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**More metanilpotent Fitting classes with
bounded chief factor ranks**

<http://www.numdam.org/item?id=RSMUP_1997__98__241_0>

More Metanilpotent Fitting Classes
with Bounded Chief Factor Ranks.

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SUMMARY - Metanilpotent Fitting classes are given consisting of groups with elementary abelian Fitting quotient and fairly wide choices of the nilpotency class of the nilpotent residual. Many of the minimal normal subgroups are non-central.

1. – Key sections.

The aim of this note is to establish further examples of Fitting classes of metanilpotent groups to further the transparence of this family of Fitting classes. Our main interest will be in further classes with groups that have nilpotent residual of bounded nilpotency class, and we will restrict ourselves to Fitting classes $\mathcal{F}$ in $S_p S_q$. In contrast to the examples given by Menth [7] and Traustason [8] the families given here do not have central socle but the central chief factors will still play an important role. For the description of the Fitting classes we will use a concept first introduced by Dark [1], the key section.

A key section of a Fitting class $\mathcal{F}$ consists, for given sets Π, Σ of primes, of all sections $O^\Sigma(G/O_\Pi(G))$ of groups $G \in \mathcal{F}$. In our case the only interesting key section is the one consisting of the sections $O^p(G/O_q(G))$, and we will denote it by $\mathcal{KF}$. It is comparatively easy to see that there is a one-to-one correspondence of nontrivial key sections $\mathcal{KF}$ to nonnilpotent Fitting classes $\mathcal{F}$ in $S_p S_q$. For a given key section of

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this family the corresponding Fitting class is achieved by applying the following two operations, beginning with a member of the key section: (*a*) normal products with *p*-groups, (*b*) subdirect products with *q*-groups.

The following two facts are not too difficult to establish:

(1) If G belongs to $\mathfrak{X}\mathfrak{F}$ and N is a *p*-perfect normal subgroup of G, then N belongs to $\mathfrak{X}\mathfrak{F}$,

(2) If M and N belong to $\mathfrak{X}\mathfrak{F}$ and their normal product MN does not possess nontrivial normal *q*-subgroups, then MN belongs to $\mathfrak{X}\mathfrak{F}$.

We are now able to restrict our attention to these key sections, and the construction of them will be our main objective. We will begin with an extension of a *p*-group by a cyclic group of order q, for details see the hypotheses below. We will construct a key section containing the given group (not necessarily the smallest one). The key section $\mathfrak{X}\mathfrak{F}$ of the Fitting class $\mathfrak{F}$ consists of the extensions of the nilpotent residual of G by its Fitting quotient $G/F(G)$, for each $G \in \mathfrak{F}$. So we will be able to make statements on nilpotent residuals (and so possible ranks of chief factors) and of Fitting quotients. The constructions will lead to key sections and Fitting classes of groups which are nilpotent by elementary abelian and will also have bounded chief factor rank (including the case of supersoluble groups). In finding the key section containing a given group the main problem will be to show that condition (2) given before is satisfied. One main tool in this connection will be the Theorem of Krull, Remak and Schmidt (see for example Huppert [5], I.12.3, p. 66).

2. – Construction of initial objects.

We begin with extensions X of a *p*-group by a cyclic group of order q. More precisely, we impose

HYPOTHESIS A. *G is a group with the following properties:*

(1) G' *is a nonabelian p-group and* G/G' *is cyclic of order* q,

(2) $[G, Z_2(G')] = Z(G')$,

(3) *Every non-central extension of* G' *by a cyclic group of order* q *is isomorphic to* G.

We collect consequences of Hypothesis A.

LEMMA 1. *Assume that G satisfies Hypothesis A. Then*

(a) *G' is not a direct product of proper subgroups.*

(b) *If $M/Z(G')$ is a direct factor of $G'/Z(G')$, then*

$$Z(M/Z(G')) \subseteq (M/Z(G'))'.$$

(c) *If there is a direct decomposition of $G'/Z(G')$, there is also a G-invariant one.*

PROOF. By (1) of Hypothesis A we have $[G, G'] = G'$ and there is no central G-quotient in G'/G''. Therefore $Z_2(G') \subseteq G''$ follows from (2), and this shows (b): should there be a proper direct factor of G', for example D, then also $Z_2(D) \subseteq D'$.

