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A Property of Generalized McLain Groups

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ABSTRACT - In this short note we show that if S is a connected unbounded poset and R a ring with no zero divisors, then a generalized McLain group $G(R,S)$ is a product of two proper normal subgroups.

1. Introduction.

McLain groups were defined in [3] for the first time. These groups are characteristically simple and locally nilpotent with some further interesting properties.

Let S be an unbounded partially ordered set (poset, in short) and R be a ring with $1 \neq 0$. Define the generalized McLain group $G(R,S)$ as in [2]. Now every element of $G(R,S)$ can be uniquely expressed as

$$1 + \sum_{i=1}^{n} a_i e_{\alpha_i \beta_i}$$

where $a_i \in R$, $\alpha_i, \beta_i \in S$, $\alpha_i < \beta_i$ for $i = 1, \ldots, r$ and $n \in \mathbb{N}$.

In [5], some properties of $G(\mathbb{F}_p, S)$ are considered for some orderings where $\mathbb{F}_p$ is the field of p elements. The generalized McLain groups are considered in a general context in [2] and the automorphism groups of these groups are considered in [1], [2] and [4].

[2, Theorem 7.1] gives a necessary and sufficient condition to be $G(R,S)$ indecomposable. In this short note we ask the following question :

Does $G(R,S)$ have proper normal subgroups K and N such that $G(R,S) = KN$?

Two elements $\alpha, \beta \in S$ are called *connected*, if there are elements $\alpha_0, \ldots, \alpha_n \in S$ such that $\alpha_0 = \alpha, \alpha_n = \beta$ and for each $0 \leq i < n$, either

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$\alpha_i \leq \alpha_{i+1}$ or $\alpha_{i+1} \leq \alpha_i$. S is called *connected* if every pair of elements in S is connected.

We shall prove the following:

THEOREM. Let S be a connected unbounded poset and R a ring with no zero divisors. Then $M := G(R,S)$ has proper normal subgroups K and N such that $M = KN$, $C_M(K) \neq 1$ and $C_M(N) = 1$. Furthermore if $1 + ce_{\xi\zeta} \in M$, then $1 + ce_{\xi\zeta} \in K$ or $1 + ce_{\xi\zeta} \in N$.

2. Proof of the Theorem.

LEMMA 2.1. *Let S be a connected unbounded poset and R a ring with no zero divisors. Then every finite family of non-trivial normal subgroups of M intersects non-trivially.*

PROOF. Obviously it is sufficient to prove the lemma for two proper non-trivial normal subgroups of M. Let N and K be such subgroups of M. Assume $N \cap K = 1$ and follow the proof of [2, Theorem 7.1] to reach a contradiction. □

PROOF OF THE THEOREM. Put $M := G(R,S)$ and let $w = 1 + ae_{\alpha\beta}$ with $0 \neq a \in R$, $K := C_M(\langle w^M \rangle)$ and $N := \langle (1 + ce_{\gamma\delta})^M : 1 + ce_{\gamma\delta} \notin K, c \in R \rangle$. Then we will prove that $M = KN$. Since $Z(M) = 1$, we have $K \neq M$. Clearly

$$\langle w^M \rangle = \left\langle (1 + ae_{\alpha\beta})^{1+\sum_{i=1}^{r} a_i e_{\lambda_i\alpha} + \sum_{j=1}^{s} b_j e_{\beta\mu_j}} : a, a_i, b_j \in R, 1 \leq i \leq r, 1 \leq j \leq s \right\rangle$$
$$= \left\langle 1 + ae_{\alpha\beta} - \sum_{i=1}^{r} aa_i e_{\lambda_i\beta} + \sum_{j=1}^{s} ab_j e_{\alpha\mu_j} + \sum_{\substack{1\leq i\leq r\\ 1\leq j\leq s}} aa_i b_j e_{\lambda_i\mu_j} : a, a_i, b_j \in R,\right.$$
$$\left. 1 \leq i \leq r, 1 \leq j \leq s \right\rangle.$$

