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NOTE ON PASTING OF TWO MARKOV PROCESSES

By Masao NAGASAWA

The author remarked(e.g. in his lectures at UCSD in 1967/68 and at Erlangen in 1973/74) that Courrege-Priouret's theorem [1] of pasting two continuous strong Markov processes is a simple corollary of the theorem of piecing out(or revival) of a Markov process which was given by the author(cf.[3]). Since this remark was published nowhere, it would be of use to explain the procedure, though Meyer gave similar one recently in [4], which contains other interesting applications.

Let U and V be open subsets of a Polish space and adjoin ∂ as an extra point to U and V as usual. Suppose we have continuous strong Markov processes $(W^U, \underline{\underline{B}}_t^U, x_t^U, P_x^U)$ on U and $(W^V, \underline{\underline{B}}_t^V, x_t^V, P_x^V)$ on V which are killed at each boundary. Suppose $U \cap V \neq \phi$ and moreover

(1) x_t^U <u>and</u> x_t^V <u>coincide</u> <u>on</u> $U \cap V$ (the consistency condition),

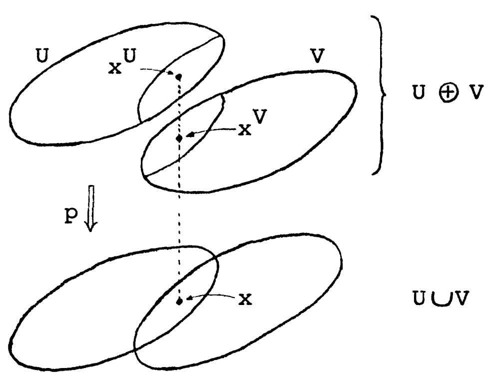
that is, if we kill the two processes at the first leaving time T from $U \cap V$, we get the same Markov process on $U \cap V$. The problem is <u>to construct a continuous strong Markov process x_t on $U \cup V$ pasting the two processes</u>, in other wards, to obtain a process x_t which moves on x_t^U in U and on x_t^V in V. This pasting technique is an important tool, for example, when we construct diffusions on a Riemanian Manifold.

(i) Let $U \oplus V$ be the direct sum of U and V. When $x \in U \cap V$, denote x^U the corresponding point in U and x^V in V as a point in $U \oplus V$ (as is clear from Figure 1). The natural identification mapping p from $U \oplus V$ onto $U \cup V$ is defined by

$$p(x^U) = p(x^V) = x,$$

and we call it the pasting operator(not projection!).

Fig.1.



First we construct a process on $U \oplus V$ (treating U and V as separated sheets) and get the one on $U \cup V$ (pasting U and V). Let us define a continuous strong Markov process $(W^\circ, \underline{\underline{B}}^\circ_t, x^\circ_t, P^\circ_x)$ on $U \oplus V$, putting

$$W^\circ = W^U \oplus W^V, \qquad \underline{\underline{B}}^\circ_t = \underline{\underline{B}}^U_t \oplus \underline{\underline{B}}^V_t,$$

$$P^\circ_x = \begin{cases} P^U_x, & x \in U, \\ P^V_x, & x \in V, \end{cases} \qquad x^\circ_t = \begin{cases} x^U_t, & \text{on } W^U, \\ x^V_t, & \text{on } W^V. \end{cases}$$

Next, let us define a kernel on $(U \oplus V) \cup \{\partial\}$ by

$$n(x,\cdot) = \begin{cases} \varepsilon_{x^V} & \text{if } x = x^U \in (\partial U) \cap V, \\ \varepsilon_{x^U} & \text{if } x = x^V \in (\partial V) \cap U, \\ \varepsilon_\partial & \text{if otherwise,} \end{cases}$$

where ε_a denote the point mass at a, and put

(2) $$N(w^\circ,\cdot) = \begin{cases} n(x^U_{\zeta-}(w^U),\cdot) & \text{if } w^\circ = w^U \in W^U, \\ n(x^V_{\zeta-}(w^V),\cdot) & \text{if } w^\circ = w^V \in W^V. \end{cases}$$

