SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

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Séminaire de probabilités (Strasbourg), tome 23 (1989), p. 186-197
<http://www.numdam.org/item?id=SPS_1989__23__186_0>

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by
L.C.G. Rogers

1. Introduction.

Let (S,d) be a complete metric space with Borel σ-field $\mathcal{S}$, and let $(X_t)_{t\geq 0}$ be an S-valued strong Markov process whose paths are right continuous with left limits. We ask

(Q) Is $P(X_{t_1} = \cdots = X_{t_k}$ for some $0 < t_1 < \cdots < t_k) > 0$?

This is equivalent to the question

(Q′) Is $P(X(\mathrm{I}_1) \cap \cdots \cap X(\mathrm{I}_k) \neq \emptyset) > 0$ for some disjoint compact intervals $\mathrm{I}_1, \ldots, \mathrm{I}_k$?

We shall find conditions sufficient to ensure that X has k-multiple points with positive probability, and we will apply this to Lévy processes, providing another proof of a result of LeGall, Rosen and Shieh [6], and its improvement due to Evans [3]. However, it is advantageous to begin with the easier question

($\bar{\text{Q}}$) Is $P(\bar{X}(\mathrm{I}_1) \cap \cdots \cap \bar{X}(\mathrm{I}_k) \neq \emptyset) > 0$ for some disjoint compact intervals $\mathrm{I}_1, \ldots, \mathrm{I}_k$?

Here, $\bar{X}(\mathrm{I}_j) \equiv$ closure $(\{X_s : s \in \mathrm{I}_j\})$, a compact subset of S. In recent years, much effort has been devoted to a study of (Q), usually in the form of constructing some non-trivial random measure on the set $\{(t_1, \ldots, t_k) : X_{t_1} = \cdots = X_{t_k}\}$ from which the existence of common points in the ranges $X(\mathrm{I}_j)$ follows immediately. We mention only the work of Dynkin [1] and Evans [2] on symmetric Markov processes, of Rosen [8], [9], Geman, Horowitz and Rosen [4], LeGall, Rosen and Shieh [6] and Evans [3] on more concrete Markov processes in $\mathbb{R}^n$, as a sample of recent activity. Typically, one studies the random variables

$$(1) \qquad Z_\varepsilon \equiv \int_C \mathrm{I}_U(X_{t_1}) F_\varepsilon(X_t)\, dt ,$$

where $C = \mathrm{I}_1 \times \cdots \times \mathrm{I}_k$, with the I_j disjoint compact intervals in $\mathbb{R}^+$, $U \in \mathcal{S}$, and

$$(2) \qquad F_\varepsilon(x_1, \ldots, x_k) \equiv \prod_{i=1}^{k-1} f_\varepsilon(x_i, x_{i+1}) ,$$

(where f_ε is some 'spike' function such that $f_\varepsilon(x,y)=0$ if $d(x,y)>\varepsilon$), and proves L^2-convergence of the Z_ε to some non-trivial limit as $\varepsilon \downarrow 0$.

This will be the approach used here, but, since we are concerned *only* with an answer to (Q), and not with the (more refined) L^2-convergence of the Z_ε, we can weaken the assumptions somewhat. In particular, we give sufficient conditions to ensure the existence of points of intersection for general (i.e. *non-symmetric*) Markov processes.

If we could prove that

(3.i) *for some* $\eta > 0$, $\{Z_\varepsilon : 0 < \varepsilon < \eta/k\}$ *is bounded in* L^2;

(3.ii) $\limsup_{\varepsilon \downarrow 0} E\, Z_\varepsilon > 0,$

then the answer to ($\overline{\mathrm{Q}}$) *is, "Yes".* The point is that $(Z_\varepsilon)_{0<\varepsilon<\eta/k}$ is then uniformly integrable; if there were no common points in the closed ranges $\overline{X}(\mathrm{I}_j)$, then the Z_ε would (almost surely) be zero for all small enough $\varepsilon > 0$, and hence the $Z_\varepsilon \to 0$ in L^1, contradicting (3.ii).

