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ON SKOROHOD EMBEDDING IN n-DIMENSIONAL BROWNIAN MOTION BY MEANS OF NATURAL STOPPING TIMES

by

Neil Falkner[1]

ABSTRACT

Let μ be a measure on $\mathbb{R}^n$ whose electrostatic potential is well-defined and not everywhere infinite. Let (B_t) be Brownian motion in $\mathbb{R}^n$ with $\text{law}(B_0) = \mu$. We give sufficient conditions for a measure ν on $\mathbb{R}^n$ to be of the form $\text{law}(B_T)$ where T is a natural (ie., non-randomized) stopping time for (B_t) which is not "too big". (If $n \geq 3$, any stopping is not "too big" but if $n = 1$ or 2 , some stopping times are "too big"). If the measure μ does not charge polar sets, the conditions we give are not only sufficient but necessary.

1. INTRODUCTION

Let $(\Omega, \mathcal{B}, \mathcal{B}_t, B_t, \theta_t, P^x)$ be Brownian motion in $\mathbb{R}^n$. In this paper we consider questions of the following sort : let μ and ν be measures on $\mathbb{R}^n$. When can one find a stopping time T such that

$$\nu(dx) = P^{\mu}(B_T \in dx) \quad ?$$

We emphasize that the fields $\mathcal{B}_t$ are the usual completed natural fields of the process (B_t) and that when we speak of a stopping time we mean a stopping time with respect to these fields. Thus the stopping times we consider are natural stopping times rather than randomized ones.

Skorohod [1] was the first to consider these questions. He considered the case $n = 1$, μ = the unit point mass at 0, ν a probability measure on $\mathbb{R}$ such that $\int x^2 \, d\nu(x) < \infty$, $\int x \, d\nu(x) = 0$, and he obtained a randomized stopping time τ such that $\nu(dx) = P^\mu(B_\tau \in dx)$ and $E^\mu(\tau) < \infty$. Dubins [1] and Root [1] considered the same case and obtained, by different methods, natural stopping times T such that $\nu(dx) = P^\mu(B_T \in dx)$ and $E^\mu(T) < \infty$. Rost [1 and 2] considered questions of this sort for general Markov processes and obtained randomized stopping times. Baxter and Chacon [1] considered the case of general n and showed that under suitable hypotheses, which included the supposition that the potential of ν be finite and continuous, it is possible to find a natural stopping time T such that $\nu(dx) = P^\mu(B_T \in dx)$. In this paper we improve the result of Baxter and Chacon by eliminating certain of their hypotheses, including the hypothesis of continuity of the potential of ν and "part of" the hypothesis of finiteness of the potential of ν. Our result is still not the best possible however, as we show by an example.

2. THE CASE OF A GREEN REGION

Throughout this section, D denotes a Green region in $\mathbb{R}^n$ with Green function G and ζ is the stopping time defined by

$$\zeta(\omega) = \inf \{t \geq 0 : B_t(\omega) \notin D\} .$$

For any measure μ in D, $G\mu : D \to [0,\infty]$ is defined by

$$G\mu(x) = \int_D G(x,y) \, d\mu(y) \qquad (x \in D) .$$

One says $G\mu$ is a potential iff it is not identically infinite on any component of D. (We remark that if μ is finite then $G\mu$ is a potential, but not conversely). If T is any stopping time then μ_T will denote the measure in D defined by

$$\mu_T(dx) = P^\mu(B_T \in dx, \; T < \zeta) .$$

That is, μ_T is the measure obtained by letting μ diffuse under Brownian motion up to the random time T where only paths of (B_t) which stay in D for the whole random time interval $[0,T]$ contribute to μ_T. The question we consider is:

Given μ such that $G\mu$ is a potential, what measures ν in D are of the form $\nu = \mu_T$ for some stopping time T ?

We shall make extensive use of classical potential theory and of the connections between Brownian motion and classical potential theory. For the former, the reader may consult Helms [1] and Brelot [1 and 2]; for the latter, Rao [1] and Blumenthal and Getoor [1].

The following lemma gives necessary conditions for ν to be of the form μ_T.

2.1. LEMMA

Let μ be a measure in D such that $G\mu$ is a potential and let T be a stopping time. Let $\nu = \mu_T$.

Then:

a) $G\mu \geq G\nu$. (Thus ν is finite on compact subsets of D).

b) There is a Borel set $C \subseteq D$ such that $\nu(Z) = \mu(Z \cap C)$ for every Borel polar set $Z \subseteq D$.

Proof :

a) For any non-negative Borel function f in D we have

$$\int f(x)\, G\mu(x)\, dx = E^{\mu}\left[\int_0^{\zeta} f(B_t)\, dt \right]$$

and

$$\int f(x)\, G\nu(x)\, dx = E^{\mu}\left[\int_{T\wedge\zeta}^{\zeta} f(B_t)\, dt \right] .$$

Hence $G\mu \geq G\nu$ a.e. [dx]. But then $G\mu \geq G\nu$ throughout D as $G\mu$ and $G\nu$ are hyperharmonic functions.

b). As T is a natural stopping time, $P^x(T=0) \in \{0,1\}$ for all x, by the zero-one law. Let $C = \{x \in D : P^x(T=0) = 1\}$. C may not be Borel but at least it is universally measurable, which is close enough. If Z is any polar subset of D then $P^\mu(B_t \in Z$ for some $t > 0) = 0$ so certainly $P^\mu(B_T \in Z, T > 0) = 0$. Thus $\nu(Z) = \mu(Z \cap C)$ for every universally measurable polar set $Z \subseteq D$.

Remarks : Clearly the proof of part a) shows that if S and T are stopping times with $S \leq T$ then $G\mu_S \geq G\mu_T$. This inequality remains valid for randomized stopping times $\sigma \leq \tau$ but for part b) we need the fact that T is a natural stopping time.

Let us also point out another way of phrasing b). Note that $\{G\nu = \infty\}$ is a polar set and ν does not charge polar subsets of $\{G\nu < \infty\}$. Thus b) can be expressed more explicitly as follows : there is a Borel set $C \subseteq D$ such that $\nu(Z) = \mu(Z \cap C)$ for all Borel sets $Z \subseteq \{G\nu = \infty\}$.

We conjecture that conditions a) and b) of the lemma imply that there exists a stopping time T such that $\nu = \mu_T$ but we are unable to prove this. We can however prove the following weaker result.

2.2. THEOREM

Let μ and ν be measures in D such that $G\mu$ and $G\nu$ are potentials. Suppose $G\mu \geq G\nu$ in D and $\mu(Z) \leq \nu(Z)$ for every Borel set $Z \subseteq \{G\nu=\infty\}$. Then there is a stopping time T such that $\nu = \mu_T$.

Remarks :Once we have T, it follows from 2.1 (b), and from the fact that ν does not charge polar subsets of $\{G\nu < \infty\}$, that $\nu(Z) = \mu(Z \cap \{G\nu = \infty\})$ for every Borel polar set $Z \subseteq D$. This is why the theorem is weaker than the conjecture. Let us remark ~~though~~, that for the case when μ does not charge polar sets, the theorem completely characterizes the measures ν which are of the form μ_T for some stopping time T.

So that the ideas in the proof of the theorem will not be lost among a mass of asides, we first establish some preliminary results.

2.3. NOTATION

A property which holds except on a polar set will be said to hold <u>quasi-everywhere</u>, abbreviated q.e. If u is a non-negative superharmonic function in D and $E \subseteq D$ then among all the non-negative superharmonic functions v in D such that $v \geq u$ q.e. on E, there is a smallest one which we denote by $\mathrm{bal}(u,E)$, read "the balayage of u over E". We write $\mathrm{reg}(E,D)$ for the set of regular points of E in D ; that is, the set

$$\{x \in D : u(x) = \mathrm{bal}(u,E)(x) \text{ for all } u \text{ superharmonic and } \geq 0 \text{ in } D\}.$$

One can show that $E\setminus\mathrm{reg}(E,D)$ is a polar set, that $\mathrm{reg}(E,D)$ is a G_δ-set, and that $\mathrm{reg}(E,D)$ is equal to the set of fine accumulation points of E in D, if $n \geq 2$, or the closure of E in D, if $n=1$.

If μ is a measure in D such that $G\mu$ is a potential then $\mathrm{bal}(G\mu,E)$ is the potential of a unique measure ν in D which we denote by $\mathrm{bal}(\mu,E)$, read "the balayage of μ onto E". One can show that $\mathrm{bal}(\mu,E)$ lives on $\mathrm{reg}(E,D)$.

T_E denotes the first hitting time of E : $T_E(\omega) = \inf\{t > 0 : B_t(\omega) \in E\}$. If E is a Borel set (or just analytic) then T_E is a-stopping time and we have the well-known formula

$$\mathrm{bal}(u,E)(x) = E^x(u(B_{T_E})\, 1_{\{T_E < \zeta\}})$$

for any non-negative superharmonic function u in D.

2.4. LEMMA

Let μ be a measure in D such that $G\mu$ is a potential. Let E be a Borel subset of D and let $T=T_E$. Let $\nu = \mu_T$. Then

a). for all $x \in D$, $G\nu(x) = E^x(G\mu(B_T)\, 1_{\{T < \zeta\}})$.

b). $\nu = \mathrm{bal}(\mu,E)$.