Consider now the set of all descriptions of G' as direct product of directly indecomposable factors. By the Theorem of Krull, Remak and Schmidt, (see Huppert [5], I.12.3, p. 66), these descriptions differ only by central automorphisms. Since $Z(G') \subseteq G''$, the number of these descriptions as (unordered) direct products is a power of p and therefore not divisible by q. This implies that there is at least one such description that is left fixed as a whole by conjugation in G. It remains to exclude that an element of order q may permute some of the direct factors. Then the orbit of the permuted direct factors consists of q members. Let z be a permuting element of order q and choose x out of one of the permuted factors, not in G'. Then $(xz)^q$ leads to a central G-chief factor in G/G', and this is impossible. So if there is a proper direct decomposition of G', there is also a proper G-invariant decomposition of G', so let $G' = U \times V$ and $G = \langle z, G' \rangle$ for some element z of order q. Then G' possesses also a non-identity automorphism of order q which is the identity on V, contrary to (3) of Hypothesis A. Thus we have shown (a), and (c) follows by analogous reasoning.

LEMMA 2. *Assume that G satisfies Hypothesis A. If T is the normal product of two groups M, N isomorphic to G, then*

$$T \cong G \quad or \quad T \cong G \times C_q \quad or \quad T \cong G \times G.$$

PROOF. It is clear that the three cases mentioned occur, and so we have to show that they are the only ones. Assume that $M \neq N$, then the Sylow q-subgroups of MN have order q^2 and they are elementary abelian. Choose one of them, say Q. Now M' is isomorphic to G'. If $C(M') \cap Q = 1$ we apply (3) and (1) of Hypothesis A to see that every nontrivial element of Q induces a fixed-point-free automorphism on M'/M'', and this is impossible. Thus $C(M') \cap Q \neq 1$. If $C(M') \cap Q \subset N$,

we have $T \cong G \times G$, in the other case we obtain $M' = N'$ and the second alternative applies.

We will have to specialize even further for our constructions; the restrictions will be in two directions, leading to two different collections of conditions. In both cases we consider a group G with nilpotent commutator subgroup G'. By G^+ we will denote the member of the ascending central series of G' with the property $G^+ = Z_s(G') \neq Z_{s+1}(G') = G'$. Clearly $G^+ \supseteq G''$.

HYPOTHESIS B1. *G satisfies Hypothesis A and is supersoluble. No two different G-chief factors of G'/G^+ are operator isomorphic.*

HYPOTHESIS B2. *G satisfies Hypothesis A and there is a group R with the following properties:*

1) R/R' is cyclic of order q and R' is an elementary abelian p-group and a direct product of two non operator isomorphic minimal normal subgroups of R;

2) There is no quotient group of G/G^+ which is isomorphic to R.

REMARK. The reader will notice that Hypotheses B1 and B2 exclude $q = 2$ and Hypothesis B2 excludes prime pairs (q, p) where $q - 1$ is the smallest natural number m such that $q | p^m - 1$. So for $q = 3$ only supersoluble cases will occur, with G'/G^+ of rank 2.

Before constructing key sections we will first have a closer look on the possibilities still given in the framework of the hypotheses just formulated. We will make use of the convention $[x, y] = [x,_1 y]$ and $[x,_m y] = [[x,_{m-1} y], y]$.

EXAMPLES. 1) Let q be an odd prime divisor of $p - 1$ and choose k such that $k^q - 1$ but not $k - 1$ is divisible by p^2. Consider

$$H = \langle a, b, c \,|\, a^q = b^p = [[b, c^i], b] = 1 \,\forall i; [b,_m c] = c^p;$$

$$a^{-1}ca = c^k;\; a^{-1}ba = b^{k^{1-n}} \rangle,$$

where for the parameter n we impose $1 < n < p$. Obviously H does not satisfy Hypothesis A if q divides $n - 1$, it satisfies Hypothesis B2 if q divides n, and it satisfies Hypothesis B1 otherwise. To show that conditions (2) and (3) of Hypothesis A are satisfied, remember $[[b,_{m-1} c], c] = c^p$ and so $\langle [b,_{m-1} c] \rangle$ is central modulo $\langle c^p \rangle = Z(\langle b, c \rangle)$.

