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Homogeneous Potentials

by

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The purpose of this note is a tentative study of supermartingales within the framework of Knight's prediction process as developed by Meyer [4] and Meyer and Yor [5] in the Seminar X. If Y is a homogeneous process which has a certain property relative to a fixed measure μ, then one might hope that, because of the homogeneity of Y, at least in some cases, this property would "propagate" along the prediction process. In Section 1 we show that the property of being a right continuous supermartingale (subject to secondary hypotheses) does, indeed, propagate along the prediction process. In particular, if Y is a μ potential, the predictable increasing process A which generates Y behaves nicely under shifts. See (2.3). In Section 3 we show that the regularity of Y also propagates.

Conceptually one may derive these facts by representing $Y_t = g(Z_t^\mu)$ where Z_t^μ is the prediction process and g is excessive for this process, and then applying standard facts for Markov processes. However, it seems to be difficult to carry out the details of this approach. In the present note we attack the problem directly, and confine ourselves to some remarks about the connection with Markov processes in Section 4.

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1. Homogeneous Supermartingales.

We assume that the reader is familiar with the papers of Meyer [4] and Meyer and Yor [5] on prediction processes. We adopt the definitions and notations of Meyer and Yor [5] without special mention except to recall the following: E is a Polish space and Ω is the space of all cadlag functions from $\mathbb{R}^+$ to $E \cup \{\partial\}$ where ∂ is an absorbing point. Also $M = M_1 \cup \{0\}$ where M_1 is the space of all probability measures on $(\Omega, \underline{\underline{F}}^0)$. Moreover Ω may be given a compact metrizable topology for which $\underline{\underline{F}}^0$ is the Borel field, and if M is given the weak topology of measures (i.e. the topology of weak (étroite) convergence on Ω), then M is a compact metrizable space and its Borel field $\underline{\underline{M}}$ is generated by the maps $\mu \to \mu(A)$ with $A \in \underline{\underline{F}}^0$. Finally $\underline{\underline{P}}$ and $\underline{\underline{O}}$ denote the predictable and optional σ-algebras over the filtration $(\Omega, \underline{\underline{F}}^0_{t+}, \underline{\underline{F}}^0)$. We adopt the convention that a stopping time is an $(\underline{\underline{F}}^0_{t+})$ stopping time unless explicitly stated otherwise. Although we shall try to give explicit references to [4] or [5] as needed, the reader of this paper should be familiar with the basic notation of [4] and [5].

A process $Y = (Y_t)$ is called homogeneous provided $Y_t \circ \theta_s = Y_{t+s}$ identically for $t,s \geq 0$. If Y is homogeneous, then $Y_t = Y_0 \circ \theta_t$ for all $t \geq 0$. (These are essentially the algebraically copredictable processes of Azéma.)

(1.1) **Proposition.** *Let $\mu \in M_1$ and let Y be a bounded, homogeneous, optional process. Suppose that (i) $t \to Y_t$ is μ almost surely right continuous on $[0, \infty[$, and (ii) Y is a supermartingale over $(\Omega, \underline{\underline{F}}^0_{t+}, \underline{\underline{F}}^0, \mu)$. Then there exists a set $\Omega_0 \in \underline{\underline{F}}^0$ with $\mu(\Omega_0) = 0$ such that if $\omega \notin \Omega_0$ and $r \geq 0$, then $\lambda = Z^\mu_r(\omega) \in M_1$ and properties (i) and (ii) hold with μ replaced by λ.*

Proof. First of all there exists a set $\Omega_1 \in \underline{\underline{F}}^0$ with $\mu(\Omega_1) = 0$ such that if $\omega \notin \Omega_1$ and $r \geq 0$, then $Z^\mu_r(\omega) \in M_1$. This follows from Lemma 2 of [4].

