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A Critical Function For The Planar Brownian Convex Hull

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Abstract: We prove that if the origin is translated so that the real axis is tangential to the (random) convex hull of a planar Brownian motion, touching at the origin, then for each positive ε $\dfrac{(\frac{\pi}{2}+\varepsilon)|x|\log^3(1/|x|)}{\log(1/|x|)}$ is an upper function for the hull but $\dfrac{(\frac{\pi}{2}-\varepsilon)|x|\log^3(1/|x|)}{\log(1/|x|)}$ is not.

Introduction

This note is concerned with the continuity properties of the convex hull of Brownian motion. This random set has been studied by Evans (1985), Cranston, Hsu and March (1989) and more recently by Burdzy and San Martin (1989). It is from the latter paper that most of the ideas in this paper are taken as well as the problem addressed. Consider a planar Brownian motion $\{(X_1(t), X_2(t)): t \geq 0\}$. Let $t_{\min}$ be the time that the minimum value of X_2 is achieved over the time interval [0, 1]. If C is the convex hull of the Brownian path over the unit interval translated by $-(X_1(t_{\min}), X_2(t_{\min}))$, then the x-axis is tangential to C at the origin. Locally at the origin the boundary of C may be represented as (x,f(x)) where f is a positive convex function. The first two papers quoted show that f is C^1. Cranston, Hsu and March (1989) showed that a non-negative function g was a lower function for f if and only if

$$\int_{0+} g(x)x^{-2}\,dx < \infty$$

and that in this case

$$\liminf_{x\to 0} \frac{f(x)}{g(x)} = \infty.$$

Burdzy and San Martin (1989) examined the limsup behaviour of f and proved that

$$\limsup_{x\to 0} \frac{f(x)}{|x|[\log(1/|x|)]^{-1}} = \infty.$$

and

$$\limsup_{x\to 0} \frac{f(x)}{|x|\log^2(1/|x|)[\log(1/|x|)]^{-1}} \leq \pi.$$

We use their approach to show

Theorem

The local representation of C, (x,f(x)) satisfies

$$\limsup_{x\to 0} \frac{f(x)}{|x|\log^3(1/|x|)[\log(1/|x|)]^{-1}} = \pi/2.$$

The lucidity of this paper has been enhanced by the e-mail of K. Burdzy for which the author is grateful.

Notation and Summary of Tools taken from Burdzy and San Martin

Throughout the paper the plane R^2 will be identified with the set of complex numbers. So i will refer to the point (0,1). Occasionally we will write a point in polar co-ordinates. We hope it will be obvious which system is in effect and no confusion will result. We will write $c_n \sim p_n$ to mean that there exist finite, strictly positive k and K so that $kc_n < p_n < Kc_n$ for all n. We will treat stopping times as random variables of various different processes. So we may define for example for a generic process X, the stopping time S to equal inf{t: |X(t)| = 1}. Then given two processes $Z^1(t)$ and $Z^2(t)$, $Z^1(S)$ will be the position at which Z^1 first hits the unit circle while $Z^2(S)$ is the position at which Z^2 hits the unit circle.

The quantities $P_x^h[A]$ will refer to the probabilities of event A for an h-process beginning at x, while $P_x[A]$ will refer to the probability of A for an unconditioned Brownian motion beginning at x. Typically when $P_x^h[A]$ is written the event A will be written in terms of a process already known to be an h-process and so the "h" suffix will be strictly speaking superfluous, nonetheless we hope its presence will make for easier reading. $P_h^{x,z}[A]$ will refer to probabilities for two independent h-processes beginning at x and z respectively.

For our purposes, the most important part of the approach of Burdzy and San Martin (1989) was the proof of Lemma 2.1. which showed that for local properties of C we could instead consider $\tilde{C}$, the convex hull of the paths of two independent h-processes Y_h^1 and Y_h^2, beginning at i for h(x,y) = $\frac{y}{\pi(x^2+y^2)}$. These h-processes are more commonly known as Brownian motions conditioned to exit H, the upper half plane, at the origin. Note this is not the fact proved in Lemma 2.1.

The following is essentially Lemma 1.1 of Burdzy and San Martin (1989).

Lemma One

Let L be a line through the origin whose slope is $\alpha\pi$ where α is in the interval (0, 1/16). Let B_r be the ball centered at the origin and radius r. If T is the first hitting time of $L \cup B_r$, then

$$P_h^i[\,|Y_h^1(T)-ri\,| < r/2] \sim r^{\alpha/(1-\alpha)}$$

Proof

A standard h-process identity gives

$$P_h^i[\,|Y_h^1(T)-ri\,| < r/2] = E^i[h(Y(T)),\ |Y(T)-ri\,| < r/2] \quad \frac{1}{r}P^i[\,|Y(T)-ri\,| < r/2],$$

where Y is an unconditioned planar Brownian motion, initially at i.

