SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

RENZO CAIROLI
JOHN B. WALSH

Some examples of holomorphic processes

Séminaire de probabilités (Strasbourg), tome 11 (1977), p. 340-348
<http://www.numdam.org/item?id=SPS_1977__11__340_0>

SOME EXAMPLES OF HOLOMORPHIC PROCESSES

by R. Cairoli and J.B. Walsh

We would like to provide some examples which complement the article [3] on holomorphic processes and which give some hints of possible new directions at the same time.

Let $\{W_z,\ z \in \mathbb{R}_+^2\}$ be the Brownian sheet and let $\mathcal{F}_z$ be the field $\sigma\{W_\zeta,\ \zeta \prec z\}$, suitably completed. We refer the reader to [3] for the definition of a holomorphic process and for the relevant notation. One comment on this definition is necessary. If Φ is holomorphic in a domain D with derivative ϕ, and if $z_1, z_2 \in D$, then if $\Gamma \subset D$ is a path from z_1 to z_2,

$$\Phi_{z_2} = \Phi_{z_1} + \int_\Gamma \phi \, \partial W. \tag{1}$$

This was only required for increasing paths in [1] and [3], but it really should have been required for a larger class of paths, namely at least for — in the terminology of [1], p. 142 — all piecewise-pure paths. The articles [1] and [3] were only concerned with the square-integrable case, where this distinction makes no difference. Indeed, in this case, once (1) holds for horizontal and vertical paths, it must hold not only for all increasing paths, but for all piecewise-pure paths. It is not clear that this remains true for non-square-integrable processes, as is indicated by example 5. Rather than to redefine holomorphic processes, we will say in this note that a holomorphic process which satisfies (1) for all piecewise-pure Γ is <u>strongly holomorphic</u>.

The only other fact about holomorphic processes we will use is that if f(x,t) satisfies the backward heat equation

$$\tfrac{1}{2} f_{xx} + f_t = 0$$

in the strip $\{(x,t): \alpha < t < \beta\}$, then $\{f(W_z, |z|), \alpha < |z| < \beta\}$ is a strongly holomorphic process with derivative $\{f_x(W_z, |z|), \alpha < |z| < \beta\}$.

So far the only holomorphic processes which have been studied are those which are square-integrable and which are defined on a fixed domain, but the above fact is a simple consequence of Ito's formula for ordinary stochastic integrals (cf. [1]) and has nothing to do with any integrability conditions.

It is clearly of interest to weaken the requirement of square-integrability, but it is perhaps even more important to study random, rather than fixed domains. Consider the following natural example.

Example 1. Let $f(x,t)$ be a function which is defined in $\{|x| < M, t \geqq 0\}$ and which solves the backward heat equation there. Then $f(W_z, |z|)$ will be holomorphic in the random region

$$A(\omega) = \{z \in \mathbb{R}^2_+ : |W_z(\omega)| < M\},$$

in the sense that if z_1 and z_2 are in $\mathbb{R}^2_+$ and if Γ is a piecewise-pure path from z_1 to z_2, then

$$f(W_{z_2}, |z_2|) - f(W_{z_1}, |z_1|) = \int_\Gamma f_x(W_\zeta, |\zeta|)\, \partial W_\zeta$$

a.s. on the set $\{\omega: \Gamma \subset A(\omega)\}$.

The set A is open in $\mathbb{R}^2_+$ and adapted, i.e. $\{z \in A\} \in \mathcal{F}_z$ for all $z \in \mathbb{R}^2_+$. A is not connected, however, and a curious fact about it is that its components are not adapted. For instance, let A_o and $(A \cap R_z)_o$ be the connected components of A and $A \cap R_z$ respectively which contain the origin. Then

$$\{z \in A_o\} = \{z \in (A \cap R_z)_o\} \cup \{z \in A_o,\ z \notin (A \cap R_z)_o\}.$$

The first set on the right is in $\mathcal{F}_z$, but the second is not, for it depends on the behavior of W in $\mathbb{R}^2_+ - R_z$.

