RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

D. AREGBA-DRIOLLET
J.-M. MERCIER

Convergence of numerical algorithms for semilinear hyperbolic system

Rendiconti del Seminario Matematico della Università di Padova, tome 102 (1999), p. 241-283
<http://www.numdam.org/item?id=RSMUP_1999__102__241_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 102 (1999)

Convergence of Numerical Algorithms for Semilinear Hyperbolic Systems.

D. AREGBA-DRIOLLET (*) - J.-M. MERCIER (**)

ABSTRACT - We study numerical schemes for some semilinear hyperbolic systems. We want to insure convergence and to be able to detect blow up phenomena. We construct finite difference schemes and define a maximal convergence time which is proved to coincide with the maximal existence time of the continuous solution. Numerical experiments are presented.

1. Introduction.

In this paper we study the convergence of numerical schemes for some semilinear hyperbolic systems. Here we focus our attention on two classes of systems for which one does not always know whether the solution of the Cauchy problem is global in time or not. In order to provide some answers to this question, we expect the numerical approximation to converge towards the solution as long as this solution exists and also to be able to detect an eventual blow up. To reach that aim we use a local convergence theory in the spirit of [7], whose definitions of stability and consistency allow us to transpose at the numerical level the wellknown properties of semilinear hyperbolic systems: local well posedness and stability for smooth data, finite propagation speed. We define finite dif-

(*) Indirizzo dell'A.: Mathématiques Appliquées de Bordeaux, Université Bordeaux 1, 351 cours de la Libération, F-33405 Talence, France.
E-mail:aregba@math.u-bordeaux.fr
(**) S.I.S.S.A, 4/7 Via Beirut, I-34014 Trieste, Italy.
E-mail: mercier@sissa.it. This author was granted by European contract ERBCHBGCT940607 and ERBFMRXCT960033 during the realisation of this paper.

ference discretizations of our systems for which we construct suitable functional spaces where stability and consistency hold. Then we are able to define a maximal convergence time which is proved to coincide with the maximal existence time of the continuous problem. Actually in the convergence proof the properties of the exact local solutions as well as those of the numerical ones are involved (see section 2). Considerations on special cases and numerical experimentation show that our schemes actually allow the detection of blow up and its qualitative analysis.

The first class of systems under consideration is a family of one dimensional first order systems with quadratic interaction:

$$\partial_t u_i + c_i \partial_x u_i = q_i(u), \qquad 1 \leq i \leq L \tag{1}$$

with

$$q_i(u) = \sum_{j,k=1}^{L} A_{jk}^i u_j u_k, \qquad 1 \leq i \leq L \tag{2}$$

The speeds c_i and the interaction coefficients A_{jk}^i are real and $u = (u_i)_{1 \leq i \leq L}$.

The second family of problems is a system of wave equations with cubic interaction:

$$P(\varepsilon_1, \varepsilon_2) \begin{cases} (\partial_t^2 - \partial_x^2)\, u_1 = \varepsilon_1 u_1 u_2^2 \\ (\partial_t^2 - \partial_x^2)\, u_2 = \varepsilon_2 u_2 u_1^2 \end{cases} \tag{3}$$

with $\varepsilon_1, \varepsilon_2 \in \{-1, +1\}$.

Systems of type (1) or (3) arise in various fields of physics, generally as a simplification of more complicated nonlinear problems. For example in kinetic theory of gases, the discrete Boltzmann equations are of type (1) (see [19] for instance). In plasma physics the 3-waves complex valued system:

$$\begin{cases} \partial_t u_1 + c_1 \partial_x u_1 = K u_2 u_3 \\ \partial_t u_2 + c_2 \partial_x u_2 = -K^* u_3^* u_1 \qquad K \in \mathbb{C} \\ \partial_t u_3 + c_3 \partial_x u_3 = -K^* u_1 u_2^* \end{cases} \tag{4}$$

can be put in form (1). If $L = 2$ the results concerning the global Cauchy problem for (1) with «small data» or «large data» are complete [2]. For $L \geq 3$ global existence theorems are known for particular cases as discrete Boltzmann equations with positive data [19], [3] or conservative systems [8] among which (4). Also some blow up results are available but there is no general classification. A lot of numerical studies are concerned with discrete velocity Boltzmann models. In [20] a fractional step method is used to prove a global existence result of Carleman's model. In [5], the fluid dynamical limit of Broadwell's model is studied numerically. In [16] uniform convergence results are proved under some Tartar's type asumptions [19]. One can see also [9]. Note that all these papers deal with global solutions while our viewpoint is the approximation of possibly blowing up solutions. Systems of type (4) have also been numerically experimented, see for example [11]. In this paper we use a fractional step method where the linear part is solved exactly and the nonlinear part is discretized by a semi-implicit method. The convergence is obtained by proving local uniform stability in L^∞ and consistency in L^∞, locally uniformly in $W^{1,\infty}$.

$P(+1, -1)$ comes from the field theory (see [13] and related references for a survey) and the associated Cauchy problem may be interpreted as an intermediate one between $P(+1, +1)$ and $P(-1, -1)$. Actually, the long time behavior of solutions in the $\varepsilon_1 \varepsilon_2 > 0$ cases are better understood [13], but the question of the global existence in time for smooth enough solutions of $P(+1, -1)$ still remains open (see [14] for partial results). A numerical experimentation of the behavior of a Klein-Gordon equation with a homogeneous and supercritical semilinearity in the three dimensional case owning spherically symmetric data is done in [18]. These authors solve numerically an equivalent one dimensional problem. They use an implicit scheme, more precisely an explicit one for the linear wave equation part and an implicit one for the nonlinear part. In the three dimensional case with spherically symmetric data, algorithms devoted to semilinear wave equations are studied also in [17] in the subcritical case of a semilinearity. Here the problem is solved directly expressing the equation in polar coordinates, but with the same type of scheme. These authors prove a convergence result in the conservative case using abstract convergence theorem. The previous works are extended in the three dimensional case without spherically assumptions over the initial data in [6]. We also use the previous ideas, but our approach is a little bit different since we are interested also in the blow up cases. First we define a semi implicit scheme : we use the same discretization as in [18] for the linear part of the wave equation, but the nonlinear part is treated in such a way that we compute an explicit solution. This avoids some technical difficulties linked to the implicit nature of a scheme, while keeping its essential property of conservation. Furthermore, we define energy type functional spaces of regular functions in which our algorithm naturally acts. The computed solution is then the values of a function onto the points of a regular grid. This improvement allows us, using an energy type approach to this problem, to give self contained proofs of convergence results.

The plan of the paper is the following: in section 2 we recall the needed theoritical results for (1) and (3) and present them in a synthetized and rather general form. The second part of this section is then devoted to a general framework of the numerical theory: we define a numerical maximal convergence time and we precise what stability and consistency conditions have to be satisfied in order to obtain the local convergence and the coincidence of numerical convergence time and theoritical maximal existence time (Theorem 2.5). Part 3 (resp. 4) is devoted to the specific study of (1) (resp. (3)). Numerical experiments are presented in the fifth part of this paper. In the annex we give some results for a three dimensional axisymmetric version of (3).

2. A general framework.

In this section we give sufficient conditions for a numerical scheme for a semilinear system of type (1) or (3) to converge. The solutions of the considered system as well as the numerical ones are involved, so that we first give some theoritical results. The local theory for semilinear hyperbolic problems is rather classical, see for example [12], [10] and references therein.

2.1. *Local existence results*

2.1.1. *Local existence for system* (1). – Consider system (1) where $u = (u_i)_{1 \leqslant i \leqslant L}$ is real and the system is hyperbolic: the speeds c_i are real. Note that the diagonality is not restrictive because the interaction remains quadratic through a linear change of function. We take initial data $\phi \in W^{1,\infty}(\mathbb{R})^L$:

$$u(0,.) = \phi(.) \tag{5}$$

For $u \in W^{1,\infty}(\mathbb{R})^L$ we denote:

$$\|u\|_\infty = \max_{1 \leq l \leq L} \|u_l\|_\infty, \qquad \|u\|_{1,\infty} = \|u\|_\infty + \|u'\|_\infty .$$

THEOREM 2.1. *Take $\phi \in L^\infty(\mathbb{R})^L$.*

1) *There exists a unique local solution u of* (1)(5), $u \in L^\infty([0, T], L^\infty(\mathbb{R})^L) \cap C^0([0, T], \mathcal{D}'(\mathbb{R})^L)$ *and there exists a constant C_1 which depends only on the coefficients A^i_{jk} of the interaction such that for all $a > \|\phi\|_\infty$ one can take*

$$T = \frac{a - \|\phi\|_\infty}{C_1 a^2} \tag{6}$$

and then

$$\sup_{t \in [0, T]} \|u(t)\|_\infty \leq a \tag{7}$$

2) *If u exists in $L^\infty([0, T_1], L^\infty(\mathbb{R})^L)$ and if moreover $\phi \in W^{1,\infty}(\mathbb{R})^L$ then $u \in L^\infty([0, T_1], W^{1,\infty}(\mathbb{R})^L) \cap C^0([0, T_1,], L^\infty(\mathbb{R})^L)$.*

3) *Let $T^* = \sup\{T, u$ exists in $L^\infty([0, T], L^\infty(\mathbb{R})^L)\}$. Then either $T^* = +\infty$ or $\lim_{t \to T^*} \|u(t)\|_\infty = +\infty$.*

2.1.2. *Local existence for system* (3) – All the results of this paragraph are available with the same statement in the three dimensional case, and we refer to [13] for a complete proof.

First we express the one dimensional Cauchy problem in the following alternate Hamiltonian form:

(8) $\quad P(\varepsilon_1, \varepsilon_2)$:

$$\left\{ \frac{d}{dt}\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} V \\ \partial_x^2 U + Q_\varepsilon(U)\,U \end{pmatrix};\ U(0, x) = U^0(x),\ \partial_t U(0, x) = V^0(x) \right\}.$$

with $\quad U = {}^T(u_1, u_2),\ V = {}^T(\partial_t u_1, \partial_t u_2), \quad$ and $\quad Q_\varepsilon(U) = \begin{pmatrix} \varepsilon_1 u_2^2 & 0 \\ 0 & \varepsilon_2 u_1^2 \end{pmatrix}.$

We denote with $W = {}^T(U, V)$ the solution of (8) for initial data $W^0 = W(0) = {}^T(U^0, V^0)$. These initial conditions are chosen in the functional space $Y^\mu = [H^\mu(\mathbb{R})]^2 \times [H^{\mu-1}(\mathbb{R})]^2$, where H^μ is the usual Sobolev space.

THEOREM 2.2. *Let* $\mu \geqslant 1$ *and* $W^0 \in Y^\mu$. *There exists* $T > 0$ *such that* $P(\varepsilon_1, \varepsilon_2)$ *defines a unique solution* $W \in C^0([0, T]; Y^\mu)$.

Let T^* be the supremum over all $T > 0$ for which $W \in C^0([0, T]; Y^\mu)$. Remark that $T^* \equiv T^*[W^0, \varepsilon, \mu]$. For instance, we deduce from Sobolev injection $\mu \leqslant \mu' \Rightarrow T^*[\mu] \geqslant T^*[\mu']$. We precise in the next theorem the behavior of a solution in a neighborhood of a presupposed finite blow up time.

THEOREM 2.3. *Let* $\mu \geqslant 1$ *and* $W^0 \in Y^\mu$. *Suppose that* $T^*[W^0, \varepsilon, \mu] < \infty$.

i) $T^*[\mu] = T^*[1] = T^*$

ii) $\limsup_{t \to T^*} \|(u_1, \partial_t u_1); H^1(\mathbb{R}) \times L^2(\mathbb{R})\|(t) = +\infty$ *and* $\limsup_{t \to T^*} \|(u_2, \partial_t u_2); H^1(\mathbb{R}) \times L^2(\mathbb{R})\|(t) = +\infty$.

iii) $\|W; Y^\mu\| \leqslant 2\|W^0; Y^\mu\|$, *for every* $t \leqslant (C_0 / \|W^0; Y^1\|^2) = T_S(\|W^0; Y^1\|)$.

2.1.3. *Synthesis* – The properties appearing in the above results can be synthetized in view of unifying the presentation of the numerical theory. We denote for (1):

$$E = W^{1,\infty}(\mathbb{R})^L, \qquad Y = Z = L^\infty(\mathbb{R})^L. \tag{9}$$

Then we define for (3):

$$\begin{aligned} &E = \{U \in Y^4, \operatorname{supp} U \text{ compact}\}, \quad Y = \{U \in Y^2, \operatorname{supp} \mathrm{U} \text{ compact}\}, \\ &Z = \{U \in Y^1, \operatorname{supp} U \text{ compact}\}. \end{aligned} \tag{10}$$

We take compactly supported data for numerical purposes (see section 4 below). We have $E \subseteq Y \subseteq Z$ with continuous imbeddings. Both systems (1) and (3) may be written in the form

$$\partial_t u + Au = f(u) \tag{11}$$

with initial condition

$$u(0, x) = \phi(x), \qquad x \in \mathbb{R}. \tag{12}$$

In all what follows (11)(12) is considered as the unified representation of (1) and (3) with related functionnal spaces defined by (9) and (10).

