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JAMES W. PITMAN

MARC YOR

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# On the lengths of excursions of some Markov processes\*

Jim Pitman<sup>(1)</sup> and Marc Yor<sup>(2)</sup>

(1) Department of Statistics, University of California, 367 Evans Hall # 3860, Berkeley, CA 94720-3860, USA

(2) Laboratoire de Probabilités, Université Paris VI, 4 Place Jussieu, 75252 Paris, France

**Abstract.** Results are obtained regarding the distribution of the ranked lengths of component intervals in the complement of the random set of times when a recurrent Markov process returns to its starting point. Various martingales are described in terms of the Lévy measure of the Poisson point process of interval lengths on the local time scale. The martingales derived from the zero set of a one-dimensional diffusion are related to martingales studied by Azéma and Rainer. Formulae are obtained which show how the distribution of interval lengths is affected when the underlying process is subjected to a Girsanov transformation. In particular, results for the zero set of an Ornstein-Uhlenbeck process or a Cox-Ingersoll-Ross process are derived from results for a Brownian motion or recurrent Bessel process, when the zero set is the range of a stable subordinator.

## 1 Introduction

Let  $Z$  be the random set of times that a recurrent diffusion process  $X$  returns to its starting state 0. For a fixed or random time  $T$ , let  $\mathbf{V}(T) = (V_1(T), V_2(T), \dots)$  where

$$V_1(T) \geq V_2(T) \geq \dots \tag{1}$$

are the ranked lengths of component intervals of the random open set  $(0, T) \setminus Z$ . Features of the distribution of the random sequence  $\mathbf{V}(T)$  have been studied by a number of authors [17, 32, 11, 15, 18, 19, 24, 25, 26]. It is well known that  $Z$  is the closure of the range of the *subordinator*  $(\tau_s, s \geq 0)$  which is the inverse of the local time process of  $X$  at zero. If  $(\tau_s)$  is a stable( $\alpha$ ) subordinator for some  $0 < \alpha < 1$ , as is the case if  $X$  is a Brownian motion without drift ( $\alpha = 1/2$ ) or a Bessel process of dimension  $2 - 2\alpha$ , it is obvious that the law of  $\mathbf{V}(t)/t$  is the same for all  $t$ , and that the law of  $\mathbf{V}(\tau_s)/\tau_s$  is the same for all  $s$ . It is less obvious, but nonetheless true [24], that the common law of  $\mathbf{V}(t)/t$  for  $t > 0$  is identical to the common law of  $\mathbf{V}(\tau_s)/\tau_s$  for all

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$s > 0$ . See [25] for a detailed study of this probability law on decreasing sequences of positive reals with sum 1, and relations between this distribution and Kingman's [11] Poisson-Dirichlet distribution on the same set of sequences.

If  $Z$  is the zero set of a real valued diffusion, the law of which is locally equivalent either to Wiener measure, or to the distribution of a Bessel process of dimension  $2 - 2\alpha$  started at 0, it follows from the identities in distribution mentioned above that for each  $t > 0$  and  $s > 0$  the laws of  $\mathbf{V}(t)/t$  and  $\mathbf{V}(\tau_s)/\tau_s$  are *equivalent*, that is to say mutually absolutely continuous. Our interest here is in describing explicitly the Radon-Nikodym densities relating these various laws, and thereby extending various aspects of our previous studies of zero sets derived from a stable( $\alpha$ ) subordinator to this more general case. We start in Section 2 by treating the example of Ornstein-Uhlenbeck processes. In particular, we obtain various generalizations of results of Truman-Williams [30, 31] and Hawkes-Truman [5] regarding the zero set of the simplest Gaussian-Ornstein-Uhlenbeck process derived from Brownian motion. The results of Section 2 lead to the study in Section 3 of various martingales associated with the range of a subordinator which arise from a change in the Lévy measure of the subordinator. Finally, in Section 4 we compare the results of Sections 2 and 3 to some relations between the stationary distribution of a recurrent Markov process and the Lévy measure of the inverse local time process at a point in the state space. While the basic relations are known to hold in great generality [20], the application of these relations to the zero sets of diffusion processes has been rather neglected in the literature.

## 2 Lengths of excursions of Ornstein-Uhlenbeck processes

The Ornstein-Uhlenbeck process  $(U_t, t \geq 0)$  with parameter  $\mu > 0$  is the solution of Langevin's equation

$$dU_t = dB_t - \mu U_t dt \quad (2)$$

where  $B$  is a Brownian motion. So far as the zero set of  $U$  is concerned, we may as well consider the process  $X := U^2$ . More generally, we consider for  $0 < \alpha < 1$  and  $\mu > 0$  the *squared OU process with dimension  $\delta = 2 - 2\alpha$  and drift parameter  $\mu$* , that is the non-negative solution  $X$  of

$$dX_t = 2\sqrt{X_t}dB_t + (\delta - 2\mu X_t) dt \quad (3)$$

where we assume  $X_0 = 0$ . Denote by  $Q^{\delta, \mu}$  the law of this process  $X$  on the usual path space  $C[0, \infty)$ . See [22, 23, 6] for further background and motivation for the study of these processes, known in mathematical finance as Cox-Ingersoll-Ross processes. Note that for a positive integer  $\delta$ , if  $U$  solves (2), where we now suppose that the equation concerns  $\mathbb{R}^\delta$ -valued processes, then  $X = |U|^2$  solves (3). Let  $Z$  denote the zero set of  $X$ , now taken to be the coordinate process on  $C[0, \infty)$ , and define  $V_n(T)$  in terms of  $Z$  as in (1). Let  $Q^\delta = Q^{\delta, 0}$ , so  $Q^\delta$  is the law of the square of a  $\delta$ -dimensional Bessel process [29, 22]. Let  $(S_t, t \geq 0)$  denote a local time process for  $X$  at zero, and let  $(\tau_s)$  be the right continuous inverse of this local time process. Then  $(\tau_s)$  is a stable ( $\alpha$ ) subordinator, and  $Q^{\delta, 0}$  almost surely the zero set  $Z$  of  $X$  is the closure of the range of  $(\tau_s)$ . Note that while the definition of both  $(S_t)$  and  $(\tau_s)$  depends on the value of  $\delta$ , this dependence is hidden in the notation.

