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A COUNTER-EXAMPLE CONCERNING A CONDITION OF OGAWA INTEGRABILITY

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Abstract

A counterexample is exibited showing that the condition of Ogawa integrability introduced in **[3]** *is not satisfied by any complete orthonormal system.*

KEY WORDS: Ogawa integral, trace, orthonormal bases.

Introduction

In the past twenty years Itô's integral has been variously generalized by several authors. A very attractive notion has been suggested by Ogawa, who in **[5]** gave a definition of a stochastic integral linked to the results of Itô-Nisio concerning the uniform convergence to the Wiener process of a suitable random walk. To describe it more fully, let $W = (W_t)_{t\in[0\,,1]}$ be a Brownian motion on the probability space $(\Omega, \mathcal{A}, P)$, and let λ be the Lebesgue measure on [0, 1]. A process H belonging to $L^2(\lambda \otimes P)$ is said to be Ogawa integrable, with respect to a given orthonormal system (e_i) of the space $L^2(\lambda)$, if the series

$$\sum_{i=1}^{\infty} \int_0^1 e_i(s)dW_s \int_0^t e_i(s)H_s ds$$

converges in probability for any t in $[0\,,1]$.

Such an integral may depend on the particular orthonormal system chosen. In **[6]** Ogawa studies the integrability of the continuous quasi-martingales and shows their integrability relatively to the trigonometric system. Later on, in **[7]**, he proves that the integrability with respect to the trigonometric system implies the integrability with respect to the Haar system. Lastly, in **[8]**, the orthonormal systems which make integrable every continuous quasi-martingale are characterized as the ones satisfying the following condition

$$\sup_n \int_0^1 \left(\sum_{i=1}^n e_i(t) \int_0^t e_i(s)ds \right)^2 dt < \infty \ . \tag{1}$$

The trigonometric system as well as Haar system are easily seen to verify condition (1). However, the problem of deciding if (1) holds for any orthonormal system of $L^2(\lambda)$ is left open (see **[3]**).

More recently some authors investigated Ogawa integrability independently from the base by restricting the class of integrands (e.g., to those processes which are regular in the sense of Malliavin derivative, as in **[3]** **[4]**, or to multiple Itô-Wiener integrals, as in **[9]**). Nevertheless these classes do not contain all continuous quasi-martingales (see the counter-example in **[1]**), so that it is still interesting to decide whether or not condition (1) holds for any orthonormal system. In the present note we give a negative answer to this question by exibiting a counter-example.

In order to construct a complete orthonormal system verifying

$$\sup_n \int_0^1 \left(\sum_{i=1}^n e_i(t) \int_0^t e_i(s)ds \right)^2 dt = \infty \ , \tag{2}$$

we consider as a starting point the easier problem (Lemma 2) of finding, for a given real number M, a finite orthogonal family of simple functions $u_1, \ldots, u_n$ such that

$$\int_0^1 \left(\sum_{i=1}^n u_i(t) \int_0^t u_i(s)ds \right)^2 dt \geq M \ .$$

In Lemma 1 the latter problem, which has a finite-dimensional nature, turns into the spectral analysis of a suitable matrix. Iterating the above construction on each interval of some countable partition of $[0,1]$ gives rise to an orthonormal system (e_i) of $L^2(\lambda)$ satisfying (2). The system (e_i) can then be completed into an orthonormal base, and, if the latter is conveniently ordered, there results that property (2) still holds.

1. Preliminaries

In this section we shall briefly sketch the proof of Ogawa integrability for continuous quasi-martingales so as to emphasize the necessity of condition (1).

Let H be a continuous quasi-martingale of the form $A + K.W$, where A is an adapted process having bounded variation trajectories, K is a bounded predictable process, and $K.W$ denotes the Itô integral of the process K. Since bounded variation processes are Ogawa integrable with respect to any orthonormal system, as it is readily seen by means of a simple integration by parts, we can restrict ourselves to the processes of the form $H = K.W$.