THEOREM 2.4. *For all* $\phi \in E$ *there exists* $T(\|\phi\|_Z) > 0$ *and a unique* *solution u of* (11)(12), $u \in L^\infty([0, T], E) \cap C^0([0, T], Z)$. *Define*

$$T^* = \sup\{T,\ u \text{ exists in } C^0([0, T], Z)\}.$$

Either $T^* = +\infty$ *or* $\limsup_{t \to T^*} \|u(t)\|_Z = +\infty$. *Moreover,* $u \in L^\infty([0, T], E)$ *for all* $T < T^*$.

DEFINITION 2.1. *If the maximal existence time T^* is finite it is called the blow up time of the solution.*

In the following we denote F the evolution operator associated to the problem: $u(t) = F_t \phi$.

2.2. *The convergence theorem.*

We discretize (11)(12) by a finite difference method. If τ and h are the time step and the space step, they are in a fixed proportion $\sigma = \tau/h$ governed by a CFL condition. We denote u^n the numerical solution at time $t_n = \tau n$ and K_τ the semidiscretization operator:

$$u^{n+1} = K_\tau u^n \tag{13}$$

DEFINITION 2.2. *Uniform convergence* [7]. *For $\phi \in Z$ the scheme K_τ converges uniformly on $[0, T]$ towards $c \in C^0([0, T], Z)$ if*

$$\lim_{N \to \infty} \sup_{t \in [0, T]} \|K_{t/N}^N \phi - c(t)\|_Z = 0. \tag{14}$$

DEFINITION 2.3. *Numerical blow up time. For $\phi \in Z$ the maximal convergence time $T^{**}(\phi)$ for the scheme K_τ is defined by:*

$$T^{**}(\phi) = \sup\{T \geqslant 0;\ K_\tau \text{ converges uniformly on } [0, T]\}.$$

If $T^{**}(\phi)$ is finite we call it the numerical blow up time.

We suppose that for $n \geqslant 0$, $u_n \in Y$ and that we can define on Y a family of norms X^τ depending on τ and satisfying

$$\alpha\|u\|_Z \leqslant \|u; X^\tau\| \leqslant \beta\|u\|_Y. \tag{15}$$

We denote

$$B_E(A) = \{\phi \in E;\ \|\phi\|_E < A\}$$

and similarly are defined $B_Y(A)$, $B_Z(A)$, $B_\tau(A)$. Our main result is the following:

THEOREM 2.5. *Consider $\phi \in E$ and the scheme* (13). *Suppose that the following conditions are satisfied:*

A) Local uniform stability.

For all $A > 0$ there exists $T_S(A) > 0$ such that:

$$\text{(16)} \qquad \forall N > 1, \forall n \leqslant N, \forall t \in [0, T_S(A)],$$

$$\sup_{(\psi_1, \psi_2) \in B_{t/N}(A)^2} \|K_{t/N}^n \psi_1 - K_{t/N}^n \psi_2; X^{t/N}\| \leqslant \gamma \|\psi_1 - \psi_2; X^{t/N}\| .$$

Here γ is a uniform constant.

B) Local uniform consistency.

For all $M > 0$, there exist $C > 0$ and $\tau_1 > 0$ such that if $\tau < \tau_1$ then:

$$\text{(17)} \qquad \sup_{\|\psi\|_E \leqslant M} \|K_\tau \psi - F_\tau \psi\|_Y < C\tau^2 .$$

Then:

(i) *The maximal convergence time and the maximal existence time for* (11)(12) *are identical.*

(ii) *Let T^* be this time. For all $T_0 < T^*$ the scheme converges uniformly on* $[0, T_0]$ *towards the solution of* (11) (12).

PROOF OF THEOREM 2.5. The plan of the proof is the following: we first prove convergence for T small enough. Then we prove convergence for every time of existence of the solution of (11)(12). Finally we prove $T^*(\phi) = T^{**}(\phi)$.

Let $T_0 \in]0, T^*[$ be a time of existence of the solution of (11) (12). Take $g_0 = \beta \sup_{t \in [0, T_0]} \|F_t \phi\|_Y$, $A > g_0$, $M = \sup_{t \in [0, T_0]} \|F_t \phi\|_E$, which is finite according to the Theorem 2. We denote $T_S' = \min(T_0, T_S(A))$.

Actually we prove a little more than (14): we prove that

$$\text{(18)} \qquad \lim_{N \to \infty} \sup_{t \in [0, T]} \|K_{t/N}^N \phi - F_t \phi; X^{t/N}\| = 0 .$$

1) *Local convergence.* We estimate the global error by introducing the approximate solutions issued from each exact $F_{(N-n)\tau}\phi$. For $N > 1$

and $t \in [0, T'_S]$ we have:

$$F_t \phi - K_\tau^N \phi = F_{N\tau} \phi - K_\tau F_{(N-1)\tau} \phi$$

$$+ \dots +$$

$$K_\tau^n F_{(N-n)\tau} \phi - K_\tau^{n+1} F_{(N-n-1)\tau} \phi$$

$$+ \dots +$$

$$K_\tau^{N-1} F_\tau \phi - K_\tau^N \phi$$

where

$$\tau = \frac{t}{N},$$

so that

$$\|F_t \phi - K_\tau^N \phi; X^\tau\| \leqslant \sum_{n=0}^{N-1} \|K_\tau^n F_{(N-n)\tau} \phi - K_\tau^n K_\tau F_{(N-n-1)\tau} \phi; X^\tau\|.$$

For $0 \leqslant n \leqslant N-1$, $F_{(N-n)\tau} \phi \in B_\tau(A)$ and $\|F_{(N-n)\tau} \phi\|_E \leqslant M$. Moreover by (17), there exists τ_1 such that for $\tau < \tau_1$

$$\|K_\tau F_{(N-n-1)\tau} \phi - F_{(N-n)\tau} \phi; X^\tau\| \leqslant \beta C \tau^2.$$

Thus $K_\tau F_{(N-n-1)\tau} \phi \in B_\tau(A)$ for N large enough, depending on the choice of A. Therefore by (16) we obtain:

$$\|F_t \phi - K_\tau^N \phi; X^\tau\| \leqslant \gamma \sum_{n=0}^{N-1} \|F_\tau F_{(N-n-1)\tau} \phi - K_\tau F_{(N-n-1)\tau} \phi; X^\tau\|.$$

Hence by (15):

$$\|F_t \phi - K_\tau^N \phi; X^\tau\| \leqslant \beta\gamma \sum_{n=0}^{N-1} \|F_\tau F_{(N-n-1)\tau} \phi - K_\tau F_{(N-n-1)\tau} \phi\|_Y.$$

Then we use (17) to conclude that there exists $\tau_2 > 0$ depending only on A and M such that:

$$\forall N > T_S/\tau_2, \ \|F_t \phi - K_{t/N}^N \phi; X^{t/N}\| \leqslant \beta\gamma C \frac{T_S^2}{N}.$$

In fact by the same argument we also have for all $N > T_S/\tau_2$:

(19) $\forall s < T_0, \quad \forall t \in [s, \min(s + T'_S, T_0)],$

$$\|F_{t-s}F_s\phi - K^N_{t-s/N}F_s\phi; X^{t-s/N}\| \leqslant \beta\gamma C\frac{T_S^2}{N}.$$

An important remark is that no estimate of the E norm of the approximated solution is needed but only the one of the exact solution.

Consequently the scheme converges uniformly on an interval of length T'_S towards the solution of (1)(5).

2) *Convergence on* $[0, T_0]$. We have to consider the case $0 < T_S < T_0$. Take $T = \min(T_0, 2T_S)$. For $N > 1$ and $t \in [0, T]$, $K^N_{t/2N}F_{t/2}\phi$ exists and

$$\|F_t\phi - K^{2N}_{t/2N}\phi; X^{t/2N}\| \leqslant$$

$$\leqslant \|F_{t/2}F_{t/2}\phi - K^N_{t/2N}F_{t/2}\phi; X^{t/2N}\| + \|K^N_{t/2N}F_{t/2}\phi - K^N_{t/2N}K^N_{t/2N}\phi; X^{t/2N}\|.$$

By (19)

$$\sup_{t\in[0,T]}\|F_{t/2}F_{t/2}\phi - K^N_{t/2N}F_{t/2}\phi; X^t_{2N}\| < \beta\gamma C\frac{T_S^2}{N}.$$

Also by the first part of the proof we know that

$$\lim_{N\to+\infty}\sup_{t\in[0,T]}\|F_{t/2}\phi - K^N_{t/2N}\phi; X^{t/2N}\| = 0.$$

Therefore, for $N \geqslant N_1(A, M)$, $K^N_{t/2N}\phi$ remains in $B_{t/2N}(A)$ and we can use (16):

$$\|K^N_{t/2N}F_{t/2}\phi - K^N_{t/2N}K^N_{t/2N}\phi; X^{t/2N}\| \leqslant \gamma\|F_{t/2}\phi - K^N_{t/2N}\phi; X^{t/2N}\|.$$

Applying (19) once again, we conclude that

$$\lim_{N\to+\infty}\sup_{t\in[0,T]}\|F_t\phi - K^{2N}_{t/2N}\phi; X^{t/2N}\| = 0.$$

By a classical argument we obtain then the uniform convergence of the scheme towards the solution of (1)(5) on $[0, T]$.

If $T_0 \leqslant 2T_S$ the convergence part of the proof is complete. Otherwise we can repeat the above argument and reach T_0 in a finite number of steps.

3) *Blow up times.* At this point, we know that $T^{**}(\phi) \geqslant T^*(\phi)$.

Let us suppose $T^{**}(\phi) > T^*(\phi)$. For $T^{**}(\phi) > T > T^*(\phi)$ there exists $c \in C^0([0, T], Z)$ satisfying (14).

By Theorem 2.4 $\limsup_{t \to T^*} \|F_t \phi\|_Z = +\infty$. For $A > \sup_{t \in [0, T]} \|c(t)\|_Z$, there exists $t_0 < T^*$ such that $\|F_{t_0} \phi\|_Z > A$. We obtain a contradiction because $c(t_0) = F_{t_0} \phi$.

3. A first order system.

In this section we study the convergence of a semi-implicit scheme for (1). Here G_i is a (not necessarily symmetric) bilinear form associated to q_i:

$$G_i(u, v) = \sum_{1 \leq j, k \leq L} A^i_{jk} u_j v_k, \qquad 1 \leq i \leq L .$$

We discretize system (1) in time on a strip $[0, T] \times \mathbb{R}$ by splitting it into an ordinary differential system and a linear hyperbolic one. For a time step $\tau = T/N$ we denote u^n the numerical solution at time $t_n = \tau n$ and K_τ the semidiscretization operator:

$$u^{n+1} = K_\tau u^n .$$

We take $u^0 = \phi$. For given $u^n = (u_i^n)_{1 \leq i \leq L}$ and $x \in \mathbb{R}$, $u^{n+1/2}(x)$ is an approximate solution at time t_{n+1} of:

$$\begin{cases} v' = G(v, v), \\ v(t_n) = u^n(x). \end{cases} \tag{20}$$

We use here a semi implicit scheme:

$$v^{n+1} = v^n + \tau[\theta G(v^n, v^{n+1}) + (1 - \theta) G(v^{n+1}, v^n)]$$

where $\theta \in [0, 1]$ is a parameter to be chosen. Then u^{n+1} is the exact solution at time t_{n+1} of the system:

$$\begin{cases} \partial_t u_i + c_i \partial_x u_i = 0 , & 1 \leq i \leq L . \\ u_i(t_n, x) = u_i^{n+1/2}(x) \end{cases}$$

Denoting $R(\tau) u^{n+1/2} = u^{n+1}$ the associated group:

$$R(\tau) u^{n+1/2}(x) = \{u_i^{n+1/2}(x - c_i \tau)\}_{1 \leq i \leq L}$$

the scheme is finally written as:

$$R(-\tau)\,u^{n+1} = \tag{21}$$

$$= u^n + \tau[\theta G(u^n, R(-\tau)\,u^{n+1}) + (1-\theta)\,G(R(-\tau)\,u^{n+1}, u^n)]\,.$$

We do not consider here the space discretization. From a computational viewpoint, let us remark that if the speeds c_i are in a rational proportion, which is usually the case, we are able to choose a space step for which we can compute the exact solution of (21) onto the points of a regular grid. In this case, notice also that the numerical domain of influence coincide with the theoretical one.

The choice of this scheme is motivated by the fact that for $L=1$, (21) provides the exact solution of the problem:

$$\begin{cases} \partial_t u + c\partial_x u = u^2 \\ u(0,.) = \phi\,. \end{cases}$$

Actually (21) gives:

$$u^{n+1}(x+c\tau) = \frac{u^n(x)}{1-\tau u^n(x)}$$

so that $u^n = u(t_n)$ and the numerical solution blows up exactly as the exact one. As we see in theorem 2.1 the blow up mechanism is due uniquely to the explosion of the L^∞ norm of the solution: an explosion of the gradient letting a weak solution exist is not possible. Moreover in all known proofs of non existence of global smooth solutions of (1)(5) the equation $y' = y^2$ is involved either as a comparison tool or to provide blowing up solitary waves (see [2] and references therein). Hence one hopes a good detection of blow up phenomena for (1).