We recall the Cameron-Martin-Girsanov relationship between  $Q^{\delta,\mu}$  and  $Q^\delta$ : for every  $t > 0$

$$\frac{dQ^{\delta,\mu}}{dQ^\delta} \Big|_{\mathcal{F}_t} = \exp \left( -\frac{\mu}{2}(X_t - \delta t) - \frac{\mu^2}{2} \int_0^t du X_u \right) \tag{4}$$

As a consequence of (4) and the recurrence of  $X$  under  $Q^{\delta,\mu}$  for every  $\mu > 0$ , we have also for every  $s > 0$  that

$$\frac{dQ^{\delta,\mu}}{dQ^\delta} \Big|_{\mathcal{F}_{\tau_s}} = \exp \left( \frac{\mu\delta\tau_s}{2} - \frac{\mu^2}{2} \int_0^{\tau_s} du X_u \right) \tag{5}$$

From this absolute continuity relation, it is immediate the zero set  $Z$  of  $X$  is represented  $Q^{\delta,\mu}$  almost surely for all  $\mu \geq 0$  as the closed range of the process  $(\tau_s)$ , which is a subordinator under  $Q^{\delta,\mu}$  for each  $\mu \geq 0$ , a subordinator that is stable for  $\mu = 0$  but not for  $\mu > 0$ . The Lévy measure of  $(\tau_s)$  under  $Q^{\delta,\mu}$  can be computed from (5) as indicated below.

**Theorem 1** *For a random time  $T$  let  $\mathcal{V}_T = \sigma(V_n(T), n = 1, 2, \dots)$ . Then for each  $t > 0$*

$$\frac{dQ^{\delta,\mu}}{dQ^\delta} \Big|_{\mathcal{V}_t} = \exp \left( \frac{\mu\delta t}{2} \right) \Sigma(\mu, t) \Pi(\mu, t)^{2-\frac{\delta}{2}} \tag{6}$$

and for each  $s > 0$

$$\frac{dQ^{\delta,\mu}}{dQ^\delta} \Big|_{\mathcal{V}_{\tau_s}} = \exp \left( \frac{\mu\delta\tau_s}{2} \right) \Pi(\mu, \tau_s)^{2-\frac{\delta}{2}} \tag{7}$$

where

$$\Sigma(\mu, t) = \sum_n \frac{1 - e^{-2\mu V_n(t)}}{2\mu t} \quad \text{and} \quad \Pi(\mu, t) = \prod_n \frac{\mu V_n(t)}{\sinh(\mu V_n(t))}$$

**Proof.** Let  $G_t = \text{sup}(Z \cap [0, t])$ . Note first that for fixed  $t$ ,

$$\mathcal{V}_t \subset \mathcal{H}_t \subset \mathcal{F}_{G_t}$$

where  $\mathcal{H}_t = \sigma(G_s, 0 \leq s \leq t)$  and  $\mathcal{F}_{G_t} = \sigma(X_s 1_{(s \leq G_t)}, 0 \leq s \leq t)$ . Moreover, for each  $s > 0$ , the random time  $\tau_s$  is an  $(\mathcal{H}_t)$  stopping time with  $\tau_s = G_{\tau_s}$  a.s., and

$$\mathcal{V}_{\tau_s} \subset \mathcal{H}_{\tau_s} \subset \mathcal{F}_{\tau_s}$$

modulo  $Q^\delta$  null sets. Consequently, we will be able to prove the formulae of the theorem by projecting the  $Q^\delta$  martingale which appears in (4), first on  $(\mathcal{F}_{G_t})$ , then on  $(\mathcal{H}_t)$ , and finally on the  $\sigma$ -field  $\mathcal{V}_t$ . (Note that  $\mathcal{V}_s$  is not contained in  $\mathcal{V}_t$  for  $s < t$ . So unlike the other families considered above, the family  $(\mathcal{V}_t, t > 0)$  does not constitute a filtration.)

*Projection on  $(\mathcal{F}_{G_t})$ .* Here we will use the fact that under  $Q^\delta$  the squared meander

$$\left( m_u^2 := \frac{1}{t - G_t} X_{G_t+u(t-G_t)}, 0 \leq u \leq 1 \right)$$

is independent of  $\mathcal{F}_{G_t}$ , and satisfies

$$(m_u^2, 0 \leq u \leq 1) \stackrel{d}{=} (\rho_u^2 + R_u^2, 0 \leq u \leq 1)$$

where  $(\rho_u, 0 \leq u \leq 1)$  is a standard Bessel bridge of dimension  $2 - \delta$ , and  $R$  is an independent 2-dimensional Bessel process. See [33, Corollary 3.9.1, page 44]. From the above description of  $(m_u^2, 0 \leq u \leq 1)$ , as a special case of the extended Lévy area formulae given in [33, (2.1) and (2.5)], and in [22, (2.k)], we easily deduce the following formula: for all  $\gamma, \nu \geq 0$

$$Q^\delta \left[ \exp \left( -\gamma m_1^2 - \frac{\nu^2}{2} \int_0^1 ds m_s^2 \right) \right] = \left( \frac{\nu}{\sinh \nu} \right)^{1-\frac{\delta}{2}} \left( \cosh \nu + \frac{2\gamma}{\nu} \sinh \nu \right)^{-1} \quad (8)$$

