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REPORT ON IGUSA'S LOCAL ZETA FUNCTION

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Igusa's local zeta functions are related to the number of solutions of congruences mod p^m and to exponential sums mod p^m. This report is a survey on what is known about these zeta functions. There are several conjectures and intriguing connections with topology and singularity theory. They will be emphasized throughout the paper (especially sections 2 and 4). The case of curves is well-understood and is explained in section 5 without any calculations. Much less is known in higher dimensions, although there is by now a lot of experimental evidence (section 6) for the monodromy conjecture which relates the poles of Igusa's zeta functions to local monodromy. Relative invariants of prehomogeneous vector spaces are discussed in section 7. They provide very interesting examples which motivated the conjectures. The adelic situation is only mentioned briefly in 7.7. We will not treat the connection with Siegel-Weil formulas, but refer for this to Igusa's book [30, Chap. 4] and his survey paper [34]. At the end we briefly discuss the theory of p-adic subanalytic sets which yields very general rationality results.

1. FIRST PROPERTIES OF LOCAL ZETA FUNCTIONS

1.1 Local zeta functions

(1.1.1) Let K be a p-adic field, i.e. $[K\colon Q_p] < \infty$. Let R be the valuation ring of K, P the maximal ideal of R, and $\overline{K} = R/P$ the residue field of K. The cardinality of $\overline{K}$ is denoted by q, thus $\overline{K} = F_q$. For $z \in K$, ord $z \in \mathbb{Z} \cup \{+\infty\}$ denotes the valuation of z, $|z| = q^{-\operatorname{ord} z}$, and $\operatorname{ac}(z) = z\pi^{-\operatorname{ord} z}$ where π is a fixed uniformizing parameter for R.

(1.1.2) Let $f(x) \in K[x], x = (x_1, \ldots, x_n), f \notin K$. Let $\Phi : K^n \to \mathbb{C}$ be a Schwartz-Bruhat function, i.e. a locally constant function with compact support. Finally, let χ be a character of $R^\times$, i.e. a homomorphism $\chi : R^\times \to \mathbb{C}^\times$ with finite image, where $R^\times$ denotes the group of units of R. We formally put $\chi(0) = 0$.

(1.1.3) To the above data one associates Igusa's local zeta function

$$Z_\Phi(s,\chi) = Z_\Phi(s,\chi,K,f) := \int_{K^n} \Phi(x)\chi(\mathrm{ac}f(x))|f(x)|^s|dx| \ ,$$

for $s \in \mathbb{C}, \mathrm{Re}(s) > 0$, where $|dx|$ denotes the Haar measure on K^n so normalized that R^n has measure 1. These zeta functions were introduced by Weil [83] and their basic properties for general f were first studied by Igusa [28], [30]. We will see below that $Z_\Phi(s,\chi)$ is rational in q^{-s}, so that it extends to a meromorphic function on $\mathbb{C}$.

We will write Z, resp. Z_0, instead of Z_Φ, when Φ is the characteristic function of R^n, resp. PR^n. Throughout this paper, we put $t = q^{-s}$. Note that $Z_\Phi(s,\chi)$ is a power series in t. The coefficient of t^m in a power series $P(t)$ is denoted by $\mathrm{Coeff}_{t^m}P(t)$. We denote the trivial character by χ_{triv} and the support of Φ by Supp Φ.

(1.1.4) *Remark.* Note that $\mathrm{ac}(f(x))$ and hence also $Z_\Phi(s,\chi,K,f)$ depend on the choice of the uniformizing parameter π. More canonically one introduces

$$Z_\Phi(w,K,f) = \int_{K^n} \Phi(x)w(f(x))|dx|$$

as a function of a quasicharacter w of $K^\times$ (i.e. a continuous homomorphism $w\colon K^\times \to \mathbb{C}^\times$). Every quasicharacter w of $K^\times$ is of the form $w(y) = \chi(\mathrm{ac}(y))|y|^s$. Thus studying $Z_\Phi(w,K,f)$ is equivalent with studying $Z_\Phi(s,\chi,K,f)$.

Sometimes it is also helpfull to think of Z_Φ as a distribution $\Phi \mapsto Z_\Phi$.

1.2 Number of solutions of congruences

(1.2.1) Suppose $f(x)$ has coefficients in R. Let N_m be the number of solutions of $f(x) \equiv 0 \bmod P^m$ in R/P^m and put $P(t) := \sum_{m=0}^{\infty} q^{-nm}N_m t^m$. The Poincaré series $P(t)$ is directly related to $Z(s,\chi_{triv})$ by the formula

$$P(t) = \frac{1 - tZ(s,\chi_{triv})}{1-t}.$$

Indeed, since $q^{-nm}N_m$ equals the measure of $\{x \in R^n | \mathrm{ord}\ f(x) \geqslant m\}$, this follows directly from

$$(1.2.2) \qquad \int_{\mathrm{ord}\ f(x) \geqslant m} \Phi(x)|dx| = Z_\Phi(0,\chi_{triv}) - \mathrm{Coeff}_{t^m}\frac{tZ_\Phi(s,\chi_{triv})}{1-t} \ .$$

To verify this last equality note that the left-hand side equals

$$\int \Phi(x)|dx| - \sum_{k\leqslant m-1} \int_{\mathrm{ord}\ f(x)=k} \Phi(x)|dx| = Z_\Phi(0,\chi_{triv}) - \sum_{k\leqslant m-1} \mathrm{Coeff}_{t^k} Z_\Phi(s,\chi_{triv}).$$

1.3 Rationality of local zeta functions

(1.3.1) *Resolutions.* Put $X = \mathrm{Spec}\ K[x]$ and $D = \mathrm{Spec}\ K[x]/(f(x))$. Choose an (embedded) resolution (Y, h) for $f^{-1}(0)$ over K, meaning that Y is an integral smooth closed subscheme of projective space over $X, h\colon Y \to X$ is the natural map, the restriction $h\colon Y \setminus h^{-1}(D) \to X \setminus D$ is an isomorphism, and the reduced scheme $(h^{-1}(D))_{red}$ associated to $h^{-1}(D)$ has only normal crossings (i.e. its irreducible components are smooth and intersect transversally, cf. [25]). Let $E_i, i \in T$, be the irreducible components of $(h^{-1}(D))_{red}$. These consist of the components $E_i, i \in T_s$, of the strict transform of D, and the exceptional divisors $E_i, i \in T \setminus T_s$. For each $i \in T$ let N_i be the multiplicity of E_i in the divisor of $f \circ h$ on Y and let $\nu_i - 1$ be the multiplicity of E_i in the divisor of $h^*(dx_1 \wedge \cdots \wedge dx_n)$. The (N_i, ν_i) are called the *numerical data* of the resolution. For $i \in T$ and $I \subset T$ we consider the schemes

$$\overset{\circ}{E}_i := E_i \setminus \bigcup_{j\neq i} E_j \ ,\ E_I := \bigcap_{i\in I} E_i \ ,\ \overset{\circ}{E}_I := E_I \setminus \bigcup_{j\in T\setminus I} E_j.$$

When $I = \emptyset$, we put $E_\emptyset = Y$. Finally let $C_f \subset X$ be the singular locus of $f\colon X \to \mathbb{A}^1_K$.

(1.3.2) Theorem (Igusa [28],[30]). *Assume the notation of 1.1 and 1.3.1, then*
(i) $Z_\Phi(s,\chi)$ *is a rational function of* q^{-s}. *Its poles are among the values* $s = -\nu_i/N_i + 2\pi\sqrt{-1}k/N_i \log_e q$ *with* $k \in \mathbb{Z}$ *and* $i \in T$ *such that the order of* χ *divides* N_i.
(ii) If $C_f \cap \mathrm{Supp}\ \Phi \subset f^{-1}(0)$, *then* $Z_\Phi(s,\chi) = 0$ *for almost all* χ.