Given an orthonormal system (e_i) of $L^2(\lambda)$, for any $i \geq 1$ and any t in $[0,1]$, we denote $E_i(t)$ the function $\int_0^t e_i(s)ds$. We have to prove that the series

$$\sum_{i=1}^{\infty} \int_0^1 e_i(s)dW_s \int_0^t H_s e_i(s)ds$$

converges in probability for any t in $[0,1]$.
Due to the integration by parts formula, there holds

$$\int_0^t H_s e_i(s)ds = E_i(t)H_t - \int_0^t E_i(s)H_s dW_s.$$

Applying the Itô formula gives

$$\sum_{i=1}^{n} \int_0^1 e_i(s)dW_s \int_0^t H_s e_i(s)ds = S_1(n) + S_2(n) + S_3(n)$$

where

$$S_1(n) = \sum_{i=1}^{n} \left\{ H_t E_i(t) \int_0^1 e_i(s)dW_s - \int_0^t E_i(s)K_s dW_s \int_0^s e_i(s')dW_{s'} \right\},$$

$$S_2(n) = -\sum_{i=1}^{n} \int_0^1 e_i(s)dW_s \int_0^t E_i(s)K_s dW_s,$$

$$S_3(n) = -\sum_{i=1}^{n} \int_0^t E_i(s)e_i(s)K_s ds.$$

Hence one easily verifies that $S_1(n)$ and $S_3(n)$ converge in probability. Moreover, if the system (e_i) verifies condition (1), then $S_2(n)$ converges in probability to $-\frac{1}{2}\int_0^t K_s ds$.

2. Construction of a counterexample

Theorem. *There exists a complete orthonormal system $\{e_i\}_{i\in\mathbf{N}}$ of $L^2(\lambda)$ such that*

$$\sup_{n\in\mathbf{N}} \int_0^1 \left(\sum_{i=1}^{n} e_i(t) \int_0^t e_i(s)ds \right)^2 dt = \infty \quad .$$

We need some lemmas.

Lemma 1. *Let $n \in \mathbf{N}$ and let V be the $n \times n$ matrix with coefficients*

$$V_{ij} =: \frac{1+(-1)^{n+\max(i,j)}}{2} \qquad 1 \le i,j \le n \quad . \tag{3}$$

Then, letting $\theta_k =: \frac{k\pi}{2n+1}$ for $k = 1,2,...n$, the eigenvalues λ_k of V and the associated eigenvectors $u^k = (u_1^k, \dots, u_n^k)$ are

$$\begin{aligned} \lambda_k &= \frac{(-1)^n}{2} \sec(2\theta_k) \\ u^k &= \rho_k \Big(-\sin\theta_k,\ -\sin(3\theta_k),\ \sin(5\theta_k), \dots, (-1)^{[\frac{n+1}{2}]} \sin((2n-1)\theta_k) \Big) \\ \rho_k &= \left(\frac{n}{2} + \frac{1}{4} \tan\theta_k \right)^{-\frac{1}{2}} \quad . \end{aligned} \tag{4}$$

Proof. First, let us observe that the inverse matrix of V writes

$$\begin{bmatrix} & \cdots & 1 & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & \cdots & 1 & 0 & 1 \\ 0 & \cdots & 0 & 0 & 1 \\ 1 & \cdots & 1 & 1 & 1 \end{bmatrix}^{-1} = (-1)^n \begin{bmatrix} -1 & 1 & & & \\ 1 & 0 & -1 & & \\ & -1 & 0 & & \\ & & & \ddots & \pm 1 \\ & & & \pm 1 & 0 \end{bmatrix}$$

or, in shorter notation, $V^{-1} = (-1)^n DJD$, where D and J are the $n \times n$ matrices with coefficients respectively $D_{ij} =: (-1)^{[\frac{i+1}{2}]}\delta_{ij}$ and $J_{ij} =: -\delta_{i1}\delta_{1j} + \delta_{|i-j|,1}$, that is

$$D = \begin{bmatrix} -1 & & & & \\ & -1 & & & \\ & & +1 & & \\ & & & +1 & \\ & & & & \ddots \end{bmatrix}, \qquad J = \begin{bmatrix} -1 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & & \\ & & & \ddots & 1 \\ & & & 1 & 0 \end{bmatrix} .$$