In particular, for  $\gamma = \nu/2$ ,

$$Q^\delta \left[ \exp \left( -\frac{\nu}{2} m_1^2 - \frac{\nu^2}{2} \int_0^1 ds m_s^2 \right) \right] = \Phi_\delta(\nu) := \left( \frac{\nu}{\sinh \nu} \right)^{1-\frac{\delta}{2}} e^{-\nu} \quad (9)$$

We deduce from (4) and (9) that

$$\frac{dQ^{\delta,\mu}}{dQ^\delta} \Big|_{\mathcal{F}_{G_t}} = \exp \left( \frac{\mu\delta t}{2} \right) \Phi_\delta(\mu(t - G_t)) \exp \left( -\frac{\mu^2}{2} \int_0^{G_t} du X_u \right) \quad (10)$$

*Projection on  $(\mathcal{H}_t)$ .* From the previous formula we obtain

$$\frac{dQ^{\delta,\mu}}{dQ^\delta} \Big|_{\mathcal{H}_t} = \exp \left( \frac{\mu\delta t}{2} \right) \Phi_\delta(\mu(t - G_t)) \Pi(\mu, G_t)^{2-\frac{\delta}{2}} \quad (11)$$

We derive (11) from (10) using the excursion theory under  $Q^\delta$ , in particular, the fact that under  $n_\delta$ , the corresponding Itô law of excursions, given that the lifetime equals  $v$ , the excursion process  $(\epsilon_u, u \leq v)$  is a Bessel bridge of dimension  $4 - \delta$ , and we have used the Lévy-type formula [22, 33]

$$Q^{4-\delta} \left( \exp -\frac{\mu^2}{2} \int_0^v ds X_s \Big| X_v = 0 \right) = \left( \frac{\mu v}{\sinh(\mu v)} \right)^{2-\frac{\delta}{2}}$$

Since

$$\Pi(\mu, G_t) = \Pi(\mu, t) \left( \frac{\sinh(\mu(t - G_t))}{\mu(t - G_t)} \right)$$

and

$$\Phi_\delta(x) \left( \frac{\sinh x}{x} \right)^{2-\frac{\delta}{2}} = \left( \frac{1 - e^{-2x}}{2x} \right)$$

we can write (11) as

$$\frac{dQ^{\delta,\mu}}{dQ^\delta} \Big|_{\mathcal{H}_t} = \exp \left( \frac{\mu\delta t}{2} \right) \left( \frac{1 - e^{-2\mu(t-G_t)}}{2\mu(t - G_t)} \right) \Pi(\mu, t)^{2-\frac{\delta}{2}} \quad (12)$$

*Projection on  $(\mathcal{V}_t)$ .* Formula (6) follows from the previous formula (12) and the result of [24] that

$$Q^\delta(t - G_t = V_n(t) | \mathcal{V}_t) = \frac{V_n(t)}{t} \quad (13)$$

□

Let  $\Lambda^{\delta,\mu}$  denote the Lévy measure of  $(\tau_s)$  under  $Q^{\delta,\mu}$ . So by definition

$$Q^{\delta,\mu}(\exp(-\theta\tau_s)) = \exp\left(-s \int_0^\infty (1 - e^{-\theta x}) \Lambda^{\delta,\mu}(dx)\right) \tag{14}$$

Write simply  $\Lambda^\delta$  for  $\Lambda^{\delta,0}$ . From Theorem 1 and the basic formula

$$\Lambda^\delta(dy) = C dy / y^{2-\frac{\delta}{2}} = C dy / y^{1+\alpha} \tag{15}$$

where  $C$  is a constant depending on the choice of normalization of local time, we obtain for  $\mu > 0$  the formula

$$\Lambda^{\delta,\mu}(dy) = C \left(\frac{\mu}{\sinh(\mu y)}\right)^2 \left(\frac{e^{2\mu y} - 1}{2\mu}\right)^{\delta/2} dy \tag{16}$$

To check, we recover (15) from (16) in the limit as  $\mu \downarrow 0$ . And for  $\delta = 1$  we recover the result of Hawkes-Truman [5] for the zero set of the Gaussian-Ornstein-Uhlenbeck process. See also Section 4 for another confirmation of the formula (16) which involves almost no calculation. By combination of (14) and (12) we obtain for  $\mu > 0$  the formula

$$Q^{\delta,\mu}(t - G_t = V_n(t) | \mathcal{V}_t) = \frac{1 - e^{-2\mu V_n(t)}}{\sum_m (1 - e^{-2\mu V_m(t)})} \tag{17}$$

which is a particular case of formula (7.d) of [24]. From the proof of Theorem 1, we extract also the following corollary, which is a particular case of more general results presented in the next section.

**Corollary 2** *Let  $G_t = \sup(Z \cap [0, t])$  where  $Z$  is the range of a stable  $(\alpha)$  subordinator and let  $\mathcal{H}_t = \sigma(G_s, 0 \leq s \leq t)$ . Then for every  $\mu > 0$*

$$\psi(\mu(t - G_t)) \exp(\mu(1 - \alpha)t) \prod_{n=1}^\infty \left(\frac{\mu V_n(t)}{\sinh(\mu V_n(t))}\right)^{1+\alpha} \tag{18}$$

is an  $(\mathcal{H}_t)$ -martingale, where  $\psi(x) := (1 - e^{-2x})/(2x)$ , and

$$\exp(\mu(1 - \alpha)\tau_s) \prod_{n=1}^\infty \left(\frac{\mu V_n(\tau_s)}{\sinh(\mu V_n(\tau_s))}\right)^{1+\alpha} \tag{19}$$

is an  $(\mathcal{H}_{\tau_s})$ -martingale.

