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Examples of Birationality of Pluricanonical Maps.

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ABSTRACT - By generalizing an Enriques construction, in $\mathbb{P}^4$ we construct a *double space* V of degree 12, whose branch locus has a 6-ple point of the type $z^6 + \dots + x^{12} + \dots + y^{12} = 0$. We demonstrate that a desingularization of V has birational invariants $q_1 = q_2 = 0$, $p_g = P_1 = 3$, $P_2 = 7$, $P_3 = 13$, $P_4 = 22$, $P_5 = 34$, $P_6 = 51$. Moreover, we prove that the m-canonical transformation has fibers that are generically finite sets if and only if $m \geqslant 2$ and it is birational if and only if $m \geqslant 6$.

Introduction.

E. Bombieri [B] proved that the m-canonical transformation of any nonsingular surface of general type is birational if $m \geqslant 5$ and $m = 5$ is the minimum for the surfaces (minimal models) with $(K^2) = 1$ and $p_g = 2$.

F. Enriques constructed a surface with $(K^2) = 1$, $p_g = 2$ (see [E] § 14, pp. 303-304); this is a desingularization of a double plane with a branch curve of degree 10, having a singular [5,5] point on it.

At a seminar, E. Stagnaro suggested generalizing the Enriques double plane to a three-dimensional *double space* for constructing new examples of threefolds, whose m-canonical transformation becomes birational if m is large enough.

This paper touches first on a demonstration of the fact that the m-canonical transformation of the Enriques example is birational if and only if $m \geqslant 5$, then such a situation is generalized, constructing a *double*

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space V. We thus have the birationality of the m-canonical transformation if and only if $m \geqslant 6$. A desingularization of V has the birational invariants $q_1 = q_2 = 0$, $p_g = P_1 = 3$, $P_2 = 7$, $P_3 = 13$, $P_4 = 22$, $P_5 = 34$, $P_6 = 51$.

We define *double space* of degree $2n$ the projective closure in $\mathbb{P}^4$ of the affine hypersurface given by $t^2 = f_{2n}(x, y, z)$, being $f_{2n}(x, y, z)$ a polynomial of degree $2n$; the surface of equation $f_{2n}(x, y, z) = 0$ is the branch locus of the double space.

We must bear in mind that a double plane with a branch curve of degree 10 with a singular [5,5] point on it is affinely represented by an equation of the type $z^2 + y^5 + \dots + x^{10} = 0$. In the following paragraphs, said situation will be generalized by constructing a double space affinely given by an equation of the type $t^2 + z^6 + \dots + x^{12} + \dots + y^{12} = 0$.

M. Chen [C] and S. Lee [L] proved that if the canonical divisor K of a threefold is «nef» and (K^3) is positive, then the m-canonical transformation is birational for $m \geqslant 6$. In the proposed example the said properties are not simultaneously satisfied, but the birationality of the m-canonical transformation holds true for $m \geqslant 6$.

In this paper we consider surfaces and threefolds on the field $\mathbb{C}$ of the complex numbers and we'll write $\mathbb{P}^N$ instead of $\mathbb{P}^N_{\mathbb{C}}$.

1. Example of a double plane S of degree 10 in $\mathbb{P}^3$ whose m-canonical transformation is birational if and only if $m \geqslant 5$.

1.1. *Description of S.*

Let us choose a generic curve C in the linear system of curves in $\mathbb{P}^2$ defined by

$$F_{10}(X_0, X_1, X_2) = aX_0^5 X_2^5 + bX_0 X_2^9 + cX_1^{10} + dX_2^{10}.$$

According to Bertini theorem, C has its unique singularity at the point $A_0 = (1, 0, 0)$. To be more precise, C has a $[5, 5]$ point at A_0, i.e. a 5-ple point with an infinitely near 5-ple point. By using the affine coordinates

$$x = \frac{X_1}{X_0}, \qquad y = \frac{X_2}{X_0}, \qquad z = \frac{X_3}{X_0}$$

we obtain the polynomial

$$f_{10}(x, y) = ay^5 + by^9 + cx^{10} + dy^{10}$$

and hence the double plane of affine equation $z^2 = f_{10}(x, y)$. Let S be its projective closure in $\mathbb{P}^3$:

$$S : X_0^8 X_3^2 - aX_0^5 X_2^5 - bX_0 X_2^9 - cX_1^{10} - dX_2^{10} = 0 .$$

S is normal and its singularities are the points $A_3 = (0, 0, 0, 1)$ and $A_0 = (1, 0, 0, 0)$. To be more precise:

– S has an 8-ple point at A_3 and four double curves r_1, r_2, r_3, r_4 infinitely near in the next neighbourhoods;

– S has a double point at A_0 with a double curve r_5, a double point P and again two double curves r_6 and r_7 infinitely near, in the next neighbourhoods.

