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Jia-An Yan

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An Asymptotic Evaluation of Heat Kernel for Short Time[1]

In Honor of P.A. Meyer and J. Neveu

J.A. Yan

Consider the following heat equation

$$\frac{\partial u}{\partial t} = (\frac{\Delta}{2} + V)u, \tag{1}$$

where Δ is the Laplacian operator on $\mathbb{R}^d$ and V is a continuous function on $\mathbb{R}^d$. Under mild assumptions on V the fundamental solution of equation (1) exists and can be expressed by the Feynman-Kac formula (cf.[2]). This fundamental solution is called the heat kernel.

The purpose of this paper is to prove the following theorem, which gives an asymptotic evaluation of the heat kernel for short time.

Theorem. Let V be a continuous function on $\mathbb{R}^d$. Assume there exist positive constants C, C_1 and C_2 such that

$$V(x)^+ \leq C(1 + |x|^2), \tag{2}$$

$$V(x)^- \leq C_1 e^{C_2|x|^2}. \tag{3}$$

Let $q(t, x, y)$ be the fundamental solution of the heat equation (1). Then we have

$$\lim_{t \downarrow 0} \frac{1}{t} \log \frac{q(t, x, y)}{p(t, x, y)} = \int_0^1 V((1-s)x + sy)\, ds, \tag{4}$$

where $p(t, x, y)$ is the transition density of a standard Brownian motion.

The main tool for proving this theorem is the Feynman-Kac formula. We recall it for the reader's convenience.

Let $\Omega = C\left([0, \infty), \mathbb{R}^d\right)$ be the collection of all continuous functions from $[0, \infty)$ to $\mathbb{R}^d$. For $\omega \in \Omega$, let $X_t(\omega) = \omega(t)$. Let $\mathcal{F}_t = \sigma\{X_s, s \leq t\}, \mathcal{F} = \sigma\{X_s, s < \infty\}$. We denote by $(\mathbb{P}_x, x \in \mathbb{R}^d)$ the unique family of probability measures on $(\Omega, \mathcal{F})$ such that $(\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \mathbb{P}_x)$ is a standard Brownian motion. Let $y \in \mathbb{R}^d$ and $t > 0$. Put

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$$Y_s(\omega) = X_s(\omega) - \frac{s \wedge t}{t}(X_t(\omega) - y) \ , \ s \geq 0.$$

Then under $\mathbb{P}_x$ the process $(Y_s, 0 \leq s \leq t)$ is a *Brownian bridge from x to y on* $[0,t]$ and $(Y_s, t \leq s < \infty)$ is a Brownian motion with $Y_t = y$. Moreover, under $\mathbb{P}_x$ these two processes are independent. We denote by $\mathbb{P}_{x,y,t}$ the distribution of the process $(Y_s, s \geq 0)$ on $(\Omega, \mathcal{F})$ under $\mathbb{P}_x$. We call $\mathbb{P}_{x,y,t}$ the $(0,x;t,y)-$*Brownian bridge measure.* Under mild asumptions on V it was shown that the heat kernel for (1) can be expressed by the following Feynman-Kac formula(cf.[2,Theorem 3.2]) :

$$\begin{aligned} q(t,x,y) &= p(t,x,y)\mathbb{E}_{x,y,t}[e^{\int_0^t V(X_s)\,ds}] \\ &= p(t,x,y)\mathbb{E}_0[e^{\int_0^t V(x+\frac{s}{t}(y-x)+X_s-\frac{s}{t}X_t)\,ds}] \\ &= p(t,x,y)\mathbb{E}_0[e^{t\int_0^1 V(x+s(y-x)+\sqrt{t}(X_s-sX_1))\,ds}], \end{aligned} \tag{5}$$

where

$$p(t,x,y) = (2\pi t)^{-\frac{d}{2}} exp\{-\frac{|x-y|^2}{2t}\}.$$

We are going to prove the theorem. To begin with we prepare a lemma.

Lemma. Let $\{\xi(\varepsilon), \varepsilon > 0\}$ be a family of integrable random variables such that $\lim_{\varepsilon \downarrow 0} \mathbb{E}[\xi(\varepsilon)]$ exists and is finite. If

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}[\xi(\varepsilon)(e^{\varepsilon\xi(\varepsilon)} - 1)] = 0, \tag{6}$$

then we have

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \log \mathbb{E}[e^{\varepsilon\xi(\varepsilon)}] = \lim_{\varepsilon \downarrow 0} \mathbb{E}[\xi(\varepsilon)]. \tag{7}$$

Proof. Since $1 + x \leq e^x \leq 1 + x + x(e^x - 1)$, we have

$$\xi(\varepsilon) \leq \frac{1}{\varepsilon}[e^{\varepsilon\xi(\varepsilon)} - 1] \leq \xi(\varepsilon) + \xi(\varepsilon)[e^{\varepsilon\xi(\varepsilon)} - 1].$$

This together with (6) imply

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}(\mathbb{E}[e^{\varepsilon\xi(\varepsilon)}] - 1) = \lim_{\varepsilon \downarrow 0} \mathbb{E}[\xi(\varepsilon)], \tag{8}$$

which is equivalent to (7). ∎

Corollary 1. If instead of (6) we assume

$$\lim_{\varepsilon \downarrow 0} \varepsilon \mathbb{E}[\xi(\varepsilon)^2(e^{\varepsilon\xi(\varepsilon)} + 1)] = 0, \tag{9}$$

then (7) holds.