<u>Proof</u> :

Clearly a) and b) are equivalent. Let us prove a). For any

$x,y \in D$, $\mathrm{bal}(G(y,.),E)(x) = \mathrm{bal}(G(x,.),E)(y)$; see Brelot [1, p. 15]. Thus

$$E^x(G\mu(B_T)\, 1_{\{T < \zeta\}}) = \int \mu(dy)\, E^x(G(y,B_T)\, 1_{\{T < \zeta\}})$$

$$= \int \mu(dy)\, \mathrm{bal}(G(y,.),E)(x) = \int \mu(dy) \mathrm{bal}(G(x,.),E)(y)$$

$$= \int \mu(dy)\, E^y(G(x,B_T)\, 1_{\{T < \zeta\}}) = E^\mu(G(x,B_T)\, 1_{\{T < \zeta\}})$$

$$= \int \nu(dz)\, G(x,z) = \int \nu(dz) G(z,x) = G\nu(x).$$

2.5. COROLLARY

Let μ be a measure in D such that $G\mu$ is a potential. Let U be a Borel finely open subset of D and let $T=T_{U^c}$. Then $\mu_T(U) = 0$.

Proof :

By 2.4, $\mu_T = \mathrm{bal}(\mu,U^c)$. Therefore μ_T lives on $\mathrm{reg}(U^c,D)$. But $\mathrm{reg}(U^c,D) \subseteq U^c$ since U is finely open.

2.6. COROLLARY

Let μ be a measure in D such that $G\mu$ is a potential. Let v be a superharmonic function in D such that $G\mu \geq v$. Let U be a Borel relatively compact subset of D and suppose there is a function h which is harmonic in a neighbourhood of $\overline{U}$ such that $G\mu \geq h \geq v$ on U. Let $T=T_{U^c}$. Then $G\mu_T \geq v$ in D.

Proof :

For $x \in \mathrm{reg}(U^c,D)$ we have $P^x(T > 0) = 0$ so $G\mu_T(x) = G\mu(x)$ by 2.4. Suppose $x \in U$. Then $P^x(T < \zeta) = 1$ since U is relatively compact in D. Also for every $\omega \in \{B_0=x\}$ and for every $t \in [0,T(\omega)) \cup \{0\}$ we have $G\mu(B_t(\omega)) \geq h(B_t(\omega))$. Now $G\mu$ need not be continuous but nevertheless

$t \mapsto G\mu(B_t(\omega))$ is continuous on $[0,\zeta(\omega))$ for P^x - a.a. ω. Therefore $G\mu(B_T) \geq h(B_T)$ P^x - a.s. . From 2.4 we then obtain $G\mu_T(x) \geq E^x(h(B_T))$. But $E^x(h(B_T)) = h(x)$ as h is harmonic in a neighbourhood of $\overline{U}$. Thus we find that $G\mu_T \geq v$ except possibly on the set $U^c \setminus reg(U^c,D)$. But this exceptional set is polar. Therefore $G\mu_T \geq v$ throughout D.

2.7. LEMMA

Let μ be a measure in D such that $G\mu$ is a potential. Let (T_i) be a sequence of stopping times converging pointwise on Ω to a random time T.

Then :

a) T is a stopping time

b) For any Borel function ϕ in D such that

$$\int |\phi(x)| G\mu(x)dx < \infty$$

we have $\int_D \phi(x) G\mu_{T_i}(x)dx \to \int_D \phi(x)\, G\mu_T(x)dx.$

(One can deduce from this that

$$\int \psi \, d\mu_{T_i} \to \int \psi \, d\mu_T$$

for any continuous function ψ with compact support in D but we shall not need this).

Proof :

a) follows from the right continuity of (B_t). Let us prove b). Consider the decreasing process

$$Z_t = \int_{t \wedge \zeta}^{\zeta} \phi(B_s)ds \quad (0 \leq t \leq \infty).$$

For any stopping time S we have

$$\int_D \phi(x) G\mu_S(x)dx = E^\mu(Z_S).$$

Also, for $\omega \in \{Z_0 < \infty\}$ the map $t \mapsto Z_t(\omega)$ is continuous on $[0,\infty]$. The proof may now be concluded by applying the Lebesgue dominated convergence theorem.

2.8. LEMMA

Let μ be a measure in D. Let U be a finely open Borel subset of D. Let (T_i) be a sequence of stopping times converging pointwise on Ω to a stopping time T. Suppose $\mu_{T_i}(U) = 0$ for all i. Then $\mu_T(U) = 0$.

Proof : First suppose U is of the form $V \cap \{v < c\}$ where V is open in D, v is superharmonic in D, and c is a real number. Suppose that $\mu_T(U) \neq 0$. Then there is an open set W which is relatively compact in V and a real number $d < c$ such that $\mu_T(W \cap \{v < d\}) \neq 0$.

Let f be a $[0,1]$ -valued continuous function in D such that f=1 on W and f=0 outside some compact subset of V, let g be a $[0,1]$ -valued continuous function on $(-\infty,\infty]$ such that g=1 on $(-\infty,d]$ and g=0 on $[c,\infty]$, and let $\phi(x) = f(x)g(v(x))$ for $x \in D$. Since ϕ vanishes outside U we have $E^\mu(\phi(B_{T_i})\, 1_{\{T_i < \zeta\}}) = 0$ for all i. Thus if we let

$$A = \{x \in D : E^x(\phi(B_{T_i})\, 1_{\{T_i<\zeta\}}) = 0 \text{ for all } i\}$$

then A is universally measurable and $\mu(A^c) = 0$.

Let $x \in A$. Then for P^x-a.a. ω the map $t \mapsto \phi(B_t(\omega))$ is continuous on $[0,\zeta(\omega))$. From this it is clear that $\phi(B_{T_i})\, 1_{\{T_i<\zeta\}} \to \phi(B_T) 1_{\{T<\zeta\}}$ P^x-a.s. on $\{T \neq \zeta\}$. This convergence also holds on $\{T=\zeta<\infty\}$ since ϕ vanishes outside a compact subset of D. If $P^x(T=\zeta=\infty) \neq 0$ then we must have $n \geq 3$ (D is a Green region so if $n \leq 2$ then $\mathbb{R}^n \setminus D$ is not polar and $P^x(\zeta<\infty) = 1$) so $||B_t|| \to \infty$ P^x-a.s. . Thus we find that $\phi(B_{T_i})\, 1_{\{T_i<\zeta\}} \to \phi(B_T)\, 1_{\{T<\zeta\}}$ P^x-a.s. on Ω in any case. Applying the Lebesgue dominated convergence theorem we conclude that $E^x(\phi(B_T)\, 1_{\{T<\zeta\}}) = 0$. As this is true for μ-a.a. $x \in D$ we have $E^\mu(\phi(B_T)\, 1_{\{T<\zeta\}}) = 0$. But $\phi \geq 1_{W \cap \{v < d\}}$ so this implies that $\mu_T(W \cap \{v < d\}) = 0$, which is a contradiction. Thus we must have $\mu_T(U) = 0$ after all.

We remark that for the proof of theorem 2.2 it suffices to have the lemma for U of the special form we have just considered.

Now suppose U is a general finely open Borel subset of D. Then $U = \bigcup_{\alpha \in \Sigma} U_\alpha$ where $(U_\alpha)_{\alpha \in \Sigma}$ is a family of finely open sets of the form considered in the first part of the proof. Next, there is a countable set $\Sigma_0 \subseteq \Sigma$ such that $Z \equiv U \setminus \bigcup_{\alpha \in \Sigma_0} U_\alpha$ is a polar set. (See Blumenthal and Getoor [1, p. 203]). By the first part of the proof we have $\mu_T(U_\alpha) = 0$ for all α. Now $\mu_T(Z) = P^\mu(B_T \in Z, T=0)$ since Z is polar. That is, $\mu_T(Z) = \int_Z \mu(dx)\, P^x(T=0)$. Let $S=\inf\{t \geq 0 : B_t \notin U\}$. Then $P^x(S > 0) = 1$ for each $x \in U$ as U is finely open. For each i we have $\int_U \mu(dx)\, P^x(T_i < S)$

$$\leq \int_U \mu(dx)\, P^x(B_{T_i} \in U,\, T_i<\zeta) \leq \mu_{T_i}(U) = 0.$$

Thus for μ-a.a. $x \in U$ we have $P^x(T \geq S) = 1$. Hence $\mu_T(Z) = 0$. This completes the proof of the lemma.

2.9. DOMINATION PRINCIPLE

Let λ be a measure in D such that $G\lambda$ is a potential. Let v be a non-negative superharmonic function in D with Riesz measure $\nu \equiv -\Delta v$. Then the following are equivalent :

a) $\nu \geq G\lambda$ throughout D.

b) $\nu \geq G\lambda$ a.e. $[\lambda]$ and $\nu(Z) \geq \lambda(Z)$ for all Borel polar sets $Z \subseteq D$.

c) $\nu \geq G\lambda$ a.e. $[\lambda]$ and $\nu(Z) \geq \lambda(Z)$ for all Borel sets $Z \subseteq \{v=\infty\}$.

d) $\nu \geq G\lambda$ a.e. $[\lambda]$ and $\nu(Z) \geq \lambda(Z)$ for all Borel sets $Z \subseteq \{G\lambda = \infty\}$

Proof : b) $\Longrightarrow$ c). Any subset of $\{v=\infty\}$ is a polar set.

c) $\Longrightarrow$ d). $\lambda(\{G\lambda=\infty, v<\infty\}) = 0$.

d) $\Longrightarrow$ a). $v = G\nu + h$ where h is a non-negative harmonic function in D. Let $P = \{G\lambda=\infty\}$ and let λ_1, λ_2 be the measures in D defined by $\lambda_1(dx) = \lambda(P^c \cap dx)$, $\lambda_2(dx) = \lambda(P \cap dx)$. Then $\lambda_2 \leq \nu$ so $\nu = \nu_1 + \lambda_2$ for some (unique) measure ν_1 in D. Let $v_1 = G\nu_1 + h$. Then $v_1 \geq G\lambda_1$ a.e. $[\lambda_1]$. Also λ_1 does not charge polar sets. Hence λ_1 lives on $E \equiv \mathrm{reg}(\{v_1 \geq G\lambda_1\}, D)$. Thus we have $v_1 \geq G\lambda_1$ throughout D. This follows from the integral representation of balayage due to Brelot [1].

a) $\Longrightarrow$ b). The only thing to prove is that ν dominates λ on polar sets. For this, see lemma 3.11 below. We remark that we have included the fact that a) $\Longrightarrow$ b) only for completeness ; it is not needed for understanding the rest of the paper.