2. The main result. We suppose that there is a σ-finite measure μ on S such that for all $x \in S$

$$\mu(B_\varepsilon(x)) > 0 \qquad \forall\ \varepsilon > 0. \tag{4}$$

Here, $B_\varepsilon(x) \equiv \{y : d(x,y) \le \varepsilon\}$. (The assumption (4) is no great restriction, since we could always confine ourselves to the closed set of x for which it is true.)

We shall suppose that the Green's functions of X have densities with respect to μ: for $0 \le a < b < \infty$, there exists $g_{a,b}(\cdot,\cdot)$ such that

$$G_{a,b}(x,A) \equiv E^x\left[\int_a^b \mathrm{I}_A(X_s)ds\right] = \int_A g_{a,b}(x,y)\,\mu(dy) \qquad (\forall\, x \in S,\ A \in \mathcal{S}). \tag{5}$$

We suppose also that there are open $U \subset V \subset S$ such that for some $\eta > 0$ the η-neighbourhood of U is contained in V, and that there are positive finite K, T such that

(A) $\mu(B_{2\varepsilon}(x)) \leq K\,\mu(B_\varepsilon(x)) \qquad \forall\ \varepsilon \in (0,\eta],\ \forall\ x \in V$;

(B) $\int_{V\times V} g_{0,T}(x,y)^k\,\mu(dx)\,\mu(dy) < \infty$;

(C) for each $\delta \in (0,2T)$,

$$\sup_{x,y\in V} g_{\delta,2T}(x,y) < \infty;$$

(D) for each $0 < a < b < \infty$, $g_{a,b}(\bullet,\bullet)$ is lower semicontinuous on $V\times V$;

(E) for some $\xi \in U$ and $\tau \in (0,T)$,

$$g_{0,\tau}(\xi,\xi) > 0.$$

Remarks on conditions (A)-(E). Condition (A) seems fairly mild; it is trivially satisfied for Lebesgue measure on Euclidean space. The purpose of (A) is to let us take

$$f_\varepsilon(x,y) \equiv \mu(B_\varepsilon(x))^{-1}\,\mathrm{I}_{\{d(x,y)\leq\varepsilon\}} \tag{6}$$

and estimate

$$\begin{aligned} f_\varepsilon(x,y) &\leq K\,\mu(B_{2\varepsilon}(x))^{-1}\,\mathrm{I}_{\{d(x,y)\leq\varepsilon\}} \\ &\leq K\,\mu(B_\varepsilon(y))^{-1}\,\mathrm{I}_{\{d(x,y)\leq\varepsilon\}} \\ &= K\,f_\varepsilon(y,x). \end{aligned} \tag{7}$$

Condition (B) is the 'folklore' condition for k-multiple points. Condition (C) may appear severe, but is frequently satisfied. Conditions (A)-(C) will give us (3.i), and conditions (D) and (E) will give us (3.ii). We may (and shall) suppose that the τ appearing in (E) is a point of increase of $g_{0,\bullet}(\xi,\xi)$.

THEOREM 1. *Assuming conditions* (A), (B), *and* (C), *the family* $\{Z_\varepsilon : 0 < \varepsilon < \eta/k\}$ *is bounded in* L^2. *Assuming also conditions* (D) *and* (E), *there exist initial distributions such that for some disjoint compact intervals* $\mathrm{I}_1, \ldots, \mathrm{I}_k$

$$P(\overline{X}(\mathrm{I}_1)\cap\cdots\cap\overline{X}(\mathrm{I}_k)\neq\varnothing) > 0.$$

Proof. (i) Let m be the law of X_0. For ease of exposition, we shall suppose that X has a transition density $p_t(\bullet,\bullet)$ with respect to μ; the result remains true without this assumption though.