Therefore, if $u \in \mathbb{R}^n$ is an eigenvector of V corresponding to the eigenvalue λ, then $\mu =: (-1)^n \lambda^{-1}$ is the eigenvalue of J relative to eigenvector $v =: Du$. Now a base of eigenvectors of J, $(v^k)_{1 \le k \le n}$, together with eigenvectors $(\mu_k)_{1 \le k \le n}$, is the one defined by letting $\theta_k =: \frac{k\pi}{2n+1}$ and

$$\begin{aligned} \mu_k &= 2\cos(2\theta_k) \\ v^k &= (\sin\theta_k,\ \sin(3\theta_k),\ \sin(5\theta_k), \dots, \sin((2n-1)\theta_k)) \ , \end{aligned} \tag{5}$$

as can be shown by direct computation (or using the argument developed in the next Remark 1). Letting $\rho_k =: \|v_k\|$, one obtains

$$\rho_k^2 = \sum_{j=1}^{n} (\sin(2j-1)\theta_k)^2 = \frac{n}{2} + \frac{1}{4}\tan\theta_k \quad ,$$

whence equations (4) follow letting $\lambda_k = (-1)^n \mu_k^{-1}, \quad u^k = \rho_k D v^k$.

Remark 1. For a more euristic computation of the spectrum of J, let us observe that equation $Jv = 2\xi v$, with v in $\mathbb{R}^n$, ξ in $\mathbb{R}$, is equivalent to:

$$\begin{cases} v = \rho\big(W_0(\xi),\, W_1(\xi), \dots, W_{n-1}(\xi)\big), & \rho \in \mathbb{R} \\ W_n(\xi) = 0\ , \end{cases}$$

where the $W_k(x)$, for $k \in \mathbf{N} \cup \{0\}$, are polynomials verifying the linear recurrence formula

$$\begin{cases} W_0(x) = 1 \\ W_1(x) = 2x + 1 \\ W_{k+1}(x) + W_{k-1}(x) = 2xW_k(x) \quad . \end{cases} \tag{6}$$

Hence one recognizes the orthogonal Jacobi polynomials $J_k^{(\alpha,\beta)}(x)$ for $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$; these admit the representation in the following closed formula, which one obtains smoothly from equations (6) (see e.g. [**2**])

$$W_m(x) = \frac{\sin(m + \frac{1}{2})\theta}{\sin(\frac{\theta}{2})}, \qquad x = \cos\theta \tag{7}$$

and one finds again equations (5), taking account that the roots of W_n are, thanks to (7), $x_k = \cos(2\theta_k),\ 1 \le k \le n$.

The theorem follows smoothly from a slightly weaker preliminar result, which we now state with the same notations as Lemma 1.

Lemma 2. *Let n be even and, for $1 \le k \le n$, let f_k be the simple functions defined by*

$$f_k(t) = \sqrt{n} \sum_{j=1}^{n} u_j^k\, \chi_{[\frac{j-1}{n}, \frac{j}{n}]}(t)\ .$$

Then f_k are orthonormal in $L^2(\lambda)$ and one has

$$\int_0^1 \left(\sum_{1 \le k \le \frac{n}{2}} f_k(t) \int_0^t f_k(s)ds \right)^2 dt \ge C_n =: \frac{1}{2} \left(\frac{\log n}{2\pi} \right)^2 \ . \tag{8}$$