For the upper bound of our lemma, we simply note that

$$P^i[\,|Y(T)-ri\,| < r/2] \le P^i[Y(T) \notin L] \sim r^{\frac{1}{1-\alpha}}.$$

For the lower bound we remark that (by the quasi-stationary behaviour of 1-dimensional Brownian motion in an interval) as r tends to zero the distributions of arg(Y(T)) conditioned on {Y(T) L} converge to a distribution with strictly positive density on $(\alpha\pi,\pi)$. See Ito and McKean(1965), page 31, for details.

We prove similarly the following reformulation of Lemma 3.2 of Burdzy and San-Martin.

Lemma Two

Let L be a line which makes angle α with the real line for α in the interval (0, 1/16) and which intersects the x-axis at the point (r,0). If T is the first hitting time of $L \cup B_{2r}$, then

$$P_h^i[\,|Y_h^1(T) - \sqrt{3}ri\,| < r/2] \sim r^{\alpha/(1-\alpha)}$$

Section One

We prove the theorem by splitting it up into two propositions. In this section, we prove the first proposition.

Proposition One

Let (x,f(x)) be a local representation of $\tilde{C}$ at the origin.

$$\limsup_{x\to 0} \frac{f(x)}{|x|\,log^3(1/|x|)[log(1/|x|)]^{-1}} \le \pi/2.$$

Proof

Fix $\gamma > \pi/2$. Let ε be a small positive number so that $\gamma > \pi/2(1+2\varepsilon)^2$. Let $r_k = e^{-(1+\varepsilon)^k}$ and L_k be the line $\{z: \arg(z) = \pi\alpha_k = \frac{(1+\varepsilon)logk}{(1+\varepsilon)^k}\pi/2\}$. For a process Y the stopping time T_k will be the first hitting time of $L_k \cup B_{r_k}$. The event A_j is defined to be $\{|Y_h^1(T_k)|$ or $|Y_h^2(T_k)| > r_j \sec(\pi\alpha_k)\}$. Note that if A_j occurs then $\tilde{C}$ contains the line segment from the origin to $(r_k, \pi\alpha_k)$. So for $r_{k+1} \le x \le r_k$ we have

$$f(x) \le x \tan\left[\frac{(1+\varepsilon)logk}{(1+\varepsilon)^k}\pi/2\right]$$

For k large, enough the right hand side will be less than $x\frac{\gamma \log^3(1/|x|)}{log(1/|x|)}$

Now by Lemma One and the independence of Y_h^1 and Y_h^2, the probability of A_k^c is less than $(Ck^{-(1+\varepsilon)/2})^2$ and the Borel-Cantelli lemma enables us to conclude that A_k must occur for all k large enough and therefore for all x small enough

$$f(x) \le x\frac{\gamma \log^3(1/|x|)}{log(1/|x|)}$$

□

Section Two

In this section we wish to prove the reverse inequality to that of Proposition One:

Proposition Two

Let (x,f(x)) be a local representation of $\tilde{C}$ at the origin. If $\gamma < \pi/2$, then

$$\limsup_{x\to 0} \frac{f(x)}{|x|\,log^3(1/|x|)[log(1/|x|)]^{-1}} > \gamma$$

Before embarking on the proof proper we will need some preliminary lemmas. We fix for this section $\varepsilon > 0$ so that $\pi/2 > (1+\varepsilon)^2\gamma$, and we define (or redefine) the quantities

$$r_j = e^{-(1+\varepsilon)^j};\ \alpha_j = \frac{(1-\varepsilon)\log j}{(1+\varepsilon)^j}\pi/2;\ R_j = e^{-(1+\varepsilon)^j/j}r_j.$$

L_j is the line through the points $(R_j,0)$ and (r_j,α_j),

T_j is the first hitting time by a process of $L_j \cup B_{r_j} \cup R$ where R is the real line.

A_j is the event $\{|Y_h^1(T_j) - ir_j| < r_j/2\} \cap \{|Y_h^2(T_j) - ir_j| < r_j/2\}$.

We now make the following observations

1) The angle α_j' made by the line L_j with the real line is decreasing in j and equal to $\alpha_j + O(\alpha_j^3)$.

2) For j large enough, and for all positive m, the line L_{j+m} meets the line L_j inside the disc B_{r_j}.

Lemma Three

The conditional probability that the point (r_j,α_j) is not in $\tilde{C}$ given that A_j occurs, $P_h^{i,i}[(r_j,\alpha_j) \quad \tilde{C} \mid A_j]$, is bounded below by a strictly positive c.