1.2. *Birationality of the m-canonical transformation for $m \geqslant 5$.*

We state the birationality of the m-canonical transformation, $m \geqslant 5$, using the theory of adjoints of Enriques. This theory has recently been revised by E. Stagnaro in [S_2]. We keep the same nomenclature and notations as are used in said paper. In our examples all the singularities satisfy the hypothesis assumed in [S_2].

The properties of a double plane are well known, but it may be useful to mention the ones that will be generalized to the hypersurface (double space) in $\mathbb{P}^4$ that we construct later on.

It is maybe less well known, however see [E], [S_1], [S_2] (a detailed calculation of the bicanonical adjoints is given in [S_1]), that the m-canonical adjoints to a double plane of affine equation $S : z^2 = f_{2n}(x, y)$, with a nonsingular branch curve $f_{2n}(x, y) = 0$, are:

$$\phi_{m(n-3)}(x, y) + z\phi_{(m-1)n-3m}(x, y) = 0 ,$$

where $\phi_i(x, y)$ denotes a polynomial of degree i in x, y.

In compliance with [S_2], let us call the m-canonical adjoints defined by $\phi_{m(n-3)}(x, y) = 0$ as *global* and the m-canonical adjoints defined by $z\phi_{(m-1)n-3m}(x, y) = 0$ as *non-global.*

Let us emphasize the following facts.

1. The m-canonical transformation $\varphi_{|mK|}$ coincides (on an open set), up to isomorphisms, with the rational transformation $\psi_{m|S}$ pro-

duced by the linear system of the m-canonical adjoints restricted to the double plane S (see [S_2], section 16).

2. If we want $\psi_{m|S}$ to be birational, it is necessary (but generally not sufficient) for at least one of the m-canonical adjoints to be of the kind $z\phi_{(m-1)n-3m}(x, y) = 0$. Conversely, the transformation is generically $2:1$, at most.

3. It is possible to prove (but we omit the demonstration) that in every m-canonical adjoint, $m \leqslant 4$, the «z» coefficient vanishes as soon as the branch curve has a $[5,5]$ point on it.

4. From 2 and 3 it follows for $m \leqslant 4$ that $\psi_{m|S}$, so $\varphi_{|mK|}$, cannot be birational. Moreover, one can prove directly that $\psi_{5|S}$ is birational and also that $\psi_{m|S}$ is birational for $m \geqslant 5$, because p_g is positive.

The idea for generalizing all this to double spaces is to transfer the properties 1, 2, 3 and 4 to a suitable double space. As a result, in the case of our example at least, the birationality holds true if and only if $m \geqslant 6$.

2. Example of a double space V of degree 12 in $\mathbb{P}^4$, whose m-canonical transformation is birational if and only if $m \geqslant 6$.

2.1. *Description of V.*

To extend the foregoing situation to $\mathbb{P}^4$, let S be a generic surface in the linear system of surfaces in $\mathbb{P}^3$ defined by

$$F_{12}(X_0, X_1, X_2, X_3) = aX_0^6X_3^6 + bX_0X_3^{11} + cX_1^{12} + dX_2^{12} + eX_3^{12}.$$

According to Bertini theorem, S has a unique singularity at the point $A_0 = (1, 0, 0, 0)$. To be more specific, S has a 6-ple point at A_0 with an infinitely near 6-ple curve. By using the affine coordinates

$$x = \frac{X_1}{X_0}, \quad y = \frac{X_2}{X_0}, \quad z = \frac{X_3}{X_0}, \quad t = \frac{X_4}{X_0}$$

we obtain the polynomial

$$f_{12}(x, y, z) = az^6 + bz^{11} + cx^{12} + dy^{12} + ez^{12}$$

and hence the hypersurface of affine equation $t^2 = f_{12}(x, y, z)$.

Let V be its projective closure in $\mathbb{P}^4$:

$$V: X_0^{10}X_4^2 - aX_0^6X_3^6 - bX_0X_3^{11} - cX_1^{12} - dX_2^{12} - eX_3^{12} = 0\,.$$

We call V a *double space*, according to our definition.

V is normal and only has singularities at $A_4 = (0, 0, 0, 0, 1)$ and at $A_0 = (1, 0, 0, 0, 0)$. To be more precise:

– V has a 10-ple point at A_4 with 5 double surfaces $\alpha_1, \ldots, \alpha_5$ infinitely near, in the next neighbourhoods,

– V has a double point at A_0 with 2 double surfaces α_6, α_7, 1 double curve s, and 2 double surfaces α_8, α_9 infinitely near, in the next neighbourhoods.

2.2. *Computation of* $p_g = P_1$ *and* P_m *of* V.

Now we calculate the genus and plurigenera of V, i.e.

$$P_m = dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(mK_X)) = dim|mK_X| + 1, \quad m \geqslant 1, \quad p_g = P_1,$$

where X denotes a nonsingular model of V.

The path chosen for constructing X consists in two sequences of relations owing to the singularities of V at A_4 and A_0.

