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ON THE LÉVY TRANSFORMATION

OF BROWNIAN MOTIONS AND CONTINUOUS MARTINGALES

L. E. Dubins[1], M. Émery[2], M. Yor[3]

[...] je vous confie aujourd'hui mes espérances, qui ne reposent encore que sur des calculs de probabilité.
É. Zola

Introduction

If $(B_t)_{t\geqslant 0}$ is a Brownian motion started at 0 and $(L_t)_{t\geqslant 0}$ its local time at 0, Lévy's characterization (see for instance [6] p. 141) implies that

$$\widehat{B} = |B| - L = \int \operatorname{sgn}(B)\, dB$$

is also a Brownian motion. In other words, the *Lévy transformation* $T : B \longrightarrow \widehat{B}$, defined almost everywhere on the Wiener space $W = \mathcal{C}(\mathbb{R}_+, \mathbb{R})$, preserves the Wiener measure μ.

Dubins & Smorodinsky [3] have established that an analogue of T for coin-tossing is ergodic. This increases the plausibility of the following conjecture:

$(\mathcal{L})$ *The Lévy transformation is ergodic,*

that is, the σ-field $\mathcal{J}$ on W of all events a.s. invariant by T is trivial.

Known since the late 70's, the problem of the ergodicity of T is mentioned as an open question in Revuz & Yor [6], page 257.

We shall see that $(\mathcal{L})$ is closely related to the question of knowing which continuous martingales $M = (M_t)_{t\geqslant 0}$ with $M_0 = 0$ have the same law as their Lévy transform $\widehat{M} = \int \operatorname{sgn}(M)\, dM$. (A discussion of this subject is begun in Exercise (2.32) page 231 of Revuz & Yor [6].) Recall that to each continuous martingale M is associated its quadratic variation $\langle M\rangle$; $\langle M\rangle$ is the continuous, non-decreasing, adapted process such that $\langle M\rangle_t = \lim_{n\to\infty} \sum_{k=1}^{2^n} \left(M_{k2^{-n}t} - M_{(k-1)2^{-n}t}\right)^2$. For simplicity, we shall deal only with continuous martingales verifying $M_0 = 0$ and $\langle M\rangle_\infty = \infty$; such processes will be called *divergent martingales*. As is well known

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(see [1] and [2]), each divergent martingale M is obtained by time-changing a Brownian motion, the time-change being given by $\langle M\rangle$. More precisely, to M is associated a (unique) Brownian motion β^M such that $M_t = (\beta^M)_{\langle M\rangle_t}$ for all t; this defines a map $\beta_M : \Omega \longrightarrow W$ transforming $\mathbb{P}$ into the Wiener measure μ. The law of M is characterized by the joint law of β^M and $\langle M\rangle$.

Changing time in the integral $\int \operatorname{sgn}(M)\,dM$ gives $\beta^{\widehat{M}} = \widehat{\beta^M}$; as $\langle\widehat{M}\rangle = \langle M\rangle$, $\widehat{M}$ has the same law as M if and only if

$$(\beta^M)_{\langle M\rangle} \overset{\text{(law)}}{=} T(\beta^M)_{\langle M\rangle}\,.$$

A sufficient condition is the independence of β^M and $\langle M\rangle$. We conjecture that this sufficient condition is also necessary.

Let us give a name to this conjecture:

($\mathcal{M}$) *A divergent martingale M has the same law as its Lévy transform $\widehat{M}$ (if and) only if the processes β^M and $\langle M\rangle$ are independent.*

A reason to believe in this conjecture is Ocone's Theorem A of [5]: The independence of β^M and $\langle M\rangle$ is a necessary and sufficient condition for M to have the same law as all integrals $\int H\,dM$, where H ranges either over all deterministic processes of the form $H = \mathbb{1}_{[0,a]} - \mathbb{1}_{(a,\infty)}$, or over all $\{-1,1\}$-valued processes that are predictable for the natural filtration of M.

The next section gives some preliminary observations about ($\mathcal{L}$). Then comes our main result, the equivalence of ($\mathcal{L}$) and ($\mathcal{M}$), established, with some further precisions, in the third section. In the last section, we try to understand ($\mathcal{L}$) better, in particular by constructing examples of martingales which are not identical in law with their Lévy transform. The appendix borrows from [5] Ocone's theorem and its proof, with a few remarks.

Preliminary remarks

The following lemma from ergodic theory is well known.

LEMMA 1. — *Let $(W,\mathcal{G},\mu)$ be a probability space and T a measurable transformation of W which preserves μ. A random variable $Z \in \mathrm{L}^2(W,\mu)$ is a. s. invariant by T if and only if*

$$\langle Y\circ T, Z\rangle_{\mathrm{L}^2} = \langle Y, Z\rangle_{\mathrm{L}^2}$$

for all $Y \in \mathrm{L}^2(W,\mu)$.

PROOF. — If Z is invariant, $Z = Z\circ T$ a. s. and $\langle Y\circ T, Z\rangle = \langle Y\circ T, Z\circ T\rangle = \langle Y, Z\rangle$ by the invariance of μ.