Proof of (i). Consider the set $Y(K)$ of K-rational points of Y as a K-analytic manifold. We have

$$Z_\Phi(s,\chi) = \int_{Y(K)} (\Phi \circ h)\chi(\mathrm{ac}(f \circ h))|f \circ h|^s |h^*(dx_1 \wedge \cdots \wedge dx_n)|.$$

Let $b \in Y(K)$ and $\{i \in T | b \in E_i\} = \{i_1, \ldots, i_r\}$. There are local coordinates $y_1, \ldots, y_n$ for $Y(K)$ centered at b such that

$$f \circ h = \epsilon y_1^{N_{i_1}} \ldots y_r^{N_{i_r}}, \quad h^*(dx_1 \wedge \cdots \wedge dx_n) = \eta y_1^{\nu_{i_1}-1} \ldots y_r^{\nu_{i_r}-1} dy_1 \wedge \cdots \wedge dy_n,$$

where ϵ and η are analytic in a neighbourhood U of b and $\epsilon(b) \neq 0$, $\eta(b) \neq 0$. Note that $|\epsilon|, |\eta|$ and $\chi(\text{ac } \epsilon)$ are constant on U when U is small enough. Because $h^{-1}(\text{Supp } \Phi)$ is compact, we see that $Z_\Phi(s,\chi)$ is a finite $\mathbb{C}$-linear combination of products of factors of the form $q^{ks}, k \in \mathbb{Z}$, or $\int_{\text{ord} z \geqslant e} \chi^{N_i}(\text{ac } z)|z|^{N_i s+\nu_i-1}|dz|$. But this last integral is zero unless the order of χ divides N_i, in which case it is a rational function of q^{-s} with denominator $1 - q^{-N_i s-\nu_i}$. This proves (i). For (ii), see [30, p. 91-96].

Remark. The rationality of $Z_\Phi(s,\chi)$ can also be proved without the use of resolution of singularities, see [10] and section 8.

1.4. Exponential sums and integration on fibers

(1.4.1) Let Ψ be the standard additive character on K, thus, for $z \in K$, $\Psi(z) = \exp(2\pi i \text{Tr}_{K/Q_p}(z))$, where Tr denotes the trace. Weil [83] introduced the following two functions

$$E_\Phi(z) = E_\Phi(z,K,f) = \int_{K^n} \Phi(x)\Psi(zf(x))|dx|,$$
$$F_\Phi(y) = F_\Phi(y,K,f) = \int_{f^{-1}(y)} \Phi(x)|\frac{dx}{df}|,$$

for $z \in K$ and $y \in K \setminus V_f$, where $V_f = f(C_f)$. The function $E_\Phi(z)$ is locally constant and bounded on K. One is interested in its behaviour for $|z| \to \infty$. The simplest case is when $C_f \cap \text{Supp}\Phi = \emptyset$, then $E_\Phi(z) = 0$ for $|z|$ large enough. The function $F_\Phi(y)$ is locally constant on $K \setminus V_f$ and has compact support. One is interested in its behaviour when y tends to a point of V_f. For a nice introduction, see Serre [73].

We will write E, F, resp. E_0, F_0, instead of E_Φ, F_Φ when Φ is the characteristic function of R^n, resp. PR^n.

(1.4.2) Suppose $f(x) \in R[x]$, $m \in \mathbb{N}\setminus\{0\}$. If $u \in R^\times$, then obviously $E(u\pi^{-m}) = q^{-nm}\sum_{x\in(R/P^m)^n} \Psi(uf(x)/\pi^m)$, which is a classical exponential sum mod P^m.

Let $a \in R\setminus V_f$ and denote by $N_m(a)$ the number of solutions in R/P^m of the congruence $f(x) \equiv a \mod P^m$. Then one verifies that $F(a) = N_m(a)/q^{(n-1)m}$, for m big enough (depending on a). This stable quotient $F(a)$ is classically known as the local singular series associated to f and a, and plays an important role in the circle method.

(1.4.3) Note that $E_\Phi(z) = \int_K F_\Phi(y)\Psi(zy)|dy|$ is the Fourier transform of $F_\Phi(y)$ on K and that $Z_\Phi(\omega, K, f) = \int_K F_\Phi(y)\omega(y)|dy|$ is the Mellin transform of $(1 - q^{-1})yF_\Phi(y)$ on $K^\times$. This gives the relation between exponential sums and local zeta functions. By decomposing Φ and translation one reduces to the case where $C_f \cap \operatorname{Supp} \Phi \subset f^{-1}(0)$. Then, due to 1.3.2 (i) and (ii), the following are related by formulas (see [28], [30]):

(i) Principal parts of the Laurent expansions of the $Z_\Phi(s,\chi)$ around their poles,

(ii) Terms of an asymptotic expansion of $E_\Phi(z)$ as $|z| \to \infty$,

(iii) Terms of an asymptotic expansion of $F_\Phi(y)$ as $y \to 0$.

Still more information is provided by the following.

(1.4.4) Proposition. *Let $u \in R^\times$ and $m \in \mathbb{Z}$. Then $E_\Phi(u\pi^{-m})$ equals*

$$Z_\Phi(0,\chi_{triv}) + \operatorname{Coeff}_{t^{m-1}} \frac{(t-q)Z_\Phi(s,\chi_{triv})}{(q-1)(1-t)} + \sum_{\chi \neq \chi_{triv}} g_{\chi^{-1}}\chi(u)\operatorname{Coeff}_{t^{m-c(\chi)}} Z_\Phi(s,\chi)$$

where $c(\chi)$ denotes the conductor of χ, i.e. the smallest $c \geqslant 1$ such that χ is trivial on $1 + P^c$, and g_χ denotes the Gaussian sum

$$g_\chi = (q-1)^{-1}q^{1-c(\chi)} \sum_{v \in (R/P^{c(\chi)})^\times} \chi(v)\Psi(v/\pi^{c(\chi)}).$$

Proof. Replacing f by uf we see that it suffices to prove the theorem for $u = 1$. We introduce for any $e \in \mathbb{N} \setminus \{0\}$ the integral

$$I_{\Phi,e}(s) = \int_{K^n} \Phi(x)\Psi(\pi^{-e}\mathrm{ac}\, f(x))|f(x)|^s|dx|.$$

Direct verification shows

$$E_\Phi(\pi^{-m}) = \int_{\operatorname{ord} f(x) \geqslant m} \Phi(x)|dx| \;+ \sum_{k \leqslant m-1} \operatorname{Coeff}_{t^k} I_{\Phi,m-k}(s).$$

The proposition follows now from 1.2.2, since Fourier transformation on $(R/P^e)^\times$ yields $I_{\Phi,e}(s) = \sum_{c(\chi)=e} g_{\chi^{-1}} Z_\Phi(s,\chi)$.

(1.4.5) Corollary (Igusa [28], [30]). *Suppose that $C_f \cap \mathrm{Supp}\ \Phi \subset f^{-1}(0)$. Then for $|z|$ big enough $E_\Phi(z)$ is a finite $\mathbb{C}$-linear combination of functions of the form $\chi(\mathrm{ac}(z))|z|^\lambda(\log_q |z|)^\beta$ with coefficients independent of z, and with $\lambda \in \mathbb{C}$ a pole of $(s+1)Z_\Phi(s, \chi_{triv})$ or of $Z_\Phi(s,\chi), \chi \neq \chi_{triv}$, and with $\beta \in \mathbb{N}, \beta \leqslant$ (multiplicity of pole λ) $- 1$. Moreover all poles λ appear effectively in this linear combination.*

Proof. This follows from 1.3.2 (i), (ii) and 1.4.4, by writing $Z_\Phi(s,\chi)$ in partial fractions.

1.5 Igusa's conjecture on exponential sums

Let F be a number field, $f(x) \in F[x]\setminus\{0\}$ a homogeneous polynomial and $\sigma \in \mathbb{R}$. Suppose $1 < \sigma < \min \nu_i/N_i$, where the minimum is taken over all i except those with $N_i = \nu_i = 1$ and the (N_i, ν_i) are the numerical data of a fixed resolution of $f^{-1}(0)$ over F. By 1.4.5, for each p-adic completion K of F there exists $C(K) \in \mathbb{R}$ satisfying $|E(z, K, f)| \leqslant c(K)|z|^{-\sigma}$ for all $z \in K$.

Conjecture (Igusa [30]). *In the above inequality one can take $c(K)$ independent of K.*

This is related to the validity of a certain Poisson formula, see [30, p. 122, 170]. Igusa [27] proved the conjecture when $C_f = \{0\}$, by using Deligne's bound [8] for exponential sums over F_q, which in turn depends on the Riemann hypothesis for varieties over F_q. He also verified it for certain relative invariants of prehomogeneous vector spaces [26], [29], [30, p. 123-127]. Recently Sperber and Denef proved the conjecture for polynomials $f(x)$ which are non-degenerate with respect to their Newton polyhedron $\Delta(f)$ (see 5.3) assuming that $\Delta(f)$ has no vertex in $\{0,1\}^n$ (and only considering toric resolutions).

1.6 The Archimedean case

Replacing K by $\mathbb{R}$ or $\mathbb{C}$ and Φ by a C^∞ function with compact support, one defines

$$Z_\Phi(s, K, f) = \int_{K^n} \Phi(x)|f(x)|^{\delta s} dx,$$

for $s \in \mathbb{C}, \mathrm{Re}\,(s) > 0$, where $\delta = 1$ if $K = \mathbb{R}$ and $\delta = 2$ if $K = \mathbb{C}$. One proves that $Z_\Phi(s, K, f)$ extends to a meromorphic function on $\mathbb{C}$ whose poles are rational, either by resolution of singularities [2],[5] or by the theory of Bernstein polynomials [4].