To solve the singularity at A_4 we have the following sequence of blow-ups:

(1) $$V_6 \subset \mathbb{P}_6 \xrightarrow{\pi_6} \mathbb{P}_5 \xrightarrow{\pi_5} \mathbb{P}_4 \xrightarrow{\pi_4} \mathbb{P}_3 \xrightarrow{\pi_3} \mathbb{P}_2 \xrightarrow{\pi_2} \mathbb{P}_1 \xrightarrow{\pi_1} \mathbb{P}^4 \supset V$$

where π_1 denotes the blow-up of $\mathbb{P}^4$ at A_4 and π_i $(2 \leqslant i \leqslant 6)$ is the blow-up of $\mathbb{P}_{i-1}$ along α_{i-1}. From (1) the relations follow:

$$\begin{cases} K_{\mathbb{P}_1} = \pi_1^*(K_{\mathbb{P}^4}) + 3E_{A_4} \\ V_1 = \pi_1^*(V) - 10E_{A_4} \end{cases} \quad \begin{cases} K_{\mathbb{P}_i} = \pi_i^*(K_{\mathbb{P}_{i-1}}) + E_{\alpha_{i-1}} \\ V_i = \pi_i^*(V_{i-1}) - 2E_{\alpha_{i-1}} \end{cases} \quad (2 \leqslant i \leqslant 6),$$

where E_{A_4}, E_{α_i} denote the exceptional divisors of the blow-ups at A_4 and α_i and V_i denotes the strict transformation of V_{i-1}.

To solve the singularity at A_0 we have the following sequence of blow-ups:

(2) $$V_{12} \subset \mathbb{P}_{12} \xrightarrow{\pi_{12}} \mathbb{P}_{11} \xrightarrow{\pi_{11}} \mathbb{P}_{10} \xrightarrow{\pi_{10}} \mathbb{P}_9 \xrightarrow{\pi_9} \mathbb{P}_8 \xrightarrow{\pi_8} \mathbb{P}_7 \xrightarrow{\pi_7} \mathbb{P}_6 \supset V_6$$

(in the following V_{12} will be X), where π_7 is the blow-up of $\mathbb{P}_6$ at A_0, π_8 and π_9 are the blow-ups of $\mathbb{P}_7$ and $\mathbb{P}_8$ along α_6 and α_7, π_{10} is the blow-up of $\mathbb{P}_9$ along s and finally π_{11} and π_{12} are the blow-ups of $\mathbb{P}_{10}$ and $\mathbb{P}_{11}$ along

α_8 and α_9. From (2) we can say that:

$$\begin{cases} K_{\mathbb{P}_7} = \pi_7^*(K_{\mathbb{P}_6}) + 3E_{A_0} \\ V_7 = \pi_7^*(V_6) - 2E_{A_0} \end{cases} \qquad \begin{cases} K_{\mathbb{P}_8} = \pi_8^*(K_{\mathbb{P}_7}) + E_{\alpha_6} \\ V_8 = \pi_8^*(V_7) - 2E_{\alpha_6} \end{cases}$$

$$\begin{cases} K_{\mathbb{P}_9} = \pi_9^*(K_{\mathbb{P}_8}) + E_{\alpha_7} \\ V_9 = \pi_9^*(V_8) - 2E_{\alpha_7} \end{cases} \qquad \begin{cases} K_{\mathbb{P}_{10}} = \pi_{10}^*(K_{\mathbb{P}_9}) + 2E_s \\ V_{10} = \pi_{10}^*(V_9) - 2E_s \end{cases}$$

$$\begin{cases} K_{\mathbb{P}_{11}} = \pi_{11}^*(K_{\mathbb{P}_{10}}) + E_{\alpha_8} \\ V_{11} = \pi_{11}^*(V_{10}) - 2E_{\alpha_8} \end{cases} \qquad \begin{cases} K_{\mathbb{P}_{12}} = \pi_{12}^*(K_{\mathbb{P}_{11}}) + E_{\alpha_9} \\ X = V_{12} = \pi_{12}^*(V_{11}) - 2E_{\alpha_9}, \end{cases}$$

where E_{A_0}, E_{α_i} and E_s denote the exceptional divisors of the blow-ups at A_0, α_i and s.

Because X is nonsingular, we can apply the adjunction formula that states: if D is a divisor linearly equivalent to $K_{\mathbb{P}_{12}} + X$, i.e. $D \equiv K_{\mathbb{P}_{12}} + X$, and if $D_{|X}$ is defined, then $D_{|X} = K_X$, where K_X is a canonical divisor on X.