Modification of Example 1: Choose a finite field F of characteristic

p and an element $\lambda \in F$ of order q which is not contained in a proper subfield of F. For $4 \leqslant n \leqslant p$, consider the group T of all $n \times n$-matrices $(a_{i,j})$ over F of the following form

$$a_{i,j} = 0 \quad \text{for all } (i, j) \text{ with } i > j ;$$

$$a_{1,1} = \lambda^{(n-3)s}, \quad \text{where } s \text{ is any integer};$$

$$a_{i,i} = \lambda^{(i-2)s} \quad \text{for } i > 1 ;$$

$$a_{1,j} = \alpha_j \quad \text{for } j > 1 ;$$

$$a_{i,i+j} = \frac{\beta^j}{j!} \quad \text{for } 1 < i < i+j \leqslant n;$$

with free choice for the elements α_j and β in F.

The centre $Z(T')$ of the commutator subgroup of this group consists of all unitriangular matrices with $\beta = \alpha_2 = \ldots = \alpha_{n-1} = 0$, the second centre $Z_2(T')$ is the set of all unitriangular matrices with only α_{n-2} and α_{n-1} different from zero, and since $a_{1,1} = a_{n-1,n-1}$ for all matrices in T, we have $[T, Z_2(T')] = Z(T')$. The set W of all unitriangular matrices in T with $a_{2,3} = 0$ is a maximal abelian normal subgroup of T, and T'/W and $Z(T')$ are operator isomorphic in T. Denote by x the diagonal matrix in T with $a_{3,3} = \lambda$, so $T = \langle x, T' \rangle$.

On the other hand, there is a direct product U of m copies of C_{p^2}, where p^m is the order of F, and there is an automorphism ω of U such that ω induces on U/U^p an automorphism analogous to that induced in F by multiplication by λ. Assume further that ω is of order q and construct $\langle v, U | v^{-1} uv = u^\omega$ for all $u \in U \rangle = V$.

Now there is an isomorphism σ mapping T/W onto $\langle v, U \rangle /U^p$ such that xW is mapped onto vU^p. We form the subdirect product of T and $\langle v, U \rangle$ using this isomorphism for a group $G \subset T \times \langle v, U \rangle$ with $G = \{(a, b) | a \in T, b \in \langle v, U \rangle, (aW)^\sigma = bU^p\}$. The socle of this group G is the direct product of two operator isomorphic minimal normal subgroups $(Z(T'), 1)$ and $(1, U^p)$. We choose a minimal normal subgroup D of G which is different from both of the previously mentioned ones (a «diagonal»). Now in the group $S = G/D$ we have that $[Z_2(S'), S] = Z(S')$, and by construction we have forced the equations $(S')^p = Z(S')$ and $\{(W, U^p)\}^p D/D = 1$.

We have to show that a noncentral extension of S' by a cyclic group of order q is isomorphic to S. For this we first look for automorphisms of order q of the quotient group $S'/[S', S''] \cong T'/[T', T'']$. This group is presented by the set of all upper unitriangular 3×3-matrices; it requires some checking to show that automorphisms of order q can be de-

scribed as conjugations by upper triangular matrices $(c_{i,j})$ with diagonal elements of order q and $c_{2,2} = 1$; and that these are conjugate under $S'/(S')_3$ to diagonal matrices. Assume $c_{3,3} = \lambda$; then looking back into our original group T we obtain $a_{i,i} = \lambda^{i-2}$ for $i > 1$ since $a_{2,3} = \ldots = = a_{n-1,n}$ for all matrices in T. The operator isomorphism of $Z(S')$ and $S'/(\{(W, U^p)\}/D)$ leads to the equation $a_{1,1} = a_{n-1,n-1}$, so also $c_{1,1} = = \lambda^{n-3}$ is established.

The group S just constructed is a possible example if q does not divide $n - 3$, if $|F| \neq p^{q-1}$ and Hypothesis B2 is fulfilled, for instance if furthermore q divides $n - 2$ (and so the two chief factors of S'/S'' are operator isomorphic in S) or if $|F|^3 < p^{q-1}$ (and pairs of non-operator isomorphic chief factors can be distinguished).

2) Assume that q is an odd prime divisor of $p + 1$. Then there is a noncentral extension of the free two-generator class 3 group of exponent p by a cyclic group of order q. This group satisfies Hypothesis B2 provided $q > 3$.