2. MONODROMY AND BERNSTEIN POLYNOMIALS

2.1 Monodromy

(2.1.1) Let $f : \mathbb{C}^n \to \mathbb{C}$ be a non constant polynomial map and fix $b \in \mathbb{C}^n$ with $f(b) = a$. Let $B \subset \mathbb{C}^n$ be a small enough ball with center b. Milnor [63] proved that the restriction $f|_B$ is a locally trivial C^∞ fibration over a small enough *pointed* disc $A \subset \mathbb{C} \setminus \{a\}$ with center a. Hence the diffeomorphism type of the so called *Milnor fiber* $F_b := f^{-1}(t) \cap B$ of f *around* b does not depend on $t \in A$ and the counter clockwise generator of the fundamental group of A induces an automorphism T of $H^{\cdot}(F_b, \mathbb{C})$ which is called the *local monodromy* of f at b. It is well known that the eigenvalues of T are roots of unity (see [22, Exposé I]). When b is an isolated critical point of f, a result of Milnor [63] states that $H^i(F_b, \mathbb{C}) = 0$ for $i \neq 0$, $n-1$, and $H^0(F_b, \mathbb{C}) = \mathbb{C}$ with trivial monodromy action.

(2.1.2) Theorem (A'Campo's formula [1]). *Suppose b is an isolated critical point of f, with $f(b) = 0$, $n \geqslant 2$. We adopt the notation of 1.3.1 with $K = \mathbb{C}$. Then the characteristic polynomial of the monodromy action on $H^{n-1}(F_b, \mathbb{C})$ equals*

$$\left((z-1)^{-1} \prod_{i \in T} (z^{N_i} - 1)^{\chi(\overset{\circ}{E}_i \cap h^{-1}(b))} \right)^{(-1)^{n-1}},$$

where χ denotes the Euler characteristic with respect to singular cohomology. In particular $\exp(2\pi\sqrt{-1}/k)$, *for $k \in \mathbb{N}, k > 1$, is an eigenvalue of the local monodromy at b if and only if $\sum_{k|N_i} \chi(\overset{\circ}{E}_i \cap h^{-1}(b)) \neq 0$.*

2.2 The Bernstein polynomial

I.N. Bernstein [4] has proved for any polynomial $f(x), x = (x_1, \ldots, x_n)$, over a field K of characteristic zero, that there exist $P \in K[x, \frac{\partial}{\partial x}, s]$, and $b(s) \in K[s] \setminus \{0\}$ which satisfy the functional equation $Pf^{s+1} = b(s)f^s$. The monic polynomial $b(s)$ of smallest degree which satisfies this functional equation is called the *Bernstein polynomial* of f and will be denoted by $b_f(s)$. If f is not constant, then $s+1$ divides $b_f(s)$. If further $f = 0$ has no singular points over the algebraic closure of K, then $b_f(s) = s+1$. A basic theorem of Kashiwara states that all roots of $b_f(s)$ are negative rational numbers. When $K = \mathbb{R}$ or $\mathbb{C}$ one easily verifies, using the functional equation and integration by parts, that the poles of $Z_\Phi(s, K, f)$ are among the values $s = \alpha - j$ with α a root of $b_f(s)$ and $j \in \mathbb{N}$. Note however that this integration by parts does not make sense in the p-adic case.

The roots of $b_f(s)$ are related to the geometry of f. Indeed by Malgrange [58], if α is a root of $b_f(s)$ then $\exp(2\pi\sqrt{-1}\alpha)$ is an eigenvalue of the local monodromy of f at some point of $f^{-1}(0)$, and all eigenvalues are obtained in this way. (Note that $b_f(s)$ is the least common multiple of all local Bernstein polynomials, see e.g. [23, Lemma 2.5.2].)

Thus, in the *Archimedean* case if s is a pole of $Z_\Phi(s, K, f)$ then $\exp(2\pi\sqrt{-1}s)$ is an eigenvalue of monodromy. (For a direct proof in the isolated singularity case, see Malgrange [57].) Moreover, Barlet [3] has proved that each eigenvalue is obtained in this way, when $K = \mathbb{C}$. We refer to Loeser [50] for information on the exact location of the poles and to Loeser [51] for an estimate on the largest pole.

2.3 The monodromy conjecture

Motivated by the situation in the Archimedean case 2.2 and the study of concrete examples, it is natural to propose the following conjectures for any polynomial $f(x)$ over a numberfield $F \subset \mathbb{C}$.

(2.3.1) Conjecture (Igusa [36]). *For almost all p-adic completions K of F , if s is a pole of $Z(s, \chi, K, f)$, then* $\mathrm{Re}(s)$ *is a root of $b_f(s)$.*

This conjecture has been verified in special cases, see 5.2.5, 5.3, 7.3 and 7.4 below.

(2.3.2) Monodromy conjecture (Igusa). *For almost all p-adic completions K of F, if s is a pole of $Z(s, \chi, K, f)$, then* $\exp(2\pi\sqrt{-1}\ \mathrm{Re}\ (s))$ *is an eigenvalue of the local monodromy of f at some complex point of $f^{-1}(0)$.*

Note that the first conjecture implies the second by what we said above. But for the second one there is by now a massive amount of experimental evidence, see section 6. Both conjectures might be true for all p-adic completions and for Z replaced by Z_Φ. In most numerical examples, theorem 1.3.2 yields a very big list of candidate poles. However, due to miraculeous cancellations, usually many of these candidates are no pole. This strange phenomenon would be explained by the monodromy conjecture.

Loeser [53] studied zeta functions $\int_{K^n} \Phi(x)|f_1(x)|^{s_1} \dots |f_k(x)|^{s_k}|dx|$ of several variables and formulated a conjecture on the existence of certain asymptotic expansions (generalizing 1.4.3 (iii)). Surprisingly this conjecture implies a relation between the polar locus of these zeta functions and the geometry of the discriminant of $f_1, \dots, f_k$. Moreover there are connections with the monodromy conjecture.

3. EXPLICIT FORMULAS

We continue to use the notation of 1.1 and 1.3.1. Reduction mod P is denoted by $^-$.

(3.1) We call a Schwartz-Bruhat function Φ *residual* if Supp $\Phi \subset R$ and $\Phi(x)$ only depends on x mod P. Such Φ induces $\overline{\Phi}\colon \overline{K}^n \to \mathbb{C}$.

(3.2) We say that the resolution (Y, h) for $f^{-1}(0)$ has *good reduction* mod$P$ if $\overline{Y}$ and all $\overline{E}_i$ are smooth, $\cup_{i\in T}\overline{E}_i$ has only normal crossings, and the schemes $\overline{E}_i$ and $\overline{E}_j$ have no common components whenever $i \neq j$ (cf. [12]). Here the reduction mod P of any closed subscheme Z of Y is denoted by $\overline{Z}$ and defined as the reduction mod P of the closure of Z in projective space over Spec $R[x]$. If in addition $N_i \notin P$ for all $i \in T$ then we say that the resolution has *tame* good reduction. When f and (Y, h) are defined over a number field F, we have good reduction for almost all completions K of F. When the resolution (Y, h) has good reduction we have $\overline{E}_I = \cap_{i\in I}\overline{E}_i$ and we put $\overset{\circ}{\overline{E}}_I = \overline{E}_I \setminus \cup_{j\in T\setminus I}\overline{E}_j$ and $\overset{\circ}{\overline{E}}_i = \overline{E}_i \setminus \cup_{j\neq i}\overline{E}_j$. Finally let $C_{\overline{f}}$ be the singular locus of $\overline{f}\colon \overline{K}^n \to \overline{K}$.

3.3 Theorem [14]. *Let $f \in R[x], \overline{f} \neq 0$. Suppose that $f^{-1}(0)$ has a resolution with tame good reduction mod P and that Φ is residual. If χ is not trivial on $1+P$ then $Z_\Phi(s,\chi)$ is constant as function of s. If moreover $C_{\overline{f}} \cap \operatorname{Supp} \overline{\Phi} \subset \overline{f}^{-1}(0)$, then $Z_\Phi(s,\chi) = 0$.*

3.4 Theorem [12], [14]. *Let $f \in R[x], \overline{f} \neq 0$. Suppose that (Y, h) is a resolution for $f^{-1}(0)$ with good reduction mod P, and that Φ is residual. Let χ be a character of $R^\times$ of order d which is trivial on $1 + P$. Then*

$$Z_\Phi(s,\chi) = q^{-n} \sum_{\substack{I\subset T \\ \forall i\in I:\ d|N_i}} c_{I,\chi,\Phi} \prod_{i\in I} \frac{q-1}{q^{N_i s+\nu_i} - 1},$$

where $c_{I,\chi,\Phi} = \sum_{a\in \overset{\circ}{\overline{E}}_I(\overline{K})} \overline{\Phi}(a)\Omega_\chi(a)$.