Conversely, if $\langle Y\circ T, Z\rangle = \langle Y, Z\rangle$ for every Y, $\langle Y\circ T, Z\rangle = \langle Y\circ T, Z\circ T\rangle$ and $Z\circ T$ is the conditional expectation of Z given $T^{-1}\mathcal{G}$; as Z and its projection $Z\circ T$ on $\mathrm{L}^2(T^{-1}\mathcal{G})$ have the same L^2-norm (invariance of μ), they must be equal. ∎

LEMMA 2. — *With the notations of the introduction, let $S : W \longrightarrow [0,\infty]$ be a stopping time for B. The stopped processes B^S and $\widehat{B}^S$ have the same law if and only if S is a. s. invariant.*

PROOF. — If S is invariant, $\widehat{B}^S = (B \circ T)^{S \circ T} = (B^S) \circ T$ has the same law as B^S.

Conversely, if B^S and $\widehat{B}^S$ have the same law, the pairs (B^S, S) and $(\widehat{B}^S, S)$ also have the same law, because S is a function of the path of B^S (for instance $S = \sup\{t \in \mathbb{Q} : B^S_t \neq B^S_\infty\}$). Now the Markov property at time S makes it possible to deduce the law of (B, S) from that of (B^S, S) and similarly for $\widehat{B}$; hence (B, S) and $(\widehat{B}, S)$ have the same law. This gives $\langle Y, e^{-S}\rangle = \langle Y \circ T, e^{-S}\rangle$ for every $Y \in L^2(W)$ and S is invariant by Lemma 1. ∎

REMARK. — The Markov property in the above proof cannot be dispensed of: if the random variable S is not a stopping time, it may happen that B^S and $\widehat{B}^S$ have the same law but S is not invariant. Take for instance any $[0,1]$-valued random variable S independent of $\mathcal{F}_1$; B^S and $\widehat{B}^S$ have the same law, namely that of a Brownian motion stopped at some independent time distributed as S.

Equivalence of $(\mathcal{L})$ and $(\mathcal{M})$

THEOREM 1. — *Let M be a divergent martingale. The following three properties are equivalent:*

(i) *M and $\widehat{M}$ have the same law;*

(ii) *$(\beta^M, \langle M\rangle)$ and $(T(\beta^M), \langle M\rangle)$ have the same law;*

(iii) *β^M and $\langle M\rangle$ are conditionally independent given the σ-field $\beta_M^{-1}(\mathcal{J})$.*

Examples of this situation are obtained by taking $\langle M\rangle$ independent of β^M; if $(\mathcal{L})$ is true, $\beta_M^{-1}(\mathcal{J})$ is trivial and there are no other examples.

PROOF. — Since $\langle M\rangle = \langle \widehat{M}\rangle$ and the law of $(\beta^M, \langle M\rangle)$ depends only on that of M, (i) implies (ii). Conversely, (ii) $\Rightarrow$ (i) follows from $M = (\beta^M)_{\langle M\rangle}$ and $\widehat{M} = T(\beta^M)_{\langle M\rangle}$.

(ii) $\Rightarrow$ (iii). Let F be a bounded random variable measurable with respect to $\sigma\{\langle M\rangle_t, t \geqslant 0\}$; there exists a bounded measurable function f on W such that $\mathbb{E}[F|\beta^M] = f(\beta^M)$. For every $g \in L^2(W)$, using the definition of f, hypothesis (ii) and again the definition of f, one can write

$$\mathbb{E}[f(\beta^M)\, g(\beta^M)] = \mathbb{E}[F\, g(\beta^M)] = \mathbb{E}[F\, g \circ T(\beta^M)] = \mathbb{E}[f(\beta^M)\, g \circ T(\beta^M)]$$

and Lemma 1 gives $f(\beta^M) = f \circ T(\beta^M)$ a.s. So f is $\mathcal{J}$-measurable and coming back to the definition of f one gets $\mathbb{E}[F|\beta^M] = \mathbb{E}[F|\beta_M^{-1}(\mathcal{J})]$, the desired result.

(iii) $\Rightarrow$ (ii). Keep the same notations and call $\mathcal{J}'$ the σ-field $\beta_M^{-1}(\mathcal{J})$. Hypothesis (iii) gives on the one hand

$$\mathbb{E}[F\, g(\beta^M)] = \mathbb{E}\big[\mathbb{E}[F|\mathcal{J}']\ \mathbb{E}[g(\beta^M)|\mathcal{J}']\big]$$

and on the other hand

$$\mathbb{E}[F\, g\circ T(\beta^M)] = \mathbb{E}\big[\mathbb{E}[F|\mathcal{J}']\ \mathbb{E}[g\circ T(\beta^M)|\mathcal{J}']\big]\ .$$

Since every $\mathcal{J}'$-measurable random variable has the form $h(\beta^M)$ where h is $\mathcal{J}$-measurable, $\mathbb{E}[g(\beta^M)|\mathcal{J}'] = \mathbb{E}[g\circ T(\beta^M)|\mathcal{J}']$ and we obtain

$$\mathbb{E}[F\, g(\beta^M)] = \mathbb{E}[F\, g\circ T(\beta^M)]\ ,$$

which means precisely that (ii) holds. ∎

Recall that a martingale M is called ***pure*** if it is divergent and if for each $t \geqslant 0$ its quadratic variation $\langle M\rangle_t$ is measurable for the σ-field $\sigma\{\beta_s^M\ ,\ s \geqslant 0\}$.