It is well known that (i) and (ii) imply that $t \to Y_t$ is, in fact, cadlag on $[0, \infty[$ almost surely μ. Let

$$A = \{\omega\colon\ t \to Y_t(\omega) \ \text{ is cadlag}\}.$$

By (18-b) page 145 of [2], $A \in \underline{\underline{F}}^0$. Therefore the process $(t, \omega) \to Z_t^{\mu}(\omega, A^c)$ is optional. If R is a stopping time, then $\int \mu(d\omega) Z_R^{\mu}(\omega, A^c) = \mu(\theta_R^{-1} A^c)$. But $\theta_R^{-1} A^c$ is the set of those ω such that $t \to Y_t(\theta_R \omega) = Y_{t+R}(\omega)$ is not cadlag. Hence $\mu(\theta_R^{-1} A^c) = 0$ and so $(t, \omega) \to Z_t^{\mu}(\omega, A^c)$ is μ evanescent. Therefore there exists $\Omega_2 \in \underline{\underline{F}}^0$ with $\mu(\Omega_2) = 0$ such that if $\omega \notin \Omega_2$ and $r \geq 0$, $t \to Y_t(\cdot)$ is cadlag $Z_r^{\mu}(\omega)$ almost surely.

Next fix t and s in $\mathbb{R}^+$. Let $F(\lambda, w)$ be the indicator of $\{(\lambda, w)\colon Z_s^{\lambda}(w, Y_t) > Y_s(w)\}$. Then F is $\underline{\underline{M}} \times \underline{\underline{F}}^0$ measurable, and so $((r, \omega), w) \to F(Z_r^{\mu}(\omega), w)$ is $\underline{\underline{O}} \times \underline{\underline{F}}^0$ measurable. Therefore

$$G(r, \omega) = \int Z_r^{\mu}(\omega, dw) F(Z_r^{\mu}(\omega), w) \tag{1.2}$$

is optional. If R is a stopping time, then using Lemma 4 and Theorem 2 of [5],

$$\int \mu(d\omega) G(R(\omega), \omega) = \int \mu(d\omega) F(Z_R^{\mu}(\omega), \theta_R \omega)$$

$$= \mu\{\omega\colon Z_s^{Z_R^{\mu}(\omega)}(\theta_R \omega, Y_t) > Y_s(\theta_R \omega)\} = \mu\{Z_{s+R}^{\mu}(\cdot, Y_t) > Y_{s+R}\}\ .$$

But μ almost surely

$$Z_{s+R}^{\mu}(\cdot, Y_t) = \mu[Y_t \circ \theta_{s+R} \,|\, \underline{\underline{F}}^0_{(s+R)+}] = \mu[Y_{t+s+R} \,|\, \underline{\underline{F}}^0_{(s+R)+}] \leq Y_{s+R}\ ,$$

and consequently the process $G(r, \omega)$ is μ evanescent. Therefore there exists $\Omega_{t,s} \in \underline{\underline{F}}^0$ with $\mu(\Omega_{t,s}) = 0$ and such that if $\omega \notin \Omega_{t,s}$ and $r \geq 0$, then letting $\lambda = Z_r^{\mu}(\omega)$ one has almost surely λ

$$\lambda[Y_{t+s}|\underline{\underline{F}}^0_{s+}] = Z^\lambda_s(\cdot, Y_t) \le Y_s . \tag{1.3}$$

(1.4) Remark. So far we have not used the fact that Y is bounded and everything we have done is valid if, for example, $Y \ge 0$ rather than bounded.

Now let $\Omega_0 = \Omega_1 \cup \Omega_2 \cup \bigcup_{t,s\in Q^+} \Omega_{t,s}$ where Q^+ denotes the nonnegative rationals. Then $\mu(\Omega_0) = 0$. Let $\omega \notin \Omega_0$ and $r \ge 0$, and set $\lambda = Z^\mu_r(\omega)$. Then $\lambda \in M_1$ and $t \to Y_t$ is cadlag almost surely λ. Given $t, s \in \mathbb{R}^+$ choose sequences of rationals (t_n) and (s_n) decreasing strictly to t and s. Then from (1.3) one has λ almost surely

$$\lambda[Y_{t_n+s_n}|\underline{\underline{F}}^0_{s_n+}] \le Y_{s_n} . \tag{1.5}$$

But $Y_{s_n} \to Y_s$ almost surely λ, and writing the leftside of (1.5) as

$$\lambda[Y_{t_n+s_n} - Y_{t+s}|\underline{\underline{F}}^0_{s_n+}] + \lambda[Y_{t+s}|\underline{\underline{F}}^0_{s_n+}]$$

it follows easily that the leftside of (1.5) approaches $\lambda[Y_{t+s}|\underline{\underline{F}}^0_{s+}]$ in $L^1(\lambda)$. Thus Y_t is a supermartingale over $(\Omega, \underline{\underline{F}}^0_{t+}, \lambda)$, completing the proof of (1.1).