Here $\Omega_\chi(a)$ is defined as follows: If $a \in \overset{\circ}{\overline{E}}_I(\overline{K})$ and $d|N_i$ for all $i \in I$ then we can write $\overline{f} \circ \overline{h} = uw^d$ with $u, u^{-1}, w \in \mathcal{O}_{\overline{Y},a}$ and we put $\Omega_\chi(a) := \chi(u(a))$.
We will write $c_{I,\chi}$, resp. $c_{I,\chi,0}$, instead of $c_{I,\chi,\Phi}$ when Φ is the characteristic function of R^n, resp. PR^n. To denote the dependence on K, we will sometimes write $c_{I,\chi,\Phi}(K)$.

3.5 Cohomological interpretation. Assume the hypothesis of 3.4 and choose a prime ℓ with $\ell \nmid q$. Note that χ induces a character of $F_q^\times$ which we denote again by χ. Let $\mathcal{L}_\chi$ be the Kummer $\overline{Q}_\ell$-sheaf on $\mathbb{A}^1_{F_q} \setminus \{0\}$ associated to this last character (see [9, Sommes Trig.]). Put $U = \overline{Y} \setminus (\overline{f} \circ \overline{h})^{-1}(0)$. Let ν be the open immersion $\nu\colon U \hookrightarrow \overline{Y}$ and $\alpha\colon U \to \mathbb{A}^1_{F_q} \setminus \{0\}$ the map induced by $\overline{f} \circ \overline{h}$. We define $\mathcal{F}_\chi := \nu_* \alpha^* \mathcal{L}_\chi$. It is easy to verify that $\mathcal{F}_\chi$ is lisse of rank one on $U_d := \overline{Y} \setminus \cup_{d \nmid N_i} \overline{E}_i$. Moreover, if $a \in U_d(F_q)$ then the action of the geometric Frobenius on the stalk of $\mathcal{F}_\chi$ at a is multiplication by $\Omega_\chi(a)$. Hence Grothendieck's trace formula yields

$$(3.5.1) \qquad c_{I,\chi} = \sum_i (-1)^i \mathrm{Tr}(F, H^i_c(\overset{\circ}{\overline{E}}_I \otimes F^a_q, \mathcal{F}_\chi)),$$

$$(3.5.2) \qquad c_{I,\chi,0} = \sum_i (-1)^i \mathrm{Tr}(F, H^i_c((\overset{\circ}{\overline{E}}_I \cap \overline{h}^{-1}(0)) \otimes F^a_q, \mathcal{F}_\chi)),$$

where F denotes the Frobenius and F^a_q the algebraic closure of F_q. For further use we still mention

3.6 Lemma. *The higher direct images $R^j\nu_*(\mathcal{F}_\chi|_U)$ are zero outside U_d for all $j \geqslant 0$. The same holds also for the open immersion $\nu_I\colon \overset{\circ}{\overline{E}}_I \hookrightarrow \overline{E}_I$.*

(3.7) We should also mention that Langlands [47] has given a formula, in terms of principal-value integrals, for the principal parts of the Laurent expansions of $Z_\Phi(s,\chi)$ around its poles.

4. CONSEQUENCES OF THE EXPLICIT FORMULAS

Unless stated otherwise, we keep the notation of 1.1 and 1.3.1. When χ is a character of $\overline{K}^\times$ we denote the induced character of $R^\times$ also by χ. We say a property holds for almost all P if it holds for almost all completions of a number field F (all data being defined over F). For any scheme V of finite type over a field $L \subset \mathbb{C}$, we denote by $\chi(V)$ the Euler characteristic of $V(\mathbb{C})$ with respect to singular cohomology.

4.1 Degree of local zeta functions

(4.1.1) Because $Z_\Phi(s,\chi)$ is a rational function of q^{-s} we can consider its degree $\deg Z_\Phi(s,\chi)$ which is defined as the degree of the numerator minus the degree of the denominator (as polynomials in q^{-s}). If the hypothesis of theorem 3.4 holds, then it is clear from the explicit formula that $\deg Z_\Phi(s,\chi) \leqslant 0$. Clearly the degree is < 0 if and only if $\lim_{s\to-\infty} Z_\Phi(s,\chi) = 0$.

(4.1.2) Proposition [12]. *For almost all P, $\deg Z_0(s,\chi_{triv}) = 0$. If moreover f is homogeneous, then $\deg Z(s,\chi_{triv}) = -\deg f$.*

Proof. From theorem 3.4 it follows that

$$q^n \lim_{s\to-\infty} Z_0(s,\chi_{triv}) \equiv \sum_{I\subset T} c_{I,\chi_{triv},0} \equiv q^n Z_0(0,\chi_{triv}) = 1 \mod q.$$

Hence $\lim_{s\to-\infty} Z_0(s,\chi_{triv}) \neq 0$. This proves the first assertion. The second assertion follows from the first by the formula $Z(s,\chi) = q^{n+s\deg f} Z_0(s,\chi)$.

(4.1.3) When $f(0) = 0$ we proved [15], using the method of vanishing cycles [22], that for almost all P and any character χ of $\overline{K}^\times$ of order d we have

$$\lim_{s\to-\infty} Z_0(s,\chi) = (1-q)q^{-n} \sum_i (-1)^i \mathrm{Tr}(\mathrm{Frob}, H^i(F_0, Q_\ell)^\chi),$$

where F_0 is the Milnor fiber of $\overline{f}$ at 0, $H^i(F_0,Q_\ell)^\chi$ denotes the component of the cohomology on which the semi-simplification of the local monodromy acts like χ, and Frob is a suitable lifting of the Frobenius. In particular this implies that $\deg Z_0(s,\chi) < 0$ when there is no eigenvalue of the local monodromy of f at 0 with order d.

4.2 The functional equation

(4.2.1) We denote by $K^{(e)}$ the unramified extension field of K of degree e, and put

$$Z(s,e,\chi) := Z(s, \chi \circ N_{K^{(e)}/K}, K^{(e)}, f),$$

where N denotes the norm. D. Meuser [61] has shown that $Z(s,e,\chi_{triv})$, as function of s and e, is a rational function of $q^{-es}, \alpha_1^e, \ldots, \alpha_r^e$ for some $\alpha_1, \ldots, \alpha_r \in$

$\mathbb{C}$. In case of good reduction, 3.4 and 3.5.1 directly imply that this remains true for $Z(s, e, \chi)$ where χ is any character of $\overline{K}^{\times}$. Because of this rationality we can canonically extend $Z(s, e, \chi)$ to a function on $\mathbb{C} \times \mathbb{Z} \setminus \{0\}$. With this notation we can state the following result of Meuser and Denef [17] (see also [62]).

(4.2.2) Theorem [17]. *If f is homogeneous, then for almost all P we have the functional equation $Z(s, -e, \chi) = q^{-es \deg f} Z(s, e, \chi^{-1})$, for all $e \in \mathbb{Z} \setminus \{0\}$.*

Idea of the proof. For a homogeneous polynomial it is possible to give an explicit formula for $Z(s, \chi)$ in terms of an embedded resolution of singularities (with good reduction) of Proj $K[x]/(f(x))$. This has the advantage that the $\overline{E}_I$ become proper. Then we can use the functional equation for the Weil zeta function of the varieties $\overline{E}_I$ to obtain the theorem when χ is trivial. In the general case we have to use Poincaré duality for the sheaf $\mathcal{F}_\chi$ on $\overline{E}_I \cap U_d$ (notations as in 3.5). This works since its cohomology equals compactly supported cohomology because of lemma 3.6.

(4.2.3) The above functional equation takes an interesting form if $Z(s, \chi_{triv})$ is *universal,* meaning that there exists $\mathcal{Z}(u, v) \in Q(u, v)$ with $Z(s, \chi_{triv}, K^{(e)}, f) = \mathcal{Z}(q^{-es}, q^{-e})$, for all $e \in \mathbb{N} \setminus \{0\}$. This happens often when f is a relative invariant of a reductive group (see 7.6). Note that the functional equation 4.2.2 takes the form $\mathcal{Z}(u^{-1}, v^{-1}) = u^{\deg f} \mathcal{Z}(u, v)$ whenever $Z(s, \chi_{triv})$ is universal. It was this form of the functional equation which was first conjectured by Igusa [38]. His conjecture was based on extensive calculations with relative invariants of prehomogeneous vector spaces.