**Remark 3** The formula (12) and the more general formula (30) presented in the next section are closely related to the studies by Azéma [1] and Rainer [27] of martingales relative to the filtration  $(\mathcal{H}_t)$  generated by the zero set of a real valued diffusion. In particular, if  $(X_t, t \geq 0)$  is a recurrent diffusion on natural scale on a subinterval of the line containing 0, and  $\Lambda = \Lambda_+ + \Lambda_-$  is the decomposition of the Lévy measure  $\Lambda$  induced by positive and negative excursions, as discussed further in Section 4, then the process

$$\mu_t := \frac{1(X_t > 0)}{\Lambda_+(t - G_t, \infty)} - \frac{1(X_t < 0)}{\Lambda_-(t - G_t, \infty)} \tag{20}$$

is an  $(\mathcal{H}_t)$  local martingale. (This is, up to a factor of 1/2, the formula at the end of the introduction of [27], after correction of a misprint in that formula as indicated at the end of the present volume.) Our martingales (18) and (30) can be recovered by application of Itô's formula. If  $(X_t)$  is Brownian motion, then  $\mu_t$  is a constant multiple of Azéma's martingale  $\text{sgn}(X_t)\sqrt{t - G_t}$ .

### 3 Change of measure formulae for subordinators

Let probability distributions  $P$  and  $Q$  on the same basic measurable space  $(\Omega, \mathcal{F})$  govern a process  $(\tau_s, s \geq 0)$  as a subordinator, with Lévy measures  $\Lambda_P$  and  $\Lambda_Q$  respectively. We assume that

$$\Lambda_Q(dy) = \Phi(y)\Lambda_P(dy) \text{ for a } \Phi \text{ such that} \tag{21}$$

$$\int_0^\infty |1 - \Phi(y)|\Lambda_P(dy) < \infty \tag{22}$$

and use the notation

$$\bar{\Lambda}_P(x) = \Lambda_P(x, \infty); \quad \bar{\Lambda}_Q(x) = \Lambda_Q(x, \infty) = \int_{(x, \infty)} \Phi(y)\Lambda_P(dy) \tag{23}$$

Let  $Z$  be the range of  $(\tau_s, V_n(T))$  as in (1). Let  $(S_t, t \geq 0)$  be the continuous local time inverse of  $(\tau_s, s \geq 0)$ .

**Theorem 4** *Under the hypothesis (22) on the function  $\Phi = d\Lambda_Q/d\Lambda_P$ , define a function  $\Psi$  and a real number  $\gamma$  by*

$$\Psi(0) = 1; \quad \Psi(x) = \frac{\bar{\Lambda}_Q(x)}{\bar{\Lambda}_P(x)} \quad (x > 0 : \bar{\Lambda}_P(x) > 0), \tag{24}$$

$$\gamma = \int_0^\infty (\Phi(x) - 1)\Lambda_P(dx) = \int_0^\infty (\Lambda_Q - \Lambda_P)(dx) \tag{25}$$

and define processes  $(\Pi_\Phi(t), t \geq 0)$  and  $(M_\Phi(t), t \geq 0)$  by

$$\Pi_\Phi(0) = 1; \quad \Pi_\Phi(t) = \prod_n \Phi(V_n(t)) \quad (t > 0) \tag{26}$$

$$M_\Phi(t) = \Psi(t - G_t)\Pi_\Phi(G_t)\exp(-\gamma S_t) \tag{27}$$

Then for each  $(\mathcal{H}_t)$ -stopping time  $T$  such that  $P(T < \infty) = Q(T < \infty) = 1$ , the law  $Q$  is absolutely continuous with respect to  $P$  on  $\mathcal{H}_T$ , with density

$$\left. \frac{dQ}{dP} \right|_{\mathcal{H}_T} = M_\Phi(T) \tag{28}$$

In particular this formula holds for every fixed time  $T$ , and for  $T = \tau_s$  for every  $s > 0$ , in which case the right side of (28) is

$$M_\Phi(\tau_s) = \Pi_\Phi(\tau_s)\exp(-\gamma s) \quad (s \geq 0) \tag{29}$$

Consequently,

$$(M_\Phi(t), t \geq 0) \text{ is an } ((\mathcal{H}_t), P)\text{-martingale} \tag{30}$$

and

$$(M_\Phi(\tau_s), s \geq 0) \text{ is an } ((\mathcal{H}_{\tau_s}), P)\text{-martingale.} \tag{31}$$

By combination of Theorem 4 with Theorem 7.1 of [24] we obtain the following:

**Corollary 5** Suppose further that  $\Lambda_P(dy) = \rho_P(y)dy$  and  $\Lambda_Q(dy) = \rho_Q(y)dy$  for some densities  $\rho_P$  and  $\rho_Q$  which are strictly positive on  $(0, \infty)$ . For  $y > 0$  and  $t > 0$  let

$$h_P(y) = \frac{\bar{\Lambda}_P(y)}{\rho_P(y)}; \quad H_P(t) = \sum_n h_P(V_n(t)); \tag{32}$$

and define  $h_Q(y)$  and  $H_Q(t)$  similarly with  $Q$  instead of  $P$ . Then

$$\Phi(x) = \frac{\rho_Q(x)}{\rho_P(x)} \quad (x > 0); \quad \gamma = \int_0^\infty (\rho_Q(y) - \rho_P(y)) dy \tag{33}$$

