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Peter V. Danchev

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Isomorphism of Modular Group Algebras of Direct Sums of Torsion-Complete Abelian p-Groups.

PETER V. DANCHEV (*)

Dedicated to the memory of my teacher Sofia Petkovska

ABSTRACT - Suppose $\mathbb{K}_p = \boldsymbol{Z}/p\boldsymbol{Z}$ is a simple field of char $\mathbb{K}_p = p$ and G is an abelian group written multiplicatively. The main result in the present paper, however, is that if G is a coproduct ($=$ direct sum) of torsion-complete p-groups such that each direct factor has cardinality not exceeding $\aleph_1$, then the $\mathbb{K}_p$-isomorphism $\mathbb{K}_p G \cong \mathbb{K}_p H$ for some group H implies $G \cong H$. This partially extends a result due to Donna Beers-Fred Richman-Elbert Walker (1983) and moreover partially settles also a question raised by Warren May (1979).

1. Introduction.

As usual, we let RG denote the group algebra of an abelian group G over a commutative ring R with identity of prime characteristic p. Besides, suppose K is a field of characteristic p and G_p is a p-torsion part of G. All other notations and terminology are standard and are in agreement with L. Fuchs [F] and G. Karpilovsky [KAR].

In 1979, W. May has asked ([M4], p. 34) whether the torsion-complete p-group G together with the K-isomorphism $KG \cong KH$ for any group H imply that H is torsion-complete, too. In the next paragraphs stated below we give a positive solution to this problem, but by cardinal restrictions on K and G, that are, $|K| = p(< \aleph_0)$ and $|G| \leqslant \aleph_1$. These restrictions on the powers of K and G can not be droped yet.

(*) Indirizzo dell'A: Department of Mathematics, University of Plovdiv, 4000 Plovdiv, Bulgaria - BG. 1991 *Mathematics Subject Classification.* Primary 20C07, Secondary 20K10, 20K21.

Moreover, D. Beers, F. Richman and E. Walker [BRW] have obtained that the $\mathbb{K}_p$-isomorphism $\mathbb{K}_p G \cong \mathbb{K}_p H$ over the finite field with p-elements $\mathbb{K}_p$ and for some group H implies that the socles $G[p]$ and $H[p]$ are isometric (i.e. are isomorphic as valuated vector spaces, where the valuation is precisely the height function). Thus if G and H (more generally, G_p and H_p) are both direct sums (i.e. restricted, bounded direct products which are called coproducts) of torsion-complete p-groups, then $\mathbb{K}_p G \cong \mathbb{K}_p H$ yields $G \cong H$ (more generally, $G_p \cong H_p$), according to the well-known result of P. Hill [H] (see also [F] and [KEEF]). Further in this article we shall generalize in some aspect the above fact by proving the same statement, but provided H is arbitrary.

2. Preliminaries.

Throughout the rest of this section (and paper) we shall denote by $V(RG)$ the group of all normalized units in RG, and its p-component by $V_p(RG) = S(RG)$. Before proving the main theorems, for the sake of completeness and for the convenience of the reader, we shall summarize (state and prove) below some needed for us facts, which are the following:

(1) If G is a direct sum of abelian p-groups with factors each of which has cardinality $\leq \aleph_1$, then G is a direct factor of $V(KG)$ with a simply presented complement, provided K is perfect [HU].

(2) As was mentioned in the introduction of the first paragraph, the socle $G[p]$ as a valuated (filtered) vector space determines up to isomorphism the direct sum G of torsion-complete abelian p-groups (cf. [H] or [F], [KEEF]).

(3) Direct summands (factors) of direct sums ($=$ coproducts) of torsion-complete primary groups are direct sums of torsion-complete groups (cf. [HI], [IRW] and [F]).

(4) For any field F, if $x \in FG$, then $x \in FT_x$ for some finite subgroup T_x of G (a well-known and elementary fact).

Now we proceed by proving some important for our good presentation statements as follows. The following two lemmas are very needed, as the first is well-known (see for example [D]) and is included here only for reference.

LEMMA 2.1. *For every ordinal α is fulfilled* $S^{p^\alpha}(RG) = S(R^{p^\alpha} G^{p^\alpha})$.