Proof of theorem 2.2. : Let $\mathcal{V}$ be a countable open base for D consisting of relatively compact subsets of D. Let $\mathcal{G}$ be the weakest topology on D which is stronger than the usual topology on D and which makes $G\nu$ continuous. Let $\mathcal{U}$ be the collection of sets of the form $V \cap \{G\nu < c\}$ where $V \in \mathcal{V}$ and c is a positive rational. Then $\mathcal{U}$ is a base for the topology $\mathcal{G}$ induces on $\{G\nu < \infty\}$. (This assertion is not used below ; indeed we do not explicitly use the topology $\mathcal{G}$, but it gives a perspective on the proof).

Now $\mathcal{U}$ is countable. Let $(U_i)_{i \geq 1}$ be a sequence in $\mathcal{U}$ in which each element of $\mathcal{U}$ occurs infinitely many times. Let $\mathcal{S}$ be the set of all stopping times S such that $G\mu_S \geq G\nu$. For each $i \geq 1$ let $H_i = \inf\{t > 0 : B_t \in U_i\}$. Let $T_0 = 0$ and for $i \geq 1$ let

$$T_i = \begin{cases} T_{i-1} + H_i \circ \theta_{T_{i-1}} & \text{if this stopping time is in } \mathcal{S} \\ T_{i-1} & \text{otherwise.} \end{cases}$$

(Note that $T_{i-1} + H_i \circ \theta_{T_{i-1}} = \inf\{t > T_{i-1} : B_t \notin U_i\}$).

Then (T_i) is an increasing sequence of stopping times in $\mathcal{S}$. Let $T = \lim_{i \to \infty} T_i$.

Then $T \in \mathcal{S}$ by 2.7. Let $\lambda = \mu_T$. We claim $\lambda = \nu$. We know $G\lambda \geq G\nu$ so it suffices to prove $G\nu \geq G\lambda$. By 2.1, λ charges polar sets less than μ. Hence $\lambda(Z) \leq \nu(Z)$ for every Borel set $Z \subseteq \{G\nu = \infty\}$. Thus by 2.9 we have only to show that $G\nu \geq G\lambda$ a.e. $[\lambda]$. Suppose $\lambda(\{G\lambda > G\nu\}) \neq 0$. Then there is a positive rational c such that $\lambda(\{G\lambda > c\} \cap \{G\nu < c\}) \neq 0$. But $\{G\lambda > c\}$ is open in D (as $G\lambda$ is lower semicontinuous) and hence is a countable union of elements of $\mathcal{V}$. Thus there exists $V \in \mathcal{V}$ such that $G\lambda > c$ on V and $\lambda(U) \neq 0$, where $U = V \cap \{G\nu < c\}$. Let $I = \{i \geq 1 : U_i = U\}$. Then I is infinite. Note that for any stopping times R and S, the strong Markov property implies that $(\mu_R)_S = \mu_{R + S \circ \theta_R}$. Suppose $i \in I$. Then

$G\mu_{T_{i-1}} > c > G\nu$ on $U_i = U$ so by 2.6, $G(\mu_{T_{i-1}})_{H_i} \geq G\nu$ in D ; hence

$T_i = T_{i-1} + H_i \circ \theta_{T_{i-1}}$. Therefore by 2.5, $\mu_{T_i}(U) = 0$. But this is true for

arbitrarily large i (as I is infinite) so by 2.8, $\mu_T(U) = 0$. That is, $\lambda(U) = 0$. This is a contradiction so we cannot have $\lambda(G\lambda > G\nu\}) \neq 0$. The theorem is proved.

2.10 EXAMPLE

Let μ be the unit point mass at the origin in $\mathbb{R}^n$. Let (r_j) be a sequence of distinct strictly positive real numbers and for $j = 1,2,\ldots$ let ν_j be the uniform unit distribution on $\{x \in \mathbb{R}^n : ||x|| = r_j\}$. Let $(p_j)_{j\geq 1}$ be a sequence of non-negative real numbers such $\Sigma_{j\geq 1} p_j = 1$ and let $\nu = \Sigma_{j\geq 1} p_j \nu_j$. we shall show that there is a stopping time T such that $\mu_T = \nu$. (When $n \geq 3$, in which case $\mathbb{R}^n$ is a Green region, and $\Sigma_{j\geq 1} p_j/r_j{}^n = \infty$ this result does not follow from theorem 2.2. Thus theorem 2.2 is not best possible.) Our method of proof does not use potential theory at all but instead relies on the following measure-theoretic result:

Lemma : Given a probability space $(\Omega, \mathcal{A}, P)$, a decreasing sequence $(\mathcal{A}_j)_{j\geq 1}$ of sub-σ-fields of $\mathcal{A}$ such that P is non-atomic on each $\mathcal{A}_j$, and a sequence $(p_j)_{j\geq 1}$ of non-negative reals such that $\Sigma_{j\geq 1} p_j = 1$, there exist disjoint $A_j \in \mathcal{A}_j$ with $P(A_j) = p_j$.

This result is taken from Dudley and Gutmann [1]. We cannot resist the temptation to sketch a proof which is much easier than the one they give. If we have chosen disjoint $A_j^k \in \mathcal{A}_j$ with $P(A_j^k) = p_j$ for $j = 1,\ldots,k$ then we can choose any $A_{k+1}^{k+1} \in \mathcal{A}_{k+1}$ with $P(A_{k+1}^{k+1}) = p_{k+1}$ and we can modify $A_k^k, A_{k-1}^k, \ldots, A_1^k$ in turn to obtain $A_k^{k+1}, A_{k-1}^{k+1}, \ldots, A_1^{k+1}$ such that $A_{k+1}^{k+1}, A_k^{k+1}, A_{k-1}^{k+1}, \ldots, A_1^{k+1}$ are disjoint and for $j = 1,\ldots,k$, $A_j^{k+1} \in \mathcal{A}_j$, $P(A_j^{k+1}) = p_j$, and $P(A_j^{k+1} \Delta A_j^k) \leq 2P(A_{k+1}^{k+1}) = 2p_{k+1}$. Then for each j, $(A_j^k)_{k\geq j}$ is a P-Cauchy sequence in $\mathcal{A}_j$, which therefore converges in P-measure to a set $A_j^\circ \in \mathcal{A}_j$. We have $P(A_j^\circ) = p_j$ and for $j_1 \neq j_2$, $P(A_{j_1}^\circ \Delta A_{j_2}^\circ) = 0$. Now let $A_j = A_j^\circ \setminus \cup_{k>j} A_k^\circ$.

Now here is how to use the lemma to construct the stopping time T. Let $\mathcal{F} = \sigma(||B_s|| : 0 \leq s < \infty)$ and for $0 \leq t < \infty$ let $\mathcal{F}_t = \sigma(||B_s|| : 0 \leq s \leq t)$. For each j let $H_j = \inf\{t : ||B_t|| = r_j\}$ (which is an $(\mathcal{F}_t)$-stopping time) and let $\mathcal{A}_j = \mathcal{F}_{H_1 \wedge \ldots \wedge H_j}$. Now $B_{H_1 \wedge \ldots \wedge H_j}$ is $\mathcal{A}_j$-measurable and its P^μ-law is the uniform unit distribution on $\{x \in \mathbb{R}^n : ||x|| = r_1 \wedge \ldots \wedge r_j\}$ so A_j is P^μ-non-atomic. Thus by the lemma there exist disjoint $A_j \in A_j$ such that $P^\mu(A_j) = p_j$. We may assume that $\bigcup_{j \geq 1} A_j = \Omega$. Now define T on Ω by $T = H_j$ on A_j $(j = 1,2,\ldots)$. Since $A_j \in F_{H_j}$ $(\supseteq A_j)$, T is an (F_t)-stopping time. Evidently $P^\mu(||B_T|| = r_j) = p_j$. Now it follows from the spherical symmetry of Brownian motion that if R is any F-measurable random time then $\mathrm{law}(B_R;P^\mu)$ is spherically symmetric. Thus $\mu_T = \nu$.

We reiterate that, as one might have hoped in view of the spherical symmetry of ν, the stopping time T we have constructed is actually a stopping time with respect to the filtration of the process $(||B_t||)$ which is of course smaller than the filtration of (B_t). Let us also point out that T has the property that $||B_T|| = \sup_{0 \leq t < \infty} ||B_{T \wedge t}||$. From this it follows that if ν is special (see section 3) then T is μ-standard (see 4.1). Hence this example also shows that in the case $n = 2$, theorem 4.12 below is not best possible (choose (r_j) and (p_j) so $\Sigma_{j \geq 1} -p_j \log r_j = \infty$).

3. POTENTIAL THEORY IN $\mathbb{R}^1$ AND $\mathbb{R}^2$

To state and prove the analoque of theorem 2.2 for the case in which the Green region D is replaced by $\mathbb{R}^1$ or $\mathbb{R}^2$ we need to develop some potential theory for $\mathbb{R}^1$ and $\mathbb{R}^2$ and we also need to discuss the notion of "standard" stopping times. The standard times are those which are not too big in a certain sense. The detailed discussion of these stopping times will be taken up in the next section. In this section we shall develop the required potential theory.