A weaker form of Conjecture $(\mathcal{M})$ is obtained by restricting to pure martingales the demand that $M \overset{\text{(law)}}{=} \widehat{M}$ if and only if β^M and $\langle M\rangle$ are independent; since $\langle M\rangle$ is measurable with respect to β^M, we get the statement:

$(\mathcal{M}')$ *A pure martingale M has the same law as its Lévy transform $\widehat{M}$ (if and) only if $\langle M\rangle$ is deterministic.*

As will be shown below, $(\mathcal{M}')$ is in fact not weaker than but equivalent to $(\mathcal{M})$.

Similarly, a weaker form of $(\mathcal{L})$ is obtained by restricting to stopping times the statement that all random variables on W invariant by T are constant:

$(\mathcal{L}')$ *On the canonical space W, every stopping time invariant by T is constant.*

THEOREM 2. — *The four conjectures $(\mathcal{L})$, $(\mathcal{L}')$, $(\mathcal{M})$ and $(\mathcal{M}')$ are equivalent.*

PROOF. — If $(\mathcal{L})$ is true, $\mathcal{J}$ is trivial and (i) $\Rightarrow$ (iii) in Theorem 1 gives $(\mathcal{M})$. In turn, $(\mathcal{M})$ trivially implies $(\mathcal{M}')$. The theorem will be proved by showing that

$$(\mathcal{L}) \text{ is false} \Longrightarrow (\mathcal{L}') \text{ is false} \Longrightarrow (\mathcal{M}') \text{ is false.}$$

Assume $(\mathcal{L})$ is false. On W endowed with Wiener measure, there exists a non-trivial invariant bounded random variable F. Call B the canonical Brownian motion on W, $\widehat{B}$ its Lévy transform, $(\mathcal{F}_t)_{t\geqslant 0}$ and $(\widehat{\mathcal{F}}_t)_{t\geqslant 0}$ their natural filtrations. Since F is a functional of $\widehat{B}$, the $(\widehat{\mathcal{F}}_t)$-martingale $M_t = \mathbb{E}[F|\widehat{\mathcal{F}}_t]$ has the form $\mathbb{E}[F] + \int_0^t \widehat{H}_s\, d\widehat{B}_s$ with $\widehat{H}$ predictable in the filtration $(\widehat{\mathcal{F}}_t)_{t\geqslant 0}$; so M is also a martingale for $(\mathcal{F}_t)_{t\geqslant 0}$ and

$$\mathbb{E}[F|\mathcal{F}_t] = \mathbb{E}[F|\widehat{\mathcal{F}}_t] = \mathbb{E}[F\circ T|\widehat{\mathcal{F}}_t] = \mathbb{E}[F|\mathcal{F}_t]\circ T\ :$$

the process M is invariant by T.

Using M, it is easy to construct a finite, non-constant stopping time S invariant by T, for instance $S = t + \mathbb{1}_\Gamma(M_t)$ where t is large enough for M_t to be non-constant and Γ is a suitable Borel set. So $(\mathcal{L}')$ is false too.

If $(\mathcal{L}')$ is false, let S be a finite, non-constant, invariant stopping time on Wiener space. For $\alpha > 0$, the increasing process

$$A_t = \int_0^t \left[\mathbb{1}_{[\![0,S]\!]}(s) + \alpha \mathbb{1}_{]\!]S,\infty[\![}(s)\right] ds$$

is not deterministic if $\alpha \neq 1$; it is also invariant, and $(B, A) \overset{\text{(law)}}{=} (\widehat{B}, A)$; the inverse of A, obtained by replacing α with α^{-1}, is adapted and each A_t is a stopping time. Consequently, $M_t = B_{A_t}$ is a martingale (for the filtration $\mathcal{G}_t = \mathcal{F}_{A_t}$), satisfying condition (ii) of theorem 1, hence $M \overset{\text{(law)}}{=} \widehat{M}$. As $\langle M \rangle = A$ is measurable with respect to $\beta^M = B$ but not deterministic, M is a counterexample to $(\mathcal{M}')$ and $(\mathcal{M}')$ does not hold. ∎

Some Remarks

a) It can be observed that the stopping time S constructed in the first part of the proof takes only two values. This leads to another variant of $(\mathcal{L})$, namely, there are no invariant stopping times taking exactly two values. Of course, this just means that each invariant event belonging to some $\mathcal{F}_t$ is trivial.

b) $(\mathcal{L}')$ amounts to stating that every non constant Brownian stopping time S is not invariant. According to Lemma 2, this means that the stopped processes B^S and $\widehat{B}^S = |B^S| - L^S$ do not have the same law. A sufficient condition is that the random variables B_S and $\widehat{B}_S = |B_S| - L_S$ have different laws. Many stopping times have this property, for instance the first hitting time of a given level by B, or by $|B|$ or by L...