Moreover (21) can be written in a simple linear way:

PROPOSITION 3.1. *For $x \in \mathrm{R}$ the solution $R(-\tau)\,u^{n+1}(x)$ of* (21) *is the solution of the linear system:*

$$u^n(x) = [I - \tau\Lambda(u^n(x))]\,R(-\tau)\,u^{n+1}(x) \tag{22}$$

where $\Lambda(u^n(x))$ is a $L\times L$ matrix with coefficients given by

$$\Lambda_{ij}(u^n(x)) = \sum_{k=1}^{L}(\theta A^i_{kj} + (1-\theta)\,A^i_{jk})\,u^n_k(x)\,. \tag{23}$$

Hence if $\|\Lambda(u^n)\|_\infty < \tau^{-1}$, the numerical solution does exist and a numerical blow up criterion is the singularity of $I - \tau A(u^n(x))$. In practice if $I - \tau A(u^n(x))$ is singular the numerical calculation must be stopped on the domain of influence of x but may be continued elsewhere in the space-time domain. By this way we construct a maximal domain which boundary is the numerical blow up curve. See for example [4], [1] for studies on the theoretical blow up curve.

LEMMA 3.1. *There exists a constant $C > 0$, depending only on the A^i_{jk} such that for $\tau < (C\|\phi\|_\infty)^{-1}$, $K_\tau \phi$ is defined. Moreover for all $g_0 > 0$ and $A > g_0$, there exist $\tau_0 > 0$ such that:*

$$\forall \tau < \tau_0, \qquad \|\phi\|_\infty \leqslant g_0 \Rightarrow \|K_\tau \phi\|_\infty < A \,. \tag{24}$$

PROOF OF LEMMA 3.1. There exists a constant C depending only on the A^i_{jk} such that

$$\forall v \in \mathbb{R}^L \qquad |A(v)|_\infty \leqslant C.\, |v|_\infty . \tag{25}$$

Hence for for $\tau < (C\|\phi\|_\infty)^{-1}$, $K_\tau \phi$ is defined and:

$$\|K_\tau \phi\|_\infty \leqslant \frac{\|\phi\|_\infty}{1 - \tau C\|\phi\|_\infty} . \tag{26}$$

Consequently we have (24) with $\tau_0 = \min(1/Cg_0, A - g_0/ACg_0)$.

Of course if G is symmetric ($A^i_{jk} = A^i_{kj}$) the value of θ is indifferent and it is the most common choice. In this case one obtains easily:

PROPOSITION 3.2. *Suppose that G is symmetric and that there exists $\alpha \in \mathbb{R}^L$ such that:*

$$\forall u \in \mathbb{R}^L \sum_{l=1}^{L} \alpha_l q_l(u) = 0 .$$

Then:

$$\sum_{l=1}^{L} \alpha_l \int_R u_l^{n+1}(x)\, dx = \sum_{l=1}^{L} \alpha_l \int_R u_l^n(x)\, dx .$$

As a particular case if one considers a discrete velocity Boltzmann system, such an α exists. The above property insures a discrete mass conservation analogous to the continuous one. Numerical approximation of discrete velocity Boltzmann equations has been considered by many authors. See for example [16] and references therein.

But one may be interested in discretizing so-called *conservative systems* [8], that are systems such that there exist $\beta_1, \dots, \beta_L > 0$ such that:

$$\forall u \in \mathbb{R}^L \sum_{l=1}^{L} \beta_l u_l q_l(u) = 0 .$$

For such an interaction the quantity $\sum \int_{\mathbb{R}} \beta_l u_l^2(t, x)\, dx$ is conserved. It is convenient to write such systems in the canonical non symmetric form:

$$G_i(u, v) = -\sum_{j=1}^{i-1} \beta_j v_j l_i^j(u) + \sum_{j=i+1}^{L} \beta_j v_j l_j^i(u) \tag{27}$$

where $\beta_j > 0$ and $l_q^p(u) = \sum_{k \leq q} a_{qk}^p u_k$. By a straightforward calculation we can see that if $\theta = 1$ then the scheme (21) is dissipative:

$$\sum_{l=1}^{L} \beta_l \int_{\mathbb{R}} (u_l^{n+1})^2\, dx \leq \sum_{l=1}^{L} \beta_l \int_{\mathbb{R}} (u_l^n)^2\, dx$$

and if $\theta = 0$ then the inverse inequality holds:

$$\sum_{l=1}^{L} \beta_l \int_{\mathbb{R}} (u_l^{n+1})^2\, dx \geq \sum_{l=1}^{L} \beta_l \int_{\mathbb{R}} (u_l^n)^2\, dx$$

so that it seems to be convenient to choose $\theta = 1/2$ although this value does not insure discrete energy conservation. Actually, numerical tests show that it is the best choice.

Let us now study the convergence of (21). The following lemma ensures that the stability requirement of theorem 2.5 is fulfilled. We claim that on a small time interval $[0, T_S]$ the scheme is L^∞ stable, uniformly locally in L^∞ and that on this interval the numerical solution depends continuously on the data in L^∞, still uniformly locally in L^∞. Moreover the estimate on T_S is the analogue of that obtained on the local existence time for the continuous problem.

LEMMA 3.2. *Local uniform stability. For all* $A > 0$ *there exists* $T_S(A) > 0$ *such that for all* $\phi, \psi \in B_\infty(A)$:

$$\forall N > 1, \quad \forall n \leq N, \quad \forall t \in [0, T_S(A)], \quad \|K_{t/N}^n \phi - K_{t/N}^n \psi\|_\infty \leq \gamma \|\phi - \psi\|_\infty \tag{28}$$

where γ *is a uniform constant. Moreover we can choose* T_S *by the follow-*

ing way:

$$T_S(A) = (2AC)^{-1} \tag{29}$$

where C is a constant which depends only on the coefficients A^i_{jk} of the interaction.

PROOF OF LEMMA 3.2. We prove first that for all $A > 0$ there exists $T_S(A) > 0$ such that:

(30) $\forall t \in [0, T_S(A)], \quad \forall N > 1, \quad \forall n \leqslant N, \quad \sup_{\phi \in B_\infty(A)} \|K^n_{t/N}\phi\|_\infty < 2A .$

Denoting $\tau = t/N$, we deduce by a recurrence on (26):

$$\|K^n_\tau \phi\|_\infty \leqslant (\|\phi\|^{-1}_\infty - Cn\tau)^{-1},$$

from which we obtain the a priori estimate:

$$\sup_{\phi \in B_\infty(A)} \|K^n_{t/N}\phi\|_\infty \leqslant (A^{-1} - Ct)^{-1}.$$

Consequently we can choose $T_S < (AC)^{-1}$. If $T_S = (2AC)^{-1}$, then the existence of $K^n_{t/N}\phi$ is ensured for $N > 1$, *ie* $\tau = (t/N) \leqslant (T_S/2)$ and (30) is satisfied.

Now, denoting $u^n = K^n_{t/N}\phi$, $v^n = K^n_{t/N}\psi$, by (22) we have:

$$u^n - v^n = (I - \tau\Lambda(u^n))(R(-\tau)\,u^{n+1} - R(-\tau)\,v^{n+1}) -$$
$$- \tau(\Lambda(u^n) - \Lambda(v^n))\,R(-\tau)\,v^{n+1}.$$

Taking into account the fact that

$$(\Lambda(u^n) - \Lambda(v^n))\,R(-\tau)\,v^{n+1} = \Lambda^{(1)}(R(-\tau)\,v^{n+1})(u^n - v^n)$$

where $\Lambda^{(1)}_{ij}(w) = \sum_{k=1}^{L} (\theta A^i_{jk} + (1-\theta)\,A^i_{kj})\,w_k$, one obtains for $N > 1$:

$$\|u^{n+1} - v^{n+1}\|_\infty \leqslant (1 + \tau 8CA)\|u^n - v^n\|_\infty$$

which leads to

$$\|u^n - v^n\|_\infty \leqslant e^4\|\phi - \psi\|_\infty$$

and ends the proof of the lemma.

Let us now study the consistency. We prove here that the scheme is L^∞-consistent with the problem (1) (5), uniformly locally in $W^{1,\infty}(\mathbb{R})^L$. Actually we prove explicitly that the scheme is first order accurate.

LEMMA 3.3. *Uniform consistency. There exist two constants B and C_2 which depend only on the coefficients A^i_{jk} and the characteristic speeds c_i such that:*

$$\forall \phi \in W^{1,\infty}(\mathbb{R})^L, \quad \forall \tau \leqslant (C_2 \|\phi\|_\infty)^{-1}, \quad \|K_\tau \phi - F_\tau \phi\|_\infty \leqslant B\tau^2 (1 + \|\phi\|_{1,\infty})^4. \tag{31}$$

PROOF OF LEMMA 3. For all $\phi \in W^{1,\infty}(\mathbb{R})^L$ and $1 \leqslant i \leqslant L$ we have:

$$u_i(\tau, x + c_i \tau) = \phi_i(x) + \int_0^\tau G_i(u(s, x + c_i s), u(s, x + c_i s))\, ds \tag{32}$$

and for $1 \leqslant j \leqslant L$:

$$u_j(s, x + c_i s) =$$

$$= \phi_j(x) + \int_0^s [(c_i - c_j)\, \phi_j'(x + (c_i - c_j)\, \sigma) + q_j(u(\sigma, x + (c_i - c_j)\, s + c_j \sigma))]\, d\sigma$$

which can be written

$$u_j(s, x + c_i s) = \phi_j(x) + \psi_j^{(i)}(s, x).$$

Choosing $\alpha = 2\|\phi\|_\infty$ in Theorem 2.1 we see there exists $B_0 > 0$ such that for $s \leqslant \tau < (4C_1 \|\phi\|_\infty)^{-1}$:

$$\|\psi^{(i)}(s)\|_\infty \leqslant B_0 s (1 + \|\phi\|_{1,\infty})^2.$$

From this and the equality

$$u_i(\tau, x + c_i \tau) = \phi_i(x) + \tau q_i(\phi(x)) +$$

$$+ \int_0^\tau [G_i(\phi(x), \psi^{(i)}(s, x)) + G_i(\psi^{(i)}(s, x), \phi(x)) + q_i(\psi^{(i)}(s, x))]\, ds$$

we obtain:

$$\|R(-\tau)\, u(\tau) - [\phi + \tau q(\phi)]\|_\infty \leqslant B_1 .\ \tau^2 (1 + \|\phi\|_{1,\infty})^4. \tag{33}$$

Let us now examine the scheme under the form (22). If C is the constant defined in (25) we have:

$$\forall \tau < (2C\|\phi\|_\infty)^{-1} \|(I - \tau \Lambda(\phi))^{-1}\|_\infty \leqslant 2\,.$$

From the identity

$$R(-\tau)\, K_\tau \phi = \phi + \tau\Lambda(\phi)\, \phi + \tau^2 \Lambda(\phi)^2 (I - \tau\Lambda(\phi))^{-1} \phi$$

and the fact that $\Lambda(\phi)\, \phi = q(\phi)$, we deduce:

$$\|R(-\tau)\, K_\tau \phi - [\phi + \tau q(\phi)]\|_\infty \leq B_2 \tau^2 \|\phi\|_\infty^3 . \tag{34}$$

By (33) and (34) the proof is complete.

Lemmas 3.2 and 3.3 prove that for system (1) with E, Y, Z given by (9) and $X^\tau = \|\,.\,\|_\infty$, the asumptions of Theorem 2.5 are satisfied. We have then the following theorem:

THEOREM 3.1. *Consider $\phi \in W^{1,\infty}(\mathbb{R})^L$ and the scheme* (21).

(i) *The maximal convergence time and the maximal existence time for* (1)(5) *are identical.*

(ii) *Let T^* be this time. For all $T_0 < T^*$ the scheme converges uniformly on $[0, T_0]$ towards the solution of* (1)(5):

$$\lim_{N \to \infty} \sup_{t \in [0, T]} \|K_{t/N}^N \phi - F_t(\phi)\|_\infty = 0 . \tag{35}$$

REMARK 3.1. *In the particular case of interaction* (27) *the following scheme*

$$R(-\tau)\, u^{n+1} = u^n + \tau G((R(-\tau)\, u^{n+1} + u^n)/2, (R(-\tau)\, u^{n+1} + u^n)/2)$$

is conservative:

$$\sum_{l=1}^{L} \beta_l \int_{\mathbb{R}} (u_l^{n+1})^2\, dx = \sum_{l=1}^{L} \beta_l \int_{\mathbb{R}} (u_l^n)^2\, dx .$$

In fact we have for $x \in \mathbb{R}$:

$$\sum_{l=1}^{L} \beta_l u_l^{n+1} (x + c_l \tau)^2 = \sum_{l=1}^{L} \beta_l u_l^n (x)^2 .$$

Hence stability is immediate and our method applies without difficulty: the scheme converges towards the solution on any strip $[0, T] \times \mathbb{R}$, $T > 0$.

4. Approximation of semilinear systems of wave equations.

In this section, we study the convergence of a semi-implicit scheme for the following semilinear system of wave equations:

$$P(\varepsilon_1, \varepsilon_2)\begin{cases} \partial_t^2 u_1 - \partial_x^2 u_1 = \varepsilon_1 u_1 u_2^2 \\ \partial_t^2 u_2 - \partial_x^2 u_2 = \varepsilon_2 u_2 u_1^2 \end{cases} \tag{36}$$

with compactly supported Cauchy data:

$$\begin{cases} u_1(0) = u_1^0;\ \partial_t u_1(0) = v_1^0 \\ u_2(0) = u_2^0;\ \partial_t u_2(0) = v_2^0. \end{cases} \tag{37}$$

This section is organized as follows: in subsection 4.1 the scheme is introduced and we state its basic properties. These last results are applied in a following subsection to prove uniform stability and consistency. In the annex the three dimensional case owning spherically symmetric data is viewed as an application of the previous case.