Suppose that $n = 1$ or 2. Define $\Phi : \mathbb{R}^n \to (\infty,\infty]$ by

$$\Phi(x) = \begin{cases} -\frac{1}{2}|x| & \text{if } n = 1, \\ -\frac{1}{2\pi}\log||x|| & \text{if } n = 2 \text{ and } x \neq 0, \\ \infty & \text{if } n = 2 \text{ and } x = 0. \end{cases}$$

If μ is a measure on $\mathbb{R}^n$ then define U_+^μ , $U_-^\mu : \mathbb{R}^n \to [0,\infty]$ by

$$U_\pm^\mu(x) = \int \Phi^\pm(x-y)\, d\mu(y)$$

and define U^μ , the potential of μ , on the set where U_+^μ and U_-^μ are not both infinite, by

$$U^\mu = U_+^\mu - U_-^\mu \ .$$

If μ is finite on compact sets and U_-^μ is finite at at least one point then we say μ is <u>special</u>. If μ is special then μ is finite, U_-^μ is Lipschitz (in particular U_-^μ is finite everywhere), U^μ is everywhere-defined and superharmonic, and $\mu = -\Delta U^\mu$ in the sense of generalized functions. It is easy to see that: if $n = 1$ then μ is special iff μ is finite and $\int_{\mathbb{R}} |x|\, d\mu(x) < \infty$; if $n = 2$ then μ is special iff μ is finite and $\int_{\mathbb{R}^2} \log^+||x||\, d\mu(x) < \infty$.

<u>3.1. LEMMA</u>

Let μ be a non-zero measure on $\mathbb{R}$ and let ξ be the centre of mass of μ . Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \mu(\mathbb{R})\Phi(x-\xi) - U^\mu$. Then $f \geq 0$, f is increasing on $(-\infty,\xi]$, f is decreasing on $[\xi,\infty)$, and $f(x) \to 0$ as $|x| \to \infty$.

The proof of this lemma is a simple computation. The two corollaries below follow immediately.

<u>3.2. COROLLARY</u>

Let μ and ν be special measures on $\mathbb{R}$ and let c be a real number. Suppose $U^\nu + c \geq U^\mu$. Then:

a) $\nu(\mathbb{R}) \leq \mu(\mathbb{R})$

b) if $\nu(\mathbb{R}) = \mu(\mathbb{R})$ then $c \geq 0$ and μ and ν have the same centre of mass.

3.3. COROLLARY

Let μ be the unit point mass at $\xi \in \mathbf{R}$ and let ν be a special probability measure on $\mathbf{R}$. Then $U^\mu \geq U^\nu$ iff the centre of mass of ν is equal to ξ.

3.4. DOMINATION PRINCIPLE FOR $\mathbf{R}$:

Let μ be a non-zero special measure on $\mathbf{R}$ and let v be a superharmonic function on $\mathbf{R}$ such that :

a) $v \geq U^\mu$ a.e. $[\mu]$

b) $\liminf_{|x| \to \infty} [v(x) - \mu(\mathbf{R})\Phi(x)] > -\infty$.

Then $v \geq U^\mu$ everywhere on $\mathbf{R}$.

Proof : A function on $\mathbf{R}$ is superharmonic iff it is finite and concave. Note that such a function is automatically continuous. Let $E = \{v \geq U^\mu\}$ and let $W = E^c$. Then W is open. Also $\mu(W) = 0$ so U^μ is harmonic in W. Since we are in dimension one, this just amounts to saying that on each component of W the graph of U^μ is a straight line. By the continuity of U^μ, this actually holds on the closure of each component of W. Suppose $p \in W$. We shall show that $v(p) \geq U^\mu(p)$. (Whence W is actually empty). Let C be the component of W containing p. Then $C = (a,b)$ where $a \in E \cup \{-\infty\}$, $b \in E \cup \{\infty\}$, and $a < p < b$. Also $E \neq \emptyset$ since $\mu \neq 0$, so at least one of a and b is finite.

Case 1 :

a and b both finite. Then $v(a) \geq U^\mu(a)$ and $v(b) \geq U^\mu(b)$. Also v is concave and U^μ is a straight-line function on $[a,b]$. Hence $v \geq U^\mu$ on $[a,b]$. In particular $v(p) \geq U^\mu(p)$.

Case 2 :

$a = -\infty$, $b \in E$. By b) and 3.1 we have $\liminf_{x \to -\infty} [v(x) - U^\mu(x)] > -\infty$. Hence $v - U^\mu$ is bounded below on $(-\infty,b]$. Now for each $x \in (-\infty,p]$ there is a unique number $c(x) \in [0,1]$ such that $p = [1-c(x)]x + c(x)b$. As $x \to -\infty$, $c(x) \to 1$.

Now

$$v(p) \geq [1-c(x)]v(x) + c(x)\, v(b)$$

$$\geq [1-c(x)]\ [v(x)-U^\mu(x)] + [1-c(x)]\ U^\mu(x) + c(x)\ U^\mu(b)$$

$$= [1-c(x)]\ [v(x) - U^\mu(x)] + U^\mu(p).$$

Letting $x \to -\infty$ we obtain $v(p) \geq U^\mu(p)$.

Case 3 :

$a \in E$, $b=\infty$. Similar to case 2.

3.5. COROLLARY

Let μ and ν be special measures on $\mathbb{R}$ and let c be a real number. Then $U +c \geq U^\mu$ on all of $\mathbb{R}$ iff $U^\nu+c \geq U^\mu$ a.e. $[\mu]$ and $\nu(\mathbb{R}) \leq \mu(\mathbb{R})$ and either $\mu \neq \nu$ or $c \geq 0$.

Proof : Combine 3.1, 3.2, and 3.4.

3.6. LEMMA

Let μ be a special measure on $\mathbb{R}^2$. Then :

a) $\lim\limits_{||x||\to\infty} [U^\mu_-(x) - \mu(\mathbb{R}^2)\ \Phi^-(x)] = 0.$

b) $\liminf\limits_{||x||\to\infty} [U^\mu(x) - \mu(\mathbb{R}^2)\ \Phi(x)] \geq 0.$

c) If μ has compact support,

$$\lim_{||x||\to\infty} [U^\mu(x) - \mu(\mathbb{R}^2)\ \Phi(x)] = 0.$$

Proof :

a). $U^\mu_-(x) - \mu(\mathbb{R}^2)\Phi^-(x) = \int \Phi^-(x-y) - \Phi^-(x)\ d\mu(y)$.

Now if $||x|| \geq 1$ and $||x-y|| \geq 1$ then

$$\Phi^-(x-y) - \Phi^-(y) = \frac{1}{2\pi}\log\frac{||x-y||}{||x||}\ .$$

Thus for each $y \in \mathbb{R}^2$, $\Phi^-(x-y) - \Phi^-(x) \to 0$ as $||x|| \to \infty$. Now $\int \log^+||x||\, d\mu(x) < \infty$ since μ is special. Hence the desired result follows from the Lebesgue dominated convergence theorem in view of the following estimate.

Claim : $|\Phi^-(x-y) - \Phi^-(x)| \leq \frac{1}{2\pi} (\log 2 + \log^+||y||)$.

First note that for all $r \geq 0$, $\log(1+r) \leq \log 2 + \log^+ r$. (Consider the two cases $0 \leq r \leq 1$ and $r \geq 1$). Now for the proof of the claim consider the four cases $||x-y|| \geq ||x|| \geq 1$, $||x|| \geq ||x-y|| \geq 1$, $||x-y|| \leq 1$, and $||x|| \leq 1$.

b). This is evident.

c). As μ has compact support, so does U^μ_+.

3.7. LEMMA

Let μ be a special measure on $\mathbb{R}^2$. For each positive real number r let D_r be the open disc of radius r centred at 0 in $\mathbb{R}^2$ and let μ_r, μ_r' be the measures on $\mathbb{R}^2$ defined by $\mu_r(dx) = \mu(D_r \cap dx)$, $\mu_r'(dx) = \mu(D_r^c \cap dx)$. Then :

a) For all $x \in \mathbb{R}^2$ and all $r > 0$,
$U^{\mu_r'}(x) \geq U^{\mu_r'}(0) - \frac{1}{2\pi}\mu(D_r^c) \log(1 + \frac{||x||}{r})$.

b) For all $\varepsilon > 0$ and all $k \geq 0$ there exists $r_0 \in (0,\infty)$ such that for $r_0 \leq r < \infty$ we have $U^\mu + \varepsilon \geq U^{\mu_r}$ on $\bar{D}_{kr}$.

Proof :

a). This follows from the estimate
$$\Phi(x-y) \geq \Phi(y) - \frac{1}{2\pi} \log(1 + \frac{||x||}{r}),$$
which holds for $x \in \mathbb{R}^2$ and $y \in D_r^c$.

b). Choose $r_0 \in [1,\infty)$ such that
$$\mu(D_{r_0}^c)\log(1+k) + \int_{D_{r_0}^c} \log||y||\, d\mu(y) \leq 2\pi\varepsilon$$

Then for any $r \geq r_0$ we have $U^{\mu'_r} \geq -\varepsilon$ on $\overline{D}_{kr}$, by a). The lemma is proved.

3.8. LEMMA

Let A be a Borel non-polar subset of $\mathbb{R}^2$. Then there is a non-zero special measure λ on $\mathbb{R}^2$ such that $\lambda(A^c) = 0$ and U^λ is bounded above on $\mathbb{R}^2$.

Proof : There is some open ball D in $\mathbb{R}^2$ such that $D \cap A$ is not polar. By the capacitability theorem, $D \cap A$ contains a compact set K which is not polar. Let u be the capacitary potential of K relative to D and let $\lambda \equiv -\Delta u$ be the Riesz measure of u. Extend λ to a measure on $\mathbb{R}^2$ by setting $\lambda(dx) = \lambda(D \cap dx)$. Then $\lambda \neq 0$ as K is not polar. Now there is a harmonic function h in D such that $U^\lambda = u+h$ in D. We have $u \leq 1$ in D, h continuous in D, U^λ continuous in K^c, and (by 3.6) $U^\lambda(x) \to -\infty$ as $||x|| \to \infty$. Hence U^λ is bounded above.