Substituting from the above relations, we obtain

$$K_{\mathbb{P}_{12}} + X = \tag{3}$$

$$\pi_{12}^*(\pi_{11}^*(\pi_{10}^*(\pi_9^*(\pi_8^*(\pi_7^*(\pi_6^*(\pi_5^*(\pi_4^*(\pi_3^*(\pi_2^*(\pi_1^*(K_{\mathbb{P}^4} + V) - 7E_{A_4}) -$$

$$E_{\alpha_1}) - E_{\alpha_2}) - E_{\alpha_3}) - E_{\alpha_4}) - E_{\alpha_5}) + E_{A_0}) - E_{\alpha_6}) - E_{\alpha_7})) - E_{\alpha_8}) - E_{\alpha_9}.$$

We now have $K_{\mathbb{P}^4} \equiv -5H$ and $V \equiv 12H$, where H is a hyperplane in $\mathbb{P}^4$. If $\Phi_7 \equiv 7H$ denotes a hypersurface of degree 7 in $\mathbb{P}^4$, we deduce from (3)

$$K_{\mathbb{P}_{12}} + X \equiv \tag{4}$$

$$\pi_{12}^*(\pi_{11}^*(\pi_{10}^*(\pi_9^*(\pi_8^*(\pi_7^*(\pi_6^*(\pi_5^*(\pi_4^*(\pi_3^*(\pi_2^*(\pi_1^*(\Phi_7) - 7E_{A_4}) - E_{\alpha_1}) - E_{\alpha_2}) -$$

$$E_{\alpha_3}) - E_{\alpha_4}) - E_{\alpha_5}) + E_{A_0}) - E_{\alpha_6}) - E_{\alpha_7})) - E_{\alpha_8}) - E_{\alpha_9} = D.$$

We see from the adjunction formula that, if $D_{|X}$ is defined, then it is a canonical divisor K'_X on X, i.e. $D_{|X} = K'_X \equiv K_X$.

If we multiply (4) by the integer $m \geqslant 1$, we obtain

$$m(K_{\mathbb{P}_{12}} + X) \equiv \tag{5}$$

$$\pi_{12}^*(\pi_{11}^*(\pi_{10}^*(\pi_9^*(\pi_8^*(\pi_7^*(\pi_6^*(\pi_5^*(\pi_4^*(\pi_3^*(\pi_2^*(\pi_1^*(\Phi_{7m}) - 7mE_{A_4}) - mE_{\alpha_1}) -$$

$$mE_{\alpha_2}) - mE_{\alpha_3}) - mE_{\alpha_4}) - mE_{\alpha_5}) + mE_{A_0}) - mE_{\alpha_6}) - mE_{\alpha_7})) - mE_{\alpha_8}) - mE_{\alpha_9} =$$

$$mD = D',$$

where Φ_{7m} is a hypersurface of degree $7m$ in $\mathbb{P}^4$.

As before we obtain $D'_{|X} \equiv mK_X$.

Let $\sigma_{|X}: X \to V$, where $\sigma = \pi_{12} \circ \ldots \circ \pi_2 \circ \pi_1$, be the desingularization of V described.

Using the theory of adjoints and pluriadjoints, we can calculate $p_g = P_1$ and P_m; again we use the nomenclature and notations of [S$_2$].

Φ_{7m}, $m \geqslant 1$, is an m-canonical adjoint to V (with respect to σ) if $D'_{|X}$ is effective, i.e. $D'_{|X} \geqslant 0$ (see [S$_2$], section 2).

We see first how the presence of the singular point A_4 characterizes the canonical and m-canonical adjoints.

The condition $\pi_1^*(\Phi_7) - 7E_{A_4} \geqslant 0$ in (4), given by A_4, says that if Φ_7 is a *global* canonical adjoint, then A_4 must be a 7-ple point for Φ_7 itself, i.e. Φ_7 is defined by a form F_7 in X_0, X_1, X_2, X_3. The further condition given by A_4

$$\pi_6^*(\pi_5^*(\pi_4^*(\pi_3^*(\pi_2^*(\pi_1^*(\Phi_7) - 7E_{A_4}) - E_{\alpha_1}) - E_{\alpha_2}) - E_{\alpha_3}) - E_{\alpha_4}) - E_{\alpha_5} \geqslant 0$$

(see (4)), implies that it is

$$F_7(X_0, X_1, X_2, X_3, X_4) = X_0^5 F_2(X_0, X_1, X_2, X_3).$$

The condition

$$[\pi_6^*(\pi_5^*(\pi_4^*(\pi_3^*(\pi_2^*(\pi_1^*(\Phi_{7m}) - 7mE_{A_4}) - mE_{\alpha_1}) - mE_{\alpha_2}) - mE_{\alpha_3}) - mE_{\alpha_4}) - mE_{\alpha_5}]_{|V_6} \geqslant 0$$

imposed by A_4 on the m-canonical adjoints (see (5)) implies that

$$F_{7m}(X_0, X_1, X_2, X_3, X_4) = X_0^{5m}[X_0^5 X_4 F_{2m-6}(X_0, X_1, X_2, X_3) + F_{2m}(X_0, X_1, X_2, X_3)].$$

So we have a situation much the same as the double plane. To be more precise, the m-canonical adjoints to a double space of affine equation $t^2 = f_{2n}(x, y, z)$, with a nonsingular branch locus $f_{2n}(x, y, z) = 0$, are:

$$\phi_{m(n-4)}(x, y, z) + t\phi_{(m-1)n-4m}(x, y, z) = 0$$

where $\phi_i(x, y, z)$ denotes a polynomial of degree i in x, y, z.