4.3 Topological zeta functions

(4.3.1) To any $f \in \mathbb{C}[x]$ and $d \in \mathbb{N} \setminus \{0\}$ Loeser and Denef [16] associate the "topological zeta function"

(4.3.1.1) $$Z_{\text{top}}^{(d)}(s) := \sum_{\substack{I \subset T \\ \forall i \in I:\ d | N_i}} \chi(\overset{\circ}{E}_I) \prod_{i \in I} \frac{1}{N_i s + \nu_i},$$

where the notation is as in 1.3.1 (for a resolution of $f^{-1}(0)$ over $\mathbb{C}$). It is a remarkable fact that this expression does not depend on the chosen resolution. Untill now the only known proof of this uses local zeta functions. To simplify, assume f has coefficients in a number field F. Then, for almost all P, formulas

3.4 and 3.5.1 yield

$$Z_{\text{top}}^{(d)}(s) = \lim_{e \to 0} Z(s, e, \chi), \tag{4.3.1.2}$$

when χ is a character of $\overline{K}^{\times}$ of order d. This shows that $Z_{\text{top}}^{(d)}(s)$ is indeed intrinsic. The limit for $e \to 0$ makes sense because one can ℓ-adically interpolate $Z(s,e,\chi)$ as a function of e (for e divisible by a suitable number, see [16]). In particular, we have

$$\lim_{e \to 0} c_{I,\chi}(K^{(e)}) = \chi_c\,(\overset{\circ}{\overline{E}}_I \otimes F_q^a, \mathcal{F}_\chi) = \chi(\overset{\circ}{E}_I), \tag{4.3.1.3}$$

for almost all P, where χ_c denotes the Euler characteristic with respect to ℓ-adic cohomology with compact support (cf. [41]). From (4.3.1.2) one also gets the following.

(4.3.2) Theorem [16]. *If ρ is a pole of $Z_{\text{top}}^{(d)}(s)$, then for almost all P and all characters χ of $\overline{K}^{\times}$ of order d there exist infinitely many unramified extensions L of K for which ρ is a pole of $Z(s, \chi \circ N_{L/K}, L, f)$.*

Thus conjecture 2.3.1 would imply that the poles of $Z_{\text{top}}^{(d)}(s)$ are roots of the Bernstein polynomial $b_f(s)$. However, even the relation with local monodromy (implied by conjecture 2.3.2) is not yet proved.

(4.3.3) Because $Z(0, \chi_{triv}) = 1$, formula 4.3.1.2 yields $Z_{\text{top}}^{(1)}(0) = 1$. A local version of this fact together with M. Artin's approximation techniques yields the following application to analytic geometry.

(4.3.4) Theorem [16]. *Let $h\colon Y \to X$ be an analytic modification of compact analytic manifolds. Suppose the exceptional locus E of h has normal crossings in Y. Let $E_i, i \in T$, be the irreducible components of E and let $\overset{\circ}{E}_i, \nu_i$ be as in 1.3.1. Then*

$$\chi(h(E)) = \sum_{\emptyset \neq I \subset T} \chi(\overset{\circ}{E}_I)/n_I,$$

where $n_I = \prod_{i \in I} \nu_i$.

It would be interesting to find a proof of this theorem which does not use local zeta functions.

4.4 Holomorphy of $Z_\Phi(s,\chi)$ and monodromy

(4.4.1) By 1.3.2, $Z_\Phi(s,\chi)$ is holomorphic on $\mathbb{C}$ when the order of χ divides no N_i. The N_i are not intrinsic, but the order (as root of unity) of any eigenvalue of the local monodromy on $f^{-1}(0)$ divides some N_i. Being very optimistic, we propose the following conjecture:

(4.4.2) Conjecture [15]. *For almost all P, when Φ is residual, $Z_\Phi(s,\chi)$ is holomorphic unless the order of χ divides the order of some eigenvalue of the local monodromy of f at some complex point of $f^{-1}(0)$.*

In fact, this might be true for all P and for any Φ. The following proposition shows that the conjecture is the best possible.

(4.4.3) Proposition [15]. *Suppose 0 is an isolated singularity of the hypersurface $f(x)=0$. For almost all P if the order $d \neq 1$ of the character χ of $\overline{K}^\times$ divides the order of some eigenvalue of the local monodromy of f at 0, then $Z_0(s,\chi\circ N_{L/K},L,f)$ is not holomorphic on $\mathbb{C}$, for infinitely many unramified extensions L of K.*

Proof. From 3.4, 3.5.2 and a variant of 4.3.1.3 we get
(4.4.3.1)

$$\lim_{e\to 0} \mathrm{Coeff}_{t^{em}}(q^e-1)^{-1}Z_0(s,\chi\circ N_{K^{(e)}/K},K^{(e)},f) = \sum_{d|N_i|m} \chi(\overset{\circ}{E}_i\cap h^{-1}(0)),$$

for all $m\in\mathbb{N}\setminus\{0\}$. The hypothesis on d and A'Campo's formula 2.1.2 imply the existence of a minimal m satisfying $d|m$ and $\sum_{N_i=m}\chi(\overset{\circ}{E}_i\cap h^{-1}(0))\neq 0$. Then this last sum equals the right-hand side of 4.4.3.1. Hence for infinitely many e the zeta function over $K^{(e)}$ is not constant, and hence not holomorphic since its degree is $\leqslant 0$.

4.5 L-functions of exponential sums mod P^m

(4.5.1) The L-function of the exponential sum mod P^m of $f\in R[x]$ is defined by

$$L_m(t,K,f) := \exp\sum_{e=1}^{\infty} E(\pi^{-m},K^{(e)},f)\ \frac{t^e}{e}\ ,$$

for $m\in\mathbb{N}\setminus\{0\}$. By adapting Dwork's method [21] one can show that $L_m(t,K,f)$ is a rational function of t. In case of tame good reduction, this can also be derived directly from 1.4.4 and section 3 (if $m\geqslant 2$). The next theorem expresses the degree of these L-functions in terms of monodromy.

(4.5.2) Theorem. *Suppose f has only isolated critical points in $\mathbb{C}^n, n \geqslant 2$. Then for almost all P and all $m \geqslant 2$ we have*

$$\deg L_m(t, K, f) = (-1)^{n-1} \sum_{\alpha} \alpha^{m-1},$$

where α runs over all eigenvalues (counting multiplicities) of the monodromy action on $H^{n-1}(F_b, \mathbb{C})$ at all critical points b of f (notation of 2.1.1).

Proof. Clearly $\deg L_m(t, K, f) = -\lim_{e\to 0} E(\pi^{-m}, K^{(e)}, f)$. If $f(0) = 0$, then 1.4.4, 3.3, 4.4.3.1 and the Hasse-Davenport relation yield

$$\lim_{e\to 0} E_0(\pi^{-m}, K^{(e)}, f) = 1 - \sum_{N_i|m-1} N_i \ \chi(\overset{\circ}{E}_i \cap h^{-1}(0)).$$

By A'Campo's formula 2.1.2 the right side of the above equals $(-1)^n \sum_\alpha \alpha^{m-1}$ where α runs over all eigenvalues of $H^{n-1}(F_0, \mathbb{C})$. The theorem follows now from remark 4.5.3 below.

For $m = 1$ the theorem remains true if f has a compactification $g\colon Y \to \mathbb{A}^1$ with $Y \setminus \mathbb{A}^n$ a divisor with normal crossings over $\mathbb{A}^1$, but it fails in general.

(4.5.3) *Remark.* Note the following completely elementary fact: If Φ is residual, $f \in R[x]$ and $C_{\overline{f}} \cap \operatorname{Supp} \overline{\Phi} = \emptyset$, then $E_\Phi(z) = 0$ when $|z| > q$.

4.6 Non-contribution of certain E_i

Theorem [14]. *Assume good reduction. Let χ be a character of $\overline{K}^\times$ of order d, and $i_0 \in T$. Suppose E_{i_0} is proper, $d|N_{i_0}$ and E_{i_0} intersects no E_j with $d|N_j, j \neq i_0$. If $\chi(\overset{\circ}{E}_{i_0}) = 0$, then E_{i_0} does not contribute to $Z(s, \chi)$, meaning that in formula 3.4 we can restrict the summation to $I \subset T \setminus \{i_0\}$.*

This is a direct consequence of 3.5.1 and a lemma stating that in the above situation $H^i_c(\overset{\circ}{\overline{E}}_{i_0} \otimes F^a_q, \mathcal{F}_\chi) = 0$ for all $i \neq n-1$. In the special case that $\overset{\circ}{\overline{E}}_{i_0}$ is affine this lemma follows from Poincaré duality because the cohomology of $\mathcal{F}_\chi$ on $\overset{\circ}{\overline{E}}_{i_0} \otimes F^a_q$ equals compactly supported cohomology by 3.6. The general case requires more work, see [14].