The next lemma was intensively used in [D].

LEMMA 2.2. *The subgroup G_p is balanced in $S(RG)$.*

PROOF. «*nice*». Following [F] (by P. Hill), the subgroup N of an abelian p-group A is said to be nice if $(A/N)^{p^\alpha} = (A^{p^\alpha} N)/N$ for all ordinals α. It is a routine matter to see that the last equality is equivalent to the next conditions: $\bigcap_{\tau<\alpha} (A^{p^\tau} N) = \left(\bigcap_{\tau<\alpha} A^{p^\tau}\right) N = A^{p^\alpha} N$ for each limit α. Further we shall consider this situation. By the presented above criterion and lemma, it remains only to show that $\bigcap_{\tau<\alpha} [S(R^{p^\tau} G^{p^\tau}) G_p] = S(R^{p^\alpha} G^{p^\alpha}) G_p$. Really, given $x \in \bigcap_{\tau<\alpha} [S(R^{p^\tau} G^{p^\tau}) G_p]$, whence $x = g_p(r_1 g_1 + \ldots + r_t g_t) = g_p'(r_1' g_1' + \ldots + r_t' g_t')$, where g_p, $g_p' \in G_p$; $r_i \in R^{p^\tau}$, $r_i' \in R^{p^\beta}$; $g_i \in G^{p^\tau}$, $g_i' \in G^{p^\beta}$; $1 \leqslant i \leqslant t$ and β is arbitrary such that $\tau < \beta < \alpha$. Because $\sum_1^t r_i g_i \in S(RG)$, then certainly there is $g_j \in G_p$ for some fixed $1 \leqslant j \leqslant t$, say $g_1 \in G_p$. On the other hand $g_p g_i = g_p' g_i'$. Apparently $g_i g_1^{-1} = g_i' g_1'^{-1} \in G^{p^\beta}$ for $2 \leqslant i \leqslant t$. Moreover $r_i = r_i' \in R^{p^\beta}$ for $1 \leqslant i \leqslant t$. That is why $x = g_p g_1^{-1}(r_1 + \ldots + r_t g_t g_1^{-1})$, where $g_p g_1^{-1} \in G_p$ and $r_1 + \ldots + r_t g_t g_1^{-1} \in S(R^{p^\beta} G^{p^\beta})$. Finally $x \in G_p S(R^{p^\alpha} G^{p^\alpha})$ and we are done.

«*isotype*». By virtue of Lemma 2.1,

$$G_p \cap S^{p^\alpha}(RG) = G_p \cap S(R^{p^\alpha} G^{p^\alpha}) = G_p \cap G^{p^\alpha} = (G^{p^\alpha})_p = (G_p)^{p^\alpha} = G_p^{p^\alpha}$$

for every ordinal α. Thus G_p is isotype in $S(RG)$.

As a final we conclude that G_p is balanced in $S(RG)$ after all. The proof is completed. ∎

Recall that for $B \leqslant G$, $I(RG; B)$ denotes the relative augmentation ideal of RG with respect to B. Besides define the set $V(RG; B) = 1 + I(RG; B)$.

REMARK 2.3. The above lemma is proved also by W. May when $R = K$ (is perfect) and G is p-torsion [MA] (cf. [HU] too); or more generally only when $R = K$ (actually, he has proved that G_p is balanced in $S(KG; G_p)$ [MAY]; but $S(KG) = S(KG; G_p)$ [D]). Besides, the used by us technique is different to these in [MA, MAY; HU] (see also [BRW], p. 49).

The next technical statement is valuable.