Here again, let us call the m-canonical adjoints given by $\phi_{m(n-4)}(x, y, z) = 0$ *global* and those given by $t\phi_{(m-1)n-4m}(x, y, z) = 0$ *non-global.*

Now let us examine the point A_0, which is a singular point for the double space because there is a 6-ple point on its branch locus.

From (4) it must be that

$$F_7(X_0, X_1, X_2, X_3, X_4) = X_0^5 X_3(a_1 X_1 + a_2 X_2 + a_3 X_3).$$

Let W_7' be the vector space of the forms defining *global* canonical adjoints and $\mathcal{W}_7'$ be the vector space of the forms defining canonical adjoints. Since $W_7' = \mathcal{W}_7'$ and $p_g = dim|K_X| + 1$ (see [S_2], section 3), it follows that

$$p_g = 3.$$

We can move on now to consider the point A_0 for calculating the m-canonical adjoints ($m > 1$). The conditions imposed by A_0 produce different results, depending on the value of m.

For $m < 6$ the vector spaces of the forms defining *global* m-canonical adjoints, W_{7m}', and those of the forms defining m-canonical adjoints, $\mathcal{W}_{7m}'$, coincide; but the equality does not hold true for $m = 6$. Indeed, being an m-canonical adjoint implies that

$$\Phi_{7m}\colon \phi_{m(6-4)}(x, y, z) + t\phi_{(m-1)6-4m}(x, y, z) = 0$$

must satisfy the condition (see (5)):

$$[\pi_{12}^*(\pi_{11}^*(\pi_{10}^*(\pi_9^*(\pi_8^*(\pi_7^*(\Phi_{7m}) + mE_{A_0}) - mE_{\alpha_6}) - mE_{\alpha_7})) - mE_{\alpha_8}) - mE_{\alpha_9})]_{|X} \geqslant 0. \tag{6}$$

Now, if $m < 6$, the degree of the «t» coefficient is too low and it satisfies the condition (6) if and only if $\phi_{(m-1)6-4m}(x, y, z)$ vanishes. So, for $m < 6$, Φ_{7m} is an m-canonical adjoint if and only if it is defined by a form

$$F_{7m}(X_0, X_1, X_2, X_3, X_4) = X_0^{5m} X_3^m F_m(X_0, X_1, X_2, X_3),$$

i.e. if and only if Φ_{7m} is really a *global* m-canonical adjoint.

To be more precise, we have

$$\mathcal{W}_{14}' = W_{14}' = \{X_0^{10} X_3^2(b_1 X_0 X_3 + b_2 X_1^2 + b_3 X_1 X_2 + b_4 X_1 X_3 + b_5 X_2^2 + b_6 X_2 X_3 + b_7 X_3^2),\ b_i \in \mathbb{C}\};$$

$$\mathcal{W}_{21}' = W_{21}' = \{X_0^{15} X_3^3(b_1 X_0 X_1 X_3 + b_2 X_0 X_2 X_3 + \cdots + b_{12} X_2 X_3^2 + b_{13} X_3^3),\ b_i \in \mathbb{C}\};$$

$$\mathcal{W}_{28}' = W_{28}' = \{X_0^{20} X_3^4(b_1 X_0^2 X_3^2 + b_2 X_0 X_1^2 X_3 + \cdots + b_{21} X_2 X_3^3 + b_{22} X_3^4),\ b_i \in \mathbb{C}\};$$

$$\mathcal{W}_{35}' = W_{35}' = \{X_0^{25} X_3^5(b_1 X_0^2 X_1 X_3^2 + b_2 X_0^2 X_2 X_3^2 + \cdots + b_{33} X_2 X_3^4 + b_{34} X_3^5),\ b_i \in \mathbb{C}\}.$$

If $m=6$, the degree of the «t» coefficient is $(m-1)6-4m=6$. This is the minimum that can satisfy condition (6) and we have the first <u>non</u>-*global* m-canonical adjoint which is affinely given by $tz^6=0$. To be more specific, Φ_{7m} is an m-canonical adjoint ($m=6$) if and only if it is defined by a form

$$F_{42}(X_0, X_1, X_2, X_3, X_4)=X_0^{30}[X_3^6 F_6(X_0, X_1, X_2, X_3)+X_0^5 X_3^6 X_4]$$

and, in affine coordinates, it has the equation

$$\phi_{42}(x, y, z, t)=z^6 \phi_6(x, y, z)+tz^6=0.$$

In a detailed expression we obtain

$$\mathcal{W}'_{42}=\{X_0^{30} X_3^6(aX_0^5 X_4+ \\ +b_1 X_0^3 X_3^3+b_2 X_0^2 X_1^2 X_3^2+\dots+b_{49} X_2 X_3^5+b_{50} X_3^6),\ a,\ b_i \in \mathbb{C}\}.$$

So we have a <u>non</u>-*global* 6-canonical adjoint defined by the form $X_0^{35} X_3^6 X_4$.