5. SPECIAL POLYNOMIALS

5.1 Polynomials of the form $f(x) + g(y)$

Let $f(x) \in K[x], g(y) \in K[y], x = (x_1, \dots, x_n), y = (y_1, \dots, y_m)$, and Φ_1, Φ_2 Schwartz-Bruhat functions on K^n, K^m. Put $A(s,\chi) = s+1$ if $\chi = \chi_{triv}$ and $A(s,\chi) = 1$ otherwise. Suppose $C_f \cap \mathrm{Supp}\ \Phi_1 \subset f^{-1}(0)$ and $C_g \cap \mathrm{Supp}\ \Phi_2 \subset g^{-1}(0)$. Then the poles of $A(s,\chi)Z_{\Phi_1\Phi_2}(s,\chi,K,f(x)+g(y))$ are of the form s_1+s_2 with s_1 resp. s_2, a pole of $A(s,\chi')Z_{\Phi_1}(s,\chi',K,f)$, resp. $A(s,\chi'')Z_{\Phi_2}(s,\chi'',K,g)$, for some $\chi'\chi'' = \chi$. Indeed, this follows directly from 1.4.5 and the obvious fact that $E_{\Phi_1\Phi_2}(z,K,f(x)+g(y)) = E_{\Phi_1}(z,K,f)E_{\Phi_2}(z,K,g)$. Concerning the monodromy conjecture, note that this is in complete agreement with the result of Thom and Sebastiani [76] on the monodromy of $f(x) + g(y)$.

5.2 Polynomials in two variables

Now we will see that conjecture 2.3.1 on the relation between $Z(s,\chi)$ and $b_f(s)$ is true for any $f \in K[x_1,x_2]$. For this we use the *canonical* embedded resolution (Y,h) of $f^{-1}(0)$ over the *algebraic closure* K^a of K, and we keep the notation of 1.3.1. In particular E_j is a projective line over K^a when $j \in T \setminus T_s$. The following theorem was first proved for analytically irreducible singularities and almost all P by Strauss [74], and further extended by Meuser [60], Igusa [32] and Loeser [52] towards the general case.

(5.2.1) Theorem. *Let $f \in K[x_1,x_2]$. If s is a pole of $Z(s,\chi)$, then* $\mathrm{Re}(s) = -\nu_j/N_j$ *for some $j \in T$, with $|E_j \setminus \overset{\circ}{E}_j| \geqslant 3$ or $j \in T_s$.*

For a converse see [79]. Usually, most $E_j \setminus \overset{\circ}{E}_j$ consist of no more than two points. This explains why so many candidate poles do not appear. All known proofs of theorem 5.2.1 are based on the following lemma.

(5.2.2) Lemma. *Fix $j \in T \setminus T_s$. Let $a_i, i \in J$, be the geometric points of $E_j \setminus \overset{\circ}{E}_j$. For $i \in J$ denote by N_i, ν_i the numerical data of the unique $E_\ell \neq E_j$ which contains a_i, and put $\alpha_i = \nu_i - N_i\nu_j/N_j$. Then*

$$\sum_{i \in J}(\alpha_i - 1) = -2 \ , \quad \textit{and} \quad \sum_{i \in J} N_i \equiv 0 \mod N_j.$$

The first proofs of this lemma were computational. Loeser [52] found a conceptual proof of the first formula, noting that the degree of ω in 5.2.4 below equals -2. We will outline in 6.1.2 a simple conceptual proof of both formulas, which is due to Veys [82].

(5.2.3) *A proof of theorem 5.2.1 (in case of tame good reduction).* For simplicity we suppose everything is defined over K. Consider the set S of all E_j with $\mathrm{Re}(s) = -\nu_j/N_j$. Suppose the theorem is false. Then $|E_j \setminus \overset{\circ}{E}_j| \leqslant 2$ and $j \notin T_s$, for all $E_j \in S$. Different $E_{j_1}, E_{j_2} \in S$ are disjoint, otherwise (applying the last lemma twice) there would be another $E_{j_3} \in S$ intersecting E_{j_2}. Iterating this would contradict the finiteness of S. Let $E_j \in S$ with N_j a multiple of the order d of χ. It suffices to show that E_j does not contribute to the pole s. Suppose $E_j \setminus \overset{\circ}{E}_j$ consists of two points a_1, a_2 (similar argument for one point). If $d \nmid N_1$, then $d \nmid N_2$ by the last lemma, and E_j does not contribute because of 4.6. Thus suppose $d|N_1, d|N_2$. Then the sheaf $\mathcal{F}_\chi$ from 3.5 is locally constant and hence geometrically constant on $\overline{E}_j$. Thus the explicit formula of section 3 implies that the contribution of E_j to the residue of $Z(s,\chi)$ at s equals (up to a factor)

$$1 + (q^{\alpha_1} - 1)^{-1} + (q^{\alpha_2} - 1)^{-1} = (q^{\alpha_1+\alpha_2} - 1)/(q^{\alpha_1} - 1)(q^{\alpha_2} - 1).$$

Since $\alpha_1 + \alpha_2 = 0$ by the last lemma, E_j does not contribute.

(5.2.4) Theorem (Loeser [52]). *Let $f \in \mathbb{C}[x_1, x_2]$ and $j \in T \backslash T_s$. If $|E_j \backslash \overset{\circ}{E}_j| \geqslant 3$, then $-\nu_j/N_j$ is a root of the Bernstein polynomial $b_f(s)$ of f.*

Idea of the proof. Assume the notation of 5.2.2 and put $S = \{a_i | i \in J\}$. We suppose $\alpha_i \neq 0$ for all $i \in J$. (Otherwise a different but easier argument is needed.) The residue of $(f \circ h)^{1-\nu_j/N_j} h^*(dx_1 \wedge dx_2)$ on $E_j = \mathbb{P}^1$ defines a meromorphic differential form ω with coefficients in a suitable rank one local system L on $\mathbb{P}^1 \setminus S$. We have (i) $|S| \geqslant 3$, (ii) ω has no zeros or poles outside S, and (iii) at each point of S the multiplicity of ω is not integral. Indeed a local calculation shows that the multiplicity of ω at a_i equals $\alpha_i - 1$ and Loeser proved by a difficult combinatorial argument that $|\alpha_i| < 1$. Then a result of Deligne and Mostow assures that ω defines a non-zero cohomology class in $H^1(\mathbb{P}^1 \setminus S, L)$. Hence there is a cycle $\gamma \in H_1(\mathbb{P}^1 \setminus S, L)$ with $\int_\gamma \omega \neq 0$. Considering a suitable etale cover of $\mathbb{P}^1 \setminus S$ and a lifting of $m\gamma$, for a suitable $m \in \mathbb{N} \setminus \{0\}$, one constructs a family of cycles $\mu(t) \in H_1(f^{-1}(t), \mathbb{C}), t \neq 0$, with

$$\lim_{t \to 0} \int_{\mu(t)} f^{1-\nu_j/N_j} \frac{dx \wedge dy}{df} = \int_{m\gamma} \omega \neq 0.$$

Hence $t^{-1+\nu_j/N_j}$ appears as the dominating term in the asymptotic expansion of $\int_{\mu(t)}(dx \wedge dy)/df$. Since $\nu_j/N_j \leqslant 1$ (see [32]), it is well known that this implies that $-\nu_j/N_j$ is a root of $b_f(s)$.

(5.2.5) Combining theorems 5.2.1 and 5.2.4 we obtain Loeser's result that for any $f \in K[x_1, x_2]$, if s is a pole of $Z(s,\chi)$, then $\mathrm{Re}(s)$ is a root of $b_f(s)$.

5.3 Non-degenerate polynomials

We treat this topic only very briefly. For the notion of a polynomial which is non-degenerate with respect to its Newton polyhedron at the origin, we refer to Varchenko [78]. For such polynomials there is a very explicit embedded resolution, which is called "toric". However, this yields a set of candidate poles for $Z(s,\chi)$ which is much too big. Lichtin and Meuser [48] have determined the actual poles in case of two variables. In general, a reasonable set of candidate poles (one value of $\mathrm{Re}(s)$ for each facet of the polyhedron) was obtained by Denef (unpublished, see [54, Thm 5.3.1]) (the method is the same as in the real case [18, I]). Loeser [54] proved that these candidate poles are indeed roots of $b_f(s)$, if some weak additional conditions are satisfied. For several results and intriguing open problems about the largest pole ($\neq -1$), we refer to [18, II].

6. THE WORK OF VEYS

6.1 Relations between numerical data

Let $f \in \mathbb{C}[x]$ and $h\colon Y \to \mathbb{C}^n$ an embedded resolution of $f^{-1}(0)$ over $\mathbb{C}$, *constructed* as in [25]. Thus in particular h is a composition of blowing-up maps. Veys [80],[81] has developed a general theory about relations between the numerical data of resolutions, generalizing lemma 5.2.2.