Proof. Since the vectors u^k defined by equations (4) are orthonormal in $\mathbb{R}^n$, the functions f_k are immediately seen to be orthonormal in $L^2(\lambda)$. So we are left with the L^2 estimate of the function

$$S(t) = \sum_{1 \le k \le \frac{n}{2}} f_k(t) \int_0^t f_k(s)ds \quad .$$

To this end let us consider the test function

$$\phi(t) = \sqrt{2}\sum_{j=1}^{n} \frac{1+(-1)^{n+j}}{2}\,\chi_{[\frac{j-1}{n},\frac{j}{n}]}(t)\ . \tag{9}$$

There results $\|\phi\|_2 = 1$, whence

$$\begin{aligned}
\|S\|_2 \geq \langle S,\phi\rangle &= \int_0^1 \left(\sum_{1\leq k\leq \frac{n}{2}} f_k(t)\int_0^t f_k(s)ds\right)\phi(t)dt \\
&= \sum_{1\leq k\leq \frac{n}{2}} \int_0^1\int_0^t f_k(t)f_k(s)\phi(t)dsdt \\
&= \frac{1}{2}\sum_{1\leq k\leq \frac{n}{2}} \int_0^1\int_0^1 f_k(t)f_k(s)\phi(\max(t,s))dtds\ ,
\end{aligned} \tag{10}$$

where the latter inequality follows from the symmetry of the integrands $f_k(t)f_k(s)\phi(\max(t,s))$ with respect to the pair (t,s).
Let us notice that the function $\phi(\max(t,s))$, using equations (3) and (9), can also be written as

$$\phi(\max(t,s)) = \sqrt{2}\sum_{i,j} V_{ij}\chi_{[\frac{i-1}{n},\frac{i}{n}]}(t)\chi_{[\frac{j-1}{n},\frac{j}{n}]}(s)$$

whence

$$\int_0^1\int_0^1 f_k(t)f_k(s)\phi(\max(t,s))dtds = \frac{\sqrt{2}}{n}\langle Vu^k,u^k\rangle = \frac{\sqrt{2}}{n}\lambda_k\ .$$

Thus equalities (10) yield

$$\frac{1}{\sqrt{2}n}\sum_{1\leq k\leq \frac{n}{2}}\lambda_k = \frac{1}{\sqrt{8}n}\sum_{1\leq k\leq \frac{n}{2}}\frac{1}{\cos\left(\frac{2k\pi}{2n+1}\right)}$$

$$= \frac{2n+1}{2\sqrt{8}n\pi}\sum_{1\leq k\leq \frac{n}{2}}\frac{1}{\cos\left(\frac{2k\pi}{2n+1}\right)}\,\frac{2\pi}{2n+1} \geq \frac{2n+1}{2\sqrt{8}n\pi}\int_0^{\frac{n\pi}{2n+1}}\frac{1}{\cos t}dt \tag{11}$$

Indeed, the sum in the left side of equation (11) is an upper Riemann estimate for the integral in the right side .
On the other hand

$$\begin{aligned}
\frac{2n+1}{2\sqrt{8}n\pi}\int_0^{\frac{n\pi}{2n+1}}\frac{1}{\cos t}dt &\geq \frac{1}{\sqrt{8}\pi}\int_0^{\frac{\pi}{2}-\frac{\pi}{4n}}\frac{1}{\cos t}dt \\
&= \frac{1}{\sqrt{8}\pi}\log\cot\left(\frac{\pi}{8n}\right) \geq \frac{1}{\sqrt{8}\pi}\log n\ .
\end{aligned}$$

Therefore one concludes

$$\|S\|_2^2 \geq \frac{1}{2}\left(\frac{\log n}{2\pi}\right)^2 .$$

Remark 2. In the summation shown in (8) we took only the first $\frac{n}{2}$ functions f_k, that is, the ones corresponding to the positive eigenvalues λ_k of the matrix V. This is due to the fact that, as it is shown in equations (11), the arithmetic mean of the positive eigenvalues of V, $\frac{2}{n}\sum_{1\leq k\leq \frac{n}{2}} \lambda_k$, is of the order (at least) of $\log n$, whereas the mean value extended to the whole spectrum is $\frac{1}{n}\sum_{1\leq k\leq n}\lambda_k = \frac{1}{n}\mathrm{Tr}(V) = \frac{1}{2}$.