This might seem to make it awkward to study processes in connected regions, but it is possible to localize as follows. Let $a \in \mathbb{R}^2_+$ and put

$$S_a(\omega) = \bigcup_{z:\ [a,z] \subset A} [a,z].$$

Then $S_a \subset A$, and S_a is simply connected, adapted and measurable. Indeed,

$$\{z \in S_a\} = \{[a,z] \subset A\} \in \mathcal{F}_z.$$

S_a is what is sometimes called a stopping neighborhood of a, and it would seem natural to study holomorphic processes in a stopping neighborhood, such as

$$\{f(W_z, |z|),\ z \in S_a\}.$$

Example 2. To see what might happen if we relax the square-integrability requirement slightly, consider the function $t^{-\frac{1}{2}}\exp(x^2/2t)$. This solves the backward heat equation in $\{(x,t): t > 0\}$, so the process

$$e_z = |z|^{-\frac{1}{2}}\exp(W_z^2/2|z|)$$

is holomorphic, even strongly holomorphic, in $\{z: |z| > 0\}$. Since $e_{s,t}$ tends to infinity as either s or t tends to zero, it can't be extended to be holomorphic on $\mathbb{R}^2_+$.

The process e_z is not square-integrable, but it satisfies the following local square-integrability property: if $a \in \mathbb{R}^2_+$ and $|a| > 0$, then for $z \succ a$, $E\{e_z^2 | \mathcal{F}_a\}$ is a.s. finite if $|z| < 2|a|$. Indeed, if $z \succ a$, then the conditional distribution of W_z given $\mathcal{F}_a$ is Gaussian with mean W_a and variance $|z| - |a|$, so that we have

$$E\{e_z^2|\mathcal{F}_a\} = \left(2\pi|z|^2(|z| - |a|)\right)^{-\frac{1}{2}} \int_{\mathbb{R}} \exp(y^2/|z|)\exp\left(-(y - W_a)^2/2(|z| - |a|)\right)dy.$$

This converges if $|z| < 2|a|$ and diverges if $|z| \geqq 2|a|$. The same cosiderations hold for the derivative e_z'. It is then easy to see that if $M > 0$, the process $\{e_z I_{\{|W_a| < M\}}, z \succ a, |z| < 2|a|\}$ is holomorphic, square-integrable, and, following [1], has an expansion in Hermite polynomials. This expansion will converge on $\{z: z \succ a, |z| < 2|a|\}$, but will diverge, in general, as soon as the process ceases to be square-integrable, namely for $|z| > 2|a|$. However the process itself can be extended to be holomorphic in all of $\{z: z \succ a\}$.

Example 3. The property of local square-integrability seems natural. For instance, if $u(x,t)$ is a positive solution of the backward heat equation in the strip $\{(x,t): 0 \leqq \alpha < t < \beta\}$, it has the representation [4]

$$u(x,t) = (\beta - t)^{-\frac{1}{2}} \int_{\mathbb{R}} \exp\left(-(y - x)^2/2(\beta - t)\right)d\mu(y), \quad \alpha < t < \beta,$$

where μ is a measure for which

$$\int_{\mathbb{R}} \exp\left(-y^2/2(\beta - t)\right)d\mu(y) < \infty, \quad \alpha < t < \beta.$$

Then, just as in example 2, the process

$$\Phi_z = u(W_z, |z|), \quad \alpha < |z| < \beta,$$

is holomorphic. While it may not be square-integrable, an argument similar to the above shows that it satisfies the local square-integrability property of the preceding example.