Consider a fixed exceptional divisor $E = E_j, j \in T \setminus T_s$, keeping the notation 1.3.1. Let $E'_i, i \in J$, be the irreducible components of $E \setminus \overset{\circ}{E}$. For $i \in J$ denote by N_i, ν_i the numerical data of the unique $E_\ell \neq E$ which contains E'_i, and put $\alpha_i = \nu_i - N_i\nu_j/N_j$. Veys' starting point is the following.

(6.1.1) Lemma [80] [81]. *Let K_E be the canonical divisor on E and E^2 the self-intersection divisor of E in Y. Then*

$$K_E = \sum_{i\in J}(\alpha_i - 1)E'_i \text{ in } \mathrm{Pic}E \otimes \mathbb{Q}, \text{ and } N_j E^2 = -\sum_{i \in J} N_i E'_i \text{ in } \mathrm{Pic}E,$$

where Pic *denotes the Picard group.*

Proof. By definition of the numerical data we have $\sum_{i\in T} N_i E_i = 0$ and $K_Y = \sum_{i\in T}(\nu_i - 1)E_i$ in Pic$Y$. Thus $N_j E = -\sum_{i\neq j} N_i E_i$ and the formula for $N_j E^2$ is obtained by intersecting with E. Moreover $K_Y + E = \nu_j E + \sum_{i\neq j}(\nu_i - 1)E_i$. Replacing $\nu_j E$ by $-\sum_{i\neq j} N_i \nu_j N_j^{-1} E_i$ we get $K_Y + E = \sum_{i\neq j}(\alpha_i - 1)E_i$. The expression for K_E follows now from the adjunction formula $K_E = (K_Y + E)\cdot E$.

(6.1.2) *Proof of Lemma 5.2.2.* This follows directly from the above lemma by taking degrees, since $\deg K_E = -2$.

(6.1.3) We now describe some of Veys' results. There are *Basic Relations* (B1 and B2) associated to the creation of E in the resolution process, generalizing 5.2.2. And there are *Additional Relations* (A) associated to each blowing-up of the resolution that "changes" E. More precisely: the variety E in the final resolution Y is in fact obtained by a finite succession of blowing-ups

$$E^0 \xleftarrow{\pi_0} E^1 \xleftarrow{\pi_1} \dots E^i \xleftarrow{\pi_i} E^{i+1} \dots \xleftarrow{\pi_{m-2}} E^{m-1} \xleftarrow{\pi_{m-1}} E^m = E$$

with irreducible nonsingular center $\mathcal{D}_i \subset E^i$ and exceptional variety $C_{i+1} \subset E^{i+1}$ for $i = 0, \dots, m-1$. The variety E^0 is created at some stage of the global resolution process as the exceptional variety of a blowing-up with center $\mathcal{D}$ and is isomorphic to a projective space bundle $\Pi\colon E^0 \to \mathcal{D}$ over $\mathcal{D}$. Let $C_i, i \in J_0$, be the irreducible components of intersections of E^0 with previously created exceptional divisors in the global resolution process or with the strict transform of $f^{-1}(0)$. Then the $E_i', i \in J$, are precisely the strict transforms of the $C_i, i \in J_0 \cup \{1, \dots, m\}$. So we put $J = J_0 \cup \{1, \dots, m\}$, E_i' = strict transform of C_i. The relations A express α_i, resp. N_i mod N_j, for $i = 1, \dots, m$, in terms of α_ℓ, resp. N_ℓ mod N_j, for $\ell \in J_0 \cup \{1, \dots, i-1\}$, see [80], [81]. The relations B1 are

$$\text{(B1)} \qquad \sum_{i\in J_0} d_i(\alpha_i - 1) = -k, \quad \text{and} \sum_{i\in J_0} d_i N_i \equiv 0 \mod N_j,$$

where $k = n - \dim \mathcal{D}$ and d_i is the degree of the intersection cycle $C_i \cdot F$ on F for a general fibre $F \cong \mathbb{P}^{k-1}$ of $\Pi\colon E^0 \to \mathcal{D}$. The relations B2 hold in Pic $\mathcal{D}$ and are more difficult to state. They are vacuous when $\mathcal{D}$ is a point. When $n = 3$ and $\mathcal{D}$ is a projective curve of genus g, the relation B2 for the α_i becomes a numerical relation by taking degrees in Pic $\mathcal{D}$, namely

$$\sum_{\substack{i\in J_0\\ d_i\neq 0}} \frac{\kappa_i}{2d_i}(\alpha_i - 1) + \sum_{\substack{i\in J_0\\ d_i = 0}} (\alpha_i - 1) = 2g - 2,$$

where κ_i is the self-intersection number of C_i in E^0. The proof of B1 depends on lemma 6.1.1 taking the degree of the intersection with F of the direct image in PicE^0 of the divisors in Pic$E$. The proofs of relations B2 and A require more work, but are also based on 6.1.1.

6.2 Applications to the monodromy conjecture

In section 5.2 we saw two configurations $\overset{\circ}{E} = \overset{\circ}{E}_j$ which do not contribute to poles, namely $\mathbb{P}^1$ minus one or two points. Using his relations 6.1, Veys extended this to higher dimensions (mostly surfaces, $n = 3$), producing a long (but not exhaustive) list of configurations $\overset{\circ}{E}$ which do not contribute to poles of $Z(s, \chi_{\text{triv}})$ (meaning that in the calculation of the residue at $s = -\nu_j/N_j$ by the explicit formula, one can omit all $\overset{\circ}{E}_I$ with $E_I \subset E$, assuming good reduction and K big enough). When $\chi(\overset{\circ}{E}) = 0$ and $\overset{\circ}{E} \subset h^{-1}(0)$, there is no contribution in A'Campo's formula 2.1.2. Thus in view of the monodromy conjecture, one expects that such an E *usually* does not contribute to poles. For $n = 3$, Veys searched for configurations $\overset{\circ}{E}$ with $\chi(\overset{\circ}{E}) = 0$, and proved for all but two of the ones he found that they do not contribute to poles. I consider this as very convincing evidence for the monodromy conjecture. Here are two examples of such non-contributing configurations with $E^0 = \mathbb{P}^2$. Example 1: $\overset{\circ}{E} = \mathbb{P}^2\setminus$ (at least two lines through the same point P and another line not through P). Example 2: $\overset{\circ}{E} = \mathbb{P}^2\setminus$ (the curves $x = 0, y = 0, z = 0$, and $y^k z = x^{k+1}$), with $k \geqslant 2$.

7. PREHOMOGENEOUS VECTOR SPACES

(7.1) We consider a regular prehomogeneous vector space (G, X) over K, consisting of a connected reductive algebraic subgroup G of GL_n defined over K, acting transitively on the complement U of an absolutely irreducible K-hypersurface V in $X = \mathbb{A}^n$. Let $f \in K[x]$ be an equation for V. Then f is homogeneous and $f(gx) = \nu(g)f(x)$ for all g in G where ν is a rational character of G. Thus f is a relative invariant of G. We have $(\det g)^2 = \nu(g)^{2\kappa}$ where $\kappa = n/\deg(f), 2\kappa \in \mathbb{N}$. Moreover the Bernstein polynomial $b_f(s)$ has degree $\deg(f)$. For all this we refer to [70]. (Actually $b_f(s)$ equals the Sato polynomial, see [23, Cor. 2.5.10].)

We will see below that the local zeta functions of such relative invariants f have very remarkable properties. This was first discovered in the Archimedean case by M. Sato and Shintani [70]. The p-adic case was first investigated by

Igusa [31]. In what follows K is p-adic field and for 7.4 and 7.5 we make the additional assumption that V has only a finite number of orbits under the action of $\ker \nu$. We first give an easy example.

(7.2) *Example.* Take for X the space of all (m, m) matrices, and $G = SL_m \times GL_m$ with action of $(g_1, g_2) \in G$ on $x \in X$ given by $g_1 x g_2^t$. Then $f(x) = \det x$, $b(s) = (s+1)(s+2)\dots(s+m)$, and $Z(s, \chi_{triv}) = \prod_{i=1}^m (1-q^{-i})/(1-q^{-s-i})$. One has also examples with f the determinant (resp. Pfaffian) on the space of symmetric (resp. antisymmetric) (m, m) matrices, or with f the discriminant on the space of binary cubic forms.