(1.6) Corollary. *Suppose, in addition to the hypotheses of* (1.1), *that* Y *is a* μ *potential; that is,* $Y \ge 0$ *and* $\mu(Y_t) \to 0$ *as* $t \to \infty$. *Then* Ω_0 *may be chosen so that* Y *is a* λ *potential for* $\lambda = Z^\mu_r(\omega)$ *when* $\omega \notin \Omega_0$ *and* $r \ge 0$.

Proof. Let Ω^* be the exceptional set Ω_0 in (1.1). If $\omega \notin \Omega^*$ and $r \ge 0$, then $Z^\mu_r(\omega, Y_t)$ decreases as t increases since (Y_t) is a supermartingale with respect to $Z^\mu_r(\omega, \cdot)$. Let $G_r(\omega) = \liminf_{n\to\infty} Z^\mu_r(\omega, Y_n)$. Then G is optional.

If R is a stopping time,

$$\mu(G_R) \leq \liminf_n \mu(Y_{n+R}) = 0 ,$$

and so G is μ-evanescent. Thus if $\Lambda^* \in \underline{\underline{F}}^0$ is such that $\mu(\Lambda^*) = 0$ and $G_r(\omega) = 0$ for all $r \geq 0$ and $\omega \notin \Lambda^*$, then $\Omega_0 = \Omega^* \cup \Lambda^*$ satisfies the conditions of (1.6).

(1.7) Remark. Proposition 1.1 remains valid if we suppose $Y \geq 0$ rather than bounded. To see this let $Y_t^n = Y_t \wedge n$. Then by (1.1) and (1.4) we may choose $\Omega_0 \in \underline{\underline{F}}^0$ with $\mu(\Omega_0) = 0$ such that if $\lambda = Z_r^{\mu}(\omega)$ with $\omega \notin \Omega_0$ and $r \geq 0$, then $t \to Y_t$ is cadlag almost surely λ and each Y^n is a supermartingale with respect to λ. Clearly this implies $\lambda[Y_{t+s} | \underline{\underline{F}}^0_{s+}] \leq Y_s$, and so Y is a λ supermartingale provided $\lambda(Y_t) < \infty$ for all t. For this it suffices that $\lambda(Y_0) < \infty$. But by a now familiar argument $\{(r, \omega): Z_r^{\mu}(\omega, Y_0) \neq Y_r(\omega)\}$ is μ evanenscent. Thus we may modify Ω_0 so that $Z_r^{\mu}(\omega, Y_0) = Y_r(\omega) < \infty$ if $\omega \notin \Omega_0$ and $r \geq 0$. This establishes the above assertion.

(1.8) Remark. Suppose $Y \geq 0$ and class (D) relative to μ rather than bounded. One would like to be able to choose the exceptional set Ω_0 so that Y is class (D) relative to all $\lambda = Z_r^{\mu}(\omega)$; $\omega \notin \Omega_0$, $r \geq 0$. Let $R_n = \inf\{t: Y_t > n\}$. Then R_n is an $(\underline{\underline{F}}_t^{\lambda})$ stopping time for each $\lambda \in M_1$, and if (Y_t) is a λ supermartingale, then Y is of class (D) relative to λ provided $\lambda(Y_{R_n}) \to 0$ as $n \to \infty$. See VI-T20 of [3]. Suppose that $t \to Y_t(\omega)$ is right continuous for each $\omega \in \Omega$. Under this assumption each R_n is an $(\underline{\underline{F}}^0_{t+})$ stopping time. If $\omega \notin \Omega_0$ and $r \geq 0$, then $Z_r^{\mu}(\omega, Y_{R_n})$ decreases with n. Since R_n is $\underline{\underline{F}}^0$ measurable $(r, \omega) \to Z_r^{\mu}(\omega, R_n)$ is optional, and hence so is $G_r(\omega) = \liminf_n Z_r^{\mu}(\omega, Y_{R_n})$. But for any stopping time R one has $R + R_n \circ \theta_R \geq R_n$, and so

$$\mu(G_R) \leq \liminf \mu(Y_{R+R_n \circ \theta_R}) \leq \liminf \mu(Y_{R_n}) = 0.$$

Therefore, in this case, G is μ evanescent, and hence we may modify Ω_0 so that Y is of class (D) relative to all $\lambda = Z_r^{\mu}(\omega)$; $\omega \notin \Omega_0$, $r \geq 0$. However, if Y is only μ almost surely right continuous, then R_n, although an $(\underline{\underline{F}}_t^*)$ stopping time, need not be $\underline{\underline{F}}^0$ measurable, and so the process G defined above may not be optional. It is still the case that $\mu(G_R) = 0$ for all stopping times R (Lemma 4 of [5]), but without knowing that G is, at least, μ indistinguishable from an optional process I do not see how to draw the desired conclusion from this fact.

