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SOME OSCILLATORY INTEGRALS AND THEIR APPLICATIONS

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We shall discuss several kinds of oscillatory integrals, and we describe briefly their applications to some questions related to the $\bar{\partial}$-Neumann problem, real-variable problems connected with certain maximal functions, and analysis on nilpotent Lie groups. The three kinds of oscillatory integrals we shall treat are in appearance quite different from each other, but in reality the ideas used to deal with each have much in common. What follows describes some joint work done with D.H. PHONG, C.D. SOGGE and F. RICCI.

1. OSCILLATORY INTEGRALS CONNECTED WITH RADON SINGULAR INTEGRALS.

For further details about the material of this section see the work of PHONG and the author [3], [4].

We place ourselves in $\mathbb{R}^n$, and we are given a real C^∞ phase function $\Phi(x,y)$, defined on $\mathbb{R}^n \times \mathbb{R}^n$. $K(x)$ will denote a function in $C^\infty(\mathbb{R}^n/\{0\})$, which is homogeneous of degree $-\mu$, $0 < \mu \leq n$. When $\mu = n$, we also demand that the mean-value of K on the unit sphere vanishes. We will also be given a fixed cut-off function, $\Psi(x,y) \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. With these and for each $\lambda > 0$, we form the family of transformations T_λ, defined by :

$$(1) \qquad T_\lambda(f)(x) = \text{P.V.} \int_{\mathbb{R}^n} e^{i\lambda\Phi(x,y)}\, \Psi(x,y)\, K(x-y)\, f(y)\, dy$$

Theorem 1. *Suppose that the rank of the Hessian of* Φ $\left(= \left\{\frac{\partial^2}{\partial x_i}\frac{\Phi}{\partial y_j}(x,y)\right\}\right)$, *is not less than* k, $1 \leq k \leq n$, *and* $n-k < \mu \leq n$. *Then* $\|T_\lambda\|$, *the norm of* T_λ *as a bounded operator on* $L^2(\mathbb{R}^n)$ *to itself, satisfies*

$$\|T_\lambda\| \leq A(1+\lambda)^{-(n-\mu)/2} .$$

Note that in the limiting case (when $k = n$, and $\mu \to 0$), and when $K(x)$ is taken to be $\equiv 1$, then T_λ corresponds to a variant of the Fourier transform which was studied by CARLESON-SJOLIN and HORMANDER [2]. The meaning of this theorem is that the different oscillations due respectively to $K(x)$ and $e^{i\lambda\Phi(x,y)}$ do not interfere , but in fact reinforce each other as concerns their beneficial effects.

The significance of the operators T_λ defined by (1) is that they occur as symbols of certain pseudodifferential operators - the singular Radon integrals - which themselves play a role in to $\bar\partial$-Neumann problem.

2. OSCILLATORY INTEGRALS CONNECTED WITH FOURIER TRANSFORMS OF SURFACE-CARRIED MEASURES.

For further details about the work described in this section see the forthcoming paper of SOGGE and the author [6].

Let S denote a C^∞ hypersurface in $\mathbb{R}^n$. We shall denote by $d\sigma(x)$ the induced (Lebesgue) measure on S, and for each $x \in S$, $K(x)$ will denote the Gaussian curvature of S at x. We fix a cut-off function Ψ, with $\Psi \in C_0^\infty(S)$.

Theorem 2.

$$\int_S e^{ix.\xi}(K(x))^{2n-2}\;\Psi(x)\,d\sigma(x)=O(|\xi|^{-(n-1)/2})$$

as $|\xi| \to \infty$.

Note that when $K(x)$ is non-vanishing in the support of Ψ, (then the factor $K(x))^{2n-2}$ is irrelevant), this is a classical result of VAN DER CORPUT when $n = 2$, and HLAWKA for $n \geq 3$. These earlier results were used to study the number of lattice points contained in dilates of S. The present result can be used in the study of the maximal function

$$\mathcal{M}(f)(x) = \sup_{0<t<\infty} \left|\int_S f(x-ty)\;\Psi(y)\;d\sigma(y)\right|.$$

It can be shown that whenever S is such that its curvature $K(x)$ does not vanish of infinite order at any $x \in S$, then there is an L^p-inequality

$$\text{(2)}\qquad \|\mathcal{M}(f)\|_{L^p(\mathbb{R}^n)} \leq A_p\,\|f\|_{L^p(\mathbb{R}^n)}\;,$$

for a range of p, $p_0 < p \leq \infty$, with $p_0 = p_0(S)$, if also $n \geq 3$. Earlier inequalities of this type for the sphere, or for S with nowhere vanishing curvature are in [8] and [1]. The maximal inequality (2) holds in particular when S is compact and real-analytic.

3. SOME OTHER OSCILLATORY INTEGRALS.

Finally we mention some results obtained jointly with F. RICCI [5], which have some connection with theorem 1 and also with some questions in harmonic analysis of nilpotent Lie groups.

We let $K(x)$ denote a function in $C^\infty(\mathbb{R}^n/\{0\})$, which is homogeneous of degree $-n$, with vanishing mean-value on the unit sphere. We denote by $P(x,y)$ a real polynomial on $\mathbb{R}^n \times \mathbb{R}^n$ of total degree $\leq d$, and consider the operator T defined by

$$(3) \qquad (Tf)(x) = \mathrm{P.V.} \int_{\mathbb{R}^n} e^{iP(x,y)} K(x-y)\, f(y)\, dy$$

Theorem 3. *The operator* T *given by (3) is bounded on* $L^2(\mathbb{R}^n)$ *to itself, with a bound which is independent of the polynomial* P, *as long as* P *is restricted to be of degree* $\leq d$, *(here* d *is arbitrary)*.

There are also similar results when one replaces $\mathbb{R}^n$ by any homogeneous nilpotent group, with $K(x-y)$ replaced by $K(x.y^{-1})$, where K has the critical degree of homogeneity, together with the vanishing of its mean-value on the unit sphere. Also the L^p results hold, when $1 < p < \infty$.

The special case of (3) when $P(x,y)$ is a billinear form in x and y, (an operator having close relation to (1)) is dealt with in [3]. Another special case, when $P(x,y) = P(x-y)$, i.e. when the operator is translation invariant, is in [7].

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