3.9. DOMINATION PRINCIPLE FOR $\mathbb{R}^2$

Let μ and ν be special measures on $\mathbb{R}^2$.

Let c be a real number. Then $U^\nu + c \geq U^\mu$ on $\mathbb{R}^2$ iff the following four conditions hold :

a) $U^\nu + c \geq U^\mu$ a.e. $[\mu]$

b) $\nu(Z) \geq \mu(Z)$ for every Borel polar set $Z \subseteq \mathbb{R}^2$.

c) $\nu(\mathbb{R}^2) \leq \mu(\mathbb{R}^2)$

d) $\mu \neq \nu$ or $c \geq 0$.

(We remark that just as in 2.9, b) can be replaced by either of the two weaker conditions :

b_1) $\nu(Z) \geq \mu(Z)$ for every Borel set $Z \subseteq \{U^\nu = \infty\}$.

b_2) $\nu(Z) \geq \mu(Z)$ for every Borel set $Z \subseteq \{U^\mu = \infty\}$).

Proof : Let us remark that for the purpose of embedding in Brownian motion in $\mathbb{R}^2$ we need only the implication ($\Longleftarrow$) of this theorem. We state the theorem in "if and only if" form for completeness. As to the proof of ($\Longrightarrow$), it is obvious that a) and d) follow from the assumption that $U^\nu+c \geq U^\mu$ everywhere. The reader who wishes to see that b) and c) also follow is referred to lemmas 3.10 and 3.11 below. These lemmas are not needed for understanding the rest of the paper.

Now let us prove ($\Longleftarrow$). We proceed by reducing to the case of a Green region. Clearly we need only consider the case in which $\mu\neq\nu$. Then from b) and c) we can conclude that there is no polar set which carries μ. Hence μ must charge the set $A\equiv\{U^\nu+c \geq U^\mu\} \cap \{U^\mu < \infty\}$. But then A is not a polar set since μ does not charge polar subsets of $\{U^\mu<\infty\}$. Choose $\varepsilon > 0$. Then by 3.8there is a non-zero special measure λ on $\mathbb{R}^2$ such that $U^\lambda \leq \frac{\varepsilon}{2}$ on $\mathbb{R}^2$ and $\lambda(A^c) = 0$. We may suppose λ has compact support. For each $r > 0$ let μ_r and D_r be as in the statement of lemma 3.7. By b) of that lemma, there exists $r_0 \in (0,\infty)$ such that $U^\mu + \frac{\varepsilon}{2} \geq U^{\mu_r}$ on $\overline{D}_r$ for all $r \geq r_0$. Choose $\alpha \in (0,1]$. Then $U^{\alpha\lambda} = \alpha U^\lambda \leq \frac{\varepsilon}{2}$ on $\mathbb{R}^2$. Thus for all r in $[r_0,\infty)$ we have $U^\mu + \varepsilon \geq U^{\mu_{r,\alpha}}$ on $\overline{D}_r$, where $\mu_{r,\alpha} = \mu_r + \alpha\lambda$. As $\alpha\lambda \neq 0$, there exists $r_1 \in [r_0,\infty)$ such that $\mu_{r_1,\alpha}(\mathbb{R}^2) > \mu(\mathbb{R}^2)$. Then for all $r \in [r_1,\infty)$ we have $\liminf\limits_{||x||\to\infty} [v(x) - \mu_{r,\alpha}(\mathbb{R}^2)\, \Phi(x)] = +\infty$, where $v=U^\nu+c$.

We may suppose r_1 is chosen so that $\text{supp}(\gamma) \subseteq D_{r_i}$.

Choose $r_2 \in [r_1,\infty)$ and let $\gamma = \mu_{r_2,\alpha}$. Then there exists $r_3 \in [r_2,\infty)$ such that on $\mathbb{R}^2 \setminus D_{r_3}$ we have $v \geq \gamma(\mathbb{R}^2)\Phi$ and $\gamma(\mathbb{R}^2)\Phi + \varepsilon \geq U^\gamma$ where the second estimate follows from 3.6 (c) because γ has compact support. Now choose $r_4 \in [r_3,\infty)$ and consider the Green region $D \equiv D_{r_4}$. Since the support of γ is a compact subset of D, U^γ is finite and continuous on ∂D. Let h be the unique continuous function on $\bar{D}$ which is harmonic in D and which agrees with U^{γ} on ∂D. Then $U^{\gamma}-h = G\gamma$ in D where G is the Green function of D. Now $v+\varepsilon-h$ is lower semicontinuous on $\bar{D}$, superharmonic in D, and non-negative on ∂D. Hence $v+\varepsilon-h$ is non-negative in D by the minimum principle. Next observe that $v+\varepsilon-h \geq U^{\mu}+\varepsilon-h$ on $\{v \geq U^{\mu}\} \cap D$, $U^{\mu}+\varepsilon-h \geq G\gamma$ in D, γ lives on $\{v \geq U^{\mu}\} \cap D$, and γ charges polar sets no more that μ does and so certainly no more than ν does. Thus $v+\varepsilon-h \geq G\gamma$ throughout D by the domination principle for a Green region (see 2.9). That is, $v+\varepsilon \geq U^{\gamma}$ in D_{r_4}. As $r_4 \in [r_3,\infty)$ was arbitrary, $v+\varepsilon \geq U^{\mu_{r_2,\alpha}}$ on all of $\mathbb{R}^2$. Now this holds for all $r_2 \in [r_1,\infty)$. Letting $r_2 \to \infty$ we obtain $v+\varepsilon \geq U^{\mu+\alpha\lambda}$, which holds for all $\alpha \in (0,1]$. Now letting $\alpha \to 0$ we get $v+\varepsilon \geq U^{\mu}$. This holds for all $\varepsilon > 0$. Hence $v \geq U^{\mu}$. The theorem is proved.

3.10 LEMMA

Let μ and ν be special measures on $\mathbb{R}^2$ and let c be a real number. Suppose $U^{\nu}+c \geq U^{\mu}$. Then :

a) $\nu(\mathbb{R}^2) \leq \mu(\mathbb{R}^2)$.

b) If $\nu(\mathbb{R}^2) = \mu(\mathbb{R}^2)$ then $c \geq 0$.

Proof : Let γ be the uniform unit distribution on $\{x \in \mathbb{R}^2 : ||x|| = 1\}$. Then $U^{\gamma} = -\phi^{-}$.

Let $\alpha = \mu*\gamma$, $\beta = \nu*\gamma$. Then $-U^{\mu}_{-} + c = U^{\gamma} * \mu + c = U^{\mu} * \gamma + c$

$$= (U^{\mu}+c) * \gamma \geq U^{\nu}*\gamma = - U^{\nu}_{-} .$$

The proof may be concluded by applying 3.6 (a).

3.11. LEMMA

Let D be an open subset of $\mathbb{R}^n$ and let u and v be superharmonic functions in D with Riesz measures $\mu = -\Delta u$ and $\nu = -\Delta v$ respectively. Suppose $v \geq u$. Then $\nu(Z) \geq \mu(Z)$ for every Borel polar set $Z \subseteq D$.

Proof : It suffices to consider Z's which are relatively compact in D. Then $Z \subseteq W \subseteq D'$ where D' is open and relatively compact in D and W is open and relatively compact in D'. Then u and v are bounded below in D', say by c, and if we let $u'=\mathrm{bal}((u-c)|D',W)$ and $v'=\mathrm{bal}((v-c)|D',W)$ then u' and v' are potentials in D' whose Riesz measures agree in W with μ and ν respectively. Thus we see that we may suppose that D is a Green region and that $u = G\mu$ and $v = G\nu$, where G is the Green function of D. Then since $G\nu \geq G\mu$, there is a randomized stopping time τ such that $\mu = \nu_\tau$, by Rost [1]. But then $\mu(Z) = \int_Z$ "$P^z(\tau=0)$" $d\nu(z)$ since Z is polar, so $\mu(Z) \leq \nu(Z)$.

We remark that it is also possible to give a proof of this lemma which uses only classical potential theory.

We conclude this section with a convergence theorem. First we need a definition.

3.12. DEFINITION

A measure γ on $\mathbf{R}^n$ will be called good iff γ has compact support and U^γ is continuous and finite.

Observe that if ϕ is a bounded compactly supported Borel function on $\mathbf{R}^n$ and $\gamma(dx) = \phi(x)dx$ then γ is good. Hence if u and v are superharmonic functions on $\mathbf{R}^n$ such that $\int u d\gamma \geq \int v d\gamma$ for all good measures γ on $\mathbf{R}^n$ then $u \geq v$ everywhere on $\mathbf{R}^n$. If $n=1$ the point masses are good measures, but not if $n=2$.

3.13. THEOREM

Let μ be a measure on $\mathbf{R}^n$ (where $n=1$ or 2) which is finite on compact sets and let $(\mu_i)_{i \in I}$ be a net of special measures on $\mathbf{R}^n$ such that $\int \phi \, d\mu_i \to \int \phi \, d\mu$ for all compactly supported continuous functions ϕ on $\mathbf{R}^n$. Then :

a) $\int U_+^{\mu_i} d\gamma \to \int U_+^{\mu} d\gamma$ for all good measures γ on $\mathbf{R}^n$

Now suppose also that the net $(\int U_-^{\mu_i} d\nu)$ converges to a finite limit for some non-zero special measure ν on $\mathbf{R}^n$. Then :

b) μ is special.

c) The net $(U_-^{\mu_i})$ converges uniformly on compact sets to $U_-^{\mu} + C$, where C is some finite non-negative constant.

d) $\int U^{\mu_i}\, d\gamma \to \int U^{\mu} - C\, d\gamma$ for all good measures γ on $\mathbf{R}^n$.