Then N is a "revival kernel" (it was called "instantaneous distribution" in [3]). Now apply the piecing out (or revival) theorem to x°_t and N, obtaining a right continuous strong Markov process $(\bar{W}, F_t, X_t, \bar{P}_x)$ on $U \oplus V$ (cf. figure 2). To obtain a continuous strong Markov process on $U \cup V$, that we are looking for, what we need to do is to apply the pasting operator p:

(3) $$x_t = p(X_t), \qquad P_x = \bar{P}_{p^{-1}(x)}, \quad x \in U \cup V.$$

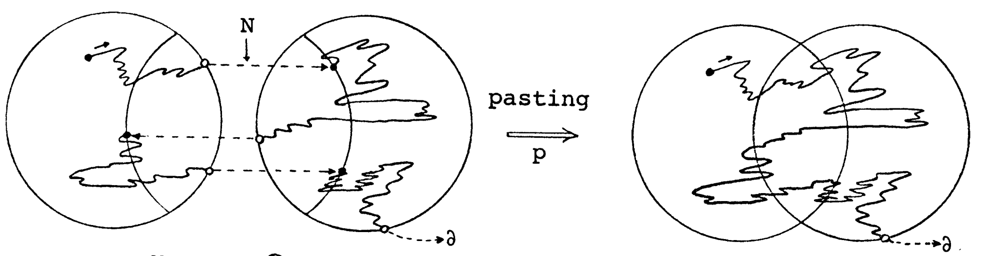


Fig. 2.

Since the pasting operator is a transformation of the state space (cf. [2], p.325), to be well-defined it is enough to prove

(4) $$\bar{E}_{x^U}[f \circ p(X_t)] = \bar{E}_{x^V}[f \circ p(X_t)], \qquad x \in U \cap V,$$

for f on $U \cup V$. The right hand side of (4) is equal to

(5) $$\bar{E}_{x^U}[f \circ p(X_t); t<T] + \bar{E}_{x^U}[\bar{E}_{X_T}[f \circ p(X_{t-s})]|_{s=T}; t \geqq T],$$

where T is the first leaving time from $U\cap V$. But from the way of construction of X_t, the first term of (5) is equal to

$$E_x^U[f(x_t^U);t<T],$$

which is equal to

$$E_x^V[f(x_t^V);t<T],$$

by the consistency condition (1), therefore is equal to

$$\bar{E}_{x^V}[f\circ p(X_t);t<T].$$

The second term of (5) is equal to

$$\bar{E}_{x^V}[\bar{E}_{X_T}[f\circ p(X_{t-s})]|_{s=T};t\geq T],$$

because

$$\bar{P}_{x^U}[X_T\in\cdot] = \bar{P}_{x^V}[X_T\in\cdot]. \tag{6}$$

Thus we have (4). It is clear that $x_t = p(X_t)$ is continuous.

(ii) Now, let x_t^U and x_t^V be strong Markov process on U and V, resp. We don't assume any continuity of paths, but assume $\exists\ x_{\zeta-}^U \in \bar{U}$ and $\exists\ x_{\zeta-}^V \in \bar{V}$, and the consistency condition (1). Let $n(x,dy)$ be a probability kernel on $\overline{(U\oplus V)}\cup\{\partial\}$ satisfying

$$\text{(7)}\qquad \begin{array}{l} \underline{\text{the support of}}\ n(x^U,\cdot)\ \underline{\text{is contained in}}\ (V\setminus U)\cup\{\partial\}, \\ \underline{\text{the support of}}\ n(x^V,\cdot)\ \underline{\text{is contained in}}\ (U\setminus V)\cup\{\partial\}. \end{array}$$