PROPOSITION 2.4. *Suppose $M \leqslant G$ and $B \leqslant G$, where B is p-torsion. Then $V(KG) = V(KM) \times V(KG; B)$ if and only if $G = M \times B$.*

PROOF. «*necessity*». Indeed, $M \cap B \subseteq V(KM) \cap V(KG; B) = 1$ and so $M \cap B = 1$. Now, for given $x \in G \subseteq V(KG)$ we write

$$x = \left(\sum_i r_i m_i\right)\left(1 + \sum_{i,j} \alpha_{ij} g_{ij}(1 - b_i)\right) = \sum_k r_k m_k + \sum_{i,j,k} r_k \alpha_{ij} m_k g_{ij}(1 - b_i),$$

where r_k, $\alpha_{ij} \in K$; $m_k \in M$, $g_{ij} \in G$, $b_i \in B$. Hence $x = mgb$ or eventually $x = m'g'$ for some (fixed) $m \in M$, $m' \in M$; $g \in G$, $g' \in G$; $b \in B$. Moreover we observe that $r_k \alpha_{ij} \neq 0$ for all k, i, j and whence that $g \in MB$ or eventually $g' \in MB$. Finally $x \in MB = M \times B$ and this finishes the proof.

«*sufficiency*». Because $G = M \times B$ we conclude that $KG = (KM)B$. Therefore for every $x \in V(KG)$ we have $x = \sum_{a \in B} x_a a$, where $x_a \in KM$. Choose $\bar{x} = \sum_{a \in B} x_a \in KM$. Apparently $x = \bar{x} + \sum_{a \in B \setminus \{1\}} x_a(a - 1)$. But B is p-torsion and obviously $x^{p^k} = \bar{x}^{p^k}$ for some natural k. Thus it is not difficult to verify that $\bar{x} \in V(KG)$ and consequently $\bar{x} \in V(KG) \cap KM = V(KM)$. Moreover select $v = 1 + \bar{x}^{-1} \cdot \sum_{a \in B \setminus \{1\}} x_a(a - 1)$. Evidently $v \in V(KG; B)$, and $x = \bar{x}v$. So $V(KG) \subseteq V(KM). V(KG; B)$. On the other hand, using [D] we conclude that $V(KM) \cap V(KG; B) = V(KM; M \cap B) = 1$ since $M \cap B = 1$ by hypothesis. As a final, $V(KG) = V(KM) \times V(KG; B)$ as desired. This gives the equality. The proposition is verified. ■

The next assertion plays a key role for the further proofs of the matters.

PROPOSITION 2.5. *Suppose $C \leqslant G$ and F is any finite field. If $FG = FH$ for some group H and $|C| \leqslant \aleph_1$, then there exists $T \leqslant H$ with $|T| \leqslant \aleph_1$ and so that $FC \subseteq FT$.*

PROOF. Take $x \in FC \subseteq FG$. Further we will differ the following cases:

1) C is finite, i.e. $|C| < \aleph_0$ and $C = \{c_1, \dots, c_m\}$. Write $x = \alpha_1 c_1 + \dots + \alpha_m c_m$ ($\alpha_i \in F$, $c_i \in C$; $1 \leqslant i \leqslant m$). Since $x \in FH$, then (4) means $x \in FT_x$ for some finite subgroup $T_x \leqslant H$. Thus $FC \subseteq FT$ for any fixed finite $T \leqslant H$; $T = \langle T_x \rangle$. If now $G \geqslant C' \supseteq C$ is an other finite group with a system of generators $\{c_1, \dots, c_l \mid l \geqslant m\} = C'$, then $FC' \subseteq FT'$ for some finite $T' = \langle T, a_i \notin T \mid 1 \leqslant i \leqslant l \rangle$, so $T' \geqslant T$; $C \subseteq C'$ leads us to $T \subseteq T'$.

2) C is countable, *i.e.* $|C| = \aleph_0$. Furthermore $C = \bigcup_{n<\omega} C_n$, where $C_n \subseteq C_{n+1}$ and all C_n are finite. Using case 1),

$$FC = F\Big(\bigcup_{n<\omega} C_n\Big) = \bigcup_{n<\omega} FC_n \subseteq \bigcup_{n<\omega} FT_n = F\Big(\bigcup_{n<\omega} T_n\Big) = FT,$$

where $T = \bigcup_{n<\omega} T_n$ and all $T_n \leqslant H$ are finite such that $T_n \subseteq T_{n+1}$. Therefore $T \leqslant H$ is a group and $|T| = \aleph_0$ by a construction. As above $C \subseteq C'$ means $T \subseteq T'$.