4.1. *The scheme and its first properties.*

Let us introduce first the following notations: h and τ hold respectively for the space and time step. $\sigma = \tau/h$ is the C.F.L. number. For a function $U(t, x)$ and a given regular grid $(n\tau, ih)_{n\in \mathbb{Z}, i\in \mathbb{Z}}$, U_i^n holds for $U(n\tau, ih)$ and U^n for $U(n\tau, .)$. We use the translation operator T_h: $T_h U(x) = U(x-h)$ and introduce the discrete space derivation operator:

$$U_x^n = \frac{U^n(.+h) - U^n(.)}{h} = \frac{T_{-h} - 1}{h} U^n; \qquad U_{\bar{x}}^n = \frac{1 - T_h}{h} U^n;$$

$$U_{\hat{x}}^n = \frac{T_{-h} - T_h}{2h} U^n$$

and the discrete time derivation operator:

$$U_t^n = \frac{U^{n+1} - U^n}{\tau}; \qquad U_{\bar{t}}^n = \frac{U^n - U^{n-1}}{\tau}; \qquad U_{\hat{t}}^n = \frac{U^{n+1} - U^{n-1}}{2\tau}.$$

With this set of notations, the discretization of (3) consists in the following scheme:

$$\left\{U^n_{t\bar{t}} - U^n_{x\bar{x}} = \frac{1}{2} Q_\varepsilon(U^n)(U^{n+1} + U^{n-1});\ U^{-1} = U^0 - \tau V^0\right\}. \tag{38}$$

Fitted in a more usual form, the scheme (38) defines (U^{n+1}, V^{n+1}) as the explicit solution of the following system:

$$\begin{cases} U^{n+1} = U^n + \tau V^n + \sigma^2[T_h + T_{-h} - 2]\, U^n + \dfrac{\tau^2}{2} Q_\varepsilon(U^n)(U^{n+1} + U^{n-1}) \\[2ex] V^{n+1} = \dfrac{U^{n+1} - U^n}{\tau} = U^n_t \end{cases} \tag{39}$$

where $U^{n-1} = U^n - \tau V^n$.

REMARK 4.1. (39) *is properly explicitly defined as long as* $(I - (\tau^2/2)\, Q_\varepsilon(U^n))$ *is invertible.*

We denote

$$W^{n+1} = (U^{n+1}, V^{n+1}) = K_\tau(U^n, V^n),$$

and

$$\mathcal{K}_\tau W^n = (U^{L,\,n+1}, V^{L,\,n+1})(\tau) =$$

$$= \left(U^n + \tau V^n + \sigma^2[T_h + T_{-h} - 2]\, U^n,\ V^n + \frac{\sigma^2}{\tau}[T_h + T_{-h} - 2]\, U^n\right)$$

the linear part of this scheme. Remark that, when properly defined, the solution of this scheme owns a finite numerical propagation speed $1/\sigma$. So, for compactly supported data W^0, the solution is compactly supported too, and we can use of the homogeneous Sobolev norm for these solutions. $K^n_{t/n}$ is the approximation of the propagator F_t. But, notice that this operator does not define an evolution group, while F_t does.

REMARK 4.2. *From a computational viewpoint* (39) *leads to compute at every point of a regular grid* $(n\tau, ih)_{n\in Z,\, i\in Z}$ *the solution of*

$$U^{n+1}_i = 2U^n_i - U^{n-1}_i + \sigma^2[U^n_{i-1} + U^n_{i+1} - 2U^n_i] +$$

$$+ \frac{\tau^2}{2} Q_\varepsilon(U^n)(U^{n+1}_i + U^{n-1}_i).$$

DISCRETE ENERGY CONSERVATION. We recall the following energy conservation, verified by any finite energy solutions of the systems of wave equations (8) (see [13] for a proof):

PROPOSITION 4.1. *Let* $\mu \geq 1$ *and* $W(t) \in C^0([0, T^*[, Y^\mu)$ *the unique solution of* (8) *with initial condition* $W^0 \in Y^\mu$. *Then for every* $0 \leq t < < T^*$

$$E[W(t)] = E[W^0] =$$

$$= -\varepsilon_1(\|v_1\|^2 + \|\partial_x u_1\|^2) - \varepsilon_2(\|v_2\|^2 + \|\partial_x u_2\|^2) + \int u_1^2 u_2^2 = C^{te}.$$

The essential property of our scheme is that a discrete functional similar to the energy functional defined above is kept constant:

PROPOSITION 4.2. *Let* $(W^n)_{0 \leq n \leq M} = (U^n, V^n)_{0 \leq n \leq M}$ *a solution of* (39), *then*

$$E_\tau[W^n] = E_\tau[W^0] = -\varepsilon_1\left(\|v_1^n\|^2 + \frac{1}{2}[\|u_{1,x}^n\|^2 + \|u_{1,x}^{n-1}\|^2] - \frac{\tau^2}{2}\|v_{1,x}^n\|^2\right) -$$

$$-\varepsilon_2\left(\|v_2^n\|^2 + \frac{1}{2}[\|u_{2,x}^n\|^2 + \|u_{2,x}^{n-1}\|^2] - \frac{\tau^2}{2}\|v_{2,x}^n\|^2\right) +$$

$$+\frac{1}{2}\int (u_1^n u_2^{n-1})^2 + (u_1^{n-1} u_2^n)^2.$$

The following lemma states the basic tools used to prove this proposition. This lemma is in the spirit of a result obtained in [17] for the discrete case.

LEMMA 4.1. *We have the following properties*:

(i) $2\langle u_{\hat{t}}, u_{t\bar{t}}\rangle = (\|u_{\bar{t}}\|^2)_t$

(ii) $2\langle u_{\hat{t}}, u\rangle = (\|u\|^2)_{\hat{t}} - (\tau^2/2)(\|u_{\bar{t}}\|^2)_t$

(iii) $\langle -u_{x\bar{x}}, u\rangle = \|u_x\|^2$

(iv) $-2\langle u_{\hat{t}}, u_{x\bar{x}}\rangle = (\|u_x\|^2)_{\hat{t}} - (\tau^2/2)(\|u_{x\bar{t}}\|^2)_t$

(v) $\langle \varepsilon U_{\hat{t}}^n, Q_\varepsilon(U^n)(U_i^{n+1} + U_i^{n-1})\rangle = \dfrac{1}{2}\left(\int (u_1^n u_2^{n-1})^2 + (u_1^{n-1}u_2^n)^2\right)_{\hat{t}}$.

PROOF OF LEMMA 4.1. Relations (i), (ii), (v) consist in direct computations, we let the demonstration to the reader.

First let us recall the following analogue of integration by part:

$$(40)\ \langle u_x, v\rangle = \left\langle \frac{u(.\ +h) - u}{h}, v\right\rangle = -\left\langle u, \frac{v - v(.\ -h)}{h}\right\rangle = -\langle u, v_{\bar{x}}\rangle .$$

So, one has just to replace $\langle u_x, v\rangle$ in (40) with $\langle -u_{x\bar{x}}, u\rangle$ in order to get relation (iii). Now, if we replace $\langle u_x, v\rangle$ in (40) with $\langle -2u_{x\bar{x}}, u_{\hat{t}}\rangle$ we get $\langle -2u_{x\bar{x}}, u_{\hat{t}}\rangle = \langle 2u_{\bar{x}}, u_{\bar{x}\hat{t}}\rangle = \|u_{\bar{x}}\|^2_{\hat{t}} - \tau^2/2\|u_{\bar{x}\hat{t}}\|^2_t$, according to relation (ii). This last expression is an equivalent form of (iv).

PROOF OF PROPOSITION 4.2. Let us take the scalar product of (38) with $2\varepsilon U_{\hat{t}}$. We integrate using the i), ii), v) properties, and we obtain

$$-\varepsilon_1\left((\|u^n_{1,\bar{t}}\|^2)_t + (\|u^n_{1,x}\|^2)_{\hat{t}} - \left(\frac{\tau^2}{2}\|u^n_{1,\bar{t}x}\|^2\right)_t\right)$$

$$-\varepsilon_2\left(\|u^n_{2,\bar{t}}\|^2)_t + (\|u^n_{2,x}\|^2)_{\hat{t}} - \left(\frac{\tau^2}{2}\|u^n_{2,\bar{t}x}\|^2\right)_t\right)$$

$$+\frac{1}{2}\left(\int (u_1^n u_2^{n-1})^2 + (u_1^{n-1}u_2^n)^2\right)_t = 0 .$$

This relation implies the proposition 4.2.

LINEAR STABILITY. Let us compare at this point the proposition 4.2 and the energy conservation given by the proposition 4.1: in the proposition 4.2, the character of the energy norm is played by

$$\|v^n\|^2 + \frac{1}{2}[\|u_x^n\|^2 + \|u_x^{n-1}\|^2] - \frac{\tau^2}{2}\|v_x^n\|^2 .$$

So that this remark introduces the following functional as a candidate for the analogue of the $Y^{\mu+1}$-norms:

DEFINITION 4.1. *Let* $W = (U, V) \in H^\mu(\mathbb{R})^4$. *We define*

$\|W; X^{\tau,\mu}\|^2 =$

$$= \|V\|^2_\mu + \frac{1}{2}[\|U_x\|^2_\mu + \|U_x^{-1}\|^2_\mu] - \frac{\tau^2}{2}\|V_x\|^2_\mu, \quad \textit{with } U^{-1} = U - \tau V .$$

We state in this section the properties of such a functional and compare it to the $Y^{\mu+1}$-norms. Mainly, we obtain that for $\sigma = \tau/h < 1$ this functional defines a norm. Remark also that it is a straightforward consequence of the proposition 4.2 that the linear part of the scheme keeps constant the functional $\|.\,; X^{\tau,0}\|^2$. This remains true for every real μ and we have

$$\|\mathcal{K}_\tau^n W^0; X^{\tau,\mu}\| = \|W^0; X^{\tau,\mu}\|, \qquad n \in \mathbb{N},\ \mu \in \mathbb{R}. \tag{41}$$

Hence, the linear part of the scheme defines a stable algorithm according to the $X^{\tau,\mu}$ norms in the $\sigma < 1$ case. In this case, the proposition 4.2 implies also a numerical stability property if the discrete energy is positive defined.

LEMMA 4.2. *Let $\mu \in \mathbb{R}$, $\sigma < 1$. Then $\|.\,; X^{\tau,\mu}\|$ defines a norm for compactly supported elements of H^μ space. Moreover, we have for a compactly supported $W = (U, V)$ function*

$$C_a \|W\|_\mu \leq \|W; X^{\tau,\mu}\| \leq (1+\sigma)\|W; Y^{\mu+1}\| \tag{42}$$

$$\frac{1-\sigma^2}{2(1+\sigma^2)}(\|V\|_\mu^2 + \|U_x\|_\mu^2) \leq \|W; X^{\tau,\mu}\|^2 \leq (1+\sigma)(\|V\|_\mu^2 + \|U_x\|_\mu^2) \tag{43}$$

$$\lim_{\tau \to 0} \|W; X^{\tau,\mu}\|^2 = \|W; Y^{\mu+1}\|^2, \qquad \sigma = \frac{\tau}{h} = C^{te}. \tag{44}$$

where C_a is a positive real constant, depending only on the size of the support of W.