Define a revival kernel $N(w^\circ,\cdot)$ by (2) using this kernel $n(x,dy)$. To keep consistency for jumps of paths of the process which will be constructed, we assume the second consistency condition: When $x\in U\cap V$,

$$\text{(8)}\qquad \begin{array}{l} P_{x^U}^U[n(x_{\zeta-}^U,\cdot);T=\zeta] = P_{x^V}^V[x_T^V\in\cdot;T<\zeta], \\ P_{x^V}^V[n(x_{\zeta-}^V,\cdot);T=\zeta] = P_{x^U}^U[x_T^U\in\cdot;T<\zeta]. \end{array}$$

Let T_U and T_V be the first leaving times from U and V, resp., then $x_t = p(X_t)$ is a strong Markov process on $U\cup V$ satisfying that x_t killed at T_U is equal to x_t^U, x_t killed at T_V is equal to x_t^V, and

$$\text{(9)}\qquad \begin{array}{ll} P_z[x_{T_U}\in\cdot\,|\,x_{T_U-}=x] = n(x,\cdot), & \text{if } x\in U, \\ P_z[x_{T_V}\in\cdot\,|\,x_{T_V-}=x] = n(x,\cdot), & \text{if } x\in V. \end{array}$$

If x_t^U and x_t^V are right continuous, x_t on $U\cup V$ is also right continuous. The equality (4) is proved in the same way. The equality (6) in this case is verified as follows: Putting $a = x^U$ and $b = x^V$, $x \in U \cap V$,

$$\begin{aligned}\bar{P}_a[X_T \in \cdot] &= \bar{P}_a[X_T \in \cdot; T=r_1] + \bar{P}_a[X_T \in \cdot; T<r_1] \\ &= P_x^U[n(x_{\zeta-}^U, \cdot); T=\zeta] + P_x^U[x_T^U \in \cdot; T<\zeta] \\ &= P_x^V[x_T^V \in \cdot; T<\zeta] + P_x^V[n(x_{\zeta-}^V, \cdot); T=\zeta] \\ &= \bar{P}_b[X_T \in \cdot; T<r_1] + \bar{P}_b[X_T \in \cdot; T=r_1] \\ &= \bar{P}_b[X_T \in \cdot],\end{aligned}$$

where r_1 is the first jumping(or revival) time of X_t. The second and the fourth equalities are by the revival theorem, and the third is by the second consistency condition (8). (9) is clear because

$$P_z[x_{T_U} \in \cdot | x_{T_U-}=x] = \bar{P}_z[X_{r_1} \in \cdot | X_{r_1-}=x] = n(x,\cdot).$$

(iii) Appendix.

DEFINITION. A probability kernel $N(w,dy)$ on $W \times \underline{\underline{B}}(S)$ is revival kernel (or instantaneous distribution) if it satisfies

(i) $N(\theta_t w, \cdot) = N(w,\cdot)$, if $t < \zeta(w)$, and

(ii) $N(w,\cdot) = \varepsilon_\partial$, if $\zeta(w) = 0$.

THEOREM (of revival or piecing out). Let $(W, \underline{\underline{B}}_t, x_t, P_x)$ $x \in S$ be a Markov process and N be a revival kernel. Then there exists a Markov process $(\bar{W}, \underline{\underline{F}}_t, X_t, \bar{P}_x)$ on S satisfying

(i) X_t, $t < r_1$, is equivalent to x_t,

(ii) $\bar{P}_x[X_{r_1} \in \cdot | \underline{\underline{F}}_{r_1-}] = N(w_1,\cdot)$,

where $\bar{W} = W^{(1,2,3,\cdots)} \ni \bar{w} = (w_1, w_2, \cdots)$ and $r_1(\bar{w}) = \zeta(w_1)$. If x_t is strong Markov , so is X_t, and if x_t is right continuous, so is X_t. If x_t is continuous and if $N(w,\cdot) = \varepsilon_{x_{\zeta-}(w)}$, then X_t is continuous.

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