The time-parameter set $C = \mathrm{I}_1 \times \cdots \times \mathrm{I}_k$ used in the definition of Z is chosen so that $j\tau$ is in the interior of I_j for each j, so that $0 < \delta \le t - s \le 2T$ if $t \in \mathrm{I}_j$, $s \in \mathrm{I}_{j-1}$ ($j = 2, \ldots, k$), and so that $|\mathrm{I}_j| < T$ for all j. Then

$$E\, Z_\varepsilon^2 \;=\; E \int_{C\times C} ds\, dt\, \mathrm{I}_U(X_{s_1})\, \mathrm{I}_U(X_{t_1})\, F_\varepsilon(\boldsymbol{X}_s)\, F_\varepsilon(\boldsymbol{X}_t)$$

$$= \sum_R \int_{C^2_\le} ds\, dt \int m(dy_0) \mathrm{I}_U(x_1) \mathrm{I}_U(y_1) F_\varepsilon(x') F_\varepsilon(y') \prod_{j=1}^{k} p_{s_j - t_{j-1}}(y_{j-1}, x_j) p_{t_j - s_j}(x_j, y_j) \mu(dx_j) \mu(dy_j)\,,$$

where $C^2_\le = \{(s,t) \in C^2 : s_i \le t_i \text{ for } i = 1, \ldots, k\}$, $t_0 \equiv 0$, the sum is taken over all subsets R of $\{1, \ldots, k\}$, and

$$x'_i = x_i\,,\quad y'_i = y_i \qquad \text{if } \; i \in R$$

$$x'_i = y_i\,,\quad y'_i = x_i \qquad \text{if } \; i \notin R\,.$$

The typical term in the sum is bounded above by some constant times

$$\int m(dy_0)\, \mathrm{I}_U(x_1) \mathrm{I}_U(y_1)\, F_\varepsilon(x')\, F_\varepsilon(y') \prod_{j=1}^{k} q\,(y_{j-1}, x_j)\, g\,(x_j, y_j)\, \mu(dx_j)\, \mu(dy_j)\,,$$

where we have made the abbreviations

$$q\,(x,y) \;\equiv\; g_{\delta, 2T}\,(x,y)\,,$$

$$g\,(x,y) \;\equiv\; g_{0,T}\,(x,y)\,.$$

By assumption (C), the factors $q\,(y_{j-1}, x_j)$ are globally bounded, because $x_1, y_1 \in \mathrm{U}$, and $d\,(x'_i, x'_{i+1}) \le \varepsilon < \eta/k$ for each i, and therefore by assumption $x_i \in \mathrm{V}$ for all $i = 1, \ldots, k$. Thus we have an upper bound in terms of

$$\int \mathrm{I}_U(x_1)\, \mathrm{I}_U(y_1)\, F_\varepsilon(x')\, F_\varepsilon(y') \prod_{j=1}^{k} g\,(x_j, y_j)\, \mu(dx_j)\, \mu(dy_j)$$

$$\le \prod_{j=1}^{k} \left(\int \mathrm{I}_U(x_1)\, \mathrm{I}_U(y_1)\, F_\varepsilon(x')\, F_\varepsilon(y')\, g\,(x_j, y_j)^k\, \mu(d\boldsymbol{x})\, \mu(d\boldsymbol{y}) \right)^{1/k},$$

by Hölder's inequality, where, of course $\mu(d\boldsymbol{x}) \equiv \prod_1^k \mu(dx_j)$. The j^{th} term in this product, raised to the power k, is bounded by

$$\int \mathrm{I}_\mathrm{V}(x_j) \mathrm{I}_\mathrm{V}(y_j)\, g\,(x_j, y_j)^k \prod_{i=1}^{k-1} f_\varepsilon(x'_i, x'_{i+1})\, f_\varepsilon(y'_i, y'_{i+1})\, \mu(d\boldsymbol{x})\, \mu(d\boldsymbol{y})\,,$$

which we deal with by integrating out successively $x_k, y_k, x_{k-1}, \ldots, x_{j+1}, y_{j+1}$, and then, exploiting (6), integrating out $x_1, y_1, \ldots, x_{j-1}, y_{j-1}$ to leave as an upper bound

$$K^{2j-2} \int \mathrm{I}_V(x_j)\, \mathrm{I}_V(y_j)\, g(x_j, y_j)^k\, \mu(dx_j)\, \mu(dy_j)$$

which is finite, by assumption (B). Hence for $0 < \varepsilon < \eta/k$, $E(Z_\varepsilon^2)$ is bounded above by a finite constant independent of ε, which proves the first statement.