For fixed  $t > 0$ , let  $\mathcal{V}_t = \sigma(V_n(t), n = 1, 2, \dots)$ . Then

$$\left. \frac{dQ}{dP} \right|_{\mathcal{V}_t} = \frac{H_Q(t)}{H_P(t)} \Pi_\Phi(t) \exp(-\gamma S_t) \tag{34}$$

**Proof of Corollary 5.** The formulae (33) are immediate. To deduce (34) from (28), it suffices to take  $T = t$  and project the density in (28) onto the  $\sigma$ -field  $\mathcal{V}_t$ , using the fact that  $\Pi_\Phi(t)$  and  $S_t$  are  $\mathcal{V}_t$ -measurable, the fact that  $\Pi_\Phi(G_t) = \Pi_\Phi(t)/\Phi(t - G_t)$  and the formula

$$E_P \left[ \left( \frac{h_Q}{h_P} \right) (t - G_t) \middle| \mathcal{V}_t \right] = \frac{H_Q(t)}{H_P(t)} \tag{35}$$

which is obtained by evaluation of the left side of (35) using the *sampling formula*

$$P(t - G_t = V_n(t) | \mathcal{V}_t) = \frac{h_P(V_n(t))}{H_P(t)} \tag{36}$$

established in Theorem 7.1 of [24]. This shows that the right side of (35) equals

$$\sum_n \frac{h_Q(V_n(t))}{h_P(V_n(t))} \frac{h_P(V_n(t))}{H_P(t)} = \frac{H_Q(t)}{H_P(t)} \tag{37}$$

**Proof of Theorem 4.**

*Step 1. Proof for  $T = \tau_s$  for fixed  $s > 0$ .* In this case we have  $\tau_s - G_{\tau_s} = 0$  a.s. so  $\Psi(\tau_s - G_{\tau_s}) = 1$ , and the task is to show that for every non-negative  $\mathcal{H}_{\tau_s}$ -measurable random variable  $X$ ,

$$E_Q(X) = E_P[X M_\Phi(\tau_s)] \text{ where } M_\Phi(\tau_s) = \Pi_\Phi(\tau_s) \exp(-\gamma s) \tag{38}$$

This is a consequence of the following variation of Campbell's formula [12, (3.35)]: for  $\Phi$  satisfying (22),

$$E_P \left[ \prod_n \Phi(V_n(\tau_s)) \right] = \exp \left( s \int_0^\infty (\Phi(x) - 1) \Lambda_P(dx) \right) = \exp(\gamma s) \tag{39}$$

Apply (39) with  $Q$  instead of  $P$  and  $g$  instead of  $\Phi$ , for non-negative  $g$  with  $\int |g - 1| d\Lambda_Q < \infty$ . Then write  $(g - 1)\Phi = (g\Phi - 1) - (\Phi - 1)$  and use (39) again twice under  $P$  to see that (38) holds for  $X = \prod_n g(V_n(\tau_s))$ . Varying  $g$  provides enough  $X$ 's to deduce that (38) holds for all non-negative  $\mathcal{V}_{\tau_s}$ -measurable  $X$ 's. But the  $\sigma$ -field  $\mathcal{V}_{\tau_s}$  is contained in  $\mathcal{H}_{\tau_s} = \sigma(\tau_u, 0 \leq u \leq s)$ , and the identity (38) extends to all non-negative



$\mathcal{H}_{\tau_s}$ -measurable  $X$  because  $P$  and  $Q$  share a common conditional distribution for  $(\tau_u, 0 \leq u \leq s)$  given  $\mathcal{V}_{\tau_s}$ , that is the unique law of an increasing process parameterized by  $[0, s]$  with exchangeable increments and jumps of the prescribed sizes  $V_n(\tau_s)$ ,  $n = 1, 2, \dots$ . (Assuming for simplicity that  $\Lambda_P$  is continuous, to avoid ties among the  $V_n(\tau_s)$ , we can write

$$\tau_u = \sum_n V_n(\tau_s) 1(\sigma_n \leq u) \quad (0 \leq u \leq s) \tag{40}$$

where  $\sigma_n$  is the a.s. unique local time  $u$  such that  $\tau_u - \tau_{u-} = V_n(\tau_s)$ . The common conditional law of  $(\tau_u, 0 \leq u \leq s)$  given  $\mathcal{V}_{\tau_s}$  is then specified by the fact that under both  $P$  and  $Q$  the  $\sigma_n$  are i.i.d random variables with uniform  $[0, s]$  distribution, independent of  $\mathcal{V}_{\tau_s}$ . See [10] regarding this decomposition of  $\tau_u$  and the corresponding result allowing  $\Lambda_P$  to have a discrete component).

*Step 2. Proof for  $T = t$  for a fixed  $t > 0$ .* From the previous result for  $T = \tau_s$ , for all  $s, t > 0$  we can compute

$$\frac{dQ}{dP} \Big|_{\mathcal{H}_t} = E_P(M_\Phi(\tau_s) | \mathcal{H}_{t \wedge \tau_s}) \quad \text{on } (t < \tau_s) \tag{41}$$

But on  $(t < \tau_s)$  we find that

$$M_\Phi(\tau_s) = \exp(-\gamma s) \prod_\Phi(G_t) \Phi(D_t - G_t) \Pi^* \tag{42}$$

where  $\Pi^*$  is the product of  $\Phi(V_n(\tau_s))$  over  $n$  corresponding to those component intervals of  $[0, \tau_s] \setminus Z$  that are contained in  $[D_t, \tau_s]$ . Let  $(S_t)$  denote the continuous local time inverse of  $(\tau_s)$ . By the strong Markov property of  $(\tau_u, u \geq 0)$  at the stopping time  $S_t$ , when  $\tau_{S_t} = D_t$ , and (39),

$$E_P \left( \Pi^* \Big| \mathcal{H}_{t \wedge \tau_s} \right) = \exp(\gamma(s - S_t)) \tag{43}$$