2. The Generating Increasing Predictable Process.

In this section we fix $\mu \in M_1$ and suppose that Y satisfies the hypotheses of (1.1) and, in addition, is a μ potential — we shall simply call such a process a bounded, homogeneous, μ potential in the sequel. Let Ω_0 be the exceptional set in Corollary 1.6. Since $\Omega_0 \in \underline{\underline{F}}^0$, $\mathbb{R}^+ \times (\Omega - \Omega_0)$ is a Borel subset of the Polish space $\mathbb{R}^+ \times \Omega$. Let M_p be the image of $\mathbb{R}^+ \times (\Omega - \Omega_0)$ in M under the map $(t, \omega) \to Z_t^{\mu}(\omega)$. Since this map is Borel (i.e. $\underline{\underline{B}}(\mathbb{R}^+) \times \underline{\underline{F}}^0$ measurable), M_p is analytic in M. Finally we add the single point μ to M_p if it is not already there. Thus M_p is an analytic subset of M such that for each $\lambda \in M_p$, Y is a λ potential and $Z_t^{\mu}(\omega) \in M_p$ for all $t \geq 0$ and $\omega \notin \Omega_0$. (The "p" in M_p is for potential.) If $\underline{\underline{M}}_p$ is the Borel σ-algebra of the subspace M_p of M, then $\underline{\underline{M}}_p$ is just the trace of $\underline{\underline{M}}$ on M_p.

If we imitate the construction of the predictable increasing process generating a class (D) potential in Dellacherie (V-T49 in [1]), keeping track at each stage of the dependence on λ, one obtains the following result.

(2.1) Proposition. There exists a positive function $A_t^{\lambda}(\omega)$ defined on $M_p \times \mathbb{R}^+ \times \Omega$ such that:

(i) For each $(\lambda, \omega) \in M_p \times \Omega$, $A_0^\lambda(\omega) = 0$ and $t \to A_t^\lambda(\omega)$ is right continuous and increasing. For each $\lambda \in M_p$, (A_t^λ) is adapted to $(\underline{\underline{F}}_{t+}^0)$.

(ii) $(\lambda, t, \omega) \to A_t^\lambda(\omega)$ is $\underline{\underline{M}}_p \times \underline{\underline{B}}(\mathbb{R}^+) \times \underline{\underline{F}}^0$ measurable and $(\lambda, \omega) \to A_t^\lambda(\omega)$ is $\underline{\underline{M}}_p \times \underline{\underline{F}}_{t+\epsilon}^0$ measurable for each $\epsilon > 0$.

(iii) For each $\lambda \in M_p$, A^λ is λ indistinguishable from the right continuous predictable (relative to the filtration $(\underline{\underline{F}}_t^\lambda)$) process generating Y with respect to λ.

It follows from (2.1-iii) that $Y_t = \lambda[A_\infty^\lambda - A_t^\lambda | \underline{\underline{F}}_{t+}^0]$ almost surely λ. The next result shows the behavior of A_t^λ under shifts and reflects the fact that Y is homogeneous.

(2.3) Proposition. Let T be a stopping time. Then the processes $t \to A_{t+T}^\mu(\omega) - A_T^\mu(\omega)$ and $t \to A_t^{Z_T^\mu(\omega)}(\theta_T\omega)$ are μ indistinguishable.