However, there also exist many stopping times (for the filtration of B) such that B_S and $\widehat{B}_S$ are not only identical in law, but a.s. equal; for instance $\inf\{t \geqslant 1 : B_t = \widehat{B}_t\}$. This stopping time is a.s. finite since the martingale $B - \widehat{B}$ is divergent (for its bracket is $4 \int \mathbb{1}_{\{B_t \leqslant 0\}} dt$).

But a finite stopping time S such that $B_S = \widehat{B}_S$ cannot be invariant unless it vanishes identically. For in that case $B_S = \widehat{B}_S = \widehat{B}_{S \circ T} = B_S \circ T$ is a function of $\widehat{B}$ only; and, since B and $-B$ have the same conditional law given $\widehat{B}$, B_S must be its own opposite and $B_S = 0$, giving $L_S = |B_S| - \widehat{B}_S = 0$ and $S = 0$. (The same argument shows more generally that a finite, non-negative random variable S measurable for $\widehat{B}$ and verifying $B_S = \phi(\widehat{B}_S)$ must vanish if ϕ is a function such that $\phi(x) \neq 0$ for every $x < 0$.)

More precisely, it can be proved that *if S is an invariant, finite, positive stopping time,* $\mathbb{P}[B_S = \widehat{B}_S] \leqslant 1/2$. Let indeed A denote the event $\{B_S = \widehat{B}_S\}$ and suppose $\mathbb{P}[A] > 1/2$. Since $\widehat{B}_S = |B_S| - L_S$ and $L_S > 0$, on A one has $\widehat{B}_S = -B_S - L_S$ and $2B_S = -L_S$; this can be rewritten $2\widehat{B}_S = \inf_{t \leqslant S} \widehat{B}_t$. But $\mathbb{P}[T^{-1}A]$ is also larger than one half, so the event $A \cap T^{-1}A$ is not negligible. On this event, one has on the one hand $2\widehat{B}_S = \widehat{I}_S$ and on the other hand $2\widehat{B}_S = -\widehat{L}_S$

(where $\widehat{I}_t = \inf_{s\leqslant t}\widehat{B}_s$ and $\widehat{L}$ is the local time at zero of $\widehat{B}$). To establish the claim, we shall prove that, if $\widehat{B}$ is a Brownian motion started at 0, the event $\{\exists t{>}0 : 2\widehat{B}_t = \widehat{I}_t = -\widehat{L}_t\}$ is negligible; replacing $\widehat{B}$ with $-\widehat{B}$, dropping the hats for typographical simplicity and letting $S_t = \sup_{s\leqslant t} B_s$, this amounts to showing that

$$\mathbb{P}[\exists t{>}0 : 2B_t = S_t = L_t] = 0 .$$

Since both processes S and L are locally constant in the random open set $\{t : B_t \neq S_t \text{ and } B_t \neq 0\}$, if equality $S = L$ holds at some time $t > 0$ such that $B_t = \frac{1}{2}S_t$, it also holds identically in some neighborhood of t and hence at some rational t; so we just have to show that $\mathbb{P}[S_t = L_t] = 0$ for each $t > 0$.

From the scaling property of Brownian motion, this probability does not depend on t; hence it is also equal to $\mathbb{P}[S_T = L_T]$ where T is an exponential random variable independent of B. Now, the joint law of (S_T, L_T) is easily computed from excursion theory arguments; in particular it is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}_+^2$, with identical exponential marginals. This proves the claim.

c) Still working in the σ-field generated by B, notice that $(\mathcal{L})$ is true if and only if, for H ranging over a total subset of L^2, $\mathbb{E}[H|\mathcal{J}] = \mathbb{E}[H]$, or equivalently by the ergodic theorem

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} H\circ T^k = \mathbb{E}[H] .$$

We shall use the subset consisting of the constant 1 (for which the above equality is trivial) and of all multiple Wiener integrals

$$H = \int_0^\infty dB_{u_1} \int_0^{u_1} dB_{u_2} \ldots \int_0^{u_{p-1}} dB_{u_p}\ f(u_1,\ldots,u_p)$$

where $p \geqslant 1$ and f satisfies

$$\int_0^\infty du_1 \int_0^{u_1} du_2 \ldots \int_0^{u_{p-1}} du_p\ f^2(u_1,\ldots,u_p) < \infty .$$

This programme can be carried out successfully in the case $p = 1$. Indeed, writing $B^{(k)} = T^k(B)$ and $\varepsilon_t^{(k)} = \operatorname{sgn} B_t^{(k)}$, Tanaka's formula gives

$$B_t^{(k)} = \int_0^t dB_s^{(k-1)}\, \varepsilon_s^{(k-1)} = \int_0^t dB_s^{(k-2)}\, \varepsilon_s^{(k-2)}\varepsilon_s^{(k-1)} = \int_0^t dB_s\, \varepsilon_s\varepsilon_s^{(1)}\ldots\varepsilon_s^{(k-1)} .$$