Also, by the last exit decomposition at  $G_t$ , on the event  $(\tau_s > t)$ , which is identical to  $(S_t < s)$ ,

$$P(D_t - G_t \in dy | \mathcal{H}_{t \wedge \tau_s}, \tau_s > t, t - G_t = x) = \Lambda_P(dy) / \bar{\Lambda}_P(x) \quad (y > x) \tag{44}$$

Combining these observations shows that

$$E_P \left( M_\Phi(\tau_s) \Big| \mathcal{H}_{t \wedge \tau_s} \right) = M_\Phi(t) \quad \text{on } (t < \tau_s) \tag{45}$$

That is to say, for every non-negative  $\mathcal{H}_t$ -measurable random variable  $H_t$

$$E_Q[H_t 1_{(\tau_s > t)}] = E_P[H_t M_\Phi(t) 1_{(\tau_s > t)}] \tag{46}$$

Now for each  $t > 0$  we can let  $s \rightarrow \infty$ , and use the fact that  $1_{(\tau_s > t)} \uparrow 1$  both  $P$  and  $Q$  a.s. to deduce  $E_Q(H_t) = E_P(H_t Z_t)$ , which is the desired result.

*Step 3. Proof for a general  $(\mathcal{H}_t)$ -stopping time  $T$  with  $P(T < \infty) = Q(T < \infty) = 1$*  This is a reprise of the previous argument, first using the optional sampling theorem for  $T \wedge t$ , then letting  $t \rightarrow \infty$ .

**Example 6** As an example of the situation described in Theorem 4 where  $\gamma \neq 0$ , following Kinkladze [13] we now consider the pair of diffusions  $B$  and  $X^{(\mu)}$ , where  $B$  is a Brownian motion, and  $X^{(\mu)}$  with law  $P^{(\mu)}$  is the solution of

$$dX_t = dB_t - \mu \operatorname{sgn}(X_t) dt$$

We have

$$P^{(\mu)} \Big|_{\mathcal{F}_t} = \exp \left( -\mu \{ |X_t| - S_t \} - \frac{1}{2} \mu^2 t \right) \cdot P \Big|_{\mathcal{F}_t} \tag{47}$$

where  $(S_t, t \geq 0)$  denotes the local time of  $X$  at 0. From (47) we deduce

$$P^{(\mu)} \Big|_{\mathcal{H}_t} = f(-\mu \sqrt{t - G_t}) \exp \left( \mu S_t - \frac{1}{2} \mu^2 t \right) \cdot P \Big|_{\mathcal{H}_t} \tag{48}$$

where  $f(\lambda) := E[\exp(\lambda m_1)]$  for  $m_1$  the value at time 1 of a Brownian meander, that is

$$f(\lambda) = \int_0^\infty r \exp(-r^2/2) \exp(\lambda r) dr$$

It follows that

$$\frac{dP^{(\mu)}}{dP} \Big|_{\mathcal{H}_t} = \Psi(t - G_t) \Pi_\Phi(G_t) \exp(-\gamma S_t)$$

where

$$\Psi(x) = f(-\mu \sqrt{x}) \exp(-\frac{1}{2} \mu^2 x); \quad \Phi(x) = \exp(-\frac{1}{2} \mu^2 x); \quad \gamma = -\mu$$

## 4 The Lévy measure of the inverse local time process

Let 0 be a recurrent point in the state-space  $E$  of a nice recurrent strong Markov process  $X$ . Let  $T_0 = \inf\{t : t > 0, X_t = 0\}$ . Assuming that 0 is regular for itself, that is  $P_0(T_0 = 0) = 1$ , it is well known that there exists a continuous increasing local time process for  $X$  at 0, say  $(L_t, t \geq 0)$ , whose right-continuous inverse, say  $(\tau_\ell, \ell \geq 0)$  is a subordinator under  $P_0$ . Let  $\Lambda$  denote the Lévy measure of this subordinator. Due to different conventions about the normalization of local time processes in different settings, let us allow an arbitrary normalization of  $(L_t)$  in this generality. So  $\Lambda$  is unique up to constant factors: multiplying  $L$  by  $c$  divides  $\Lambda$  by  $c$ . It is known [4] that such a Markov process  $X$  admits a  $\sigma$ -finite invariant measure  $m$  such that  $P^m(T_0 = \infty) = 0$ . As a consequence of a general Palm formula for excursions of stationary (not necessarily Markovian) processes established in [20], this  $m$  is unique up to constant multiples and there is the identity

$$P^m(T_0 \in da) = m\{0\} \delta_0(da) + c \Lambda(a, \infty) da \quad (a \geq 0) \tag{49}$$

for some  $c > 0$  depending on the choice of  $m$  and the choice of normalization of local time. That is to say, the  $P^m$  distribution of  $T_0$  has an atom at 0 of magnitude  $m\{0\}$ , and has a density on  $(0, \infty)$  given by  $c \Lambda(a, \infty)$  for  $0 < a < \infty$ .