Proof. For each $\lambda \in M_p$ and $h > 0$, $Z_t^\lambda(\cdot, Y_h)$ is an optional version of the supermartingale $\lambda[Y_{t+h} | \underline{\underline{F}}_{t+}^0]$. Introduce the approximate Laplacians (see V-53 of [1]),

$$A_t^{\lambda,h} = \frac{1}{h} \int_0^t [Y_s - Z_s^\lambda(\cdot, Y_h)] ds . \tag{2.4}$$

Let T be a stopping time. Then (T-54 of [1]),

$$A_T^{\lambda,h} \to A_T^\lambda \text{ in } \sigma(L^1(\lambda), L^\infty(\lambda)) \text{ as } h \to 0 . \tag{2.5}$$

Clearly $A_t^{\lambda,h}$ is $\underline{\underline{F}}_{t+}^0$ measurable, and so $A_t^{\lambda,h} \circ \theta_T$ is $\underline{\underline{F}}_{(t+T)+}^0$ measurable.

Also $A_T^{\lambda,h}$ is $\underline{\underline{F}}^0_{T+}$ measurable. Now fix t and T and let F be a bounded $\underline{\underline{F}}^0$ measurable function. Then

$$(2.6) \qquad J(h) = \mu[(A^{\mu,h}_{t+T} - A^{\mu,h}_T)F] = \mu\{(A^{\mu,h}_{t+T} - A^{\mu,h}_T)\mu[F\,|\,\underline{\underline{F}}^0_{(t+T)+}]\}.$$

Let U be a bounded $\underline{\underline{F}}^0_{(t+T)+}$ measurable version of $\mu(F\,|\,\underline{\underline{F}}^0_{(t+T)+})$; for example, $U = K^{\mu}_{t+T}(\cdot, F)$. Then by Theorem 1 of [5] there exists a bounded function $\overline{U}(\omega, w)$ on $\Omega \times \Omega$ such that (i) $\overline{U}$ is $\underline{\underline{F}}^0_{T-} \times \underline{\underline{F}}^0$ measurable, (ii) for each ω, $\overline{U}(\omega, \cdot)$ is $\underline{\underline{F}}^0_{t+}$ measurable, and (iii) $U(\omega) = \overline{U}(\omega, \theta_T\omega)$ identically. Therefore

$$J(h) = \int \mu(d\omega)[A^{\mu,h}_{t+T}(\omega) - A^{\mu,h}_T(\omega)]\overline{U}(\omega, \theta_T\omega) ,$$

while from (2.4)

$$(2.7) \qquad A^{\mu,h}_{t+T} - A^{\mu,h}_T = \frac{1}{h}\int_0^t [Y_{s+T} - Z^{\mu}_{s+T}(\cdot, Y_n)]ds$$

$$= \frac{1}{h}\int_0^t [Y_s \circ \theta_T - Z_s^{Z^{\mu}_T}(\theta_T\cdot, h)]ds$$

where the last equality holds μ almost surely because of Theorem 2 in [5]. But the last integral in (2.7) is just $A_t^{Z^{\mu}_T(\omega),h}(\theta_T\omega)$, and so conditioning with respect to $\underline{\underline{F}}^0_{T+}$ we may write

$$(2.8) \qquad J(h) = \int \mu(d\omega) \int Z^{\mu}_T(\omega, dw)A_t^{Z^{\mu}_T(\omega),h}(w)\overline{U}(\omega, w) .$$

Now μ almost surely $Z^{\mu}_T(\omega) \in M_p$ and so by (2.5) the integral over w

in (2.8) approaches

$$\int Z_T^\mu(\omega,\ dw) A_t^{Z_T^\mu(\omega)}(w)\overline{U}(\omega,\ w)$$

as $h \to 0$ almost surely μ in ω. Majorize the integral over w in (2.8) by ($\|\overline{U}\| = \sup \overline{U}$)

$$\|\overline{U}\| \int Z_T^\mu(\omega,\ dw) A_\infty^{Z_T^\mu(\omega),h}(w) \le \|\overline{U}\| Z_T^\mu(\omega,\ Y_0)\ ,$$

since the potential generated by $A^{\lambda,h}$ is dominated by that generated by A^λ for each $\lambda \in M_p$. But $Z_T^\mu(\cdot,\ Y_0)$ is μ integrable and hence from (2.8)

$$(2.9)\ J(h) \to \int \mu(d\omega) \int Z_T^\mu(\omega,\ dw) A_t^{Z_T^\mu(\omega)}(w)\overline{U}(\omega,\ w) = \int \mu(d\omega) A_t^{Z_T^\mu(\omega)}(\theta_T\omega) U(\omega)$$

as $h \to 0$.