Let $\alpha < \sigma < \tau < \beta$, then we have $1 + de_{\sigma\tau} \in K$ with $0 \neq a \in R$ by [2, Lemma 2.2]. Since a generator $1 + ce_{\gamma\delta}$ of N is not contained in K, it must be of the form $1 + ce_{\beta\delta}$ or $1 + ce_{\mu\delta}$ ($\mu > \beta$) or $1 + ce_{\gamma\alpha}$ or $1 + ce_{\gamma\lambda}$ ($\lambda < \alpha$) and its conjugates must be of the form:

$$(1 + ce_{\beta\delta})^{1+\sum_{i=1}^{r} a_i e_{\lambda_i\beta} + \sum_{j=1}^{s} b_j e_{\delta\mu_j}} = 1 + ce_{\beta\delta} - \sum_{i=1}^{r} ca_i e_{\lambda_i\delta} + \sum_{j=1}^{s} cb_j e_{\beta\mu_j} + \sum_{\substack{1\leq i\leq r\\ 1\leq j\leq s}} ca_i b_j e_{\lambda_i\mu_j}$$

or

$$(1+ce_{\mu\delta})^{1+\sum_{i=1}^{r}a_ie_{\lambda_i\mu}+\sum_{j=1}^{s}b_je_{\delta\mu_j}}=1+ce_{\mu\delta}-\sum_{i=1}^{r}a_ie_{\lambda_i\delta}+\sum_{j=1}^{s}cb_je_{\mu\mu_j}+\sum_{\substack{1\le i\le r\\1\le j\le s}}ca_ib_je_{\lambda_i\mu_j}$$

or

$$(1+ce_{\gamma\alpha})^{1+\sum_{i=1}^{r}a_ie_{\lambda_i\gamma}+\sum_{j=1}^{s}b_je_{\alpha\mu_j}}=1+ce_{\gamma\alpha}-\sum_{i=1}^{r}ca_ie_{\lambda_i\alpha}+\sum_{j=1}^{s}cb_je_{\gamma\mu_j}+\sum_{\substack{1\le i\le r\\1\le j\le s}}ca_ib_je_{\lambda_i\mu_j}$$

or

$$(1+ce_{\gamma\lambda})^{1+\sum_{i=1}^{r}a_ie_{\lambda_i\gamma}+\sum_{j=1}^{s}b_je_{\lambda\mu_j}}=1+ce_{\gamma\lambda}-\sum_{i=1}^{r}ca_ie_{\lambda_i\lambda}+\sum_{j=1}^{s}b_je_{\gamma\mu_j}+\sum_{\substack{1\le i\le r\\1\le j\le s}}ca_ib_je_{\lambda_i\mu_j}.$$

For all terms $e_{\theta\varepsilon}$ that appear in each case, $\theta \notin [\alpha,\beta]$ or $\varepsilon \notin [\alpha,\beta]$. Hence we have that there is no product of these elements which equals $1+e_{\sigma\tau}$, i.e., $1+e_{\sigma\tau} \notin N$. Hence $N \neq M$ and obviously $M = KN$.

Clearly we have $C_M(K) \neq 1$. Assume $C_M(N) \neq 1$, then $C_M(K) \cap \cap C_M(N) \neq 1$ by Lemma 2.1. But since $Z(M) = 1$, this is a contradiction. The final part of the theorem follows by the construction of K and N. Now the proof is complete. □

Corollary 2.2. *Let S be a connected unbounded poset and R a ring with no zero divisors. Put $M := G(R,S)$ and let N be the subgroup defined in the theorem. Then*

$$C_M(\langle x^M\rangle)/C_M(\langle x^M\rangle)\cap N$$

is perfect for every generator x of M of the form $1+ae_{\alpha\beta}$ ($a \in R$).

Corollary 2.3. *Let S be a connected unbounded poset and R a ring with no zero divisors. Put $M := G(R,S)$. Then M has a decomposable non-trivial epimorphic image.*

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