The connection between the invariant measure  $m$  on the state-space of  $X$  and the Lévy measure  $\Lambda$  on  $(0, \infty)$  is made via Itô's law  $n$  for excursions  $\varepsilon$  of  $X$  away from 0. Assume that an excursion  $\varepsilon = (\varepsilon_t, t \geq 0)$  is absorbed at 0 at time  $T_0 = T_0(\varepsilon) = \inf\{t :$

$t > 0, \varepsilon_t = 0$ }. And assume for simplicity that  $m\{0\} = 0$ , which is to say that the Lebesgue measure of the zero set of  $X$  is 0 a.s.  $P^x$  for all  $x \in E$ . By definition of  $n$  [7, 28]), the Lévy measure  $\Lambda$  of the inverse local time process at 0 is the  $n$  distribution of  $T_0$ :

$$\Lambda(a, \infty) = n(T_0 > a) \quad (a > 0) \quad (50)$$

Also, the formula

$$mf = \int n(d\varepsilon) \int_0^{T_0} f(\varepsilon_t) dt \quad (f \geq 0) \quad (51)$$

for non-negative measurable functions  $f$  on  $E$  defines an invariant measure  $m$  for  $X$  [4, 20], and if we take this  $m$  in (49) the constant  $c$  is forced equal to 1. That is to say, for  $m$  defined by (51)

$$P^m(T_0 \in da) = \Lambda(a, \infty) da \quad (a \geq 0) \quad (52)$$

As shown in [20], this identity is a consequence of the following more general identity. Let  $n^*$  denote Maisonneuve's exit law for state 0, that is the distribution on path-space under which  $(X_t, 0 \leq t \leq T_0)$  and  $(X_{T_0+u}, 0 \leq u \leq \infty)$  are independent with laws  $n$  and  $P_0$  respectively. Then for an arbitrary non-negative measurable  $Y$  defined on path-space

$$P^m(Y) = n^* \left( \int_0^{T_0} Y(\theta_t) dt \right) \quad (53)$$

where  $\theta_t$  is the usual shift operator on path space, so

$$Y(\theta_t) = Y(X_{t+u}, 0 \leq u < \infty)$$

Taking  $Y = f(X_0)$  yields (51), while taking  $Y = h(T_0)$  for a non-negative measurable  $h$  on  $(0, \infty)$  and using (50) yields (49).

Suppose now that  $X$  is a recurrent diffusion process on a subinterval of the line containing 0. Let  $m_+$  and  $m_-$  denote the restrictions of  $m$  to  $(0, \infty)$  and  $(-\infty, 0)$  respectively, so  $m = m_+ + m_-$ . By path continuity of  $X$ , each excursion is either positive or negative, and there are corresponding decompositions  $n = n_+ + n_-$  and  $\Lambda = \Lambda_+ + \Lambda_-$  which imply via (53) that (49) holds just as well with  $m$  replaced by  $m_{\pm}$  and  $\Lambda$  replaced by  $\Lambda_{\pm}$ , where  $\pm$  is either  $+$  or  $-$ .

The decomposition  $\Lambda = \Lambda_+ + \Lambda_-$  and reflection through 0 reduces computation of  $\Lambda$  to computation of  $\Lambda_+$ .

Put another way, there no loss of generality in assuming, as we shall from now on, that the statespace  $E$  of the diffusion is either  $[0, \infty)$  or  $[0, b]$  for some  $b > 0$ . To be definite, assume  $E = [0, \infty)$ .

**Example 7** It is known [22] and easily checked that if  $X$  has distribution  $Q^{\delta, 0}$ , then the process  $X^{\delta, \mu}$  defined by

$$X^{\delta, \mu}(t) = e^{-2\mu t} X(e^{2\mu t}/2\mu) \quad (-\infty < t < \infty) \quad (54)$$

is a two sided stationary process governed by the stochastic differential equation (3) for  $t > 0$ . Let  $\eta = \eta^{\delta, \mu}$  denote the  $Q^{\delta, 0}$  distribution of

$$X^{\delta, \mu}(0) = X(1/2\mu) \stackrel{d}{=} (2\mu)^{-1} X(1) \stackrel{d}{=} \mu^{-1} Z_{\delta/2} \quad (55)$$

where  $Z_a$  denotes a gamma( $a$ ) variable. Then the  $P^\eta$  distribution of  $T_0$  considered in (49) is immediately identified in this example with the  $Q^{\delta,0}$  distribution of

$$\inf\{t > 0 : X^{\delta,\mu}(t) = 0\} = \inf\{t > 0 : X(e^{2\mu t}/2\mu) = 0\} \tag{56}$$

$$= \frac{1}{2\mu} \log(2\mu D_{1/2\mu}) \stackrel{d}{=} \frac{1}{2\mu} \log(D_1) \tag{57}$$

where  $D_t = \inf\{u > t : X(u) = 0\}$ . Since the distribution of  $D_t$  for a stable( $\alpha$ ) zero set is given by

$$D_t \stackrel{d}{=} tD_1 \stackrel{d}{=} \frac{t}{G_1} \stackrel{d}{=} \frac{t}{Z_{\alpha,1-\alpha}} \tag{58}$$

where  $Z_{a,b}$  denotes a beta( $a, b$ ) variable [3, 17], a simple change of variables yields the following formula for the density of  $(2\mu)^{-1} \log(D_1)$  in (57), hence for  $\Lambda(a, \infty)$  in (52):

$$\Lambda^{\mu,\delta}(a, \infty) = \frac{P[(2\mu)^{-1}(\log D_1) \in da]}{da} = \frac{2\mu}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{e^{-2\mu\alpha a}}{(1 - e^{-2\mu a})^\alpha} \tag{59}$$

where  $\alpha = 1 - \delta/2$ . It is easily verified that this formula is consistent with the previous formula (16).