Proof : For any good measure γ , U_+^{γ} is compactly supported and continuous. Part a) follows immediately from this, upon interchanging orders of integration. Now suppose ν is an in the statement of the theorem. Then for some i_0, $\sup_{i \geq i_0} \int U_-^{\nu}\, d\mu_i < \infty$. Hence

$$\lim_{r \to \infty} \Big[\sup_{i \geq i_0} \mu_i(\{x \in \mathbf{R}^n : ||x|| \geq r\}) \Big] = 0.$$

Also, as $\mu_i \to \mu$ vaguely, for each compact set $K \subseteq \mathbf{R}^n$ there exists i_0' such that $\sup_{i \geq i_0'} \mu_i(K) < \infty$. Combining these two observations we find that $\sup_{i \geq i_1} \mu_i(\mathbf{R}^n) < \infty$ for some $i_1 \geq i_0$, $\mu(\mathbf{R}^n) < \infty$, and $\int f\, d\mu_i \to \int f d\mu$ for all bounded continuous functions f on $\mathbf{R}^n$. Next, it is easy to show that :

$$(*) \qquad U_-^{\mu} \leq \liminf U_-^{\mu_i}.$$

Thus by Fatou's lemma, $\int U^{\mu}\, d\nu \leq \lim \int U^{\mu_i}\, d\nu$. Hence $\int U_-^{\mu}\, d\nu < \infty$. Therefore, as $\nu \neq 0$, U_-^{μ} is not identically infinite. Hence U_-^{μ} is finite everywhere and μ is special. Also, for any $x \in \mathbf{R}^n$ the function $y \mapsto U_-^{\nu}(y) - \nu(\mathbf{R}^n)\Phi^-(x-y)$ is bounded and continuous. It follows that the net $(U_-^{\mu_i}(x))$ converges to a finite limit $u_-(x)$ for each x in $\mathbf{R}^n$. But $|U_-^{\mu_i}(x) - U_-^{\mu_i}(y)| \leq \mu_i(\mathbf{R}^n)||x-y||$ so $\{U_-^{\mu_i} : i \geq i_1\}$ is equicontinuous. Hence u_- is continuous and $U_-^{\mu_i} \to u_-$ uniformly on compact sets. Now for any $x \in \mathbb{R}^n$ the function $y \mapsto \Phi^-(x-y) - \Phi^-(y)$ is bounded and continuous so $[U_-^\mu(x) - U_-^{\mu_i}(x)] - [U_-^\mu(0) - U_-^{\mu_i}(0)]$

$$= \int \Phi^-(x-y) - \Phi^-(y) d\mu(y) - \int \Phi^-(x-y) - \Phi^-(y)\, d\mu_i(y)$$

$$\to 0.$$

Thus $u_- = U_-^\mu + C$ where C is some constant. As $U_-^\mu \leq u_-$, C is non-negative.

Remark : The constant C need not be zero. For instance on $\mathbb{R}^1$, let μ_i be the point mass at $2i$ of total mass $1/i$ for $i=1,2,3,\ldots$. Then (μ_i) converges vaguely to 0 but $U^{\mu_i} \to -1$. In this case, $C=1$. The next result gives a useful condition under which C will be 0.

3.14. COROLLARY

Let μ be a measure on $\mathbb{R}^n$ (where $n=1$ or 2) which is finite on compact sets and let $(\mu_i)_{i \in I}$ be a net of special measures on $\mathbb{R}^n$ such that $\int \phi d\mu_i \to \int \phi d\mu$ for all compactly supported continuous functions ϕ on $\mathbb{R}^n$. Suppose there is a special measure α on $\mathbb{R}^n$ such that $U^{\mu_i} \geq U^\alpha$ and $\mu_i(\mathbb{R}^n) = \alpha(\mathbb{R}^n)$ for all i. Then μ is special, $(U_-^{\mu_i})$ converges uniformly on compact sets to U_-^μ, and $\int U^{\mu_i} d\gamma \to \int U^\mu\, d\gamma$ for all good measures γ on $\mathbb{R}^n$.

Proof : It suffices to show that every subnet of (μ_i) has a further subnet for which the conclusions of the corollary hold. But if $n=2$ then $U_-^{\mu_i} = -U^{\mu_i} * \sigma \leq -U^\alpha * \sigma = U_-^\alpha$, where σ is the uniform unit distribution on $\{x \in \mathbb{R}^n : ||x|| = 1\}$. Thus whatever n is, the net $(U_-^{\mu_i}(0))$ is bounded. Thus we may reduce to the case in which this net converges to a finite limit. But then by the theorem, with ν = the unit point mass at 0, μ is special and there is a constant

$C \in [0,\infty)$ such that $U_-^{\mu_i} \to U_-^{\mu} + C$ uniformly on compact sets and $\int U^{\mu_i}\, d\gamma \to \int U^{\mu}\, d\gamma$ for all good measures γ on $\mathbb{R}^n$. Then $U_-^{\mu} + C \leq U_-^{\mu}$.

Also $\mu(\mathbb{R}^2) = \alpha(\mathbb{R}^2)$. Now apply 3.6 (a), if $n=2$, or 3.2, if $n=1$, to see that C must be zero.

4. THE CASE OF BROWNIAN MOTION IN $\mathbb{R}^1$ OR $\mathbb{R}^2$

In this section, if μ is a measure on $\mathbb{R}^n$ and T is a stopping time then μ_T will denote the measure on $\mathbb{R}^n$ defined by $\mu_T(dx) = P^{\mu}(B_T \in dx)$. If $n \geq 3$, we may think of μ_T as being obtained from μ by letting μ "spread out" under Brownian motion up to time T. The measure μ_T is more spread out than μ in the sense that its electrostatic potential is lower. As soon as $n \leq 2$ this need no longer be true. For example let $E = \{x \in \mathbb{R}^n : ||x|| = 2\}$, let $F = \{x \in \mathbb{R}^n : ||x|| = 1\}$, let μ be the uniform unit distribution on E, and let $T = \inf\{t \geq 0 : B_t \in F\}$. If $n \geq 3$ then $P^{\mu}(T=\infty) > 0$ and the mass of μ_T is sufficiently smaller than that of μ so that the potential of μ_T is $\leq$ that of μ. If $n=1$ or 2 though, then any non-polar set is hit in finite time with probability one and so μ_T is the uniform unit distribution on F, whence $U^{\mu} < U^{\mu_T}$ on $\{x \in \mathbb{R}^n : ||x|| < 2\}$.

Here is another example, due to Doob. Let $n=1$. Let μ and ν be two arbitrary probability measures on $\mathbb{R}$. Let $\rho = \text{law}(B_1 ; P^{\mu})$. Then ρ is μ convolved with a Gaussian, so ρ has no atoms. Hence there is a Borel function $f : \mathbb{R} \to \mathbb{R}$ such that $\nu = \text{law}(f ;\rho)$. Let $T = \inf\{t \geq 1 : B_t = f(B_1)\}$. Then T is a stopping time and, since the paths of (B_t) are continuous and unbounded above and below P^{μ}- a.s., $B_T = f(B_1)$ P^{μ}-a.s. . In particular $\text{law}(B_T ; P^{\mu}) = \text{law}(f(B_1);P^{\mu})=\nu$. Thus in the one dimensional case the possibilities for μ_T are unrestricted if no restriction is placed on T !

If, however, we restrict our attention to small enough stopping times then examples of the sort described above do not occur. Traditionally the stopping times considered small enough have been those with finite expectation; see Skorohod [1], Dubins [1], Root [1], and Baxter and Chacon [1]. This is too stringent a restriction though since it implies that μ_T has a finite variance if μ does. What we want is a theorem analagous to 2.2, which would say that if μ and ν are special measures with $U^\mu \geq U^\nu$ and μ and ν well-related on polar sets then $\nu = \mu_T$ for some "small enough" stopping time T. For us, the stopping times which are small enough are the ones we call standard following Chacon [1].

4.1. DEFINITION

Let μ be a special measure on $\mathbb{R}^n$, where $n = 1$ or 2. Let T be a stopping time. We shall say T is μ-standard iff whenever R and S are stopping times with $R \leq S \leq T$ then μ_R and μ_S are special and $U^{\mu_R} \geq U^{\mu_S}$.

Remarks : From 3.2 (if $n = 1$) or 3.10 (if $n = 2$) we see that if T is μ-standard then $P^\mu(T = \infty) = 0$. Also note that if T is μ-standard then any stopping time smaller than T is also μ-standard.

4.2. LEMMA

Let μ be a special measure on $\mathbb{R}^n$, where $n = 1$ or 2, and let T be a bounded stopping time. Then T is μ-standard and

$$E^\mu\left[\int_0^T f(B_t)\, dt \right] = \int \left[U^\mu(x) - U^{\mu_T}(x) \right] f(x)\, dx$$

for any non-negative Borel function f on $\mathbb{R}^n$.

(Remark : We are working with Brownian motion normalized so that $E^x(||B_t-x||^2) = 2nt$).

Proof : First let f be a non-negative compactly supported C^2 function on $\mathbb{R}^n$ and let $u=U^\gamma$, where $\gamma(dx) = f(x)dx$. Then u is C^2 and u and its partials up to order 2 don't grow too fast at infinity so by Dynkin's formula the process

$$M_t \equiv u(B_t) - \int_0^t \Delta u(B_s)ds \quad (0 \leq t < \infty)$$

is a martingale over $(\Omega, \mathcal{B}, \mathcal{B}_t, P^x)$ for every $x \in \mathbb{R}^n$. Now $\Delta u = -f$ so $\int_0^t \Delta u(B_s)ds$ is bounded. Also $E^\mu(|u(B_t)|) < \infty$ as μ is special. Thus (M_t) is a martingale over $(\Omega, \mathcal{B}, \mathcal{B}_t, P^\mu)$. Also the sample paths of (M_t) are continuous. Thus for any bounded stopping time R, M_R is P^μ-integrable and $E^\mu(M_R) = E^\mu(M_0)$.