PROOF OF LEMMA 4.2. Consider a $W = (U, V)$ function. It suffices to prove the $\mu = 0$ case of the Lemma 4.2. Remark first that for a real valued function u:

$$|\widehat{u_x}(\xi)|^2 = \left| \frac{1 - e^{ih\xi}}{h} \widehat{u}(\xi) \right|^2 = \frac{4\sin^2(h\xi/2)}{h^2} |\widehat{u}(\xi)|^2.$$

Hence, denoting $U^{-1} = U - \tau V$:

$$\|W; X^{\tau,0}\|^2 = \int_{\mathbb{R}} |\widehat{V}|^2 + \frac{4\sin^2(h\xi/2)}{2h^2}[\,|\widehat{U}|^2 + |\widehat{U^{-1}}|^2 - \tau^2 |\widehat{V}|^2]\, d\xi\,.$$

We deduce from this last expression, for any real $0 < \theta^2$,

$$(45)\qquad \|W; X^{\tau,0}\|^2 = \int_{\mathbb{R}} (1 - 2\theta^2\sigma^2 \sin^2(h\xi/2)) |\widehat{V}|^2 d\xi +$$

$$+ \int_{\mathbb{R}} \frac{2\sin^2(h\xi/2)}{h^2} \left[2\left(1 - \frac{1}{2\theta^2}\right) |\widehat{U}|^2 + \left| \theta\tau\widehat{V} - \frac{1}{\theta}\widehat{U} \right|^2 \right] d\xi .$$

Assigning $\theta^2 = 1/(1+\sigma^2)$, we obtain

$$(46)\qquad \|W; X^{\tau,0}\|^2 = \int_{\mathbb{R}} \left(1 - \frac{2\sigma^2}{1+\sigma^2}\sin^2(h\xi/2)\right) |\widehat{V}|^2 d\xi +$$

$$+ \int_{\mathbb{R}} \frac{2\sin^2(h\xi/2)}{h^2} \left[(1-\sigma^2) |\widehat{U}|^2 + |\theta\tau\widehat{V} - \frac{1}{\theta}\widehat{U}|^2 \right] d\xi .$$

Hence, for a compactly supported W function and for $\sigma < 1$, (46) shows immediately the following relations: $\|W; X^{\tau,0}\|^2 \geqslant 0$, $\|W; X^{\tau,0}\|^2 = 0 \Rightarrow W \equiv 0$, and the triangular inequality $\|W_1 + W_2; X^{\tau,0}\| \leqslant \|W_1; X^{\tau,0}\| + \|W_2; X^{\tau,0}\|$. So that $\|.; X^{\tau,0}\|$ is a norm in the case $\sigma < 1$. ■

Now, for $W = (U, V) \in Y^1$, (46) proves, using a dominated convergence argument

$$\lim_{\tau \to 0} \|W; X^{\tau,0}\|^2 = \|\partial_x U\|^2 + \|V\|^2.$$

This is the lemma assertion (44) in the $\mu = 0$ case. ■

Now, remark that (46) implies the following relation

$$\|W; X^{\tau,0}\|^2 \geqslant \frac{1-\sigma^2}{1+\sigma^2}\left(\int_{\mathbb{R}} |\widehat{V}|^2 d\xi + \int_{\mathbb{R}} \frac{2\sin^2(h\xi/2)}{h^2} |\widehat{U}|^2 d\xi \right) \geqslant$$

$$\geqslant \frac{1-\sigma^2}{2(1+\sigma^2)} (\|V\|^2 + \|U_x\|^2).$$

We have also, due to (45), the relation

$$\|W; X^{\tau,0}\|^2 \leq (1+2\sigma^2\theta^2)\int_{\mathbb{R}} |\widehat{V}|^2 d\xi + \left(1+\frac{1}{2\theta^2}\right)\int_{\mathbb{R}} \frac{4\sin^2(h\xi/2)}{h^2}|\widehat{U}|^2 d\xi .$$

So the particular setting $2\theta^2 = 1/\sigma$ gives us

$$\|W; X^{\tau,0}\|^2 \leq (1+\sigma)(\|V\|^2 + \|U_x\|^2).$$

Hence we have proved (43) for the case $\mu = 0$. ∎

Now, in order to prove (42), we have to compare the norm $\|u_x\|$ versus the Sobolev norm $\|u\|_1$. So we consider $\phi \in H^1(\mathbb{R})$ a compactly supported function with support

$$\operatorname{supp}\phi \subset [-a, a], \qquad a > 0 .$$

First, using the Parseval identity we obtain

$$\|\varphi_x\| = \left\| \frac{1-T_h}{h}\varphi \right\| = \left\| \frac{2}{h}\sin\left(\frac{h\xi}{2}\right)|\widehat{\varphi}| \right\| \leq \| |\xi| \, |\widehat{\varphi}| \| = \|\varphi\|_1 .$$

This last estimate, using (43) shows that $\|W; X^{\tau,0}\|^2 \leq (1+\sigma)(\|V\|^2 + \|U\|_1^2)$ and the right inequality of (42) for the $\mu = 0$ case. ∎

Second, we use the following analogue of the Poincaré inequality

$$\|\phi\| \leq a\|\phi_x\|$$

in order to prove, using (43)

$$(\|V\|^2 + \|U\|^2) \leq C_a(\|V\|^2 + \|U_x\|^2) \leq C_a\|W; X^{\tau,0}\|^2$$

which is the left inequality of (42) for the $\mu = 0$ case. ∎

4.2. *Main result.*

We state in this section our main convergence result.

THEOREM 4.1. *Let $\sigma < 1$, and $W^0 \in Y^4$. Then $T^{**} = T^*$. Furthermore for all $T < T^*$ the scheme $K_{t/N}^N W^0$ converges uniformly on $[0, T]$ towards $F_t W^0$:*

$$\lim_{N\to\infty} \sup_{0\leq t\leq T} \|F_t W^0 - K_{t/N}^N W^0; Y^1\| = 0 . \tag{47}$$

This theorem is the statement of the general convergence result (theorem 13) for the particular setting given in (10). To prove it, we verify in the next section the stability and consistence results as stated in the theorem (13). For that aim, we recall the following definition

$$B_{\tau,\mu}(A) = \left\{W \in (H^{\mu}(\mathbb{R}))^4,\ \text{supp } W \text{ compact},\ \|W; X^{\tau,\mu}\| < A\right\}.$$

REMARK 4.3. *Actually we prove a better result than the one stated: we get*

$$\forall T < T^*, \qquad \lim_{N \to \infty} \sup_{0 \leq t \leq T} \|F_t W^0 - K_{t/N}^N W^0; X^{t/N,1}\| = 0.$$

This has to be compared with (44). *We note also that the straightforward generalization of the scheme* (38) *to the n dimensional case is stable under the C.F.L condition* $\sigma < 1/\sqrt{n}$, *using a similar demonstration. Due to this fact and from Sobolev injections we think that this result remains true in the higher dimensions under this C.F.L. condition and with added regularity assumptions over the initial conditions (see the annex for a result in this direction).*

4.2.1. *Stability.* – We show in this part the stability of the scheme (39) uniformly with respect to the $X^{\tau,1}$ norms.

LEMMA 4.3. *Let* $\sigma < 1$. *For all* $A > 0$ *there exists* $T_s(A) = C_0/A^2$ *such that for all* $N \geq 1$, $1 \leq n \leq N$ *and* $0 \leq t \leq T_s(A)$,

$$\sup_{(W_a^0, W_b^0) \in B_{t/N,1}(A)^2} \|K_{t/N}^n W_a^0 - K_{t/N}^n W_a^0; X^{t/N,1}\| \leq C\|W_a^0 - W_b^0; X^{t/N,1}\|.$$

PROOF OF LEMMA 4.3. Before proving this lemma, we shows the following stability result

(48) *for all* $1 \leq n \leq N$

$$\textit{and } 0 \leq t \leq T_s(A), \quad \sup_{W^0 \in B_{t/N,1}(A)} \|K_{t/N}^n W^0; X^{t/N,1}\| \leq 2\|W^0; X^{t/N,1}\|.$$

To prove this relation, we study first a one step in time $\tau = t/N$ of our scheme, for any data $W^i = (U^i, V^i)$ belonging to $B_{\tau,1}(A)$. We note $W^{i+1} = (U^{i+1}, V^{i+1}) = K_\tau W^i$, $i \leq n$, which solves, according to (39), the equation

$$W^{i+1} = \mathcal{K}_\tau W^i + {}^T(\tau^2 H^i, \tau H^i),$$

where H^i stands for the two components vector valued function $\frac{1}{2}Q_\varepsilon(U^i)(U^{i+1}+U^{i-1})$. Now, we estimate the $X^{\tau,1}$ norm of W^{i+1}:

1: according to (41) we have $\|\mathcal{K}_\tau W^i; X^{\tau,1}\| = \|W^i; X^{\tau,1}\|$.

2: going back to the expression of the $X^{\tau,\mu}$ norms, we compute

$$\|(\tau^2 H^i, \tau H^i); X^{\tau,1}\|^2 = \|\tau H^i\|_1^2.$$

So we obtain

$$\|W^{i+1}; X^{\tau,1}\| \leq \|W^i; X^{\tau,1}\| + \tau\|H^i\|_1. \tag{49}$$

Using the algebra's properties of the $H^\mu(\mathbb{R})$ spaces, $\mu > 1/2$, and the Lemma 4.2, we estimate the non linear part with:

$$\|H^i\|_1 \leq (\|U^{i-1}\|_1 + \|U^{i+1}\|_1)\|U^i\|_1^2 \leq$$

$$\leq C(\|W^{i+1}; X^{\tau,1}\| + \|W^i; X^{\tau,1}\|)\|W^i; X^{\tau,1}\|^2.$$

So we include these estimates in (49) in order to obtain:

$$\|W^{i+1}; X^{\tau,1}\| \leq \|W^i; X^{\tau,1}\| \frac{(1 + C\tau\|W^i; X^{\tau,1}\|^2)}{(1 - C\tau\|W^i; X^{\tau,1}\|^2)}. \tag{50}$$

So, hypothesizing $W^i \in B_{\tau,1}(A_i)$, we get $W^i \in B_{\tau,1}(A_{i+1})$, with $A_{i+1} = A_i(1 + C\tau A_i^2)/(1 - C\tau A_i^2)$. Thus, we obtain recursively from this last point that $A_N \leq 2A_0$, provided that $t \leq 1/16CA_0^2$. ■

We turn out to the demonstration to the stability lemma. First choose data $W_a^i = (U_a^i, V_a^i)$ and $W_b^i = (U_b^i, V_b^i)$ belonging to $B_{\tau,1}(A)$. We denote $W_a^{i+1} = (U_a^{i+1}, V_a^{i+1}) = K_\tau W_a^i$, $H_a^i = \frac{1}{2}(U_a^{i+1} + U_a^{i-1})\, Q_\varepsilon(U_a^i)$, and the same with the subscript b. We denote in the following $W^i = W_a^i - W_b^i$, $H^i = H_a^i - H_b^i$. Now, we estimate the $X^{\tau,1}$ norm of W^{i+1}, with $\tau = t/N$, and we obtain, as for (49):

$$\|W^{i+1}; X^{\tau,1}\| \leq \|W^i; X^{\tau,1}\| + \tau\|H^i\|_1. \tag{51}$$

We factorize H^i in the following way:

$$H_a^i - H_b^i = \frac{1}{4}(Q_\varepsilon(U_a^i) + Q_\varepsilon(U_b^i))(U_a^{i+1} - U_b^{i+1} + U_a^{i-1} - U_b^{i-1}) + \tag{52}$$

$$+ \frac{1}{4}(Q_\varepsilon(U_a^i) - Q_\varepsilon(U_b^i))(U_a^{i+1} + U_a^{i-1} + U_b^{i+1} + U_b^{i-1}).$$

This factorization allows us to estimate:

$$\tau\|H^i\|_1 \leq C\tau(\|W_a^i; X^{\tau,1}\|^2 + \|W_b^i; X^{\tau,1}\|^2)(\|W^{i+1}; X^{\tau,1}\| + \|W^i; X^{\tau,1}\|).$$

Now, suppose that $tA^2 \leq C_0$ and consider $i < N$. We estimate $C\tau(\|W_a^i; X^{\tau,1}\|^2 + \|W_b^i; X^{\tau,1}\|^2) \leq C/N$, using (3.2). So we get inductively $\|W^{i+1}; X^{\tau,1}\| \leq e^C \|W^0; X^{\tau,1}\|$. ■

4.2.2. *Consistency.* – We show in this part the consistency of the scheme (39). This consistency is stated uniformly according to Y^2 space, locally uniformly for data belonging to the Y^4 space:

LEMMA 4.4. *Let $\sigma < 1$. For every $W^0 \in Y^4$, and $\tau \leq T_s(\|W^0; Y^2\|)$*

$$\|K_\tau W^0 - F_\tau W^0; Y^2\| \leq C\tau^2 \|W^0; Y^4\|(1 + \|W^0; Y^2\|^2).$$

PROOF OF LEMMA 4.4. In this section, $*$ is the convolution operator relatively to the space variable. Moreover, for a bounded set Ω, $\chi_\Omega(x)$ is the truncated function associated to Ω and we denote the function $\phi_\Omega(x) = \chi_\Omega / \mathrm{mes}(\Omega)$.

First, remark that the condition $\tau \leq T_s(\|W^0; X^{\tau,1}\|)$ allows us to use the propositions 2.3, and 4.3.

