Friedrich Hirzebruch

Intersection numbers of curves on Hilbert modular surfaces

Séminaire Delange-Pisot-Poitou. Théorie des nombres, tome 17, nº 1 (1975-1976), exp. nº 22, p. 1-3
<http://www.numdam.org/item?id=SDPP_1975-1976__17_1_A20_0>

Séminaire DELANGE-PISOT-POITOU
(Théorie des nombres)
17e année, 1975/76, n° 22, 3 p.

15 mars 1976

INTERSECTION NUMBERS OF CURVES ON HILBERT MODULAR SURFACES

by Friedrich HIRZEBRUCH

The lecture concerned joint work with D. ZAGIER which in the meantime has appeaered under the title "Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus" [1].

We give here a short résumé.

Let $p \equiv 1 \pmod 4$ be a prime, $\mathcal{O}$ the ring of integers of $\mathbb{Q}(\sqrt{p})$. The group $SL_2(\mathcal{O})$ operates on the product of the upper half-plane with itself by

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}(z_1 , z_2) = \left(\frac{\alpha z_1 + \beta}{\gamma z_1 + \delta} , \frac{\alpha' z_2 + \beta'}{\gamma' z_2 + \delta'}\right) , \quad (z_1 , z_2) \in \mathfrak{H}^2 .$$

The quotient is a non-compact complex surface X with finitely many quotient singularities. On X we define a series of curves $T_1 , T_2 , \ldots$ as follows : given N , we consider all points $(z_1 , z_2) \in \mathfrak{H}^2$ satisfying some equation of the form

$$a\sqrt{p} z_1 z_2 + \lambda z_2 - \lambda' z_1 + b\sqrt{p} = 0$$

with $a , b \in \mathbb{Z}$, $\lambda \in \mathcal{O}$, $\lambda\lambda' + abp = N$; this set is an invariant under $SL_2(\mathcal{O})$, and T_N denotes its image in X. The curves T_M and T_N meet transversally if MN is not a square. This assumed, the intersection number of T_M and T_N in X is given by the following formula (Suppose that the exponent $\nu_p(N)$ of p in N is less or equal to the exponent $\nu_p(M)$ of p in M).

$$(1) \qquad (T_M T_N)_X = \frac{1}{2} \sum_{d|(M,N)} (d\chi_p(d) + d\chi_p(\tfrac{N}{d})) H_p(MN/d^2)$$

where $\chi_p(d) = (\frac{d}{p})$ and

$$H_p(N) = \sum_{x \in \mathbb{Z}, x^2 \leq 4N, x^2 \equiv 4N \pmod p} H(\frac{4N - x^2}{p})$$

and

$$H(n) = \sum_{d^2 | n} h(- n/d^2) \quad \text{if} \quad n > 0 , \text{ and } H(0) = - 1/12 .$$

Here d runs through those natural numbers such that $- n/d^2$ is a discriminant and h is the class number with the understanding that $h(- 4) = 1/2$, $h(-3) = 1/3$.

To prove the formula (1) which is one of the main results of the paper we study those points $\mathfrak{z}$ on X which are special in the sense that there exist two curves

T_M, T_N which meet transversally in $\mathfrak{z}$. For such a point $\mathfrak{z}$ we choose a representative $(z_1, z_2) \in \mathfrak{H}^2$ and consider the binary lattice $\mathfrak{M}$ of all skew-hermitian matrices

$$A = \begin{pmatrix} a\sqrt{p} & \lambda \\ -\lambda' & b\sqrt{p} \end{pmatrix} \qquad (a, b \in \mathbb{Z}, \lambda \in \mathcal{O})$$

with $a\sqrt{p}z_1 z_2 + \lambda z_2 - \lambda' z_1 + b\sqrt{p} = 0$. On $\mathfrak{M}$, we have the binary quadratic form

$$A \mapsto abp + \lambda\lambda'$$

which is positive definite, has a discriminant divisible by p and represents only quadratic residues modulo p. It may have a content $m > 1$ (content = greatest common divisor of all values taken by the quadratic form). To prove (1), we wish to know how often a positive definite quadratic form φ of discriminant $\Delta \equiv 0 \pmod p$, and content m occurs as the quadratic form associated to a special point $\mathfrak{z} \in X$. Let $s(\varphi)$ be the "number" of special points in X having a quadratic form which is $SL_2(\mathbb{Z})$-equivalent to φ. Then