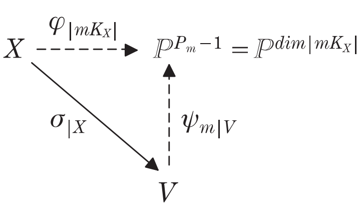
In particular, the plurigenera $P_i=dim|iK_X|+1$, $i \geqslant 1$ (see [S₂]), are

$$p_g=P_1=3, \quad P_2=7, \quad P_3=13, \quad P_4=22, \quad P_5=34, \quad P_6=51.$$

2.3. *The m-canonical transformations* $\varphi_{|mK_X|}$, $1 \leqslant m \leqslant 5$.

In this paragraph, we prove that $\varphi_{|mK_X|}$ is a generically $2:1$ map for $2 \leqslant m \leqslant 5$.

Let us consider the following triangle



where $\sigma_{|X}$ is the desingularization of V and $\psi_{m|V}$ is the rational transformation, restricted to V, defined by the linear system of bicanonical adjoints to V. The foregoing diagram is commutative because the divisors of $|mK_X|$ are of the kind

$$[\pi_{12}^*(\pi_{11}^*\cdots(\pi_1^*(\Phi_{7m})-7mE_{A_4})\cdots-mE_{\alpha_8})-mE_{\alpha_9}]_{|X}.$$

To prove that $\varphi_{|mK_X|}$ is generically $2:1$, it sufficies to consider such a transformation on an open set of X. σ is a sequence of blow-ups and so it is an isomorphism outside the exceptional divisors of the single blow-ups; so, on an open set of X, $\sigma_{|X}$ is an isomorphism. As a result, to say that $\varphi_{|mK_X|}$ is generically $2:1$ means that $\psi_{m|V}$ generically $2:1$.

Now let us demonstrate that $\psi_{2|V}$ is generically $2:1$.

Bearing in mind that

$$\mathcal{W}'_{14} = W'_{14} = \{X_0^{10} X_3^2 (b_1 X_0 X_3 + b_2 X_1^2 + b_3 X_1 X_2 + \\ + b_4 X_1 X_3 + b_5 X_2^2 + b_6 X_2 X_3 + b_7 X_3^2),\ b_i \in \mathbb{C}\},$$

we shall have

$$\begin{array}{ccc} V \subset \mathbb{P}^4 & \overset{\psi_2}{\to} & \mathbb{P}^6 \\ (X_0, X_1, X_2, X_3, X_4) & \mapsto & (Y_0, \dots, Y_6) \end{array}$$

defined by

$$\begin{cases} Y_0 = (X_0^{10} X_3^2)\, X_0 X_3 \\ Y_1 = (X_0^{10} X_3^2)\, X_1^2 \\ Y_2 = (X_0^{10} X_3^2)\, X_1 X_2 \\ Y_3 = (X_0^{10} X_3^2)\, X_1 X_3 \\ Y_4 = (X_0^{10} X_3^2)\, X_2^2 \\ Y_5 = (X_0^{10} X_3^2)\, X_2 X_3 \\ Y_6 = (X_0^{10} X_3^2)\, X_3^2 . \end{cases}$$

Let $U = \mathbb{P}^4 - \{X_0 = X_1 = X_3 = 0\}$ be the affine open set chosen in $\mathbb{P}^4$, with the coordinates

$$x = \frac{X_0}{X_1}, \quad y = \frac{X_2}{X_1}, \quad z = \frac{X_3}{X_1}, \quad t = \frac{X_4}{X_1}.$$

Let $T = \mathbb{P}^6 - \{Y_1 = Y_3 = 0\}$ be the affine open set in $\mathbb{P}^6$ with the coordinates

$$y_1 = \frac{Y_0}{Y_1}, \quad y_2 = \frac{Y_2}{Y_1}, \dots, \quad y_6 = \frac{Y_6}{Y_1}.$$

We shall thus have

$$\begin{array}{ccc} \psi_{2|U}\colon U & \to & T \\ (x, y, z, t) & \mapsto & (y_1, \dots, y_6) \end{array} : \begin{cases} y_1 = xz \\ y_2 = y \\ y_3 = z \\ y_4 = y^2 \\ y_5 = yz \\ y_6 = z^2 . \end{cases}$$

Let $\overline{P} = (\overline{y}_1, \ldots, \overline{y}_6)$ be a generic point of $Im\psi_{2|U}$; the fiber on $\overline{P}$ is

$$\psi_{2|U}^{-1}(\overline{P}) = \left\{ (x, y, z, t) : \begin{bmatrix} xz = \overline{y}_1 \\ y = \overline{y}_2 \\ z = \overline{y}_3 \\ y^2 = \overline{y}_4 \\ yz = \overline{y}_5 \\ z^2 = \overline{y}_6 \end{bmatrix} \right\} = \left\{ (x, y, z, t) : \begin{bmatrix} xz = \overline{y}_1 \\ y = \overline{y}_2 \\ z = \overline{y}_3 \end{bmatrix} \right\}.$$

