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THE HEAT EQUATION IN RIEMANNIAN GEOMETRY

[after PATODI, GILKEY, etc.]

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1. Heat Equation approach to the index theorem.

As explained in the preceding lecture the idea of using the heat equation to solve the index problem is as follows. Let $D : C^\infty(X,E) \to C^\infty(X,F)$ be an elliptic operator on the compact manifold X. Using metrics on E,F and X we may form its adjoint D^* and the self-adjoint operators

$$\Delta_E = D^*D \quad , \quad \Delta_F = DD^* \quad .$$

For $t > 0$ the heat operators $e^{-t\Delta_E}, e^{-t\Delta_F}$ have C^∞ kernels so that their traces are well-defined. Now except for their zero eigenspaces the operators Δ_E and Δ_F are conjugate ($\Delta_E = D^{-1}\Delta_F D$), while their zero-eigenspaces coincide with the null spaces $\mathcal{N}(D)$, $\mathcal{N}(D^*)$ of D and D^* respectively. Hence, for all $t > 0$,

$$(1.1) \qquad \operatorname{Tr} e^{-t\Delta_E} - \operatorname{Tr} e^{-t\Delta_F} = \dim \mathcal{N}(D) - \dim \mathcal{N}(D^*) = \operatorname{index} D .$$

But we have asymptotic expansions as $t \to 0$ for the terms on the left, namely

$$(1.2) \qquad \operatorname{Tr} e^{-t\Delta_E} \sim \sum_{k=-n}^{\infty} a_k \, t^{k/m}$$

where $n = \dim X$, $m = \text{order } \Delta_E$. If b_k are the coefficients of the corresponding expansion for Δ_F we deduce from (1.1)

$$a_k = b_k \quad \text{for} \quad k \neq 0$$

(1.3) $$a_0 - b_0 = \text{index } D\ .$$

Now the point about the expansion (1.2) is that the coefficients a_k are locally computable. More precisely we have

$$a_k = \int_X \alpha_k(x)$$

where $\alpha_k(x)$ is a C^∞ measure on X given (in principle) as an explicit function of the coefficients of Δ_E. In terms of a complete orthonormal set $\{\phi_n\}$ of eigenfunctions of Δ_E with eigenvalues λ_n, the α_k are defined by an asymptotic expansion

$$\Sigma\, e^{-\lambda_n t} \, |\phi_n(x)|^2 \, |dx| \sim \sum_{k=-n}^{\infty} \alpha_k(x)^{k/m}$$

(as $t \to 0$). Thus we derive, in principle, a general formula

(1.4) $$\text{index } D = \int_X \{\alpha_0(x) - \beta_0(x)\}$$

where $\beta_k(x)$ is the measure associated to b_k.

This approach to the index problem was put forward in [1] but it was not until the recent work of Patodi [6] that any progress was made. The difficulty lies in the fact that the α_k are very complicated functions of the coefficients of Δ_E. In particular they involve many derivatives. Patodi's contribution was to show that, in the particular case $D = d + d^*$: even forms $\to$ odd forms (so that index D = Euler characteristic of X), remarkable cancellation took place between α_0 and β_0 so that finally (1.4) reduced to the Gauss-Bonnet formula.

Subsequently Gilkey [5] gave an alternative proof of Patodi's result based on an intrinsic identification of the integrand in (1.4). This was later extended to cover the important case of the Hirzebruch signature theorem. [2] presents a simplified version of Gilkey's argument and generalizes it to cover more general operators arising in Riemannian geometry. This provides enough examples of the index theorem so that the general case can be deduced from them by topological arguments (K-theory). Here we follow the presentation in [2] .

§2. The Hirzebruch signature theorem.

Suppose now X is a compact oriented Riemannian manifold of dimension 2ℓ, and let $d : \Omega^i \to \Omega^{i+1}$ $d^* : \Omega^i \to \Omega^{i-1}$ be the exterior derivative on forms and its adjoint. Then $d + d^*$ acts on the space Ω of all forms and anti-commutes with the involution τ defined by

$$\tau(\alpha) = i^{p(p-1)+\ell} * \alpha \qquad \alpha \in \Omega^p$$

where $* : \Omega^p \to \Omega^{2\ell-p}$ is the duality operator. Thus $d + d^*$ switches the ± 1-eigenspaces $\Omega_\pm$ of τ and hence is of the form $\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$ where $A : \Omega_+ \to \Omega_-$ is called the signature operator. One verifies easily (using Hodge theory) that, if $\ell = 2k$, index A = sign (X) is the signature of the quadratic form on $H^{2k}(X)$. The heat equation method of §1 then gives us an integral formula