Thus $\int |u| \, d\mu_R = E^\mu(|u(B_R)|) < \infty$ (so μ_R is special - if we take $\int f(x)dx = 1$ then $u + \Phi^-$ is bounded) and

$$E^\mu\left[\int_0^R f(B_s)ds\right] = E^\mu(u(B_0)) - E^\mu(u(B_R))$$

$$= \int u \, d\mu - \int u \, d\mu_R$$

$$= \int \left[U^\mu(x) - U^{\mu_R}(x)\right] f(x)dx.$$

Now by a monotone class argument one can show that this equality holds if f is any non-negative Borel function.

From this we see that if R and S are stopping times with $R \leq S \leq T$ then μ_R and μ_S are special and for any non-negative Borel function f on $\mathbb{R}^n$,

$$\int \left[U^{\mu_R}(x) - U^{\mu_S}(x)\right] f(x)dx = E^\mu\left[\int_R^S f(B_s)ds\right] \geq 0 \ ;$$

hence $U^{\mu_R} \geq U^{\mu_S}$. Thus T is μ-standard.

4.3. LEMMA

Let M be a family of special measures on $\mathbb{R}^n$, where n=1 or 2. Suppose $\sup_{\mu \in M} \mu(\mathbb{R}^n) < \infty$ and $\{x \in \mathbb{R}^n : \inf_{\mu \in M} U^\mu(x) > -\infty\}$ is not polar. Then $\lim_{r\to\infty} \sup_{\mu \in M} \mu(\{x \in \mathbb{R}^n : ||x|| \geq r\}) = 0$.

Proof : Suppose not. Then there is a countable set $N \subseteq M$ such that $\lim_{r\to\infty} \sup_{\mu\in N} \mu(\{x \in \mathbb{R}^n : ||x|| \geq r\}) > 0$. Let $A_k = \{x \in \mathbb{R}^n : \inf_{\mu \in N} U^\mu(x) \geq -k\}$, for positive integers k. Then each A_k is Borel and for some k_0, A_{k_0} is not polar. But then by 3.8, there is a non-zero special measure λ on $\mathbb{R}^n$ such that U^λ is bounded above and $\lambda(A^c_{k_0}) = 0$. Then for each $\mu \in N$, $\int U^\lambda d\mu = \int U^\mu \, d\lambda \geq - k_0 \lambda(A_{k_0})$.

$$\text{Thus} \sup_{\mu\in N} \int (U^\lambda)^- \, d\mu \leq k_0 \lambda(A_{k_0}) + \sup_{\mu \in N} \int (U^\lambda)^+ \, d\mu < \infty$$

Now we may suppose λ has compact support. Then $(U^\lambda)^-(x) \to +\infty$ as $||x|| \to \infty$ so we conclude that $\lim_{r\to\infty} \sup_{\mu \in N} \mu(\{x \in \mathbb{R}^n : ||x|| \geq r\ ) = 0$ after all.

4.4. LEMMA

Let μ be a finite measure on $\mathbb{R}^n$.

Let T be a collection of stopping times such that if $T \in T$ and S is a stopping time satisfying $S \leq T$ then $S \in T$. Suppose $\lim_{r\to\infty} \sup_{T \in T} \mu_T(\{x \in \mathbb{R}^n : ||x|| \geq r\}) = 0$. Then $\lim_{t\to\infty} \sup_{T\in T} P^\mu(T \geq t) = 0$.

Proof : For i=1,2,... let $R_i = \inf\{t \geq 0 : ||B_t|| \geq i\}$. Then $P^\mu(R_i = \infty) = 0$. If $T \in T$ then $T \wedge R_i \in T$ and $P^\mu(T \geq R_i) = \mu_{T \wedge R_i}(\{x \in \mathbb{R}^n : || x|| \geq i\})$.

Thus $\lim_{i\to\infty} \sup_{T\in \mathcal{T}} P^{\mu}(T \geq R_i) = 0$. Now fix $\varepsilon > 0$. Then for some i, $\sup_{T\in\mathcal{T}} P^{\mu}(T \geq R_i) \leq \frac{\varepsilon}{2}$. Next for some $t \in [0,\infty)$, $P^{\mu}(R_i \geq t) \leq \frac{\varepsilon}{2}$. Then $P^{\mu}(T \geq t) \leq \varepsilon$ for all $T \in \mathcal{T}$.

4.5. PROPOSITION

Let μ be a special measure on $\mathbf{R}^n$, where $n=1$ or 2. Let Γ be a set of good measures on $\mathbf{R}^n$ such that whenever α and β are good measures on $\mathbf{R}^n$ such that $\int U^{\alpha}\, d\gamma \geq \int U^{\beta}\, d\gamma$ for all $\gamma \in \Gamma$, then $U^{\alpha} \geq U^{\beta}$. Suppose (T_i) is a sequence of μ-standard stopping times converging pointwise on Ω to a stopping time T. Consider the following statements :

a) There is a special measure σ on $\mathbf{R}^n$ such that $\sigma(\mathbf{R}^n) = \mu(\mathbf{R}^n)$ and $U^{\mu_{T_i}} \geq U^{\sigma}$ for all i.

b) μ_T is special and $\int U^{\mu_{T_i}}\, d\gamma \to \int U^{\mu_T}\, d\gamma$ for all $\gamma \in \Gamma$.

c) T is μ-standard.

Then a) $\Longrightarrow$ b) $\Longrightarrow$ c).

Proof : a) $\Longrightarrow$ b). Let $\mathcal{H}$ be the set of stopping times H such that $H \leq T_i$ for some i. Then for all $H \in \mathcal{H}$, μ_H is special and $U^{\mu_H} \geq U^{\sigma}$, since each T_i is μ-standard. Therefore

$\lim_{r\to\infty} \sup_{H\in\mathcal{H}} \mu_H(\{x \in \mathbf{R}^n : ||x|| \geq r\}) = 0$, by 4.3. Hence

$\lim_{t\to\infty} \sup_{i} P^{\mu}(T_i \geq t) = 0$, by 4.4. Therefore T is P^{μ}-a.s. finite so for each bounded continuous function ϕ on $\mathbf{R}^n$, $\int \phi\, d\mu_{T_i} \to \int \phi\, d\mu_T$, since (B_t) is continuous on $[0,\infty)$. The statement b) now follows from 3.14.

b) $\Longrightarrow$ c). If Q is any bounded stopping time then $U^{\mu_{T_i \wedge Q}} \geq U^{\mu_Q}$ for all i, so by the argument of a) $\Longrightarrow$ b) we find that $\int U^{\mu_{T_i \wedge Q}} d\gamma \to \int U^{\mu_{T \wedge Q}} d\gamma$ for all $\gamma \in \Gamma$. Now for each i, $U^{\mu_{T_i \wedge Q}} \geq U^{\mu_{T_i}}$ as T_i is μ-standard. Since $\int U^{\mu_{T_i}} d\gamma \to \int U^{\mu_T} d\gamma$ for all $\gamma \in \Gamma$, it follows that $\int U^{\mu_{T \wedge Q}} d\gamma \geq \int U^{\mu_T} d\gamma$ for all $\gamma \in \Gamma$, whence $U^{\mu_{T \wedge Q}} \geq U^{\mu_T}$. In particular, if R and S are stopping times satisfying $R \leq S$ then $U^{\mu_{T \wedge R \wedge t}} \geq U^{\mu_{T \wedge S \wedge t}} \geq U^{\mu_T}$. (Also $U^{\mu} \geq U^{\mu_T}$ so $\mu(\mathbf{R}^n) \leq \mu_T(\mathbf{R}^n)$, whence $P^{\mu}(T=\infty) = 0$). If in addition $S \leq T$, then $U^{\mu_{R \wedge t}} \geq U^{\mu_{S \wedge t}} \geq U^{\mu_T}$. Letting $t \to \infty$ and applying 3.14 we obtain $U^{\mu_R} \geq U^{\mu_S}$. Thus T is μ-standard.

4.6. COROLLARY

Let μ be a special measure on $\mathbf{R}^n$, where $n=1$ or 2, and let T be a stopping time. Then T is μ-standard iff μ_T is special and $U^{\mu_{T \wedge t}} \geq U^{\mu_T}$ for all $t \in [0,\infty)$.

4.7. COROLLARY

Let μ be a special measure on $\mathbf{R}^n$, where $n=1$ or 2, and let T be a stopping time. Let m denote Lebesgue measure on $\mathbf{R}^n$. Then :

a) For any finite measure ν on $\mathbf{R}^n$, $U^{\mu} - U^{\nu}$ is defined a.e. $[m]$ and its m-integral over any compact subset of $\mathbf{R}^n$ makes sense, though it may be $+\infty$.

b) T is μ-standard iff for each compact subset $K \subseteq \mathbf{R}^n$, $\int_K U^{\mu} - U^{\mu_T} dm$ is finite and equal to $E^{\mu}[\int_0^T 1_K(B_s)ds]$

Proof :

a) is trivial

b) Combine 3.14, 4.2, and 4.6.

Remark : In order that T be μ-standard it is not enough that $E^{\mu}[\int_0^T 1_K(B_s)ds]$ be finite for each compact set $K \subseteq \mathbf{R}^n$. An example showing this is furnished by taking μ to be the uniform unit distribution on $\{x \in \mathbf{R}^n : ||x|| = 2\}$ and T to be $\inf\{t \geq 0 : ||B_t|| = 1\}$.