Now, if $H = \int_0^\infty dB_u\, f(u)$, where $f \in \mathrm{L}^2$, we have

$$\frac{1}{n}\sum_{k=1}^{n} H\circ T^k = \int_0^\infty dB_u\ f(u) \Big(\frac{1}{n}\sum_{k=1}^{n} \varepsilon_u\varepsilon_u^{(1)}\ldots\varepsilon_u^{(k-1)}\Big)$$

so that

$$\mathbb{E}\Big[\Big(\frac{1}{n}\sum_{k=1}^{n} H\circ T^k\Big)^2\Big] = \int_0^\infty du\ f^2(u)\, \mathbb{E}\Big[\Big(\frac{1}{n}\sum_{k=1}^{n} \varepsilon_u\varepsilon_u^{(1)}\ldots\varepsilon_u^{(k-1)}\Big)^2\Big] .$$

For fixed u, $\varepsilon_u^{(\ell)}$ is independent of $B^{(\ell+1)}$ and hence also of all the $\varepsilon_u^{(m)}$ for $m > \ell$; so the sequence $\varepsilon_u, \varepsilon_u^{(1)}, \varepsilon_u^{(2)}, \ldots$ of Bernoulli variables is independent, whence

$$\mathbb{E}\Big[\Big(\frac{1}{n}\sum_{k=1}^{n}\varepsilon_u\varepsilon_u^{(1)}\ldots\varepsilon_u^{(k-1)}\Big)^2\Big] = \frac{1}{n},$$

and $\frac{1}{n}\sum_{k=1}^{n} H \circ T^k$ does tend to zero.

For random variables H belonging to chaoses of higher order, the same method, and the well-known isometry between the p^{th} chaos and the space of square-integrable symmetric functions of p variables, reduce $(\mathcal{L})$ to an equivalent property.

PROPOSITION. — *$(\mathcal{L})$ is true if and only if for each $p > 1$*

$$\lim_{n\to\infty}\int_0^1 du_1\int_0^{u_1} du_2\ldots\int_0^{u_{p-1}} du_p\,\mathbb{E}\Big[\Big(\frac{1}{n}\sum_{k=1}^{n}\prod_{\substack{0\leqslant\ell<k\\1\leqslant m\leqslant p}}\varepsilon_{u_m}^{(\ell)}\Big)^2\Big] = 0\,.$$

Hence, the first step in that direction would be to take $p = 2$ and to get a good estimate of

$$\mathbb{E}\Big[\Big(\frac{1}{n}\sum_{k=1}^{n}\varepsilon_u\varepsilon_v\varepsilon_u^{(1)}\varepsilon_v^{(1)}\ldots\varepsilon_u^{(k-1)}\varepsilon_v^{(k-1)}\Big)^2\Big]\,.$$

APPENDIX: Ocone's Theorem

Ocone's Theorem A of [5] consists of the equivalence between (ii), (iii) and (iv) below. His setting is more general than the following rephrasing: he deals with local martingales (and further extends his results to the càdlàg case).

THEOREM. — *Let M be a continuous, divergent martingale with natural filtration $\mathcal{F} = (\mathcal{F}_t)_{t\geqslant 0}$. The following five statements are equivalent:*

(i) *the processes β^M and $\langle M\rangle$ are independent;*

(ii) *for every $\mathcal{F}$-predictable process H taking values in $\{-1, 1\}$, the pairs of processes $\big(\int H\,dM, \langle M\rangle\big)$ and $\big(M, \langle M\rangle\big)$ have the same law (in particular the martingales $\int H\,dM$ and M have the same law);*

(iii) *for every deterministic function h of the form $\mathbb{1}_{[0,a]} - \mathbb{1}_{(a,\infty)}$, the martingale $\int h\,dM$ has the same law as M;*

(iv) *for every $\mathcal{F}$-predictable process H measurable for the product σ-field $\mathcal{B}(\mathbb{R}_+)\otimes\sigma(\langle M\rangle)$ and such that $\int_0^\infty H_s^2\,d\langle M\rangle_s < \infty$ a. s.,*

$$\mathbb{E}\Big[\exp\Big(i\int_0^\infty H_s\,dM_s\Big)\,\Big|\,\langle M\rangle\Big] = \exp\Big(-\tfrac{1}{2}\int_0^\infty H_s^2\,d\langle M\rangle_s\Big)\,;$$

(v) *for every deterministic function h of the form $\sum_{j=1}^{n}\lambda_j\,\mathbb{1}_{[0,a_j]}$,*

$$\mathbb{E}\Big[\exp\Big(i\int_0^\infty h(s)\,dM_s\Big)\Big] = \mathbb{E}\Big[\exp\Big(-\tfrac{1}{2}\int_0^\infty h^2(s)\,d\langle M\rangle_s\Big)\Big]\,.$$

Notice that (iii) is a symmetry assumption: given the past of M, the conditional law of the future increments is symmetric; (iv) says that, conditionally given $\langle M\rangle$, M is a Gaussian martingale with variance $\langle M\rangle$.