Proof

First note that $P_h^{i,i}[(r_j,\alpha_j) \notin \tilde{C} \mid A_j] \geq P_h^{i,i}[L_j$ is not hit by Y_h^1 or $Y_h^2] = P_h^i[L_j$ is not hit by $Y_h^1]^2$. Secondly note $P_h^{i,i}[arg(Y_h^1(T_j))$ and $arg(Y_h^2(T_j)) \in (\pi/2,2\pi/3) \mid A_j]$ are bounded away from 0. We now investigate the term $P_h^i[L_j$ is not hit by $Y_h^1]$.

Let the stopping time S_j be defined to be $\inf\{t\colon X(t) \in R$ or L_j or $|X(t) - (R_j,0)| = 2R_j\}$ where again R is the real line. It is trivial that $P_h^i[L_j$ is not hit by $Y_h^1 \mid arg(Y_h^1(S_j)) \in (\pi/2,2\pi/3)] > k$ for some strictly positive k. Equally by our second remark it is clear that

$$P_h^i[arg(Y_h^1(S_j)) \in (\pi/2,2\pi/3) \mid A_j] \quad \text{is of the order} \quad \frac{h(R_j i)}{h(r_j i)}P^{r_j i}[argX(S_j) \in (\pi/2,2\pi/3)]$$

for an unconditioned Brownian motion X. The latter term is of the order

$$\frac{r_j}{R_j}\left(\frac{R_j}{r_j}\right)^{\frac{1}{1-\alpha_j'}} \sim 1$$

This proves the lemma.

□

Lemma Four

Let j and m be positive integers:

i) $P_h^i[A_j] \sim \dfrac{1}{j^{1-\varepsilon}}$

ii) $P_h^i[A_{j+m} \mid A_j] \sim \left(\dfrac{1}{(j+m)^{1-\varepsilon}}\right)^{1-(1+\varepsilon)^{-m}}$

Proof

The lemma follows simply from Lemma Two and the Strong Markov Property.

□

In exactly the same way, the corollary below follows.

Corollary

Let z be a fixed point in the upper half plane with arg(z) $\in$ $(\pi/2, 2\pi/3)$. There exist finite strictly positive C and c so that for j large enough and all positive m we have

i) $P_h^z[A_j] > \dfrac{c}{j^{1-\varepsilon}}$

ii) $P_h^z[A_{j+m} \mid A_j] < C \left[\dfrac{1}{(j+m)^{1-\varepsilon}}\right]^{1-(1+\varepsilon)^{-m}}$

Proof of Proposition Two

For a process X define the stopping time D_n as $\inf\{t: |X(t)| = r_n\}$. For the two h-processes Y_h^1 and Y_h^2, define the filtration $\{F_n\}_{n=0}^{\infty}$ by

$$F_n = \sigma(Y_h^1(t), t \le D_n) \, V \, \sigma(Y_h^2(t), t \le D_n).$$

Given Corollary One and the fact that before times D_n the Y_h^1 and Y_h^2 processes are bounded away from the x-axis, it is easily seen that for $\arg(Y_h^1(D_n))$, $\arg(Y_h^2(D_n)) \in (\pi/2, 2\pi/3)$ and j large enough we have

i) $P_h[A_j \mid F_n] > \dfrac{c}{j^{1-\varepsilon}}$

ii) $P_h[A_{j+m} \mid A_j, F_n] < C \left[\dfrac{1}{(j+m)^{1-\varepsilon}}\right]^{1-(1+\varepsilon)^{-m}}$.

Now take $n_j = [j^{\frac{1}{1-\varepsilon}}]$. We can choose j large enough so that for all k, l in [j, 2j] (k < l) we have

a) $P_h[A_{n_k} \mid F_n] > \dfrac{c'}{j}$

b) $P_h[A_{n_l} \cap A_{n_k} \mid F_n] < \dfrac{C'}{j^2}$.

for strictly positive c' and C'.

This means that if we define the random variable $W_j = \sum_{k=j}^{2j} I_{A_{n_k}}$, then for j large enough $E[W_j \mid F_n] > c'$ and $E[W_j^2 \mid F_n] < C'$. This implies that there is a $\delta > 0$, so that whenever $Y_h^1(D_n)$ and $Y_h^2(D_n)$ both have argument in the interval $(\pi/2, 2\pi/3)$, $P_h[\bigcup_{j>n} A_j \mid F_n] > \delta]$. Thus with probability one $\limsup_{n\to\infty} P_h[\bigcup_{j>n} A_j \mid F_n] > \delta$. This in turn implies that $P_h^{i,i}[\limsup_{j\to\infty} A_j] = 1$. By Lemma Three this means that $P_h^{i,i}[(r_j, \alpha_j) \notin \tilde{C}$ for infinitely many j $] = 1$, which completes the proof of Proposition Two and hence the proof of the Theorem. □

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