Example 4. We modify example 2 slightly. Let

$$e_z^\tau = ||z| - \tau|^{-\frac{1}{2}}\exp\bigl(W_z^2/2(|z| - \tau)\bigr).$$

Then $\{e_z^\tau, |z| > \tau\}$ is holomorphic and locally square-integrable. It blows up as $|z|$ approaches τ, so it cannot be extended into $\{z: |z| \leqq \tau\}$. Thus $\{z:|z| > \tau\}$ is evidently a region of holomorphy for this process, but it is not one of the regions of holomorphy described in [3]. Thus the regions of holomorphy change when the requirements of square-integrability are relaxed.

Example 5. If we regard e_z^τ for $|z| < \tau$, we get a quite different behavior. The process $\{e_z^\tau, |z| < \tau\}$ can be extended past the hyperbola $|z| = \tau$. Indeed, the function

$$h^\tau(x,t) = \begin{cases} (\tau - t)^{-\frac{1}{2}}\exp\bigl(x^2/2(t - \tau)\bigr) & \text{if } t < \tau, \\ 0 & \text{if } t \geqq \tau, \end{cases}$$

satisfies the backward heat equation except at the single point $x = 0$, $t = \tau$. Thus if we set

$$h_z^\tau = h^\tau(W_z, |z|),$$

then h_z^τ will be strongly holomorphic in the non-simply connected random domain $D = \mathbb{R}_+^2 - \{z: |z| = \tau, W_z = 0\}$. Surprisingly enough, h_z^τ is holomorphic on all of $\mathbb{R}_+^2$. Indeed, if Γ is any increasing path, the probability that Γ encounters any singularity of h_z^τ is zero. To see this, just note that if $z_0 = \Gamma \cap \{z:|z|=\tau\}$, then h_z^τ is singular at z_0 if and only if $W_{z_0} = 0$. Since W_{z_0} is Gaussian with variance τ, $P\{W_{z_0} = 0\} = 0$. It follows that h_z^τ satisfies (1) on Γ. Is h_z^τ strongly holomorphic on all of $\mathbb{R}_+^2$? We do not know the answer to this. We point out that along the path $\{z: |z| = \tau\}$, which is of pure type, the process is identically zero, so that (1) certainly holds. However, it is possible to conceive of more complicated paths of pure type, along which the process may not be continuous.

Example 6. Here is an example which indicates that one should perhaps consider integration along random paths as well as fixed paths when defining holomorphic processes.

Let $h^{\tau}(x,t)$ be the function defined in Example 5, let $T(\omega) = 1+|W_{1,1}(\omega)|$, and define a process Φ by

$$\Phi_z(\omega) = h^{T(\omega)}(W_z(\omega), |z|), \quad z \succ (1,1).$$

This process is locally square-integrable, and the set of its singularities is $S(\omega) = \{z: z \succ (1,1), |z| = T(\omega), W_z(\omega) = 0\}$.

We claim that if $A \subset \mathbb{R}_+^2$ is a set of Lebesgue measure zero, $P\{A \cap S = \emptyset\} = 1$. Indeed, $P\{A \cap S = \emptyset\} = E\{P\{A \cap S = \emptyset | \mathcal{F}_{1,1}\}\}$. Now T is $\mathcal{F}_{1,1}$-measurable and, on the set $\{T = t\}$, $P\{A \cap S = \emptyset | \mathcal{F}_{1,1}\} = 1$ if the Lebesgue measure of $A \cap \{z: |z| = t\}$ is zero. By Fubini, this is true for a.e. (Lebesgue) t, and this implies our claim, since the distribution of T is diffuse.

In particular, the probability that any given piecewise-pure path encounters a singularity of Φ is zero, so that Φ is holomorphic, even strongly holomorphic in $\{z: z \succ (1,1)\}$. However, if we allow random paths, the path $L(\omega) = \{z: |z| = T(\omega)\}$ is a random path of pure type which passes through all the singularities of Φ.