For each $\varepsilon > 0$, $(\lambda,\ \omega) \to A_t^\lambda(\omega)$ is $\underline{\underline{M}}_p \times \underline{\underline{F}}_{t+\varepsilon}^0$ measurable and so $(\lambda,\ \omega) \to A_t^\lambda(\theta_T\omega)$ is $\underline{\underline{M}}_p \times \underline{\underline{F}}_{(t+T+\varepsilon)+}^0$ measurable. Hence $\omega \to A_t^{Z_T^\mu(\omega)}(\theta_T\omega)$ is $\underline{\underline{F}}_{t+T+\varepsilon}^\mu$ for each $\varepsilon > 0$ and consequently it is $\underline{\underline{F}}_{t+T}^\mu$ measurable. Therefore, recalling the definition of U, we obtain from (2.9)

$$\lim_{h\to 0} J(h) = \int \mu(d\omega) A_t^{Z_T^\mu(\omega)}(\theta_T\omega) F(\omega)\ .$$

On the other hand from the first equality in (2.6) and (2.5) we obtain

$$\lim_{h\to 0} J(h) = \int \mu(d\omega) [A_{t+T}^\mu(\omega) - A_T^\mu(\omega)] F(\omega)\ .$$

Since F was an arbitrary bounded $\underline{\underline{F}}^0$ measurable function this gives

$$A^{\mu}_{t+T}(\omega) - A^{\mu}_{T}(\omega) = A_t^{Z^{\mu}_T(\omega)}(\omega)$$

almost surely μ. But $Z^{\mu}_T(\omega) \in M_p$ almost surely μ and so both sides are right continuous in t almost surely μ. This completes the proof of (2.3).

(2.10) Remark. In this section we have made no explicit use of the boundedness of Y. Thus in light of (1.8) the results of this section are valid if we replace the assumption that Y is bounded by the assumption that Y is a class (D) relative to μ and that $t \to Y_t(\omega)$ is right continuous for each $\omega \in \Omega$.

3. Regularity.

In this section we suppose that Y is a bounded, homogeneous, μ potential and that Y is μ regular. The fact that Y is μ regular is equivalent to the statement that $t \to A^{\mu}_t$ is continuous μ almost surely. Let Ω_0, M_p, and A^{λ}_t be as in Section 2. We shall show that we can modify Ω_0 and M_p so that Y is λ regular for each $\lambda \in M_p$.

To this end first note that $\underline{\underline{M}}_p \times \underline{\underline{F}}^0$ is the trace of $\underline{\underline{M}} \times \underline{\underline{F}}^0$ on $M_p \times \Omega$. Let

$$D = \{(\lambda, \omega) \in M_p \times \Omega\colon\ t \to A^{\lambda}_t(\omega) \text{ is not continuous}\}\ .$$

In light of (2.1-ii), $G = 1_D$ is $\underline{\underline{M}}_p \times \underline{\underline{F}}^0$ measurable, and hence there exists a $G^*\colon M \times \Omega \to [0, 1]$ which is $\underline{\underline{M}} \times \underline{\underline{F}}^0$ measurable and such that G is the restriction of G^* to $M_p \times \Omega$. Then $G(Z^{\mu}_t(\omega), \omega)$ and $G^*(Z^{\mu}_t(\omega), \omega)$ are μ indistinguishable since μ almost surely for all $t \ge 0$, $Z^{\mu}_t(\omega) \in M_p$. Consequently

$$H_t(\omega) = \int Z^{\mu}_t(\omega, dw) G(Z^{\mu}_t(\omega), w) \tag{3.1}$$

is μ indistinguishable from an optional process (i.e. $\underline{\underline{0}}^{\mu}$ measurable), because if one replaces G by G^* in (3.1) the corresponding process is optional. If T is a stopping time, then using Lemma 4 of [5] we have

$$\mu(H_T) = \int \mu(d\omega)G(Z_T^{\mu}(\omega), \theta_T\omega)$$

$$= \mu\{\omega: t \to A_t^{Z_T^{\mu}(\omega)}(\theta_T\omega) \text{ is not continuous}\}$$

$$= \mu\{\omega: t \to A_{t+T}^{\mu}(\omega) - A_T^{\mu}(\omega) \text{ is not continuous}\},$$

where the last equality follows from (2.3). But the last displayed probability is zero since Y is μ regular, and so H is μ evanescent. _Therefore we may modify_ Ω_0 _and correspondingly_ M_p _so that_ Y _is a_ λ _regular potential for all_ $\lambda \in M_p$ _and such that_ Ω_0 _and_ M_p _still have the properties set forth in the first paragraph of Section_ 2.