3) Hypothesis B2 is always satisfied whenever G satisfies Hypothesis A and G'/G^+ is a minimal normal subgroup of G/G^+, not of rank $q - 1$. In this case the sylow q-subgroup of $\mathrm{Aut}(G)$ must be cyclic. As a family of such examples, begin with a field F of order $p^m < p^{q-1}$ with element λ of multiplicative order q which is not contained in a proper subfield of F. Choose a divisor $h \neq m$ of m so that $m = hk$, and another integer t such that $tk < p - 3$. Put $r = p^h$. Consider the group of all upper triangular $(tk + 2) \times (tk + 2)$-matrices $a_{i,j}$ such that

$$a_{i,j} = 0 \quad \text{for } i > j;$$

and

$$a_{i,i} = \lambda^{nr^{i-1}},$$

$$a_{i,i+j} = a_j^{r^{i-1}}$$

for all i, j satisfying $0 < i < i + j \leqslant tk + 2$. The largest p-perfect normal subgroup of this group satisfies Hypothesis B2. (Of course, Example 2 can be considered a special case of this, with $t = 1$, $k = m = 2$).

4) Choose two groups $X, Y \in S_p S_q$ such that X satisfies Hypothesis B1 or B2 and Y satisfies Hypothesis A, and the nilpotency class of Y' should be smaller than the nilpotency class of X'. Fix minimal normal subgroups K of X and L of Y. There is a subgroup of W of index q in the direct product $X \times Y$ such that the minimal normal subgroups $(K, 1)$ and $(1, L)$ become operator isomorphic in W by some isomorphism σ. Then $W/\{(t, \sigma(t)), t \in K\} = V$ satisfies the same hypothesis as X pro-

vided that condition (3) of Hypothesis A is still fulfilled. For instance, if $X'/Z(X')$ and $Y'/Z(Y')$ are directly irreducible, the method just described can not be repeated more than $q - 1$ times. Notice that $V/V^+ \cong$ $\cong X/X^+$ since Y' has lower nilpotency class.

3. – Construction of key sections.

Having explained in some detail which sorts of groups we consider as initial objects, we are now ready to describe the key sections connected with them. They will, in each case, contain the given initial object; we shall not decide whether they are minimal in this respect.

THEOREM. *Assume that X satisfies Hypothesis B1 or B2. The family of groups T satisfying the conditions below is a key section of some Fitting class in $S_p S_q$:*

(1) *T does not possess nontrivial p-groups as epimorphic images,*

(2) *T does not possess nontrivial normal q-subgroups,*

(3) *T/T' is an elementary abelian q-group,*

(4) *$T'/Z(T')$ is a direct product of T-invariant subgroups $D/Z(T')$ where $D \cong X \times U$; $U \subset Z(T')$, and $Z(T')$ is a direct product of $Z(T)$ and the intersection $D' \cap Z(T')$.*

PROOF. We have to show that the two statements on key sections mentioned in the introduction are fulfilled. The first is easy: A direct factor $D/Z(T')$ is either antinilpotent (i.e. $[T, D]Z(T') = D$) or contained in the hypercentre, the second case does not arise in a group without nontrivial p-groups as epimorphic images. Also normal subgroups of groups without nontrivial normal q-subgroups do not possess nontrivial normal q-subgroups. So by passing to corresponding normal subgroups the conditions (1)-(4) remain true.

Now consider the normal product of two groups A and B satisfying conditions (1)-(4) of the theorem. Clearly (1) is satisfied by AB, we assume that (2) is also satisfied. Choose a Sylow q-subgroup Q of AB. Then $Q \cap B$ is a Sylow q-subgroup of B. By conjugation, the elements of $Q \cap B$ operate on all (unordered) sets of direct products of directly irreducible factors of $A'/Z(A')$. As in Lemma 1 the number of these sets is a power of p and so not divisible by q. So there must be (at least) one (unordered) set which is invariant under $Q \cap B$.

Assume now that there is an element $b \in Q \cap B$ which permutes some direct factors of maximal nilpotency class of $A'/Z(A')$, for in-

stance that $W/Z(A')$ is such a factor. So, if A' is nilpotent of class n exactly, there are elements $w_1, \ldots, w_n$ in W such that $[\ldots[w_1, w_2], \ldots, w_n] \neq 1$. We have to argue differently according to which hypothesis is satisfied.