Example 7. In the square-integrable case, we gave an example of a holomorphic process in a domain of holomorphy which couldn't be extended to be square-integrable and holomorphic in any larger domain (cf. [3]). In view of example 2, one might ask whether such a process could be extended to be holomorphic without being square-integrable. We will construct a square-integrable holomorphic process on the domain of holomorphy

$$D = \{z: z \succ z_o, \; |z| < \tau\},$$

where $z_o \in \mathbb{R}^2_+$ and $\tau > |z_o|$ are given, which cannot be extended to be strongly holomorphic in any larger domain.

Let g be a lower semi-continuous function on $\mathbb{R}$ such that

(a) $\int_{\mathbb{R}} g(x)dx \leqq 1$;

(b) the set $\{x: g(x) = \infty\}$ is dense.

Define, for $t < \tau$,

$$f(x,t) = (\tau - t)^{-\frac{1}{2}} \int_{\mathbb{R}} g(y)\exp\big(-(y - x)^2/2(\tau - t)\big)dy. \tag{2}$$

Then $f(x,t) \leqq (\tau - t)^{-\frac{1}{2}}$ and f satisfies the backward heat equation on $\{(x,t): 0 \leqq t < \tau\}$. Furthermore, for each $x \in \mathbb{R}$,

$$\liminf_{y\to x,\; t\uparrow\tau} f(y,t) \geqq g(x). \tag{3}$$

This can be seen by first noting that if h is bounded and continuous on $\mathbb{R}$ and $0 \leqq h \leqq g$, then $\liminf_{y\to x,\; t\uparrow\tau} f(y,t) \geqq h(x)$, and then using the fact g is an increasing limit of such functions.

Let $L = \{z: |z| = \tau\}$ and $K = \{z: |z| < \tau\}$. The process $\{f(W_z, |z|), z \in K\}$ is square-integrable — even bounded for fixed z — and strongly holomorphic. Furthermore, there exists a dense random subset of $z \in L$ such that

$$\liminf_{\zeta\to z,\; \zeta \in K} f(W_\zeta, |\zeta|) = \infty.$$

This does not quite verify that L is a natural boundary. However, if it is not, there exists an open disc U for which $U \cap L \neq \emptyset$, and a process Φ, strongly holomorphic in U, for which $\Phi_z = f(W_z, |z|)$ if $z \in U \cap K$. Now Φ must be a.s. continuous on a given horizontal line, for it is the stochastic integral of its derivative. Thus, by (3), $\Phi_z \geqq g(W_z)$ a.s. for each $z \in U \cap L$, hence for a.e. ω, $\Phi_z(\omega) \geqq g(W_z(\omega))$ for a.e. $z \in U \cap L$. There is then no way that the equation (1) can hold over any portion Γ of the path of pure type $U \cap L$. Indeed, a stochastic integral is a continuous function of its upper limit, whereas Φ is everywhere discontinuous along L.

We remark in passing that a minor modification of this argument shows that there can be no extension, even to a random neighborhood.

Finally, to get the desired example of a process Ψ which has D as its domain of holomorphy, we need only put

$$\Psi_z = W_{z_0} + f(W_z, |z|), \quad z \in D.$$

Ψ is square-integrable, but can not be extended. Indeed, the boundaries of D are the line L and the horizontal and vertical lines H and V, respectively, which pass through z_0. We have just seen that Ψ can't be extended across L because of the singularities. On the other hand, it can't be extended across either H or V and remain adapted (cf. [3], § 5).

REFERENCES

[1] R. Cairoli and J.B. Walsh. Stochastic integrals in the plane. Acta mathematica, Vol. 134, 1975, p. 111-183.

[2] R. Cairoli and J.B. Walsh. Martingale representations and holomorphic processes. Annals of Probability (to appear).

[3] R. Cairoli and J.B. Walsh. Prolongements de processus holomorphes. Cas "carré intégrable". This Séminaire.

[4] D. V. Widder. Positive temperatures on an infinite rod. TAMS, 55, 1944, p. 85-95.