If one carefully imitates the corresponding arguments in the case of a continuous additive functional of a Markov process one can prove the following result.

(3.2) **Proposition.** _For_ $\lambda \in M_p$ _let_ $T^{\lambda} = \inf\{t: A_t^{\lambda} > 0\}$. _Let_ $F = \{\lambda \in M_p: \lambda(T^{\lambda} = 0) = 1\}$ _and_ $D^{\lambda} = \inf\{t > 0: Z_t^{\lambda} \in F\}$. _Then_ μ _almost surely_ $T^{\mu} = D^{\mu}$ _and the support of the measure_ dA_t^{μ} _on_ $\mathbb{R}^+$ _is the closure in_ $\mathbb{R}^+$ _of_ $\{t: Z_t^{\mu} \in F\}$.

4. Concluding Remarks.

For each $\mu \in M_1$, the process $(Z_t^{\mu}, \underline{\underline{F}}_{t+}^0, \mu)$ is strong Markov with state space M and semigroup

$$J_t g(\lambda) = \lambda[g(Z_t^\lambda)] \quad (4.1)$$

Here $g\colon M \to \mathbb{R}$ is $\underline{\underline{M}}$ measurable and bounded or positive. See Theorem 3 of [4] and Theorem 2 of [5]. Suppose Y is a bounded, homogeneous, optional process. Define $g\colon M \to \mathbb{R}$ by $g(\lambda) = \lambda(Y_0)$. Then for each $\lambda \in M_1$,

$$g(Z_t^\lambda) = Z_t^\lambda(\cdot, Y_0) = \lambda[Y_t \mid \underline{\underline{F}}_{t+}^0] = Y_t \quad (4.2)$$

almost surely λ. Since (4.2) is valid with t replaced by a stopping time T and both sides of (4.2) are optional, it follows that Y_t and $g(Z_t^\lambda)$ are λ indistinguishable. Now fix μ in M_1 and suppose that Y is a bounded, homogeneous, μ potential. If $\lambda \in M_p$, then using (4.2)

$$J_t g(\lambda) = \lambda[g(Z_t^\lambda)] = \lambda(Y_t) \le \lambda(Y_0) = g(\lambda) \quad (4.3)$$

since Y is a λ potential. Clearly $J_t g(\lambda) \to g(\lambda)$ as $t \to 0$ when $\lambda \in M_p$, and so g is excessive for the semigroup (J_t) — except that we have $J_t g \le g$ and $J_t g \to g$ only on M_p. Since Y_t is μ indistinguishable from $g(Z_t^\mu)$, the results in the earlier sections just mirror the well known facts concerning excessive functions of a Markov process.

This becomes even clearer if we introduce the space $\tilde{\Omega} = M \times \Omega$ and define

$$Z_t(\tilde{\omega}) = Z_t(\lambda, \omega) = Z_t^\lambda(\omega)$$

$$\tilde{\theta}_t(\tilde{\omega}) = \tilde{\theta}_t(\lambda, \omega) = (Z_t^\lambda(\omega), \theta_t\omega) .$$

It is immediate that for each $\lambda \in M_1$ the process (Z_t) is Markov with respect to the law $\epsilon_\lambda \times \lambda$ on $\tilde{\Omega}$ and has semigroup (J_t). Now Lemma 7 of [4] becomes

(4.4) $$Z_s \circ \tilde{\theta}_t = Z_{t+s} ,$$

and, in the context of (2.3), writing $A_t(\tilde{\omega}) = A_t(\lambda, \omega) = A_t^\lambda(\omega)$ we have

(4.5) $$A_{t+s} = A_t + A_s \circ \tilde{\theta}_t .$$

Thus A is an "additive functional" for the process Z. Unfortunately (4.4) and (4.5) are not identities. For each λ and t they are identities in s up to λ indistinguishability. It is not clear to me that the exceptional set can be chosen independent of t let alone λ.

This suggests that, perhaps, the theory of Markov processes should be re-worked in enough generality to cover situations of this type. On the other hand, most likely, it is simpler to use ad hoc methods as in this note.

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