Remark 3. For any given subinterval $[a,b]$ of $[0,1]$, one can also choose the functions of Lemma 2 with supports in $[a,b]$. Actually, it is sufficient to consider $\tilde{f}_k(t) =: (b-a)^{-\frac{1}{2}} f_k\left(\frac{t-a}{b-a}\right)$ if $t\in[a,b]$, and $\tilde{f}_k(t) = 0$ if $t\notin[a,b]$. Then inequality (7) holds for $\tilde{f}_k$ relatively to the constant $\widetilde{C}_n =: (b-a)C_n$.

Lemma 3. *Let $g_1,\ldots,g_p$ be orthonormal functions in $L^2(\lambda)$. There holds*

$$\int_0^1 \left(\sum_{i=1}^p g_i(t)\int_0^t g_i(s)ds\right)^2 dt \leq p\,.$$

Proof. The Fourier coefficients of $\chi_{[0,t]}$ with respect to g_i is $\int_0^t g_i(s)ds$. Then it follows, using Schwarz inequality and Bessel inequality

$$\begin{aligned}
\int_0^1 \left(\sum_{i=1}^p g_i(t)\int_0^t g_i(s)ds\right)^2 dt &= \int_0^1 \left(\sum_{i=1}^p g_i(t)\langle g_i,\,\chi_{[0,t]}\rangle\right)^2 dt \\
&\leq \int_0^1 \left(\sum_{i=1}^p g_i(t)^2\right)\left(\sum_{i=1}^p \langle g_i,\,\chi_{[0,t]}\rangle^2\right) dt \\
&\leq \int_0^1 \left(\sum_{i=1}^p g_i(t)^2\right)\|\chi_{[0,t]}\|_2^2 dt \\
&\leq \int_0^1 \sum_{i=1}^p g_i(t)^2 dt = p\,.
\end{aligned}$$

Proof of the Theorem. Let $\{I_i\}$ be a countable family of disjoint subintervals of $[0,1]$. For any index $i \in \mathbf{N}$, applying Lemma 2 and Remark 3, we can find n_i simple functions $\{f_{ij}\}_{1\leq j\leq n_i}$ such that

$$\langle f_{ij}, f_{ik}\rangle = \delta_{jk}, \quad 1 \leq j,k \leq n_i \tag{12}$$

$$\operatorname{supp}(f_{ij}) \subset I_i \tag{13}$$

$$\int_0^1 \left(\sum_{j=1}^{n_i} f_{ij}(t)\int_0^t f_{ij}(s)ds\right)^2 dt \geq 4 \tag{14}$$

The family $\{f_{ij},\, i \in \mathbf{N},\, 1 \leq j \leq n_i\}$ is clearly orthonormal, since f_{ij} and f_{lk} verify (12) whenever $i = l$, while have disjoint supports if $i \neq l$, thanks to (13).

Next let us consider an orthonormal base $\{f_{i0}\}_{i\in\mathbf{N}}$ which completes system $\{f_{ij},\, i \in \mathbf{N},\, 1 \leq j \leq n_i\}$ to an orthonormal base $\{f_{ij},\, i \in \mathbf{N},\, 0 \leq j \leq n_i\}$; then re-indicize the latter by means of the position

$$e_k = f_{ij} \quad \text{if and only if} \quad k = i + j + \sum_{0\leq l<i} n_l \quad \text{and} \quad 0 \leq j < n_i\,.$$