(7.3) M. Sato and Kimura [69] have given a complete classification in 29 types of all K-split irreducible (as representation) regular prehomogeneous vector spaces, and Kimura [42] has determined their Bernstein polynomial. For 20 out of these 29 types[1], Igusa [26],[29],[31],[37] has explicitly calculated $Z(s, \chi_{triv})$. These are tabulated in [36]. His calculations don't use resolution of singularities, but exploit the symmetry of the group structure. In all these cases the formulas show that the real parts of the poles are indeed roots of $b_f(s)$. This was the first evidence for conjecture 2.3.1. Also it was on the basis of these formulas that Igusa conjectured proposition 4.1.2 on the degree of $Z(s, \chi_{triv})$.

(7.4) Igusa [31] has found a finite list of candidate poles for $Z_\Phi(s, \chi)$ which only involves group theoretical data associated to (G, X). A weakened version of his result is the following.

Theorem (Igusa [31]). *If s is a pole of $Z_\Phi(s, \chi)$, then there exists $a \in V(K)$ such that $|\nu(h)|^{\kappa + \mathrm{Re}(s)} = \Delta_H(h)$ for all $h \in H$ where H is the fixer of a in $G(K)$ and Δ_H the modulus of H (i.e. $d(h^{-1}uh) = \Delta_H(h)du$ for any Haar measure du on H).*

More recently Kimura, F. Sato and Zhu [46] proved (using microlocal analysis) that the real parts of the above candidate poles are roots of $b_f(s)$ when (G, X) is irreducible and reduced (in the sense of [69]).

(7.5) By a theorem of Borel and Serre, $U(K)$ splits into a finite number of $G(K)$ orbits, say $U_1, \dots U_\ell$. For $i = 1, \dots, \ell$ one defines the functions $Z_{i,\Phi}(s, \chi)$ in the same way as $Z_\Phi(s, \chi)$ but integrating now over U_i instead of K^n. These

[1] Very recently Igusa [40] calculated $Z(s, \chi_{triv})$ for 4 more types.

are rational functions of q^{-s} (see section 8) and satisfy the following functional equations.

Theorem (Igusa [31]). *For $i = 1, \ldots, \ell$ and all Φ we have*

$$Z_{i,\Phi^*}(s,\chi) = \sum_{j=1}^{\ell} \gamma_{ij}(s,\chi) Z_{j,\Phi}(-s-\kappa,\ \chi^{-1}),$$

where Φ^* is a Fourier transform of Φ and the γ_{ij} are rational functions of q^{-s} which are independent of Φ. Moreover, 7.4 holds also for the Z_i and γ_{ij}.

In this theorem we have tacitly assumed that there exists an involution of $\mathrm{End}(X)$ defined over K under which G is stable (one can often take transposition). For a generalization of this result, see Kimura [44]. When $\ell = 1$ the above functional equation takes a nice form. Igusa has classified the (G, X) with $\ell = 1$ when G is irreducible and K-split, and has calculated γ_{11} explicitly in these cases [33]. In the Archimedean case much more is known about the γ_{ij}, see [70].

(7.6) For the 20 types of prehomogeneous vector spaces mentioned in 7.3, Igusa [38] found that $Z(s, \chi_{triv})$ is *universal* in the sense of 4.2.3. This led him to the following conjecture.

Conjecture. *If (G, X) is defined over a number field F, then $Z(s, \chi_{triv}, f, K)$ is universal for almost all completions K of F, provided G splits over K.*

We recall from 4.2.3 that universality implies the functional equation $\mathcal{Z}(u^{-1}, v^{-1}) = u^{\deg f}\mathcal{Z}(u, v)$, which was first proved by Igusa [38] for the above mentioned 20 types by explicit calculation. For some more conjectures, see Gyoja [24].

(7.7) Ono [65] has shown, for any non-constant absolutely irreducible polynomial f over a number field F, that the product of $(1-q^{-1})^{-1} Z(s, \chi_{triv})$ over all p-adic completions K of F is convergent and holomorphic for $\mathrm{Re}(s) > 0$. In the case of irreducible regular prehomogeneous vector spaces with only finitely many *adelic* open orbits, Igusa [35] proved that this product has a meromorphic continuation to the whole s-plane and satisfies a functional equation (assuming a mild condition which was removed in [44], [45]). However, it seems very unlikely that this remains always true when there are infinitely many adelic open orbits. For some other adelic results, see Datskovsky and Wright [7].

8. INTEGRATION OVER SUBANALYTIC SETS

Igusa's result on the rationality of $Z_\Phi(s,\chi)$ can be much generalized. We briefly survey some of the results. As always, we assume that K, R and P are as in 1.1.

A subset S of K^n is called *semi-algebraic* if it can be described by a finite Boolean combination of conditions of the form $g(x) = 0, \text{ord } g(x) \leqslant \text{ord } h(x)$, or $\exists y \in K\colon g(x) = y^m$, with $g, h \in K[x], m \in \mathbb{N}$. The subset S is called *semi-analytic* if locally at each point of K^n it can be described by such conditions where we allow now g, h to be analytic functions.

Macintyre's remarkable theorem [55] states that the projection (from K^{n+k} to K^n) of a semi-algebraic set is again semi-algebraic. His proof is based on results from mathematical logic and the work of Ax-Kochen-Ersov. For different proofs, see the references in [19, (0.14)]. Iterating the operations of projection and complementation, one sees that many sets, which arise in practice, are semi-algebraic. For example, the p-adic orbits of a p-adic algebraic group action are semi-algebraic.

Using Igusa's method 1.3 one shows that $\int_S |f(x)|^s |dx|$ is a rational function of q^{-s} whenever S is semi-algebraic and bounded and $f \in K[x]$, see [10] (this extends [59]). However, this can also be proved without resolution of singularities, using p-adic cell decomposition [10] instead. This method is based on partitioning S in semi-algebraic "cells" on which $|f(x)|$ has a simple description so that the integral can be evaluated using separation of variables and induction on n. For other applications of this method, see [11] and [68].

We now return to the analytic case. The projection of a bounded semi-analytic set is not semi-analytic in general. This motivates the following definition. A subset S of K^n is called subanalytic if locally at each point of K^n it is the projection of some bounded semi-analytic set. If we replace K by $\mathbb{R}$ then this agrees with the classical notion of real subanalytic sets, introduced by Lojasiewicz, Gabrielov and Hironaka [75]. Van den Dries and Denef [19] developed the theory of p-adic subanalytic sets. Some of their ideas were inspired by mathematical logic. A first basic theorem is that the complement of a subanalytic subset of K^n is again subanalytic. Another basic result is the following.

8.1 Theorem (Uniformization of p-adic subanalytic sets [19]). *Let $S \subset K^n$ be subanalytic and bounded. Then there exists a K-analytic manifold M and an* *analytic map $h\colon M \to K^n$, which is a composition of finitely many blowing-up maps with respect to closed submanifolds of codimension $\geqslant 2$, such that $h^{-1}(S)$ is semi-analytic in M.*

Since integrals over semi-analytic sets can be evaluated by using Igusa's method, the above theorem yields the following.

8.2 Theorem. *Let $S \subset K^n$ be subanalytic and bounded, and $f\colon K^n \to K$ analytic. Then, $\int_S |f(x)|^s |dx|$ is a rational function of q^{-s}.*

Actually this remains valid when f is any function whose graph is subanalytic [13]. A first application of theorem 8.2 is the following.

8.3 Corollary [19]. *Let f be a power series over K which converges on R^n. For $m \in \mathbb{N}$, denote by A_m the cardinality of the reduction $\bmod P^m$ of $V = \{x \in R^n \mid f(x) = 0\}$. Then $R(t) := \sum_{m=0}^{\infty} A_m t^m$ is a rational function of t.*

This corollary was conjectured by Serre and Oesterlé [72],[64] and was our main motivation to investigate p-adic subanalytic sets. Corollary 8.3 follows from 8.2 by expressing $R(t)$ in terms of the integral $\int_D |w|^s |dx||dw|$ where $D = \{(x, w) \in R^{n+1} \mid \text{distance from } x \text{ to } V \text{ is} \leqslant |w|\}$. Note that D is indeed the projection of a semi-analytic set. We refer to Oesterlé [64] for fundamental results on the asymptotics of A_m. When f has only two variables, $R(t)$ has been explicitly calculated by Bollaerts [6].
Another application of theorem 8.2 is the following remarkable result of du Sautoy [20].

8.4 Theorem. *Let G be a compact p-adic analytic group. For $m \in \mathbb{N}$ denote by C_m the number of open subgroups of index p^m in G. Then $\sum_{m=0}^{\infty} C_m t^m$ is rational in t.*

Finally we mention some further developments. The main results about p-adic semi-algebraic and subanalytic sets are not uniform in p. Uniform versions have been obtained by Pas [66],[67] and Van den Dries [77]. Subanalytic sets in the context of rigid analytic geometry have been studied by Lipshitz [49] and Schoutens [71].

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