We remind now the following integral property of the solution of (8): let $(U, V)(t) = F_t W^0$, given by the Lemma 2.2 for data (U^0, V^0) belonging to Y^1, then $W(\tau)$ solves the integral equation given by the Duhamel principle:

$$F_\tau W^0 = \mathscr{F}_\tau W^0 + \begin{bmatrix} \dfrac{1}{2} \displaystyle\int_{\Delta_{\tau,.}} Q_\varepsilon(U)\, U \\ \dfrac{1}{2\sqrt{2}} \displaystyle\int_{\Lambda_{\tau,.}} Q_\varepsilon(U)\, U \end{bmatrix}$$

where $\mathscr{F}_\tau W^0 = {}^T(\frac{1}{2}[T_\tau + T_{-\tau}]\, U^0 + \frac{1}{2}\chi_{[-\tau,\tau]}\frac{1}{2}V^0;\ \frac{1}{2}[T_\tau + T_{-\tau}]\, V^0 + \frac{1}{2}[T_{-\tau} - T_\tau]\, \partial_x U^0)$ holds for the free wave equation, and $\Delta_{\tau,x} = \{(t, y)/t > 0, t + |x - y| \leq |\tau|\}$, the wave cone issued from a (τ, x) point, $\Lambda_{\tau,x} =$

$= \{(t, y)/t > 0,\, t + |x - y| = |\tau|\}$ the upper edge of this cone. So we have to estimate the Y^2 norms of the difference

$$K_\tau W^0 - F_\tau W^0 =$$

$$= \mathcal{K}_\tau W^0 - \mathcal{F}_\tau W^0 + \begin{bmatrix} \dfrac{\tau^2}{2}(U^1 + U^{-1})\, Q_\varepsilon(U^0) - \dfrac{1}{2} \displaystyle\int_{\Delta_{\tau,\cdot}} Q_\varepsilon(U)\, U \\ \dfrac{\tau}{2}(U^1 + U^{-1})\, Q_\varepsilon(U^0) - \dfrac{1}{2\sqrt{2}} \displaystyle\int_{\Delta_{\tau,\cdot}} Q_\varepsilon(U)\, U \end{bmatrix}.$$

Let us hold first with the linear part. Using the Fourier transform, we get

$$\widehat{\mathcal{F}_\tau W^0} - \widehat{\mathcal{K}_\tau W^0} =$$

$$= \tau^2 \begin{bmatrix} \left(2\dfrac{\sin^2 \tau\xi/2}{\tau^2} - 4\sigma^2 \dfrac{\sin^2 \tau\xi/2\sigma}{\tau^2}\right) & \dfrac{1}{\tau}\left(\dfrac{\sin \tau\xi}{\tau\xi} - 1\right) \\ \left(\dfrac{\xi}{\tau^2}\sin \tau\xi - \dfrac{4\sigma^2}{\tau^3}\sin^2 \dfrac{\tau\xi}{2\sigma}\right) & -2\dfrac{\sin^2 \tau\xi/2}{\tau^2} \end{bmatrix} \begin{pmatrix} \widehat{U^0} \\ \widehat{V^0} \end{pmatrix}.$$

The following functions are uniformly bounded as τ tends to zero

$$\xi \to |\xi|^{-2}\left(2\frac{\sin^2 \tau\xi/2}{\tau^2} - 4\sigma^2 \frac{\sin^2 \tau\xi/2\sigma}{\tau^2}\right); \qquad \xi \to \frac{|\xi|^{-1}}{\tau}\left(\frac{\sin \tau\xi}{\tau\xi} - 1\right)$$

$$\xi \to |\xi|^{-3}\left[\frac{\xi}{\tau^2}\sin \tau\xi - \frac{4\sigma^2}{\tau^3}\sin^2 \frac{\tau\xi}{2\sigma}\right]; \qquad \xi \to -2|\xi|^{-2}\frac{\sin^2 \tau\xi/2}{\tau^2}.$$

So we can estimate

$$\|\mathcal{K}_\tau W^0 - \mathcal{F}_\tau W^0;\, Y^2\| \leq C\tau^2 \|W^0;\, Y^4\| \leq C\tau^2 \|W^0;\, Y^4\|.$$

For the nonlinear part, we estimate first roughly, using the proposition 2.3 iii),

$$\left\| \frac{1}{2} \int_{\Delta_{\tau,\cdot}} Q_\varepsilon(U)\, U \right\|_2 \leq C\tau^2 \sup_{0 \leq s \leq \tau} \|Q(U)\, U\|_2 \leq C\tau^2 \|W^0;\, Y^2\|^3$$

and we have also

$$\|Q_\varepsilon(U^0)(U^1 + U^{-1})\|_2 \leq C\|W^0;\, Y^2\|^2 \|W^0;\, Y^3\|.$$

Now let us hold with the second non linear estimate. We estimate its H^1

norm by

$$a_\tau \leqslant C\tau \sup_{0\leqslant s\leqslant \tau} \left\| Q_\varepsilon(U)U - \frac{1}{2} Q_\varepsilon(U^0)(U^1 + U^{-1}) \right\|_1 .$$

So

$$a_\tau \leqslant C\tau^2 \left(\sup_{0\leqslant s\leqslant \tau} \left\| \frac{Q_\varepsilon(U)\,U - Q_\varepsilon(U^0)\,U^0}{\tau} \right\|_1 + \left\| \frac{Q_\varepsilon(U^0)(2U^0 - U^1 - U^{-1})}{2\tau} \right\|_1 \right) =$$

$$= C\tau^2(a_1^\tau + a_2^\tau).$$

We estimate, using the factorization (52),

$$a_1^\tau \leqslant C\left(\|U^0\|_1^2 + \sup_{0\leqslant s\leqslant \tau} \|U\|_1^2\right) \sup_{0\leqslant s\leqslant \tau} \left\| \frac{U - U^0}{\tau} \right\|_1 .$$

We estimate $\sup_{0\leqslant s\leqslant \tau} \|U - U^0/\tau\|_1 \leqslant C\|U;\, C^1([0, \tau];\, H^1)\| \leqslant C \sup_{0\leqslant s\leqslant \tau} \|U\|_2$. This last estimate comes from the property of the continuous problem. So we estimate, using the proposition 2.3 iii),

$$a_1^\tau \leqslant C\|W^0;\, Y^1\|^2 \|W^0;\, Y^2\| .$$

For a_2^τ, we estimate first $a_2^\tau \leqslant C\tau\|Q_\varepsilon(U^0)\|_1 \|(2U^0 - U^1 - U^{-1})/2\tau^2\|_1 = C\tau\|Q_\varepsilon(U^0)\|_1 \|U^0_{t\bar t}\|_1$. As $\tau\|Q_\varepsilon(U^0)\|_1 \leqslant C\tau\|U^0\|_1^2 \leqslant C\tau\|W^0;\, Y^1\|^2 \leqslant C$, we have $a_2^\tau \leqslant C\|U^0_{t\bar t}\|_1$. Now, using the expression of the scheme (38), we estimate

$$a_2^\tau \leqslant C\|U^0_{t\bar t}\|_1 \leqslant C(\|U^0_{x\bar x}\|_1 + \|U^0\|_1^2(\|U^1\|_1 + \|U^{-1}\|_1)) \leqslant C(\|U^0\|_3 + \|W^0;\, Y^2\|^3).$$

Blending all of these partial results and using the Sobolev injections, we obtain

$$\|K_\tau W^0 - F_\tau W^0;\, Y^2\| \leqslant C\tau^2 \|W^0;\, Y^4\|(1 + \|W^0;\, Y^2\|^2). \qquad \blacksquare$$

5. Numerical experiments.

5.1. *The first order system.*

In all the experiments presented here we have fixed $\theta = 1/2$. We present three different tests on (1). First we consider a simple model of discrete Boltzmann equation, the Broadwell system:

$$
\begin{cases} \partial_t u_1 - \partial_x u_1 = u_2^2 - u_1 u_3 \\ \partial_t u_2 \qquad\qquad = -u_2^2 + u_1 u_3 \\ \partial_t u_3 + \partial_x u_3 = u_2^2 - u_1 u_3 . \end{cases} \tag{53}
$$

It is well known that for positive data in L^∞_{loc} the solution of the Cauchy problem is global in time. But one can also construct exact explosive solutions by a solitary wave method, see [2] and references therein. We look for a solution of form:

$$u(t, x) = \phi(x - ct).\, \lambda$$

were $\lambda \in \mathbb{R}^3$ is constant and $c \in]-\infty, -1[\cup]+1, +\infty[$. In our case we obtain:

$$\lambda_1 = -a/(c+1), \qquad \lambda_2 = a/c, \qquad \lambda_3 = -a/(c-1)$$

where $a = -c^2(c^2-1)$, and then ϕ is solution of the blowing up equation

$$\phi' = \phi^2. \tag{54}$$

By truncation and regularization one obtains a smooth compactly supported initial condition which is equal to $\phi.\, \lambda$ on a bounded interval, so that the solution of the problem is $\phi(x-ct).\, \lambda$ in the associated dependance domain. In this particular case the determinant of the matrix $I - \tau A(u^n(x))$ in (22) is $1 + h(u_1 + 2u_2 + u_3)/2$ where one recognizes the density $u_1 + 2u_2 + u_3$. Let us take as a blow up criterion:

$$u_1 + 2u_2 + u_3 = -2/h .$$

If u is the exact solution $\phi.\, \lambda$ then this condition reads as:

$$(\lambda_1 + 2\lambda_2 + \lambda_3)\, \phi(x - cT) = -2/h .$$

As $\phi = (\phi_0^{-1} - (x - cT))^{-1}$, this criterion furnishes the following blow up value:

$$x - cT = \phi_0^{-1} + hc$$

instead of the theoretical $x - cT = \phi_0^{-1}$. Consequently such a criterion provides an error of order h at least.

Figures 1 and 2 are the comparison between exact and numerical density $u_1 + 2u_2 + u_3$ at time $T = 2$. Here $c = 2$, $h = 0.02$. In figure 1 we focus on the smooth part of the solution far from the blow up point. The L^1 error has been computed to be less than 1% in this zone. In fig. 2 we show the blow up zone. $u_1 + 2u_2 + u_3 = -2/h$ for $x = -5.66$ while the

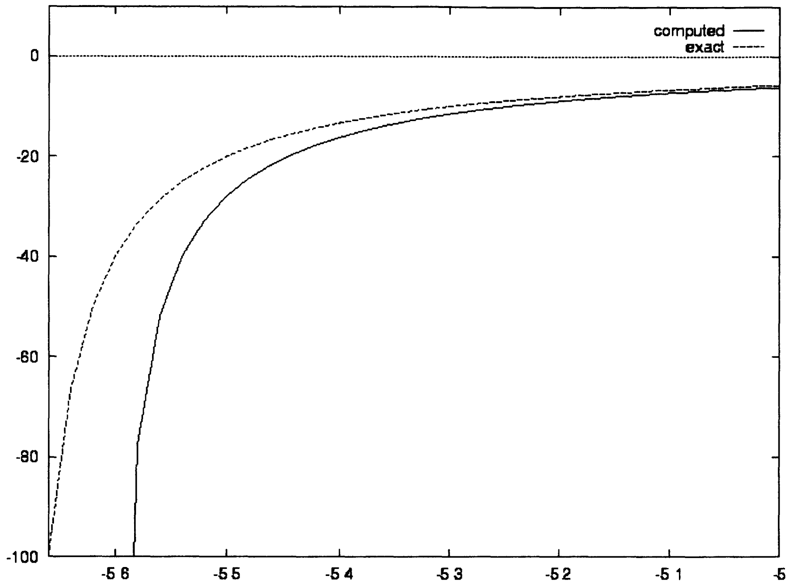

Fig. 1. – Broadwell's equations: smooth part of the solution at $T = 2$.

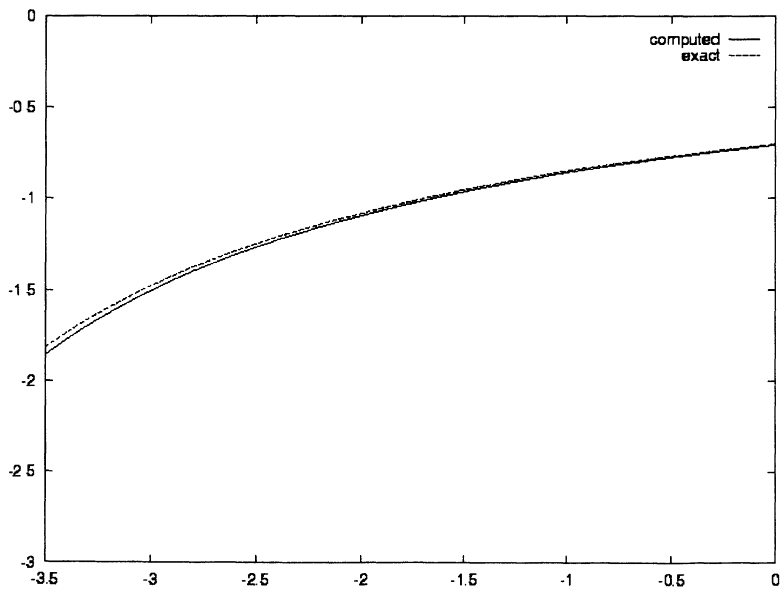

Fig. 2. – Broadwell's equations: Stiff part of the solution at $T = 2$.

theoretical blow up is at $x = -5.7$. We find the computed value $x = -$ -5.58, that is an error of 2%.

Let us now study the blow up for:

$$\begin{cases} \partial_t u_1 = u_1 u_2 \\ \partial_t u_2 + \partial_x u_2 = u_1^2 \end{cases} \qquad u(0) = \varepsilon\phi \,. \tag{55}$$

There exists a smooth compactly supported function ϕ such that for all $\varepsilon > 0$ the solution of (55) blows up in finite time T_ε^*. Moreover there exists two constants C_1, C_2 such that [2]:

$$C_1 \leqslant \varepsilon^2 T_\varepsilon^* \leqslant C_2.$$

If ϕ is chosen like in [2] with support in [-1,1], we know that

$$1 - \varepsilon - \varepsilon^2 \leqslant \varepsilon^2 T_\varepsilon^* \leqslant 2.3 + \varepsilon^2.$$

Taking the blow up criterion:

$$T_\varepsilon^* = \min\{t_n, \max(u_1, u_2) \geqslant h^{-1}\}$$

we obtain the following table:

ε	$\varepsilon^2 T_\varepsilon^*$, $h = 0.1$	$\varepsilon^2 T_\varepsilon^*$, $h = 0.02$
1	1.	1.08
2^{-1}	0.82	0.84
2^{-2}	0.89	0.91
2^{-3}	0.99	1.
2^{-4}	1.05	1.06
2^{-5}	1.09	1.09

For the same problem ($\varepsilon = 1$) we represent the space time isolines of u_1 and u_2. The maximal value for the isolines has been taken to 12.5 while $h = 0.02$. This process allows us to exhibit an approximate maximal existence domain and a blow up curve of the solution (figures 3 and 4).