$$\text{(2.1)} \qquad \operatorname{sign}(X) = \int_X \omega .$$

Here ω is a measure depending skew-symmetrically on the orientation of X (because reversing orientation takes τ into $-\tau$, hence A into $-A$) and hence may be regarded as a 4k-form canonically associated to the metric g on X. The explicit results for second-order operators assert that $\omega(g)$ is given by a universal polynomial in the g_{ij}, their derivatives of various orders and $(\det g)^{-\frac{1}{2}}$. Moreover under a change of scale in the metric $g \mapsto k^2 g$ (k a positive constant) $A \mapsto k^{-1} g$ and so the constant terms in the expansion of e^{-tA^*A}, e^{-tAA^*} are unaltered. Hence $\omega(k^2 g) = \omega(g)$. Now the main discovery of Gilkey was that these qualitative properties of $\omega(g)$ are enough to identify it as a Pontrjagin form. More precisely we have

THEOREM (Gilkey) *Let* $\omega(g)$ *be any differential form functionally associated to a Riemannian metric* g *and such that*

(i) $\omega(g)$ *is given in local coordinates by a universal polynomial in* g_{ij}, *their derivatives of all orders and* $(\det g_{ij})^{-\frac{1}{2}}$

(ii) $\omega(k^2 g) = \omega(g)$.

Then $\omega(g) = f(p_1, \ldots, p_\ell)$ *where* $p_i(g)$ *are the coefficients of the characteristic polynomial of the curvature matrix* $\mathcal{R}$:

$$\det(1 + t\mathcal{R}) = \Sigma\, p_i t^{2i}$$

Applying this theorem to the form $\omega(g)$ in (2.1) we deduce that

(2.2) $$\text{sign}(X) = \int_X f(p_1, \ldots, p_\ell)$$

for some suitable polynomial f. Computing sufficiently many examples (products of complex projective spaces) then enables us to identify f with the Hirzebruch L-polynomial (after normalizing the p_i by appropriate powers of 2π) exactly as in Hirzebruch's original proof. Thus Gilkey's theorem plays in local Riemannian geometry the part played by Thom's cobordism theory: it characterizes the Pontrjagin numbers.

We now indicate how Gilkey's theorem is proved. There are three steps of which the first two are classical and well-known:

(1) By using geodesic coordinates we argue that $\omega(g)$ must be a polynomial in the components of the curvature tensor and its covariant derivatives together with $(\det g)^{-\frac{1}{2}}$

(2) By the theorem on tensorial invariants of the orthogonal group we deduce that $\omega(g)$ is a linear continuation of basic 'monomials' of the form

$$m(R) = \Sigma_q^* R_{\alpha_1} R_{\alpha_2} \ldots R_{\alpha_r} .$$

Here each R_{α_i} denotes a covariant derivative (of order ε_i) of the curvature tensor, Σ_q means that we <u>contract</u> all but q of the suffixes and $*$ indicates that we skew-symmetrize the q suffixes left (so as to get a skew-symmetric q-tensor where $q = \deg \omega$).

(3) The homogeneity condition (ii) on $\omega(g)$ implies that we only need monomials $m(R)$ with $q = 2r + \varepsilon$, where $\varepsilon = \Sigma\varepsilon_i$ is the total number of covariant derivative indices. If $\varepsilon > 0$ this implies that in some factor R_{α_i} we must skew-symmetrize at least three out of the first five indices. The Bianchi identities then show that $m(R) = 0$. If $\varepsilon = 0$, so that $m(R)$ involves only the curvature tensor itself (and no higher derivatives), a similar calculation shows that we must use precisely two suffixes in each R_{α_i} for contraction. The symmetries of R show that we may take these as the first two suffixes. It is now clear that $m(R)$, as a form, is an exterior product

$$(\mathrm{Tr}\ R)^{k_1} \wedge (\mathrm{Tr}\ R)^{k_2} \wedge \ldots$$

and so is a Pontrjagin form.

<u>Note</u> The square root factor involving det g allowed in the hypothesis of the Gilkey theorem turns out to be illusory in the end. However it is necessary to allow for it because it appears, a priori, in the heat equation formula. This point was in fact overlooked in [2] as pointed out by Colin de Verdière. In fact in the Gauss-Bonnet theorem $(\det g)^{\frac{1}{2}}$ enters into the final formula but that is because we have there a measure independent of orientation (so not a form) and so the theorem on 'tensorial' invariants has to be modified. However the characterization of the "Euler form" is more complex than

that of the Pontrjagin forms and we shall avoid it.