3) $|C| = \aleph_1$, whence $C = \bigcup_{\alpha<\omega_1=\Omega} C_\alpha$, where C_α are both countable and $C_\alpha \subseteq C_{\alpha+1}$. Consequently as in the above scheme, $FC \subseteq FT$, where $H \geqslant T = \bigcup_{\alpha<\omega_1} T_\alpha$, $T_\alpha \subseteq T_{\alpha+1}$ and $|T_\alpha| = \aleph_0$. So T is a group of $|T| = \aleph_1$, completing the proof. ■

It is not difficult to verify that the proposition is true even when $|F| \leqslant \aleph_1$.

Now we are in position to attack

PROPOSITION 2.6. *Suppose $G = \coprod_{i\in I} G_i$ is p-torsion, where $|G_i| \leqslant \aleph_1$ for all $i \in I$. Then $KG = KH$ for any group H and finite K (or more generally when $|K| \leqslant \aleph_1$) implies $H = \coprod_{i\in I} H_i$ with $|H_i| \leqslant \aleph_1$.*

PROOF. Further we will differ the following basic cases.

Case 1. $|G| \leqslant \aleph_1$. Since $|G| = \dim_K KG = \dim_K KH = |H|$, then $|H| \leqslant \aleph_1$ and $H = \coprod_{i\in I} H_i$, where $H_j = H$ for any fixed $j \in I$, and $H_i = 1$ for all other $j \neq i \in I$.

Case 2. We may presume that $|G| > \aleph_1$, whence $|G| = |I|$. Let λ be the smallest ordinal with $|G| = |\lambda|$; so put $I = \lambda$ and as a consequence $G = \coprod_{\mu<\lambda} G_\mu$. Choose arbitrary $\alpha < \lambda$ and in this direction, take $B_\alpha = \coprod_{\mu<\alpha} G_\mu$. Clearly $B_{\alpha+1} = B_\alpha \times G_\alpha$ and owing to Proposition 2.4 we derive $V(KB_{\alpha+1}) = V(KB_\alpha) \times V(KB_{\alpha+1}; G_\alpha)$. It is easily seen that $V(KG) = \coprod_{\alpha<\lambda} V(KB_{\alpha+1}; G_\alpha) = V(KH)$. Since $|G_\alpha| \leqslant \aleph_1$, then applying Proposition 2.5, we get that there is $H_\alpha \leqslant H$ with $|H_\alpha| \leqslant \aleph_1$ and $KG_\alpha \subseteq KH_\alpha$. By the same procedure $KH_\alpha \subseteq KG^{(\alpha)}$ with $G^{(\alpha)} \leqslant G$ and $|G^{(\alpha)}| \leqslant \aleph_1$.

Without loss of generality we may assume that $G^{(\alpha)} = G_\alpha$ or that the obtaining by the method described in Proposition 2.5 groups H_α have intersections equal to 1 (i.e. their products are coproducts) and so $KG_\alpha = KH_\alpha$, owing to the special direct decomposition of G. Besides, because α is arbitrary, by a standard transfinite induction on α and Proposition 2.5 (we omit the details), $KB_{\alpha+1} = KC_{\alpha+1}$ for some subgroup $C_{\alpha+1} \subseteq H$ which is a direct sum of groups of cardinalities less than or equal to $\aleph_1$. It is a simple matter to see that $H_\alpha \subseteq C_{\alpha+1}$ since $G_\alpha \subseteq B_{\alpha+1}$. On the other hand $KG_\alpha = KH_\alpha$ does imply that $I(KG_\alpha; G_\alpha) = I(KH_\alpha; H_\alpha)$ and so $I(KB_{\alpha+1}; G_\alpha) = KB_{\alpha+1}. I(KG_\alpha; G_\alpha) = KC_{\alpha+1}. I(KH_\alpha; H_\alpha) = I(KC_{\alpha+1}; H_\alpha)$, i.e. $V(KB_{\alpha+1}; G_\alpha) = V(KC_{\alpha+1}; H_\alpha)$. Therefore $V(KH) = \coprod_{\alpha<\lambda} V(KC_{\alpha+1}; H_\alpha)$. This means that H is a direct sum of groups of powers $\leq \aleph_1$ according again to Proposition 2.4, thus completing the proof. ■

3. The main result and its corollaries.

Our central results, however, are the following.