4.8. COROLLARY

Let μ be a special measure on $\mathbf{R}^n$, where $n=1$ or 2. Let R be a μ-standard stopping time and let S be a μ_R-standard stopping time. Then $R+S\circ\theta_R$ is a μ-standard stopping time.

Proof : Apply 4.7 in conjunction with the strong Markov property.

4.9. PROPOSITION :

Let μ be a special measure on $\mathbf{R}^n$, where $n=1$ or 2, and let T be a stopping time. Then the following are equivalent :

a) T is μ-standard

b) T is P^{μ}-a.s. finite, $E^{\mu}(\Phi^-(B_T)) < \infty$, and whenever S is a stopping time satisfying $S \leq T$ then
$\Phi^-(B_S) \leq E^{\mu}(\Phi^-(B_T)|\mathcal{B}_S)$.

c) The collection of random variables of the form $\Phi^-(B_S)$, where S is a stopping time satisfying $S \leq T$ and $P^{\mu}(S=\infty) = 0$, is P^{μ}-uniformly integrable.

d) $\{\Phi^-(B_{T\wedge t}) : 0 \leq t < \infty\}$ is P^{μ}-uniformly integrable.

(Remark : We remind the reader that if $n=1$, $\Phi^-(x) = \frac{1}{2}|x|$ while if $n=2$, $\Phi^-(x) = \frac{1}{2\pi}\log^+||x||$).

Proof : a) $\Longrightarrow$ b). $E^{\mu}(\Phi^-(B_S)) = U_-^{\mu_S}(0)$

$$= -(U^{\mu_S} * \sigma)(0) \leq -(U^{\mu_T} * \sigma)(0) = E^{\mu}(\Phi^-(B_T)),$$

where σ is the unit point mass at 0 if $n=1$ or the uniform unit distribution on $\{x \in \mathbb{R}^n : ||x|| = 1\}$ if $n=2$. Therefore $E^\mu(\Phi^-(B_T)) < \infty$ and if $A \in \mathcal{B}_S$ then $E^\mu(\Phi^-(B_{S'})) \leq E^\mu(\Phi^-(B_T))$ where S' is the stopping time $S1_A + T\,1_{A^c}$. It follows that $E^\mu(\Phi^-(B_S)1_A) \leq E^\mu(\Phi^-(B_T)1_A)$ for all $A \in \mathcal{B}_S$, whence $\Phi^-(B_S) \leq E^\mu(\Phi^-(B_T)|\mathcal{B}_S)$.

b) $\Longrightarrow$ c) $\Longrightarrow$ d). Clear.

d) Let $\mathcal{S}$ be the set of bounded stopping times S such that $S \leq T$. If $S \in \mathcal{S}$ then there exists $t \in [0,\infty)$ such that $S \leq T \wedge t$; then $\Phi^-(B_S) \leq E^\mu(\Phi^-(B_{T\wedge t})|\mathcal{B}_S)$ since $T \wedge t$ is μ-standard. It follows that $\{\Phi^-(B_S) : S \in \mathcal{S}\}$ is P^μ-uniformly integrable. But then $\lim_{r\to\infty} \sup_{S\in\mathcal{S}} \mu_S(\{x \in \mathbb{R}^n : ||x|| \geq r\}) = 0$. Hence $\lim_{t\to\infty} \sup_{S\in\mathcal{S}} P^\mu(S \geq t) = 0$, by 4.4. It follows that $P^\mu(T=\infty) = 0$. Hence $\int \phi d\mu_{T\wedge t} \to \int \phi d\mu_T$ as $t \to \infty$, for every bounded continuous function ϕ on $\mathbb{R}^n$. Also $\lim_{t\to\infty} E^\mu(\Phi^-(B_{T\wedge t}))$ exists, is finite, and is equal to $E^\mu(\Phi^-(B_T))$; that is, $U_-^{\mu_T}(0) = \lim_{t\to\infty} U_-^{\mu_{T\wedge t}}(0) < \infty$. Thus by 3.13, μ_T is special and $\int U^{\mu_{T\wedge t}}\, d\gamma \to \int U^{\mu_T}\, d\gamma$ for every good measure γ on $\mathbb{R}^n$. Hence by 4.5, T is μ-standard.

4.10. COROLLARY .

Let μ be a special measure on $\mathbb{R}^n$, where $n=1$ or 2. Let A be a bounded Borel subset of $\mathbb{R}^n$ and let $T = \inf\{t > 0 : B_t \notin A\}$. Then T is μ-standard.

Proof : Let d be the diameter of A. Then $||B_{T\wedge t} - B_0|| \leq d$ so $\Phi^-(B_{T\wedge t}) \leq \Phi^-(B_0) + d$. Thus $\{\Phi^-(B_{T\wedge t}) : 0 \leq t < \infty\}$ is not only P^μ-uniformly integrable, but is actually bounded by a fixed

P^μ-integrable function. Hence T is μ-standard by d) $\Longrightarrow$ a) of the proposition.

4.11. COROLLARY

Let μ be a special measure on $\mathbb{R}$ and let T be a stopping time. Then T is μ-standard iff $P^\mu(T=\infty) = 0$ and whenever S is a stopping time satisfying $S \leq T$ then $E^\mu(|B_S|) < \infty$ and $E^\mu(B_S) = E^\mu(B_0)$.

Proof : ($\Longrightarrow$) $(B_{T \wedge t})$ is a martingale over $(\Omega, \mathcal{B}, \mathcal{B}_t, P^\mu)$ and by a) $\Longrightarrow$ d) of the proposition, it is uniformly integrable.

($\Longleftarrow$) Let S be a stopping time $\leq T$, let $A \in \mathcal{B}_S$, and consider the stopping time $S' = S\, 1_A + T\, 1_{A^c}$. Then $E^\mu(B_{S'}) = E^\mu(B_0) = E^\mu(B_T)$. Hence $E^\mu(B_S\, 1_A) = E^\mu(B_T\, 1_A)$. It follows that $B_S = E^\mu(B_T | \mathcal{B}_S)$, so $|B_S| \leq E^\mu(|B_T|\ |\mathcal{B}_S)$. Thus T is μ-standard by b) $\Longrightarrow$ a) of the proposition.

Remark : In Monroe [1] it is shown, for the case when μ is the unit point mass at 0 in $\mathbb{R}$, that if T is a stopping time then $(B_{T \wedge t})$ is P^μ-uniformly integrable iff $P^\mu(T=\infty) = 0$, $E^\mu(|B_T|) < \infty$, $E^\mu(B_T) = 0$, and T is minimal in the sense that if S is a stopping time such that $S \leq T$ and $\mathrm{law}(B_S\ ;\ P^\mu) = \mathrm{law}(B_T\ ;\ P^\mu)$ then $S=T$ P^μ-a.s. Monroe's proof of the forward implication here is not difficult and is based on the fact that the paths of Brownian motion are a.s. without intervals of constancy. (Another way of proving this implication is to use the fact that $E^\mu(\int_0^T f(B_s)ds] = \int [U^\mu(x) - U^{\mu T}(x)] f(x)dx$ for any non-negative Borel function f on $\mathbb{R}$; see 4.7). Monroe's proof of the reverse implication is not simple. Recently Chacon and Ghoussoub [1] have found an elegant and easy proof of this implication. The analogous result for the case of $\mathbb{R}^2$ remains an open question on account of difficulties with polar sets. (For the three dimensional case, or more generally for case of a Green region, there is nothing to prove since all stopping times in this case are "standard" and minimal ; see 2.1 (a)).

4.12. THEOREM

Let μ and ν be special measures on $\mathbf{R}^n$, where $n=1$ or 2. Suppose :

a) $U^\mu \geq U^\nu$.

b) $\mu(Z) \leq \nu(Z)$ for every Borel set $Z \subseteq \{U^\nu = \infty\}$.

c) $\mu(\mathbf{R}^n) = \nu(\mathbf{R}^n)$.

Then there is a μ-standard stopping time T such that $\mu_T = \nu$.

(Remark : It then follows that actually $\nu(Z) = \mu(Z \cap \{U^\nu=\infty\})$ for every Borel polar set $Z \subseteq \mathbf{R}^n$).

Proof : As the proof of this theorem is very similar to the proof of theorem 2.2, we shall limit ourselves to a few comments. First, the only real difference is that here we must check that we are working with standard stopping times. This is easy, given the results about standard stopping times which we have already developed in this section.

Second, we need the analogues of the preliminary results of section 2, for the case where the Green region D is replaced by $\mathbf{R}^1$ or $\mathbf{R}^2$. Most of these are easy to establish. We remark only that the required analogue of the convergence lemma 2.7 is 3.14 and the required analogue of the domination principle 2.9 is 3.4 (if n=1) or 3.9 (if n=2).

Remarks :

a). If μ does not charge polar sets then the condition b) of the theorem is vacuous. In this case, ν is of the form μ_T where T is a μ-standard stopping time iff $U^\mu \geq U^\nu$ and $\mu(\mathbf{R}^n) = \nu(\mathbf{R}^n)$.

b). If $n=1$ and μ is the unit point mass at 0 then ν is special iff $\int |x| \, d\nu(x) < \infty$, and if ν is special and $\nu(\mathbf{R}) = 1$

then $U^\mu \geq U^\nu$ iff $\int x \, d\mu(x) = 0$, by 3.3. In this case the measures of the form μ_T , where T is a μ-standard stopping time, are precisely the probability measures on $\mathbb{R}$ whose centre of mass is defined and equal to 0 .

c) It follows from 4.7(b), together with lemma 5 of Baxter and Chacon [1], that if μ is a special measure on $\mathbb{R}^n$ and T is a μ-standard stopping time then

$$\int x^2 \, d\mu_T(x) = \int x^2 \, d\mu(x) + 2nE^\mu(T) ;$$

in particular; if μ_T has a finite second moment then T has finite P^μ-expectation (and μ has a finite second moment too).

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Footnote

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