(ii) We next exploit (D) and (E) to give us (3.ii). By the choice of the set C, we have that for some small enough $\theta > 0$,

$$C \supseteq C_0 = \{(t_1, \ldots, t_k) : |t_i - t_{i-1} - \tau| < \theta \quad \text{for} \quad i = 1, \ldots, k\},$$

where $t_0 = 0$. Hence

$$\begin{aligned} EZ_\varepsilon &\geq E\left[\int_{C_0} dt\, \mathrm{I}_U(X_{t_1})\, F_\varepsilon(\boldsymbol{X}_t)\right] \\ &= \int m(dx_0)\, \mathrm{I}_U(x_1) \prod_{i=1}^{k} g(x_{i-1}, x_i) \prod_{i=1}^{k-1} f_\varepsilon(x_i, x_{i+1})\, \mu(d\boldsymbol{x}), \end{aligned}$$

where we write g as an abbreviation for $g_{\tau-\theta,\, \tau+\theta}$. Since τ is a point of increase of $g_{0,\cdot}(\xi,\xi)$, we know that $g(\xi,\xi) > 0$. Thus

$$EZ_\varepsilon \geq \int m(dx_0)\, \mathrm{I}_U(x_1)\, g(x_0, x_1)\, \underline{g}_\varepsilon(x_1)^{k-1} \prod_{i=1}^{k-1} f_\varepsilon(x_i, x_{i+1})\, \mu(d\boldsymbol{x}), \tag{8}$$

where

$$\underline{g}_\varepsilon(x_1) \equiv \inf\{g(x,y) : d(x, x_1) \leq k\varepsilon,\, d(y, x_1) \leq k\varepsilon\},$$

which, in view of (D), increases as $\varepsilon \downarrow 0$ to $g(x_1, x_1)$. By integrating out the variables $x_k, x_{k-1}, \ldots, x_2$ in (8), we obtain the lower bound

$$EZ_\varepsilon \geq \int m(dx_0)\, \mathrm{I}_U(x_1)\, g(x_0, x_1)\, \underline{g}_\varepsilon(x_1)^{k-1}\, \mu(dx_1),$$

and hence the estimate

$$\liminf_{\varepsilon \downarrow 0} EZ_\varepsilon \geq \int m(dx_0)\, \mathrm{I}_U(x_1)\, g(x_0, x_1)\, g(x_1, x_1)^{k-1}\, \mu(dx_1).$$

By lower semi-continuity and the fact that $g(\xi,\xi) > 0$, we know that $g(x,y)$ is positive in a neighbourhood of (ξ,ξ) and so taking $m = \delta_\xi$, for example, yields

$$\liminf_{\varepsilon \downarrow 0} EZ_\varepsilon > 0. \qquad \Diamond$$

We now turn to the more difficult question Q. Let us suppose further that *every singleton is polar*:

(F) $P^x(X_t = y \text{ for some } t > 0) = 0 \qquad \forall\ x,y \in S$,

and that

(G) *for each* $\mu \in Pr(S)$, *for each previsible stopping time* $\tau > 0$ *we have*

$$X_\tau = X_{\tau-} \qquad P^\mu - \text{a.s. on } \{\tau < \infty\}.$$

For example, if S is locally compact and separable, and the process X is Feller-Dynkin, then (G) holds; see Rogers and Williams [7], Theorem VI.15.1.