(This amounts to give the set of pairs (i,j) its lexicographic order.) For any $p \in \mathbf{N}$, let $m =: p + \sum_{0\leq l<p} n_l$. Thus one gets

$$\begin{aligned}\sum_{i=1}^{m} e_i(t)\int_0^t e_i(s)ds &= \sum_{i=1}^{p}\sum_{j=0}^{n_i} f_{ij}(t)\int_0^t f_{ij}(s)ds \\ &= \sum_{i=1}^{p} f_{i0}(t)\int_0^t f_{i0}(s)ds + \sum_{i=1}^{p}\sum_{j=1}^{n_i} f_{ij}(t)\int_0^t f_{ij}(s)ds\,.\end{aligned}$$

There follows, using the elementary inequality $(a+b)^2 \geq \frac{a^2}{2} - b^2$,

$$\begin{aligned}&\int_0^1 \left(\sum_{i=1}^{m} e_i(t)\int_0^t e_i(s)ds\right)^2 dt \qquad (15)\\ &\geq \frac{1}{2}\int_0^1 \left(\sum_{i=1}^{p}\sum_{j=1}^{n_i} f_{ij}(t)\int_0^t f_{ij}(s)ds\right)^2 dt - \int_0^1 \left(\sum_{i=1}^{p} f_{i0}(t)\int_0^t f_{i0}(s)ds\right)^2 dt \\ &= \frac{1}{2}\sum_{i=1}^{p}\int_0^1 \left(\sum_{j=1}^{n_i} f_{ij}(t)\int_0^t f_{ij}(s)ds\right)^2 dt - \int_0^1 \left(\sum_{i=1}^{p} f_{i0}(t)\int_0^t f_{i0}(s)ds\right)^2 dt\,,\end{aligned}$$

where the latter equality is a consequence of the fact that the functions f_{ij} with different indices i have disjoint support, as we remarked above. Due to inequality (14) and to Lemma 3 one has respectively

$$\frac{1}{2}\sum_{i=1}^{p}\int_0^1\left(\sum_{j=1}^{n_i} f_{ij}(t)\int_0^t f_{ij}(s)ds\right)^2 dt \geq 2p$$

$$\int_0^1\left(\sum_{i=1}^{p} f_{i0}(t)\int_0^t f_{i0}(s)ds\right)^2 dt \leq p\,,$$

and we conclude from (15)

$$\int_0^1\left(\sum_{i=1}^{m} e_i(t)\int_0^t e_i(s)ds\right)^2 dt \geq p.$$

The claim follows for p being arbitrary.

References

[1] M.E. MANCINO : *Su alcuni integrali stocastici anticipativi.* PhD Thesis, Univ. di Trento (1995)

[2] I.P. NATANSON : *Constructive Function Theory.* Vol II - Frederick Ungar Publishing Co. NY (1965)

[3] D. NUALART : *Noncausal stochastic integrals and calculus.* L.N.M. 1316

[4] D. NUALART, M. ZAKAI : *Generalized stochastic integrals and the Malliavin calculus.* Prob.Th.Rel.Fields 73 (1986)

[5] S. OGAWA : *Sur le produit direct du bruit blanc par lui-même.* C. R. Acad. Sc. Paris, t.288, Série A-359 (Février 1979)

[6] S. OGAWA : *Quelques propriétés de l'intégrale stochastique du type noncasual.* Japan J.Appl. Math. vol.1 (1984)

[7] S. OGAWA : *Une remarque sur l'approximation de l'intégrale stochastique du type noncasual par une suite d' intégrales de Stieltjes.* Tôhuku Math.J. vol.36 (1984)

[8] S. OGAWA : *The stochastic integral of noncasual type as an extension of the symmetric integrals.* Japan J.Appl. Math. vol.2 (1985)

[9] J. ROSINSKI : *On stochastic integration by series of Wiener integrals.* Appl. Math. Optim. vol.19 (1989)