LEMMA 1. — *If a right-continuous process X is a martingale for some (non necessarily right-continuous) filtration $(\mathcal{G}_t)_{t\geqslant 0}$, it is also a martingale for its right-continuous enlargment $(\mathcal{G}_{t+})_{t\geqslant 0}$.*

PROOF. — When $\varepsilon > 0$ tends to 0, $X_{s+\varepsilon}$ tends to X_s in L^1 by uniform integrability; so, for $s < t$, $\mathbb{E}[X_t - X_s|\mathcal{G}_{s+}] = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \mathbb{E}[X_t - X_{s+\varepsilon}|\mathcal{G}_{s+}] = 0.$ ∎

LEMMA 2. — *Let β be a Brownian motion with natural filtration $\mathcal{B}$ and $\mathcal{G}$ a σ-field independent of β. If a process H taking values in $\{-1,1\}$ is predictable for the filtration $\mathcal{E}$ defined by $\mathcal{E}_t = \bigcap_{\varepsilon>0}(\mathcal{B}_{t+\varepsilon}\vee\mathcal{G})$, the process $\int H\,d\beta$ is a Brownian motion independent of $\mathcal{G}$.*

PROOF. — Lemma 1 and the independence of β and $\mathcal{G}$ imply that β is a $\mathcal{E}$-Brownian motion. By Lévy's characterization, $\int H\,d\beta$ is also a $\mathcal{E}$-Brownian motion; so it is independent of $\mathcal{E}_0$, hence of $\mathcal{G}$. ∎

PROOF OF OCONE'S THEOREM. — We shall show that (i) ⇒ (ii) ⇒ (iii) ⇒ (i) and (i) ⇒ (iv) ⇒ (v) ⇒ (i). Implications (ii) ⇒ (iii) and (iv) ⇒ (v) are trivial.

(i) ⇒ (ii) and (iv). We suppose that $\beta = \beta^M$ and $\langle M\rangle$ are independent. Let $A_t = \inf\{s : \langle M\rangle_s > t\}$, so that, with the convention $A_{0-} = 0$, its left-limit is $A_{t-} = \inf\{s : \langle M\rangle_s \geqslant t\}$; denote by $\mathcal{B}$ the natural filtration of β and by $\mathcal{E}$ the right-continuous enlargement of the filtration $\mathcal{B}_t\vee\sigma(\langle M\rangle)$. If T is a $\mathcal{F}$-stopping time, $\langle M\rangle_T$ is a $\mathcal{E}$-stopping time since

$$\{\langle M\rangle_T \leqslant t\} = \{T \leqslant A_t\} \in \mathcal{F}_{A_t} \subset \bigcap_{\varepsilon>0}\sigma(M^{A_{t+\varepsilon}}) \subset \bigcap_{\varepsilon>0}\,[\mathcal{B}_{t+\varepsilon}\vee\sigma(\langle M\rangle)] = \mathcal{E}_t\,.$$

If H is a bounded, $\mathcal{F}$-predictable process, $K_t = H_{A_{t-}}$ is bounded and $\mathcal{E}$-predictable and $\int_0^t H_s\,dM_s = \int_0^{\langle M\rangle_t} K_u\,d\beta_u$ (when $H = \mathbb{1}_{[\![0,T]\!]}$ with T a $\mathcal{F}$-stopping time, $K = \mathbb{1}_{[\![0,\langle M\rangle_T]\!]}$ and both integrals agree since $M_{T\wedge t} = \beta_{\langle M\rangle_{T\wedge t}}$; the general case follows by a monotone class argument).

If furthermore H takes values in $\{-1,1\}$, Lemma 2 with $\mathcal{G} = \sigma(\langle M\rangle)$ says that $\gamma = \int K\,d\beta$ is a Brownian motion independent of $\langle M\rangle$ (as is β). Consequently, both processes $M = \beta_{\langle M\rangle}$ and $\int H\,dM = \gamma_{\langle M\rangle}$ have the same conditional law given $\langle M\rangle$; this proves (ii).

Taking now H bounded, $\mathcal{F}$-predictable, $[\mathcal{B}(\mathbb{R}_+)\otimes\sigma(\langle M\rangle)]$-measurable and such that $\int_0^\infty H_s^2\,d\langle M\rangle_s < \infty$, K is also $[\mathcal{B}(\mathbb{R}_+)\otimes\sigma(\langle M\rangle)]$-measurable,

$\int_0^\infty K_s^2\,ds < \infty$ and

$$\mathbb{E}\Big[\exp\Big(i\int_0^\infty H_s\,dM_s\Big)\,\Big|\,\langle M\rangle\Big] = \mathbb{E}\Big[\exp\Big(i\int_0^\infty K_s\,d\beta_s\Big)\,\Big|\,\langle M\rangle\Big]$$
$$= \exp\Big(-\tfrac12\int_0^\infty K_s^2\,ds\Big) = \exp\Big(-\tfrac12\int_0^\infty H^2_{A_s-}\,ds\Big)$$
$$= \exp\Big(-\tfrac12\int_0^\infty H_s^2\,d\langle M\rangle_s\Big)\,.$$

This proves (iv) when H is bounded; the general case follows by taking limits.