The last test is concerned with global solutions. Figures 5 and 6 rep-

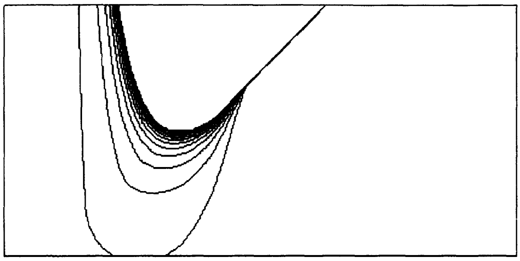

Fig. 3. – Space time isolines and blow up curve for the solution of a 2×2 system: first component.

resent the space time isolines of the solutions u_1 and u_2 of the conservative system

$$\left\{ \begin{array}{l} \partial_t u_1 = - u_1 u_2 \\ \partial_t u_2 + \partial_x u_2 = u_1^2 \end{array} \right. \qquad u(0) = \phi \tag{56}$$

with the same initial data as above and the same time step. This system arises in nonlinear optics or plasmas physics. We observe a strong interaction zone out of which the waves propagate linearly.

5.2. *The system of wave equations.*

We present in this part computed solutions of $P(\varepsilon_1, \varepsilon_2)$ systems, using the schemes (39), (63), and the remark (4.2).

We have chosen for tests a C.F.L constant σ equal to $\frac{1}{2}$. We compute the solution over regular dichotomous grids with shape $V_i = (t^j, x_k)_{0 \leq j \leq 2^{i+1}}^{0 \leq k \leq 2^{i+1}} = (\sigma j 2^{-i}, k 2^{-i})_{0 \leq j \leq 2^{i+1}}^{0 \leq k \leq 2^{i+1}}$ that mesh both the domains

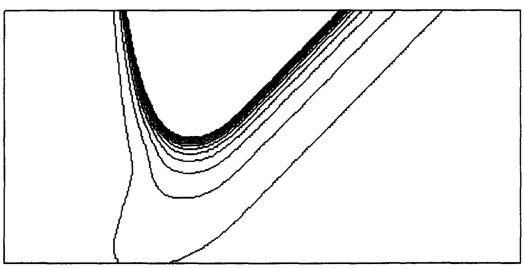

Fig. 4. – Space time isolines and blow up curve for the solution of a 2×2 system: second component.

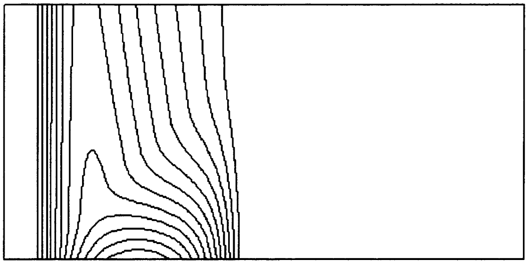

Fig. 5. – Space time isolines for the solution of a 2×2 conservative system: first component.

$(t, x) \in [0, 1] \times [0, 2]$ for 1-D computations or $(t, r) \in [0, 1] \times B(0, 2)$ for the spherical symmetric data 3-D case, where $B(0, 1)$ holds for the unit ball. When properly defined, the computed solution on such grids is noted $((u_1)_j^k, (u_2)_j^k)_{0 \leq j \leq 2^{i+1}}^{0 \leq k \leq 2^{i+1}}$. We need to compute the solution in a bounded domain, so we use Dirichlet condition at boundaries $(t^j, 0)_{0 \leq j \leq 2^i}$ and $(t^j, 2)_{0 \leq j \leq 2^i}$. The support of data has been chosen in order to ensure, using the numerical propagation speed of our scheme, that these boundary Dirichlet conditions do not perturb the solution.

Because we would like to have some indications on the time existence of solutions of $P(+1, -1)$, we define a numerical explosion criteria. It seems natural to define

$$T^{*,i} = \inf\left\{t^k / \sup_{0 \leq j \leq 2^{i+1}} \tau^2[(u_{1,j}^k)^2 + (u_{2,j}^k)^2] > \frac{1}{2}\right\},$$

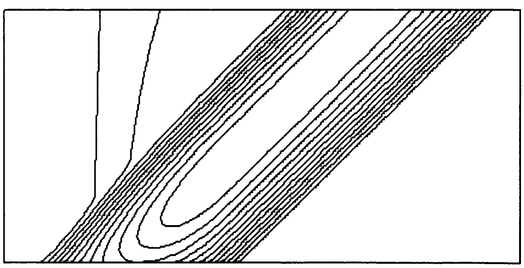

Fig. 6. – Space time isolines for the solution of a 2×2 conservative system: second component.

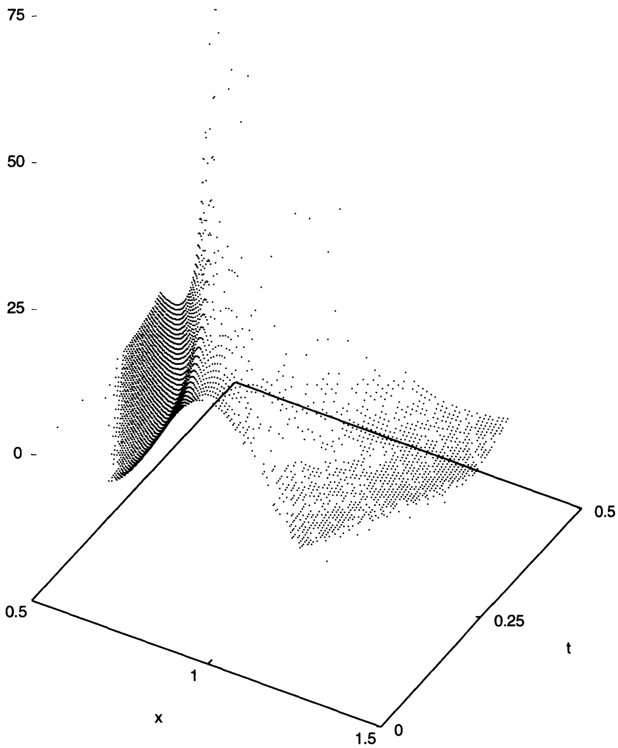

Fig. 7. – A one dimensional blow up cse of $P(+1, +1)$.

since for lower times, the solution can be explicitly deduced from the expression of our scheme (39).

The solution is represented on a three dimensional $(x, t, u(t, x))$ graphics for pictures 7 and 8.

We illustrate in picture 7 the behavior of the solution in a blow-up case of the $P(+1, +1)$ system in one dimension for which we know that $E[W^0] > 0 \Rightarrow T^* < +\infty$ ([13] and related bibliography). So we choose large enough initial data that lead to a blow up solution for the continuous problem in order to check that our explosion criteria is able to detect a finite existence time. For this computation we have chosen the same in-

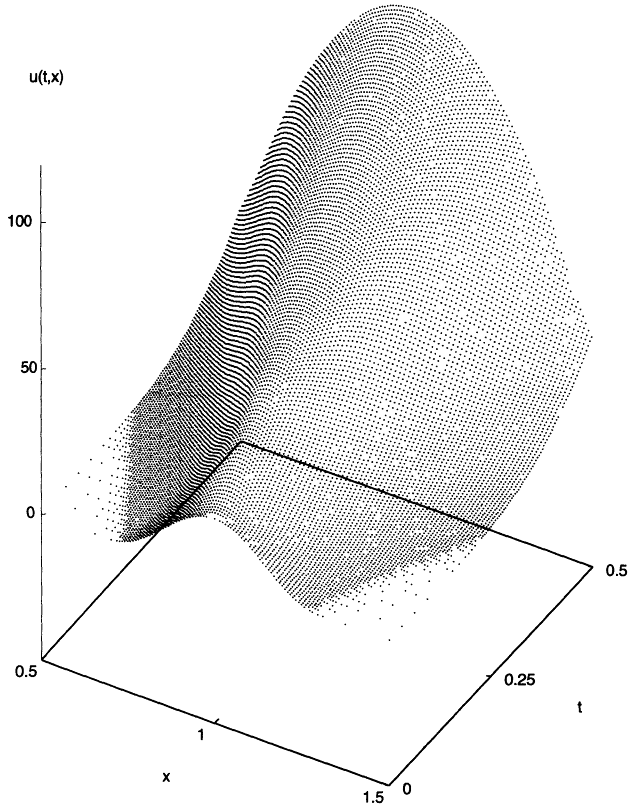

Fig. 8. - u solution of $P(+1,\ -1)$ in the one dimensional case.

itial data for u and v, so that the solution is a particular case of the scalar semilinear wave equation $\{\partial_t^2 u - \partial_x^2 u = +u^3\}$. Numerical results seem to show that $(T^{*,\,i})_{i\in\mathbb{N}}$ converges toward the blow up time of the continuous problem T^*. On picture 7, the numerical finite propagation speed of the solution is used to compute the solution after the blow up time.

The figure 8 is a computed solution of the system $P(+1,\ -1)$ in the one dimensional case. For this computation and also for the next one, we have chosen the same initial data for u_1 and u_2, but only the u_1 solution is showed here. This because the u_2 solution has an heavy oscillatory behavior that is hard to present with such a graphic. Interpretation of these oscillations are provided if one think that u_2 satisfies a Klein Gordon type equation with variable mass $(u_1)^2$.

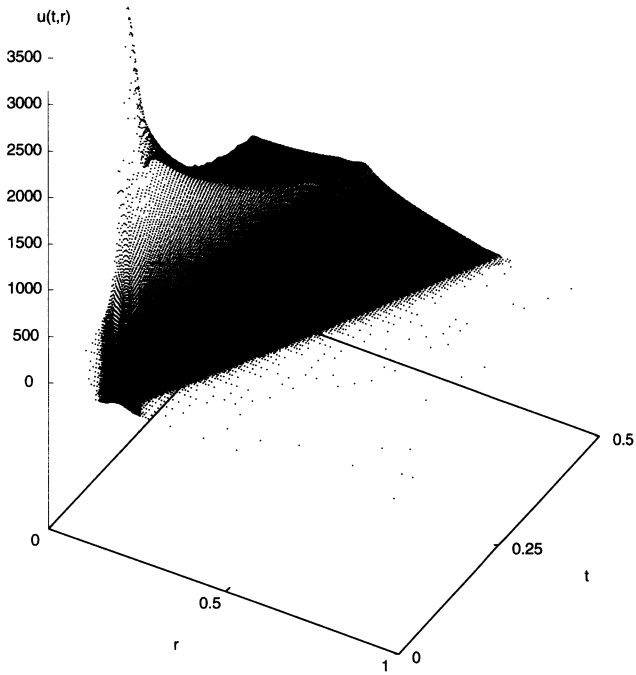

Fig. 9. – Spherically symmetric u solution of $P(+1, -1)$ in the three dimensional case.

For these computations, it seems that $(T^{*,i})_{i \in \mathbb{N}}$ goes toward infinity. So we believe that for this case, every smooth enough data leads to define a global solution. Furthermore, this picture seems to show that the solution u_1 stabilize itself linearly over large time.

In figure 9, we present a spherically symmetrical u solution of $P(+ +1, -1)$ in the three dimensional case owning spherically symmetric data, for which the numerical study is overviewed in the annex of this paper. The solution is represented on a three dimensional $(r, t, u_r(t, r))$ graphics, where r denote the spherical variable. We have chosen a «large» initial data in order to illustrate the behavior of heavily interacting solutions of the $P(+1, -1)$ system. For that aim, we use an adaptive mesher because for such initial data we have recorded finite blow up time on every regular grid V^i allowed by our computing capability. This adaptive mesher has the particularity to keep the C.F.L ratio constant throughout the computation. We refer to ([13]) for a description of the

one dimensional principle of this mesher, and indicate that a multi-dimensional version is available ([15]).
We haven't recorded any explosion for solutions computed with this adaptive mesher. So we believe that the three dimensional case still generate an unique globally defined solution what large ever the initial data are. Furthermore, this picture seems to show that the solution u stabilize itself to a non zero constant.

6. Conclusion and remarks.

The approach proposed in [7] appears to be well adapted to the numerical study of some properties of semilinear hyperbolic systems. We analyze in a similar manner the continuous problem and the approximated one: the local existence time is like the local stability one, we retrieve for the discretization the continuous dependance on data property and we are able to define a maximal convergence time which is proved to coincide with the maximal existence time of the continuous solution.We find for two distinct applications, a first order and a second order system, a discretization and a functional setting for this study. We want to point out that for our wave equation systems, we include the case where the natural space in which the evolution group acts is unstable for the discretization operator. For this problem, we had to introduce a specific norm.In fact, we own now a local convergence theory, insuring us that the scheme converges as long as the continuous solution exists. Moreover for the schemes we construct, we are able to define a numerical blow up time which appears to be coherent with the theoritical known results, and also to define numerical blow up curves. It would be of interest to prove the convergence of these numerical blow up times towards the theoretical one.