THEOREM 2. *Assuming conditions* (A)-(G), *there exist initial distributions such that for some disjoint compact intervals* $I_1, \ldots, I_k$

$$P(X(I_1)\cap \cdots \cap X(I_k) \neq \varnothing) > 0.$$

Proof. The proof uses Theorem 1, and proceeds by induction on k. For $k = 1$, the result is trivial. We suppose the result is true for $k \leq K$, and, using Theorem 1, take some initial distribution, and disjoint compact intervals $I_1, \ldots, I_{K+1}$ such that I_{j+1} is to the right of I_j for each j, and

(9) $$P(\bar{R}_K \cap \bar{X}(I_{K+1}) \neq \varnothing) > 0,$$

where $\bar{R}_K = \bar{X}(I_1)\cap \cdots \cap \bar{X}(I_K)$. Let $R_K = X(I_1)\cap \cdots \cap X(I_K)$. Then

$$P(\bar{R}_K \cap X(I_{K+1}) \neq \varnothing) > 0,$$

because, if not, from (9), the previsible time set

$$\{t \in I_{K+1} : X_{t-} \in \bar{R}_K\}$$

is non-empty with positive probability and can therefore be sectioned by a previsible time τ; but, by (G), $X_\tau = X_{\tau-} \in \bar{R}_K$.

Finally we deduce that

$$P(R_K \cap X(I_{K+1}) \neq \varnothing) > 0,$$

for if not, we would have to have

(10) $$P((\bar{R}_K \backslash R_K) \cap X(I_{K+1}) \neq \varnothing) > 0;$$

since $\overline{R}_K \backslash R_K \subset \bigcup_{j=1}^{K} (\overline{X}(\mathrm{I}_j) \backslash X(\mathrm{I}_j))$, and $\overline{X}(\mathrm{I}_j) \backslash X(\mathrm{I}_j)$ is contained in the (countable) set of left endpoints of jumps of X during time interval I_j, it follows from (F) that the set $\overline{R}_K \backslash R_K$ is *polar*, contradicting (10). ◊

3. Multiple points of Lévy processes. Let X be a Lévy process in $\mathbb{R}^n$, with resolvent $(U_\lambda)_{\lambda > 0}$. We shall assume that the resolvent is strong Feller (equivalently, that each $U_\lambda(x,.)$ has a density with respect to Lebesgue measure - see Hawkes [5]), in which case there is for each $\lambda > 0$ a λ-excessive lower semi-continuous function u_λ such that

$$U_\lambda f(x) = \int u_\lambda(y) f(y+x)\, dy .$$

To establish sufficient conditions for k-multiple points, we shall need three lemmas on Lévy processes of interest in their own right.

LEMMA 1. *The resolvent $(U_\lambda)_{\lambda>0}$ is strong Feller if and only if for every $0 \le a < b < \infty$ the kernel $G_{a,b}$ has a density $g_{a,b}$.*

If this happens, the densities $g_{a,b}(.)$ may be chosen so that

(i) *$g_{a,b}(.)$ is lower semicontinuous for each $0 \le a < b < \infty$;*

(ii) *$(a,b) \to g_{a,b}(x)$ is left-continuous increasing in b and right-continuous decreasing in a for each x;*

(iii) *for all $0 \le a < b < \infty$ and all $x \in \mathbb{R}^n$*

$$g_{a,b}(x) = \lim_{\delta \downarrow 0} \delta^{-1} \int g_{0,\delta}(y)\, g_{a,b-\delta}(x-y)\, dy .$$

LEMMA 2. *For a Lévy process with a strong Feller resolvent, the following are equivalent:*

(i) *for some $\varepsilon, T > 0$,*

$$\int_{\{|x| \le \varepsilon\}} g_{0,T}(x)^k\, dx < \infty ;$$

(ii) *for some* $T > 0$, $\quad g_{0,T} \in L^k$;

(iii) *for some* $\lambda > 0$, $\quad u_\lambda \in L^k$;

(iv) *for some* $\varepsilon, \lambda > 0$,

$$\int_{\{|x| \le \varepsilon\}} u_\lambda(x)^k \, dx < \infty .$$

LEMMA 3. *Let X be a Lévy process with a strong Feller resolvent such that* $g_{0,T}(0) > 0$ *for some T, and* $\{\xi\}$ *is non-polar for some* $\xi \in \mathbb{R}^n$. *Then* $\{x\}$ *is non-polar for every* $x \in \mathbb{R}^n$.