The fiber on $\overline{P}$ intersects $V_U = V \cap U$ at two points; indeed,

$$V_U \cap \psi_{2|U}^{-1}(\overline{P}) = \begin{cases} x^{10} t^2 - ax^6 z^6 - bxz^{11} - c - dy^{12} - ez^{12} = 0 \\ xz = \overline{y}_1 \\ y = \overline{y}_2 \\ z = \overline{y}_3 \end{cases} =$$

$$\begin{cases} \left(\dfrac{\overline{y}_1}{\overline{y}_3} \right)^{10} t^2 = a\overline{y}_1^6 + b\overline{y}_1\overline{y}_3^{10} + c + d\overline{y}_2^{12} + e\overline{y}_3^{12} \\ y = \overline{y}_2 \\ z = \overline{y}_3 \\ x = \dfrac{\overline{y}_1}{\overline{y}_3}. \end{cases}$$

This means that $\psi_{2|V}: V \to \mathbb{P}^6$, so $\varphi_{|2K_X|}: X \to \mathbb{P}^6$, is generically $2:1$. In particular, we find that V is of general type (Kodaira dimension 3). It follows that $\varphi_{|mK_X|}$, $m > 2$, is also generically $n:1$, with $n \leqslant 2$.

Let us consider an effective canonical divisor $\overline{K}$, which exists because p_g is positive; putting $n\overline{K} + |2K_X| = \{n\overline{K} + D, D \in |2K_X|\}$ for $n = 1, 2, \ldots$ ($n\overline{K}$ fixed part of the linear system), we consider the linear systems

$$\overline{K} + |2K_X| \subset |3K_X|, 2\overline{K} + |2K_X| \subset |4K_X|, \ldots, (m-2)\overline{K} + |2K_X| \subset |mK_X|, \ldots.$$

All these linear systems $\overline{K} + |2K_X|$, $2\overline{K} + |2K_X|$, … give rise to rational transformations which are generically $n:1$, $n \leqslant 2$, and so are the transformations $\varphi_{|mK_X|}$, $m \geqslant 2$.

If $2 \leqslant m \leqslant 5$, the absence of any *non-global* m-canonical adjoint implies that $n = 2$, which is the statement.

REMARK 1. We said previously that the canonical transformation $\varphi_{|K_X|}$ coincides, up to isomorphisms, with $\psi_{1|V}$ on an open set. We can now note that $\psi_{1|V}$ is generically the projection map of V from the straight line $X_1 = X_2 = X_3 = 0$ on a plane.

2.4. *The 6-canonical transformation* $\varphi_{|6K_X|}$.

Our aim is to prove that $\varphi_{|6K_X|}$ is birational. Unlike the foregoing cases, this will be based on the existence of the *non-global* 6-canonical adjoint defined by the form $G_7 = X_0^{35} X_3^6 X_4$.

As we did previously, we choose a canonical effective divisor $\overline{K}$ (e.g. let $\overline{K}$ be given by $L = X_0^5 X_3 X_1$) and we construct the linear system $4\overline{K} + |2K_X| \subset |6K_X|$. The linear system $4\overline{K} + |2K_X| \subset |6K_X|$ defines a rational transformation which coincides with $\varphi_{|2K_X|}$ on an open set, so it defines a generically 2:1 transformation. Now let's consider the *non-global* 6-canonical adjoint given by G_7 and let $\overline{D}$ be the divisor on X defined by it. Note that $\overline{D} \equiv 6K_X$. Let Σ be the linear system

$$\{L^4(\lambda_0 F_0 + \cdots + \lambda_6 F_6) + \lambda_7 G_7 = 0,\ \lambda_i \in \mathbb{C}\},$$

with $F_0 = (X_0^{10} X_3^2)\, X_0 X_3$, $F_1 = (X_0^{10} X_3^2)\, X_1^2$, $F_2 = (X_0^{10} X_3^2)\, X_1 X_2$, $F_3 = (X_0^{10} X_3^2)\, X_1 X_3$, $F_4 = (X_0^{10} X_3^2)\, X_2^2$, $F_5 = (X_0^{10} X_3^2)\, X_2 X_3$, $F_6 = (X_0^{10} X_3^2)\, X_3^2$. Note that $F_0, \ldots, F_6$ span $W_{14}' = \mathcal{W}_{14}'$ and $L^4 F_0, \ldots, L^4 F_6, G_7$ span a vector subspace of $\mathcal{W}_{42}'$. We obtain $4\overline{K} + |2K_X| \subset \Sigma \subset |6K_X|$. The linear system Σ defines a rational transformation

$$\begin{array}{ccc} V \subset \mathbb{P}^4 & \stackrel{\psi}{\rightarrow} & \mathbb{P}^7 \\ (X_0, X_1, X_2, X_3, X_4) & \mapsto & (Y_0, \ldots, Y_7) \end{array}$$