7. Annex.

The three dimensional case with spherically symmetric data. – We overview in this part the numerical study devoted to the three dimensional version of (36):

$$P(\varepsilon_1, \varepsilon_2)\begin{cases} \partial_t^2 u_1 - \Delta u_1 = \varepsilon_1 u_1 u_2^2 \\ \partial_t^2 u_2 - \Delta u_2 = \varepsilon_2 u_2 u_1^2 \end{cases} \tag{57}$$

for Cauchy data $\{u_1(0)=u_1^0;\ \partial_t u_1(0)=v_1^0,\ u_2(0)=u_2^0;\ \partial_t u_2(0)=v_2^0\}$ compactly supported and spherically symmetric. We assert in this case a theorem analogous to Theorem 4 for the three-dimensional case. For that aim, we study our systems in an equivalent form in the spirit of ([18]). We show that this equivalent form allows to deal with a C.F.L. condition $\sigma<1$, that seems to us better that the usual C.F.L. condition $\sigma<1/\sqrt{3}$ needed to study the three dimensional case ([17], [6]).

First, let us give some notations: we denote with $H_r^\mu(\mathbb{R}^3)$ the closed subspace of $H^\mu(\mathbb{R}^3)$ of spherically symmetric functions. We denote $\overline{H}^\mu(\mathbb{R})$ the half Sobolev space made of even functions that belong to $H^\mu(\mathbb{R})$, and $\overline{Y}^\mu=\overline{H}^\mu\times\overline{H}^{\mu-1}(\mathbb{R})$. Let $r(x)=|x|=[(x_1)^2+(x_2)^2+ (x_3)^2]^{1/2}$. For a function $u\in H_r^\mu(\mathbb{R}^3)$, we denote $u_r(r(x))=u(x)$, and consider it as an odd function of the real variable r. In the sequel, $\Delta_r=\partial_r^2+(2/r)\,\partial_r$ is the Laplacian expressed in polar coordinates.

We recall also the following property of the Laplacian operating on spherically symmetric functions:

$$r(x)\,\Delta u(x)=(\partial_r^2(ru_r))(r(x))\,. \tag{58}$$

The equivalent problem. – A direct computation on the (57) using the property of commutation (58) leads formally to the following one dimensional system, with $w_i=ru_{i,\,r}$, $i=1,2$:

$$P_r(\varepsilon_1,\varepsilon_2)\begin{cases}\partial_t^2 w_1-\partial_r^2 w_2=\dfrac{\varepsilon_1}{r^2}w_1w_2^2\\[2ex] \partial_t^2 w_2-\partial_r^2 w_2=\dfrac{\varepsilon_2}{r^2}w_2w_1^2\end{cases} \tag{59}$$

with even Cauchy data. In the following lemma, we provide the basic tools used in this annex, studying the mapping $R\colon u(x)\to ru_r(r)$.

LEMMA 7.1. *Let $\mu\geq 0$ an integer. Let us consider $H_r^\mu(\mathbb{R}_x^3)$ and $\overline{H}^\mu(\mathbb{R}_r)$ with the norms*

$$\|u;H^\mu(\mathbb{R}^3)\|^2=\frac{1}{8\pi}\sum_{\alpha\leq\mu}\|(\Delta)^{\alpha/2}u;L^2(\mathbb{R}^3)\|^2;$$

$$\|v;H^\mu(\mathbb{R})\|^2=\sum_{\alpha\leq\mu}\|\partial_r^\alpha v;L^2(\mathbb{R})\|^2.$$

i) *$R\colon u(x)\to ru_r(r)$ realizes an isometrical bijection between $H_r^\mu(\mathbb{R}_x^3)$ and $\overline{H}^\mu(\mathbb{R}_r)$.*

ii) *We have the following estimations*:

$$\left\| \frac{w(r)}{r} ; L^{\infty}(\mathbb{R}) \right\| \leq C \|w; H^2(\mathbb{R})\|, \qquad w \in \overline{H}^{\mu}(\mathbb{R}),\ \mu \geq 2. \tag{60}$$

$$\left\| \frac{w_1 w_2 w_2}{r^2} ; H^{\mu}(\mathbb{R}) \right\| \leq C \prod_{i=1,2,3} \|w_i; H^{\mu}(\mathbb{R})\|, \qquad w_i \in \overline{H}^{\mu}(\mathbb{R}), \tag{61}$$

$$i = 1, 2, 3,\ \mu \geq 2.$$

Such a lemma must be known, but we haven't find a precise reference.We let for short its demonstration. Now suppose that $(u_1, u_2) \in C^0([0, T[, H_r^{\mu}(\mathbb{R}_x^3))$ is a solution of (57), with $\mu \geq 2$. According to the lemma, $(w_1, w_2) = R(u_1, u_2) \in C^0([0, T[, \overline{H}^{\mu}(\mathbb{R}))$ satisfy (59) in a strong meaning. Conversely, if $(w_1, w_2) \in C^0([0, T[, \overline{H}^{\mu}(\mathbb{R}))$ satisfies (59), then $(u_1, u_2) = R^{-1}(w_1, w_2) \in C^0([0, T[, H_r^{\mu}(\mathbb{R}_x^3))$ satisfy (57). So these problems are equivalent.

The scheme, main result. – First we express the equivalent Hamiltonian formulation of (59), as in (8):

$$P_r(\varepsilon_1, \varepsilon_2): \left\{ \frac{d}{dt}\begin{pmatrix} U \\ V \end{pmatrix} = \right. \tag{62}$$

$$\left. = \begin{pmatrix} V \\ \partial_r^2 U + Q_{\varepsilon, r}(U)\, U \end{pmatrix};\ U(0) = {}^T(Ru_1^0, Ru_2^0),\ V(0) = {}^T(Rv_1^0, Rv_2^0) \right\}.$$

with $Q_{\varepsilon, r}(U) = \dfrac{1}{r^2}\begin{pmatrix} \varepsilon_1 u_2^2 & 0 \\ 0 & \varepsilon_2 u_1^2 \end{pmatrix}$. We denote in the sequel $F_{t, r}$ the evolution group linked to the problem (62).

Considering data $W^n = (U^n, V^n)$, we use the following scheme in order to solve (62), which is similar to (38):

$$\left\{ U_{t\bar{t}}^n - U_{x\bar{x}}^n = \frac{1}{2} Q_{\varepsilon, r}(U^n)(U^{n+1} + U^{n-1}) \right\} \tag{63}$$

where $U^{n-1} = U^n - \tau V^n$.

This scheme defines $W^{n+1} = (U^{n+1}, V^{n+1})$ as the solution of:

$$
(64) \quad \begin{cases} U^{n+1} = U^n + \tau V^n + \sigma^2 [T_h + T_{-h} - 2] U^n + \dfrac{\tau^2}{2} Q_{\varepsilon, r}(U^n)(U^{n+1} + U^{n-1}) \\ V^{n+1} = \dfrac{U^{n+1} - U^n}{\tau} = U_t^n . \end{cases}
$$

Remark here that the solution of this scheme is even and explicitly defined as long as $(1 - (\tau^2/2)\, Q_{\varepsilon, r}(U^n))$ is invertible. This is achieved for even data $W^n \in X^{\tau, \mu}$, $\mu \geqslant 2$, and small enough τ, using the estimation (60) of the lemma 7.1 and (42). We denote in this case $W^{n+1} = (U^{n+1}, V^{n+1}) = K_{\tau, r}(W^n)$ the defined algorithm.

For numerical application, it is enough to consider an homogeneous Dirichlet condition at $x = 0$ for W^{n+1} in order to compute the solution of (64) for $r \geqslant 0$.

The analogous of the energy conservation proposition 4.2 is

Let $(W^n)_{0 \leqslant n \leqslant M}$ *a solution of* (64), *then*

$$
\begin{aligned}
E_\tau[W^0] = E_\tau[W^n] = & - \varepsilon_1 \left(\|v_1^n\|^2 + \frac{1}{2} [\|u_{1,x}^n\|^2 + \|u_{1,x}^{n-1}\|^2] - \frac{\tau^2}{2} \|v_{1,x}^n\|^2 \right) - \\
& - \varepsilon_2 \left(\|v_2^n\|^2 + \frac{1}{2} [\|u_{2,x}^n\|^2 + \|u_{2,x}^{n-1}\|^2] - \frac{\tau^2}{2} \|v_{2,x}^n\|^2 \right) + \\
& + \frac{1}{2} \int_{\mathbf{R}} \frac{[(u_1^n u_2^{n-1})^2 + (u_1^{n-1} u_2^n)^2](r)}{r^2} \, dr .
\end{aligned}
$$

We state now our main result, which is similar to the Theorem 4.1 with an added hypothesis of regularity over the initial conditions.

THEOREM 7.1. *Let* $\sigma < 1$, *and* $W^0 \in \overline{Y}^5$. *Then* $T^{**} = T^*$. *Furthermore*

$$
\forall T < T^*, \lim_{N \to \infty} \sup_{0 \leqslant t \leqslant T} \|F_{t, r} W^0 - K_{t/N, r}^N W^0 ; X^{t/N, 2}\| = 0 .
$$

The demonstration of this result follows exactly the same line that the Theorem 4.1: we show first stability for data that belong to $\{W = (U, V)$ *even*, $\|W; X^{\tau, 2}\| < A\}$ for a stability time $T_S(A)$. Then we prove consistency for data in $\overline{Y}^5$. The demonstrations are the same as for the Lemmas 4.4 and 4.3 except that the relation (61) is used to estimate non linear terms. This allows us to prove local and global convergence of the scheme towards the solution of (59).

REFERENCES

[1] D. AREGBA-DRIOLLET, *The blow up curve for a semilinear hyperbolic system*, preprint 95008 Mathématiques Appliquées de Bordeaux (1995).

[2] D. AREGBA-DRIOLLET - B. HANOUZET, *Cauchy problem for one-dimensional semilinear hyperbolic systems: global existence, blow up*, J. Differential Equations, **125** (1996), pp. 1-26.

[3] J. M. BONY, *Existence globale à données de pour les modèles discrets de l'équation de Boltzmann*, Comm. in partial differential equations, **16** (1991).

[4] L. A. CAFFARELLI - A. FRIEDMAN, *The blow-up boundary for nonlinear wave equations*, Trans. Amer. Math. Soc., **297** (1986), pp. 223-241.

[5] R. E. CAFLISCH - S. JIN - G. RUSSO, *Uniformly accurate schemes for hyperbolic systems with relaxation. Technical report*, Università dell'Aquila (1994).

[6] F. CASTILLO-ARANGUREN - H. JUAREZ-VALENCIA - A. NICOLAS-CARRIZOSA, *Theoritical and numerical aspects of some semilinear hyperbolic problems*, Calcolo (1994), pp. 337-354.

[7] A. J. CHORIN - T. JR. HUGUES - M. F. MCCRACKEN - J. E. MARSDEN, *Product formulas and numerical algorithms*, CPAM, **31** (1978), p. 205-256.

[8] D. DRIOLLET - B. HANOUZET, *Systèmes hyperboliques semi-linéaires conservatifs 1-d*, C.R. Acad. Sc. Paris, **307**, serie I (1988), pp. 231-234.

[9] E. GABETTA - L. PARESCHI, *Approximating the Broadwell model in a strip*, Math. Models and Methods in Appl. Sci., **2** (1992), pp. 1-19.

[10] T. KATO, *Nonlinear equations of evolution in banach spaces, Proc. Sympos. Pure Math.*, **45** (2) (1986), pp. 9-23.

[11] D. J. KAUP - A. REIMAN - A. BERS, *Space-time evolution of nonlinear three-wave interaction. 1. interaction in a homogeneous medium*, Reviews of Modern Physics, **51** (1979).

[12] A. MAJDA, *Compressible fluid flow and systems of conservation laws in several space variables*, Springer Verlag, New York (1984).

[13] J.-M. MERCIER, *Sur des Systèmes d'équations des ondes semi-linéaires.* PhD thesis, université Bordeaux, 1 (1996).

[14] J.-M. MERCIER, *Global existence and long time estimation for an integro-differential system*, Preprint dell' universita di Pisa 2.275.1047, to be published in Ricerche di Matematica (1997).

[15] J. M. MERCIER, *Note over a multigrid adaptive mesh refinement technique for hyperbolic problems*, preprint SISSA 53/98/M (1998).

[16] R. NATALINI - B. RUBINO, *A discrete approximation for hyperbolic systems with quadratic interaction term*, Comm. Appl. Nonlinear Anal., **3** (1996), pp. 1-21.

[17] KUO PEN-YU - L. VASQUEZ, *Numerical solution of a nonlinear wave equation in polar coordinates*, Appl. Math. Comput., **14** (1984), pp. 313-329.

[18] W.-A. STRAUSS - L. VASQUEZ, *Numerical solution of a non-linear equation*, J. Comput. Phys., **28** (1978), pp. 271-278.

[19] L. TARTAR, *Some existence theorems for semilinear hyperbolic systems in one space variable*, Technical Report 2164, University of Wisconsin-Madison (1981).

[20] R. TEMAM, *Sur la résolution exacte et approchée d'un problème hyperbolique non-linéaire de T. Carleman*, Arch. Rational Mech. Anal., **35** (1969), pp. 351-362.

Manoscritto pervenuto in redazione il 15 marzo 1998.