We defer the proofs of these lemmas so as to show how to deduce the following result from them and Theorem 2. Fix some integer $k > 1$.

THEOREM 3 (LeGall-Rosen-Shieh; Evans). *Assuming that the Lévy process X has a strong Feller resolvent, the conditions*

(11.i) *for some* $\varepsilon, T > 0$

$$\int_{\{|x| \le \varepsilon\}} g_{0,T}(x)^k \, dx < \infty ;$$

(11.ii) *for some* $T > 0$, $\quad g_{0,T}(0) > 0$

are sufficient to ensure that the paths of X have points of multiplicity k almost surely.

Proof. In view of Lemma 3, we may assume that every singleton is polar, for, if not, every singleton is non-polar, and the existence of multiple points is trivial! To apply Theorem 2, we must check conditions (A)-(G); (A) is immediate, (B) is guaranteed by (11.i), (D) follows from Lemma 1, (E) comes from (11.ii), (F) is by assumption, and (G) is valid because the Lévy process is a Feller-Dynkin process. Finally, to check (C), (11.i) implies that $g_{0,T}$ is square-integrable in a neighbourhood of 0, so, by Lemma 2, $g_{0,T} \in L^2$. Hence $g_{0,T} * g_{0,T}$ is bounded and continuous. But for $f \ge 0$ measurable, of compact support, and $0 < \delta < T$

$$\begin{aligned} \int g_{0,T} * g_{0,T}(x) f(x) \, dx &= \int_0^T dt \int_0^T ds \, P_{t+s} f(0) \\ &\ge \delta\sqrt{2} \int_\delta^{2T-\delta} P_t f(0) \, dt \\ &= \delta\sqrt{2} \int g_{\delta,\, 2T-\delta}(x) f(x) \, dx , \end{aligned}$$

whence $g_{\delta,T}(.)$ is bounded globally (exploiting lower semi-continuity).

This completes the proof that (11.i-ii) implies that X has k-multiple points with positive probability, and hence, by Borel-Cantelli, there are almost surely k-multiple points.

Proof of Lemma 1. The arguments used are similar to those of Hawkes [5], so we will just give an outline. The first statement of the lemma is immediate. To get good versions of the densities $g_{a,b}$, firstly take any densities $g'_{p,q}(.)$ for $G_{p,q}$, $0 \le p < q < \infty$ rational, then define

$$g''_{a,b}(x) \;\equiv\; \sup\,\{g'_{p,q}(x) \;:\; a < p < q < b\}\,,$$

which have property (ii) (which remains preserved under the subsequent modifications). Next, for $n > (b-a)^{-1}$ define

$$\tilde{g}^{n}_{a,b}(x) \;=\; n\int g_{0,\delta}(y)\, g_{a,b-\delta}(x-y)\,dy\,, \qquad (\delta \equiv n^{-1})$$

which is lower semicontinuous in x (it is the increasing limit as $M \uparrow \infty$ of

$$n\int g_{0,\delta}(y)\,(M \wedge g_{a,b-\delta}(x-y))\,dy\,,$$

which are continuous by the strong Feller property of $G_{0,\delta}$). Finally, we take

$$g_{a,b}(.) \equiv \sup\,\{\tilde{g}^{n}_{a,b}(.) \;:\; n \;>\; (b-a)^{-1}\}\,.$$

Since, for fixed $a < b$, $\tilde{g}^{n}_{a,b}$ is increasing almost everywhere to a version of the density of $G_{a,b}$, this provides a version with the desirable properties (i) - (iii). ◊