given by:

$$\begin{cases} Y_0 = (X_0^5 X_3 X_1)^4 (X_0^{10} X_3^2)\, X_0 X_3 \\ Y_1 = (X_0^5 X_3 X_1)^4 (X_0^{10} X_3^2)\, X_1^2 \\ Y_2 = (X_0^5 X_3 X_1)^4 (X_0^{10} X_3^2)\, X_1 X_2 \\ Y_3 = (X_0^5 X_3 X_1)^4 (X_0^{10} X_3^2)\, X_1 X_3 \end{cases} \quad \begin{cases} Y_4 = (X_0^5 X_3 X_1)^4 (X_0^{10} X_3^2)\, X_2^2 \\ Y_5 = (X_0^5 X_3 X_1)^4 (X_0^{10} X_3^2)\, X_2 X_3 \\ Y_6 = (X_0^5 X_3 X_1)^4 (X_0^{10} X_3^2)\, X_3^2 \\ Y_7 = X_0^{35} X_3^6 X_4. \end{cases}$$

Let us now consider the open affine set $U = \mathbb{P}^4 - \{X_0 = X_1 = X_3 = 0\}$ in $\mathbb{P}^4$ with the coordinates

$$x = \frac{X_0}{X_1}, \quad y = \frac{X_2}{X_1}, \quad z = \frac{X_3}{X_1}, \quad t = \frac{X_4}{X_1}$$

and the open affine set $T = \mathbb{P}^7 - \{Y_1 = Y_3 = 0\}$ in $\mathbb{P}^7$ with the coordinates

$$y_1 = \frac{Y_0}{Y_1}, \ldots, y_7 = \frac{Y_7}{Y_1}.$$

We obtain:

$$\begin{array}{rcl} \psi_{|U}: U & \to & T \\ (x, y, z, t) & \mapsto & (y_1, \ldots, y_7) \end{array} : \begin{cases} y_1 = xz \\ y_2 = y \\ y_3 = z \\ y_4 = y^2 \\ y_5 = yz \\ y_6 = z^2 \\ y_7 = x^5 t . \end{cases}$$

$\psi_{|U}$ is 1:1. Indeed let $P_1(x_1, y_1, z_1, t_1)$ and $P_2(x_2, y_2, z_2, t_2)$ be two points on U such that $\psi_{|U}(P_1) = \psi_{|U}(P_2)$, i.e.

$$x_1 z_1 = x_2 z_2, \quad y_1 = y_2, \quad z_1 = z_2, \ldots, \quad x_1^5 t_1 = x_2^5 t_2.$$

From $y_1 = y_2$ and $z_1 = z_2$, it follows that $x_1 = x_2$ and finally that $t_1 = t_2$. This proves that ψ, so $\varphi_{|6K_X|}$ is birational.

The birationality of $\varphi_{|mK_X|}$, $m > 6$, follows from this last fact. Indeed, let us consider an effective canonical divisor $\overline{K}$, and let us construct the linear systems $\overline{K} + |6K_X| \subset |7K_X|$, $2\overline{K} + |6K_X| \subset |8K_X|$, …. All these linear systems give rise to rational transformations which are generically 1:1. So all the transformations $\varphi_{|mK_X|}$, $m \geq 6$, are birational.

REMARK 2. Note that if we «delete» $y_7 = x^5 t$ in the expression of $\psi_{|U}: U \to T$, we obtain the $\psi_{2|U}$ of section 2.3. So we have obtained all the informations we need on the pluricanonical transformations only considering the linear system of bicanonical adjoints to V and the *non-global* 6-canonical adjoint given by $X_0^{35} X_3^6 X_4$.

2.5. *Irregularities of V.*

We have to show that the following two relations hold true:

$$q_1(X) = dim_{\mathbb{C}} H^1(X, \mathcal{O}_X) = 0, \quad q_2(X) = dim_{\mathbb{C}} H^2(X, \mathcal{O}_X) = 0.$$

To do this, we use the arguments of [S$_2$], section 4. We consider the surface of degree 12 $\mathcal{S} = \sigma^{-1}(H \cap V)$, where H is the generic hyperplane in $\mathbb{P}^4$. Since A_0 and A_4 are isolated singular points on V, then $H \cap V$, and so $\mathcal{S}$, is nonsingular. Thus it is well known (and easy to see, cf. for instance formula (36)), that $q(\mathcal{S}) = 0$. We deduce from remark 8 that

$$q_1(X) = q(\mathcal{S}) = 0.$$

In addition from formula (36), we have

$$q_2(X) = p_g(X) + p_g(\mathcal{S}) - dim_{\mathbb{C}} W_8,$$

where W_8 is the vector space of the forms defining *global* adjoints to V in $\mathbb{P}^4$ of degree 8. Thus

$$q_2(X) = 3 + 165 - 168 = 0.$$

This proves the statement.

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