Proof. Immediate from the fact that $x(e^x - 1) \le x^2(e^x + 1)$. We leave the proof of this fact to the reader. ∎

Corollary 2. Let $\{\xi(\varepsilon), \varepsilon > 0\}$ be a family of integrable random variables such that $\lim_{\varepsilon \downarrow 0} \mathbb{E}[\xi(\varepsilon)]$ exists and is finite. If there exists an $\varepsilon_0 > 0$ such that $\{\xi(\varepsilon), 0 < \varepsilon \le \varepsilon_0\}$ is uniformly integrable (u.i. for short) and $\mathbb{E}[e^{\delta \sup_{0<\varepsilon\le\varepsilon_0} \xi(\varepsilon)^+}] < \infty$ for some $\delta > 0$, then (6) is satisfied. In particular, we have (7).

Proof. For any $c > 0$, we have

$$\xi(\varepsilon)^+ e^{\varepsilon\xi(\varepsilon)^+} \le \frac{1}{c} e^{(\varepsilon+c)\xi(\varepsilon)^+}.$$

Thus, by assumption we can find an $\varepsilon_1 > 0$ with $\varepsilon_1 \le \varepsilon_0$ such that $\{\xi(\varepsilon)^+ e^{\varepsilon\xi(\varepsilon)^+}, 0 < \varepsilon \le \varepsilon_1\}$ is u.i.. On the other hand, we have

$$|\xi(\varepsilon)| e^{\varepsilon\xi(\varepsilon)} \le \xi(\varepsilon)^+ e^{\varepsilon\xi(\varepsilon)^+} + |\xi(\varepsilon)|.$$

Consequently, $\{\xi(\varepsilon)[e^{\varepsilon\xi(\varepsilon)} - 1], \varepsilon_1 \ge \varepsilon > 0\}$ is u.i.. Therefore, (6) holds, because $\varepsilon\xi(\varepsilon)$ tends to 0 in probability as ε tends to 0. ∎

Now we are in a position to prove our theorem. Put

$$\xi(\varepsilon) = \int_0^1 V(x + s(y-x) + \sqrt{\varepsilon}(X_s - sX_1))\, ds. \tag{10}$$

We may assume $C_1 \ge C, C_2 \ge 1$. Then by (2) and (3) we get

$$\begin{aligned} \sup_{0\le\varepsilon\le\varepsilon_0} |\xi(\varepsilon)| &\le C_1 \int_0^1 e^{C_2 \sup_{0\le\varepsilon\le\varepsilon_0} |x+s(y-x)+\sqrt{\varepsilon}(X_s - sX_1)|^2}\, ds \\ &\le C_1 \int_0^1 e^{2C_2|x+s(y-x)|^2}\, ds e^{4C_2\varepsilon_0 \sup_{0\le s\le 1}|X_s|^2}. \end{aligned}$$

Thus by Fernique's theorem ([1]) for sufficiently small $\varepsilon_0 > 0, \{\xi(\varepsilon), 0 < \varepsilon \le \varepsilon_0\}$ is u.i. and we have

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}[\xi(\varepsilon)] = \int_0^1 V(x + s(y-x))\, ds.$$

On the other hand, by (10) and (2) we have

$$\begin{aligned} & e^{\sup_{0\le\varepsilon\le\varepsilon_0} \xi(\varepsilon)^+} \\ \le\ & e^{\int_0^1 \sup_{0\le\varepsilon\le\varepsilon_0} C(1+|x+s(y-x)+\sqrt{\varepsilon}(X_s - sX_1)|^2)\, ds} \\ \le\ & e^{\int_0^1 C(1+2|x+s(y-x)|^2)\, ds} e^{4C\varepsilon_0 \sup_{0\le s\le 1}|X_s|^2}. \end{aligned}$$

Thus, once again by Fernique's theorem, for sufficiently small ε_0, $\mathbb{E}[e^{\sup_{0\le\varepsilon\le\varepsilon_0} \xi(\varepsilon)^+}] < \infty$. Consequently, in view of (5) and (10) we can apply Corollary 2 to conclude the theorem.

Refrences

[1] X. Fernique, Intégrabilité des vecteurs Gaussiens, C.R.A.S. Paris 270, Series A(1970), 1698-1699.

[2] J.A. Yan, From Feynman-Kac formula to Feynman integrals via analytic continuation, *Stoch. Processes and their Applications*, 54 (1994), 215 - 232.

J.A. Yan
Institute of Applied Mathematics
Academia Sinica
P.O. Box 2734
Beijing 100080, P.C. China
Email: jayan@bepc2.ihep.ac.cn