Proof of Lemma 2. The implications (iii) => (iv) => (i) are trivial. The implication (ii) => (iii) follows easily from the estimate

$$\begin{aligned}
\int g_{a,a+T}(x)^k\,dx &= \int \Big(\int P_a(dy)\, g_{0,T}(x-y)\Big)^k\,dx \\
&\le \int dx \int P_a(dy)\, g_{0,T}(x-y)^k \\
&= \int g_{0,T}(z)^k\,dz\,.
\end{aligned}$$

So, finally, we assume (i) and prove (ii). Specifically, let K denote the cube

$$K \;\equiv\; \{x \in \mathbb{R}^n \;:\; |x_i| \;\le\; \tfrac{1}{2} \quad \text{for} \quad i = 1, \ldots, n\}\,,$$

and assume without loss of generality that

$$\int_{K+K} g(x)^k \, dx \; < \; \infty ,$$

where we have abbreviated $g_{0,T}$ to g. For $j \in \mathbb{Z}^n$, let

$$\tau_j \;\equiv\; \inf\{t > 0 \;:\; X_t \in j + K\} .$$

Then for $x \in j + K$, we have from the strong Markov property at τ_j that

$$g(x) \;\le\; \int_{j+K} P(\tau_j < T,\, X(\tau_j) \in dy)\, g(x-y) ,$$

from which

$$g(x)^k \;\le\; P(\tau_j < T)^{k-1} \int_{j+K} P(\tau_j < T,\, X(\tau_j) \in dy)\, g(x-y)^k ,$$

and, integrating,

$$\int_{j+K} g(x)^k \, dx \;\le\; P(\tau_j < T)^k \int_{K+K} g(z)^k \, dz .$$

The proof is finished if we can show that

$$\phi(T) \;\equiv\; \sum_j P(\tau_j < T) \;<\; \infty .$$

Since ϕ is evidently increasing, it is enough to prove that

$$\int_0^\infty \lambda\, e^{-\lambda T}\, \phi(T)\, dT \;=\; \sum_j P(\tau_j < \zeta) \;<\; \infty ,$$

where ζ is an exp(λ) random variable independent of X. But we have the lower bound

$$\int_{j+K+K} u_\lambda(x)\, dx \;\ge\; P(\tau_j < \zeta) \int_K u_\lambda(x)\, dx . \tag{12}$$

The sum over $j \in \mathbb{Z}^n$ of the left-hand sides of (12) is clearly finite, and $\int_K u_\lambda(x)\, dx > 0$, so the proof is finished.

Proof of Lemma 3. If $\{\xi\}$ is non-polar, the resolvent density $u_\lambda(\bullet)$ must be bounded, since

$$E^x \exp(-\lambda H_\xi) \;=\; c_\lambda \cdot u_\lambda(\xi - x)$$

for some constant c_λ. (Here, $H_\xi = \inf\{t > 0 : X_t = \xi\}$.) By lower semicontinuity, $u_\lambda(0) > 0$ implies that $u_\lambda > 0$ in some neighbourhood of zero and hence, by the resolvent equation, $u_\lambda > 0$ everywhere. Thus $P^x(H_\xi < \infty) \;>\; 0$ for every x, and translation invariance implies that every point is non-polar. ◊

Remarks. (i) It is evident that (11.ii) is equivalent to the condition

(9.ii) for some $\lambda > 0$, $u_\lambda(0) > 0$.

Hence, in view of Lemma 2, the conditions (11) are equivalent to those imposed by Evans [3].

(ii) Similar techniques can be used to study the problem of the existence of common points in the ranges of k independent Markov processes, a technically easier problem.

Acknowledgements. It is a pleasure to thank my hosts at the Laboratoire de Probabilités, especially Marc Yor, for numerous stimulating discussions on these and other subjects during my visit to Paris in October 1987; and a referee for helpful criticisms on the first draft of this paper.

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