langle X\rangle_T < \langle X\rangle_\infty] > 0$.*

Definition 5.2 *A process X in BMO has the property QS (quasi-stationary), if for each proper stopping time T for X, we can find another proper stopping time $S \ge T$ $P-a.s.$ for X, s.t. ${}^S X 1_{\{^S X \not\equiv 0\}}/P[\{^S X \not\equiv 0\}] \sim X$ hold. Here $\sim$ stands for equality in law.*

Our next lemma shows that - not very surprisingly - for QS processes the BMO-norm "does not decline", no matter when the process is started.

Lemma 5.1 *Let X be in BMO with the property QS. Then for all proper stopping times R we have $||{}^R X||_{BMO_2} = ||X||_{BMO_2}$*

Proof: Let U be a proper stopping time s.t. $U \ge R$ $P-a.s.$ and ${}^U X 1_{\{^U X \not\equiv 0\}}/P[\{^U X \not\equiv 0\}] \sim X$ hold. We get

$$||{}^R X||^2_{BMO_2} = \sup_T \frac{E[({}^R X_\infty - {}^R X_T)^2]}{P[T < \infty]} = \sup_T \frac{E[(X_\infty - X_{T \vee R})^2]}{P[T < \infty]} \ge$$
$$\sup_{T \ge R} \frac{E[(X_\infty - X_T)^2]}{P[T < \infty]} \ge \sup_{T \ge U} \frac{E[(X_\infty - X_T)^2]}{P[T < \infty]} = ||X||^2_{BMO_2}.$$

The reverse inequality follows from Lemma 2.1. □
Using a result proved by W. Schachermayer in [5], which characterizes the distance of a given martingale to H^∞ in $||\cdot||_{BMO_2}$, we get our final result.

Theorem 5.1 *Let X be in BMO with the property QS. Then we have*

$$\inf_{Z \in H^\infty} ||X - Z||_{BMO_2} = ||X||_{BMO_2}$$

Proof: Assuming the contrary, namely

$$\inf_{Z\in H^\infty} ||X - Z||_{BMO_2} < ||X||_{BMO_2},$$

yields, by applying Theorem 1.1 of [5], a finite increasing sequence of stopping times

$$0 = T_0 \leq T_1 \leq ... \leq T_N \leq T_{N+1} = \infty$$

s.t.

$$||^{T_n}X^{T_{n+1}}||_{BMO_2} < ||X||_{BMO_2} \qquad n = 0, ..., N$$

(Without loss of generality we may assume that T_N is a proper stopping time for X.) In particular we find

$$||^{T_N}X||_{BMO_2} < ||X||_{BMO_2},$$

which is a contradiction to Lemma 5.1. □

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