(iii) $\Rightarrow$ (i). Let $s < t$. Since $M' = \int \big(\mathbb{1}_{[0,s]} - \mathbb{1}_{(s,\infty)}\big)\,dM$ has the same law as M, the triple $(M^s, \langle M\rangle, M_t - M_s)$ has the same law as $(M'^s, \langle M'\rangle, M'_t - M'_s)$. But the stopped processes M^s and M'^s are the same, the quadratic variations $\langle M\rangle$ and $\langle M'\rangle$ are equal and $M'_t - M'_s = -(M_t - M_s)$, yielding

$$\big(M^s, \langle M\rangle, M_t - M_s\big) \overset{(\text{law})}{=} \big(M^s, \langle M\rangle, -(M_t - M_s)\big)\,.$$

Denoting by $\mathcal{G}_s$ the σ-field generated by the processes M^s and $\langle M\rangle$ and the null events, this implies that M is a martingale for the filtration $\mathcal{G}$, whence also for its right-continuous enlargement $\mathcal{H}$ (Lemma 1).

The random variables $A_t = \inf\{s : \langle M\rangle_s > t\}$ are stopping times for the filtration $\mathcal{H}$ (they are $\mathcal{H}_0$-measurable!); the stopped martingales M^{A_t} are square-integrable and one has $\langle M\rangle_{A_t} = t$. Introducing the filtration $\mathcal{K}_t = \mathcal{H}_{A_t}$ and the Brownian motion $\beta = \beta^M$, one can write

$$\mathbb{E}[\beta_t - \beta_s | \mathcal{K}_s] = \mathbb{E}[\beta_{\langle M\rangle_{A_t}} - \beta_{\langle M\rangle_{A_s}} | \mathcal{H}_{A_s}] = \mathbb{E}[M_{A_t} - M_{A_s} | \mathcal{H}_{A_s}] = 0\,.$$

Consequently β is a $\mathcal{K}$-martingale, hence (Lévy's characterization) a $\mathcal{K}$-Brownian motion; so it is independent of $\mathcal{K}_0$, and a fortiori of $\mathcal{G}_0 = \sigma(\langle M\rangle)$.

(v) $\Rightarrow$ (i). Let B be a Brownian motion independent of $\langle M\rangle$. Applying (i) $\Rightarrow$ (v) to the martingale $N = B_{\langle M\rangle}$ and remarking that, since $\langle M\rangle = \langle N\rangle$, the right-hand side of (v) is the same for M and N, we see that M and N have the same law. Consequently $\big(\beta^M, \langle M\rangle\big)$ and $\big(\beta^N, \langle N\rangle\big) = \big(B, \langle M\rangle\big)$ also have the same law and β^M is independent of $\langle M\rangle$. ∎

REMARKS. — a) The hypotheses that $\mathcal{B}$ is the natural filtration of β in Lemma 2 and $\mathcal{F}$ that of M in (ii) cannot be dropped.

If one supposes only that $\mathcal{B}$ is a filtration such that β is a $\mathcal{B}$-Brownian motion, Lemma 2 becomes false: Take a Brownian motion B with natural filtration $\mathcal{B}$, call $\beta = \int \operatorname{sgn}(B)\,dB$ the Lévy transform of B and $\mathcal{G}$ the σ-field generated by $\operatorname{sgn}(B_1)$. Now $H = \operatorname{sgn}(B)$ is $\mathcal{B}$-predictable and a fortiori $\mathcal{E}$-predictable, where $\mathcal{E}_t = \bigcap_{\varepsilon>0}(\mathcal{B}_{t+\varepsilon} \vee \mathcal{G})$. But $\int H\,d\beta = B$ is certainly not independent of $\operatorname{sgn}(B_1)$. (What makes this example work is that for $t < 1$ both random variables $\operatorname{sgn}(B_t)$ and $\operatorname{sgn}(B_1)$ are independent of β, but the pair $(\operatorname{sgn}(B_t), \operatorname{sgn}(B_1))$ is not.)

Similarly, the theorem becomes false if $\mathcal{F}$ is no longer the natural filtration of M, but only some filtration for which M is a martingale. In that case, (i), (iii), (iv) and (v) are still equivalent, but (ii) may become stronger, as shown by the following example. Take as above B with natural filtration $\mathcal{B}$ and Lévy transform β; define an increasing process A independent of β by $A_t = t$ for $t \leqslant 1$ and $A_t = 1 + [u\mathbb{1}_{\{B_1>0\}} + v\mathbb{1}_{\{B_1\leqslant 0\}}](t-1)$ for $t > 1$, where u and v are strictly positive real numbers. Our martingale verifying (i) will be the Lévy transform $M_t = \beta_{A_t}$ of B_{A_t}; as the latter is a martingale for the filtration $\mathcal{F}_t = \mathcal{B}_{A_t}$, so is also M. The process

$$H_t = \begin{cases} \operatorname{sgn}(B_t) & \text{if } t \leqslant 1 \\ \operatorname{sgn}(B_1)\operatorname{sgn}(B_{A_t}) & \text{if } t > 1 \end{cases}$$