$$(2) \qquad s(\varphi) = \frac{1}{2}(1 + \chi_p(\varphi_0))\, \beta_p(m)\, h(\Delta/p)$$

where φ_0 is the primitive form corresponding to φ,

$$\chi_p(\varphi_0) = 0 \text{ if } p \nmid (\Delta/m^2), \quad \chi_p(\varphi_0) = 1 \text{ if } p \mid (\Delta/m^2),$$

and φ_0 represents only quadratic residues (mod p), and $\chi_p(\varphi_0) = -1$ otherwise. The number $\beta_p(m)$ in (2) equals $\prod_{q|m} (1 + \chi_p(q))$ if $p^2 \nmid m$ and is 0 if $p^2 \mid m$.

The proof of (1) by (2) is a complicated calculation using elementary facts on the representation of binary quadratic forms by binary quadratic forms.

We also consider the compact surface $\tilde{X}$ obtained by adding to X the "cusps" and resolving the singularities thus created. The compactification of the curve T_N represents a cycle in the middle homology group $H_2(\tilde{X})$. This group decomposes canonically as the direct sum of the image of $H_2(X)$ and the subspace generated by the homology cycles of the curves of the cusp resolution, we denote by T_N^c the component of T_N in the first summand. We wish to calculate the intersection number

$$T_M^c\, T_N^c = T_M\, T_N^c = T_M^c \cdot T_N$$

in all cases (also when T_M and T_N have common components). For this, we need complete information how the curves T_N pass through the resolved cusps. The result is the following. We assume again $\nu_p(N) \leq \nu_p(M)$. Then

$$(3) \qquad T_N^c\, T_N^c = \frac{1}{2}\sum_{d|(M,N)} (d\chi_p(d) + d\chi_p(\tfrac{N}{d}))(H_p(\tfrac{MN}{d^2}) + I_p(\tfrac{MN}{d^2}))$$

where $I_p(N) = \frac{1}{\sqrt{p}}\sum_{\lambda\in\mathcal{O},\lambda>>0,\lambda\lambda'=N} \min(\lambda, \lambda')$. (This is a convergent series.)

Interest in the intersection numbers arose from the classification problem for Hilbert modular surfaces where special configurations of curves where needed. Later it was conjectured that the $T_M^C T_N^C$ are Fourier coefficients of modular forms. In fact, the following holds. For $N > 0$, consider the Fourier series $(q = \exp 2\pi i z)$

$$(4) \qquad -\frac{1}{24}\sum_{d|N} (\chi_p(d) + \chi_p(N/d))\, d + \sum_{M=1}^{\infty} (T_M^C\, T_N^C)\, q^M .$$

This is a modular form for $\Gamma_0(p)$ of Nebentypus, indeed it belongs to the subspace $M_2^+(\Gamma_0(p), \chi_p)$ (of half the dimension) of modular forms having the property that the N-th Fourier coefficient is 0 whenever $(\frac{N}{p}) = -1$. The basic result due to ZAGIER is that (4) is such a modular form for $N = 1$. The result in general, then follows by the use of Hecke operators.

Because

$$\dim M_2^+(\Gamma_0(p), \chi_p) = \left[\frac{p+19}{24}\right] ,$$

the infinite matrix $(T_M^C\, T_N^C)$ has rank $\leq \left[\frac{p+19}{24}\right]$. This implies class number relations. Moreover, the natural map from the subspace of $H(\tilde{X}, \underset{\sim}{C})$ generated over $\underset{\sim}{C}$ by the T_N^C to $M_2^+(\Gamma_0(p), \chi_p)$ is injective, and we conjecture it to be subjective, or equivalently, we conjecture that the infinite matrix $(T_M^C\, T_N^C)$ has rank equal to $\left[\frac{p+19}{24}\right]$. This has been verified by computer for $p < 200$.

BIBLIOGRAPHY

[1] ZAGIER (D.). - Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus, Inventiones math., t. 36, 1976, p. 57-113.

(Texte reçu le 15 octobre 1976)

Friedrich HIRZEBRUCH
7 Endenicher Allee
D-53 BONN 1
(Allemagne fédérale)