THEOREM 3.1. *Let G be a direct sum of torsion-complete abelian p-groups with the cardinality of each factor not exceeding $\aleph_1$. Then $\mathbb{K}_p H \cong \mathbb{K}_p G$ as $\mathbb{K}_p$-algebras for some group H if and only if $H \cong G$.*

PROOF. We may harmlessly assume that $\mathbb{K}_p H = \mathbb{K}_p G$. Owing to (2) along with the preceding above result due to Beers-Richman-Walker, it is sufficient to show only that H is a direct sum of torsion-complete p-groups. In fact, we now employ Proposition 2.6 and (1) to infer that $G \times V(\mathbb{K}_p G)/G \cong H \times V(\mathbb{K}_p H)/H$, since $\mathbb{K}_p$ as a finite field is perfect. On the other hand (1), Lemma 2.2 and the fact that G is separable yield $V(\mathbb{K}_p G)/G$ is separable simply presented p-group, i.e. in the other words, $V(\mathbb{K}_p G)/G$ is a direct sum of cyclics [F]. Thus H is a direct factor of a direct sum of torsion-complete p-groups, and as a final (3) guarantees that H is indeed so as claimed. The proof is fulfilled after all. ■

Begin in this paragraph with

COROLLARY 3.2. *Suppose G is an abelian group of cardinality at most $\aleph_1$ such that G is torsion-complete (semi-complete). Then $\mathbb{K}_p H \cong \mathbb{K}_p G$ as $\mathbb{K}_p$-algebras for some group H implies $H \cong G$.*

REMARK 3.3. The last corollary gives a partial affirmative answer of a problem posed by W. May [M].

REMARK 3.4. In the above assertion we examine a torsion-complete G with cardinality $\aleph_1$. As an example, this is possible when B is an unbounded basic subgroup of G ($\Leftrightarrow G$ is unbounded) of power $\leqslant \aleph_1$, say $B = \bigoplus_{n=1}^{\infty} \oplus_{\aleph_0} \boldsymbol{Z}(p^n)$ or $B = \bigoplus_{n=1}^{\infty} \oplus_{\aleph_1} \boldsymbol{Z}(p^n)$, and moreover assume that the continuum hypothesis (CH) holds. Therefore owing to ([F], p. 29, Exercise 7), $|G| = |B|^{\aleph_0} = \aleph_0^{\aleph_0} = 2^{\aleph_0} = \aleph_1^{\aleph_0} = \aleph_1$, as required. An example of a direct sum G of torsion-complete p-groups which has cardinality $\aleph_1$ is the following: $G = \bigoplus_{n=1}^{\infty} \oplus_{\aleph_1} G_n$ or $G = \bigoplus_{n=1}^{\infty} G_n$, where $|G_n| = \aleph_1$ for all naturals n. ■

REMARK 3.5. If G is a semi-complete p-group [KOL], then in [DA] is obtained that G is a direct factor of $V(RG)$, since G is a direct sum of torsion-complete p-groups. On the other hand, if G_p is a direct sum of cyclics, then $S(RG)$ contains G_p as a direct factor when R has no nilpotents, and besides, the complement is also a direct sum of cyclics [D]. In this way, if G_p is torsion-complete, then by ([F], p. 25, Theorem 68.4 of Kulikov-Papp) and Lemma 2.2 follows obviously that G_p is a direct factor of $S(RG)$, but the major complementary factor is still unknown. Of some interest is then the problem of whether the semi-complete group G_p is a direct factor of $S(RG)$? What is the structure of the complement? ■

4. Concluding discussion.

There are some questions and left-open problems which immediately arise. First, if G_p is torsion-complete (semi-complete [KOL]; direct sums of torsion-complete groups), is then $S(RG)/G_p$ a direct sum of cyclics? This is probably so (the separability holds by making use of Lemma 2.2). Thus G_p semi-complete (a direct sum of torsion-complete groups) yields that $S(RG)$ is semi-complete (a direct sum of torsion-complete groups). More generally, does $S(RG)$ is semi-complete (a direct sum of torsion-complete groups) if and only if G_p is?

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