**Some general formulae for diffusions.** In the case of one-dimensional diffusion processes, there is an alternative *local formula* for  $\Lambda$  which has been known for much longer than the *global formula* (52). Assuming for simplicity that the statespace is  $[0, \infty)$ , the local formula for  $\Lambda$  is

$$\Lambda(a, \infty) = c \lim_{x \downarrow 0} \frac{P^x(T_0 > a)}{s(x) - s(0)} \tag{60}$$

where  $s$  is the scale function of the diffusion and  $c$  is a constant depending on normalization conventions for the scale function and the local time process. This formula appears in Section 6.2 of Itô-McKean[8], along with various Laplace transformed expressions of this formula now discussed. There are also corresponding local formulae for Itô's excursion law  $n$  and for Maisonneuve's exit law  $n^*$  in this setting, for instance  $n^*(Y) = c \lim_{x \downarrow 0} \frac{P^x(Y)}{s(x) - s(0)}$  for appropriately regular  $Y$ . See e.g. Section 3 of [22] for further discussion and other descriptions of  $n$ .

So far as the zero set of  $X$  is concerned, there is no loss of generality in replacing  $X$  by  $s(X)$  where  $s$  is a scale function for  $X$  chosen so that  $s(0) = 0$ , such a choice being possible due to the assumed recurrence of the boundary state 0. So let us assume that  $X$  is already on natural scale, i.e. that  $s(x) = x$ , so the generator  $G$  of  $X$ , acting on smooth functions vanishing in a neighbourhood of 0 is

$$G = \frac{1}{2} \frac{d}{dm} \frac{d}{dx} \tag{61}$$

where  $m$  is the *speed measure* of  $X$  on  $[0, \infty)$  and we assume for simplicity that  $m\{0\} = 0$ . Now in (60) we obtain

$$\Lambda(a, \infty) = \frac{1}{2} \frac{d}{dx} P^x(T_0 > a) \Big|_{x=0+} \tag{62}$$

provided the local time process  $(L_t)$  at 0 is defined as  $L_t = L_t^0$  where

$$(L_t^x; t \geq 0, x \geq 0)$$

is a jointly continuous version of the local times normalized as occupation densities relative to the speed measure  $m$  of  $X$ . See e.g. [8].

In terms of the Laplace exponent

$$\Theta(\lambda) := \int_0^\infty (1 - e^{-\lambda x})\Lambda(dx) = \lambda \int_0^\infty e^{-\lambda a}\Lambda(a, \infty) da \tag{63}$$

taking a Laplace transform converts (62) into

$$\Theta(\lambda) = -\frac{1}{2} \left. \frac{d}{dx} \phi_\lambda(x) \right|_{x=0+} \tag{64}$$

where

$$\phi_\lambda(x) = P^x(e^{-\lambda T_0}) \tag{65}$$

is well known to be the unique solution  $\phi$  of the Sturm-Liouville equation

$$G\phi = \lambda\phi \text{ on } (0, \infty) \text{ with } \phi(0) = 1, 0 \leq \phi \leq 1, \tag{66}$$

which can be written alternatively as

$$\frac{1}{2}\phi'' = \lambda m \cdot \phi \text{ on } (0, \infty) \text{ with } \phi(0) = 1, 0 \leq \phi \leq 1. \tag{67}$$

Another well known formula in this setting is

$$\Theta(\lambda)^{-1} = \int_0^\infty e^{-\lambda t} p(t, 0, 0) dt \tag{68}$$

where  $p(t, x, y)$  is a smooth transition density for  $X$  relative to  $m$ , that is  $P^x(X_t \in dy) = p(t, x, y)m(dy)$ . See our papers [22, 23, 21] regarding the relation between the above formulae, the Ray-Knight theorems for Brownian local times, and the distribution of quadratic functionals of Bessel processes, and see the work of Knight [14] and Kotani-Watanabe [16] regarding the relation of these formulae to Krein's spectral theory for vibrating strings [9, 2]. Since the speed measure  $m$  is an invariant measure for  $X$ , in this setting the global formula (52) gives

$$\Lambda(a, \infty) = P^m(T_0 \in da)/da \tag{69}$$

which when Laplace transformed amounts via (63) to

$$\Theta(\lambda) = \lambda P^m(e^{-\lambda T_0}) = \lambda \int_0^\infty \phi_\lambda(x)m(dx) \tag{70}$$

Note that this formula holds just as well in the general Markov setting discussed earlier. Comparison of (64) and (70) shows that the agreement of the local and global formulae for  $\Lambda$  amounts to the following about the unique solution  $\phi_\lambda$  of the Sturm-Liouville equation (67):

$$-\left. \frac{1}{2} \frac{d}{dx} \phi_\lambda(x) \right|_{x=0+} = \lambda \int_0^\infty \phi_\lambda(x)m(dx) \tag{71}$$

This is easily checked from (67), since from that equation the right side of (71) is

$$\frac{1}{2} \int_0^\infty dx \phi_\lambda''(x) = \frac{1}{2} (\phi_\lambda'(\infty) - \phi_\lambda'(0+)) \quad (72)$$

and since  $\phi_\lambda'$  is an increasing function of  $x$  the constraint that  $\phi_\lambda$  is bounded forces  $\phi_\lambda'(\infty) = 0$ . The formula (71) is a generalization of an identity of Truman-Williams [30, (77) and (92)].

**Example 8 Reflecting BM.** Let  $X$  be RBM on  $[0, \infty)$ . We take  $m(dx) = dx$ , local time at zero is occupation density at  $0+$  relative to  $dx$ . The Laplace exponent is  $\Theta(\lambda) = \sqrt{2\lambda}/2$ , and we find  $\phi_\lambda(x) = \sqrt{2\lambda}x$ ,  $\phi_\lambda'(0+) = \sqrt{2\lambda}$  and  $\lambda \int_0^\infty \phi_\lambda(x) dx = \lambda/\sqrt{2\lambda}$ .

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