§3. The general index theorem.

If ξ is a complex vector bundle over X with a hermitian inner product and a unitary connection we can define an elliptic operator A_ξ, acting on forms with coefficients in ξ, which generalizes the signature operator A of §2. To compute index A_ξ by the heat equation method we need to extend Gilkey's theorem to include auxiliary vector bundles. Essentially the same proof as before enables us now to characterize the mixed Pontrjagin-Chern forms of (g,ξ) as the only functorially defined forms satisfying the analogues of (i) and (ii) in the Gilkey theorem: we now require invariance also under change of scale in the inner product of ξ.

Again the explicit polynomial in the Pontrjagin classes of X and the Chern classes of ξ must be found by computing some suitable examples.

We now have sufficiently many explicit examples to deduce the general index theorem. Roughly speaking every elliptic operator can be deformed into a generalized signature operator A_ξ. More precisely we recall (see [4]) that every elliptic operator P defines via its symbol an element $\sigma_P \in K(TX)$, where K denotes the K-theory with compact supports and TX is the cotangent bundle of X. The stability of the index under deformation, and the existence of (pseudo-differential) operators with given symbol, imply that $P \mapsto$ index P induces a homomorphism

(3.1) $\quad$ index : $K(TX) \to \mathbf{Z}$.

Now $K(TX)$ is a module over $K(X)$ in such a way that

$$\xi \cdot \sigma_A = \sigma_{A_\xi} .$$

Moreover, modulo 2-primary groups, $K(TX)$ is a free $K(X)$-module with generator σ_A. Hence (3.1) is determined by its values on σ_{A_ξ} , for all ξ . Thus the generalized signature theorem implies the general index theorem for X even-dimensional and oriented. The other cases follow easily by passing to the oriented double covering and multiplying by a circle.

Included in the general index theorem is the Riemann-Roch theorem for compact complex manifolds. For Kähler manifolds it can be given a direct treatment on the lines of the signature theorem. One uses the Dirac operator of the associated Spin^c-structure. It should be emphasized that this is still essentially a theorem about Riemannian manifolds: it is not necessary to try to extend the Gilkey theorem to complex Hermitian manifolds. The first heat equation proof of the Riemann-Roch theorem is due to Patodi [7].

§4. Further developments.

The heat equation proof of the signature theorem (and others of the same type) gives more information than previous proofs, in that the integrand is given an analytical interpretation. In other words we have a local version of the signature theorem, from which the global version follows by integration. Not surprisingly this local version gives rise to interesting results for manifolds with boundary [3]. To explain these results let us consider a compact oriented 4k-dimensional Riemannian manifold X with boundary Y and assume that, near the boundary, X is isometric to the product $Y \times I$. We define sign (X) now as the signature of the quadratic form induced by the cup-product on the image of $H^{2k}(X,Y)$ in $H^{2k}(X)$. Then the difference

$$\text{sign}(X) - \int_X L$$

(where L denotes the L-polynomial in the Pontrjagin forms) need not vanish but, as an easy consequence of the signature theorem for closed manifolds, it depends only on Y (as oriented Riemannian manifold). Heat equation methods enable us to identify this invariant of Y as a 'spectral invariant', that is as a function of the eigenvalues of the Laplacian on forms. Precisely define the self-adjoint operator B on even forms on Y by

$$B\phi = (-1)^{k+p+1}(d* - *d)\phi \qquad (\phi \in \Omega^{2p})$$

and put

$$\eta(s) = \sum_{\lambda \neq 0} \text{sign}\,\lambda\, |\lambda|^{-s} \tag{4.1}$$

where λ runs over the eigenvalues of B (with multiplicities counted). Note that B^2 is just the Laplacian Δ so that λ^2 runs over the eigenvalues of Δ.

For $\mathrm{Re}(s)$ large (4.1) converges absolutely. One proves that $\eta(s)$ can then be continued meromorphically to the entire s-plane, that $s = 0$ is not a pole and finally that

$$(4.2) \qquad \mathrm{sign}\,(X) - \int_X L = -\,\eta(0)$$

There are similar results for the other operators arising in Riemannian geometry. Moreover as a consequence of these Riemannian cases one deduces, by topological methods, that for <u>any</u> elliptic self-adjoint operator the η-function analogous to (4.1) is finite at $s = 0$.

Formula (4.2) has many interesting extensions and applications for which we refer to [3].

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