is $\mathcal{F}$-predictable, but the random variables

$$M_2 = \begin{cases} \beta_{1+u} & \text{if } B_1 > 0 \\ \beta_{1+v} & \text{if } B_1 \leqslant 0 \end{cases} \quad \text{and} \quad \int_0^2 H_s\,dM_s = \begin{cases} B_{1+u} & \text{if } B_1 > 0 \\ 2B_1 - B_{1+v} & \text{if } B_1 \leqslant 0 \end{cases}$$

do not have the same law in general. Indeed, on the one hand the law of M_2 is symmetric (β is independent of $\operatorname{sgn}(B_1)$) and on the other hand, if u is chosen large and v small, $\mathbb{P}[B_{1+u} > 0 \text{ and } B_1 > 0]$ is close to 1/4 and $\mathbb{P}[2B_1 - B_{1+v} > 0 \text{ and } B_1 \leqslant 0]$ to 0, yielding by addition

$$\mathbb{P}\big[\textstyle\int_0^2 H_s\,dM_s > 0\big] \approx \frac{1}{4} \neq \frac{1}{2} = \mathbb{P}[M_2 > 0]\ .$$

This shows that the filtration $\mathcal{F}$ is too large for (ii) to hold.

b) *If M is a martingale on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ for a filtration $\mathcal{F} = (\mathcal{F}_t)_{t\geqslant 0}$ and if $\mathcal{C}$ is a sub-σ-field of $\mathcal{A}$, then M is still a martingale for the enlarged filtration $(\mathcal{F}_t \vee \mathcal{C})$ if and only if $\mathbb{E}[\int_0^t H_s\,dM_s|\mathcal{C}] = 0$ for each t and each simple, $\mathcal{F}$-predictable process H verifying $|H| = 1$.* Indeed, if M is a martingale for $(\mathcal{F}_t \vee \mathcal{C})$, so is also $\int H\,dM$, yielding $\mathbb{E}[\int_0^t H_s\,dM_s|\mathcal{F}_0 \vee \mathcal{C}] = 0$. Conversely, if the condition holds, $\mathbb{E}\big[(M_t - M_s)U\mathbb{1}_C\big] = 0$ for each $C \in \mathcal{C}$ and each $\mathcal{F}_s$-measurable U with values in $\{-1, 1\}$; but for $A \in \mathcal{F}_s$, $2\mathbb{1}_A = (\mathbb{1}_A - \mathbb{1}_{A^c}) + 1$ is the sum of two such U's, whence $\mathbb{E}\big[(M_t - M_s)\mathbb{1}_A\mathbb{1}_C\big] = 0$, and M is a martingale for the large filtration.

c) *If $\mathcal{F} = (\mathcal{F}_t)_{t\geqslant 0}$ is the natural filtration of some Brownian motion B and if $\mathcal{C}$ is a non-trivial sub-σ-field of $\bigvee_t \mathcal{F}_t$, no $\mathcal{F}$-Brownian motion can be a martingale for $(\mathcal{F}_t \vee \mathcal{C})$.* For such a Brownian motion β would have the form $\int H\,dB$ for an $\mathcal{F}$-predictable H with $|H| = 1$; so $B = \int H\,d\beta$ would also be a $(\mathcal{F}_t \vee \mathcal{C})$-martingale, hence a $(\mathcal{F}_t \vee \mathcal{C})$-Brownian motion and would be independent of $\mathcal{F}_0 \vee \mathcal{C} = \mathcal{C}$.

d) Yet, keeping the notations of c), *there exist a non-trivial sub-σ-field $\mathcal{C}$ of $\bigvee_t \mathcal{F}_t$ and a process that is both a $(\mathcal{F}_t)$- and a $(\mathcal{F}_t \vee \mathcal{C})$-martingale,* for instance the σ-field $\mathcal{C} = \mathcal{F}_1$ and the process $\int h\,dB$, with $h = \mathbb{1}_{[1,\infty)}$. This example generalizes as follows: Let A be a $(\mathcal{F}_t)$-predictable set and assume, for simplicity, that (almost) all sections $A^c(\omega)$ of its complementary have infinite Lebesgue measure. Let $M = \int \mathbb{1}_A \, dB$, $N = \int \mathbb{1}_{A^c} \, dB$ and denote by $\mathcal{C}$ the σ-field generated by the Brownian motion β^N. Time-changing by $\langle N \rangle = \int \mathbb{1}_{A^c} \, dt$ the predictable representation property with respect to β^N shows that every square-integrable, $\mathcal{C}$-measurable random variable assumes the form

$$U = \mathbb{E}[U] + \int_0^\infty K_s \mathbb{1}_{A^c}(s) \, dB_s \, ,$$

where K is predictable and such that $\mathbb{E}\left[\int_0^\infty K_s^2 \mathbb{1}_{A^c}(s) \, ds\right] < \infty$. This easily implies that $M = \int \mathbb{1}_A \, dB$ satisfies the condition in b) above, showing that M is a $(\mathcal{F}_t \vee \mathcal{C})$-martingale.

It seems worthwhile to present such examples here as they play an important rôle in some martingale proofs of the Ray-Knight theorems for Brownian local times (see, for instance, Exercises (2.8) and (2.9) pages 426–427 of Revuz & Yor [6] and Jeulin [4]).

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