Case 1: Hypothesis B1 is satisfied.
Put $p^i = \exp(W)$ and choose k such that $k/ \equiv k^q \equiv 1$ modulo p^i. Then

$$f(w_j) = \prod_{m=0}^{q-1} b^{-m} w_j^{k^m} b^m$$

is contained in $[b, A'] \subseteq A' \cap B'$. Since the orbit of W under y consists of q groups which form a direct product, we have that $[\ldots[f(w_1), f(w_2)], \ldots, f(w_n)] \neq 1$, since the first component is different from 1. On the other hand, $f(w_1)$ and $f(w_2)$ generate operator isomorphic chief factors with respect to b. In a direct decomposition of $B'/Z(B')$, considered componentwise, we find that $[f(w_1), f(w_2)]$ will be contained in $[B', B'']$ and so the whole commutator has to be trivial. This contradiction shows that such an element b does not exist.

Case 2: Hypothesis B2 is satisfied.
Here we have to modify Case 1 by choosing functions $f_j(w_j)$ such that the first component does not become trivial and the chief factors generated by $f_1(w_1)$ and $f_2(w_2)$ are not b-operator-isomorphic. This is always possible: we work essentially in the group ring of C_q over C_{p^i}. Since by Hypothesis B2 certain combinations of two p-modules do not occur in quotient groups of X/X^+, f_1 and f_2 can be chosen as belonging to such a combination. Again we deduce that the commutator $[f_1(w_1), f_2(w_2)]$ belongs to $[B', B'']$, so

$$[\ldots[f_1(w_1), f_2(w_2)], \ldots, f_n(w_n)] = 1$$

contrary to the fact that $x_1, \ldots, x_n$ were chosen such that

$$[\ldots[x_1, x_2], \ldots, x_n] \neq 1,$$

making the first component of the commutator formed by the elements $f_i(x_i)$ different from 1. Again we see that an element b does not exist.

In both cases we have the result that the irreducible direct factors of maximal nilpotency class of $A'/Z(A')$ can all be chosen $(Q \cap B)$-invariant. But then, since X is directly irreducible, all direct factors belonging to one factor $D/Z(A')$ are left invariant and also $C(D') \cap (Q \cap \cap A)$ is left invariant by b, for all factors $D/Z(A')$. Since $|(Q \cap$

$\cap A$): $C(D') \cap (Q \cap A)| = q$ and $(Q \cap A) \cap C(A') = 1$, we have that b centralizes $Q \cap A$, for all b in $Q \cap B$. So

$$Q \cap B \subseteq Z((Q \cap B)(Q \cap A)) = Z(Q),$$

and by symmetry in A and B, Q is abelian and also of exponent q. This shows (3).

We have seen now that $A'/Z(A')$ and $B'/Z(B')$ are direct products of Q-invariant direct factors $U_i/Z(A')$ and $V_j/Z(B')$ respectively with $U_i/Z(A') \cong V_j/Z(B') \cong X'/Z(X')$ for all i, j. They are distinguished as to whether U_i', V_j' centralize $Q \cap A$ or $Q \cap B$ or none of them. In this way, collecting the corresponding direct factors, $A'/Z(A')$ is split into a factor $A^+/Z(A')$ such that $[A^+, (B \cap Q)] \subseteq Z(A')$ and $A^*/Z(A')$ with $[A^*, (B \cap Q)]Z(A') = A^*$, and likewise for $B'/Z(B')$, with the corresponding notation. By argument on the element orders it is clear that $[A^*, (B \cap Q)] = [[A^*, (B \cap Q)], (B \cap Q)] \subseteq B'$, and so $[A^*, (B \cap Q)] = [B^*, (A \cap Q)]$.

Take x from some U_i contained in A^+ and y from some V_j contained in B^+. Then $[x, y]$ is contained in A^* and B^*, and $[[x, y], y] \in Z(B')$ as well as $[[x, y], x] \in Z(A')$. On the other hand, since the conjugation by y will change the direct factors of $A'/Z(A')$ in the same way a central automorphism would do, we have that $[U_i, y] \subseteq Z_2(A')$. We know that $[U_i, (A \cap Q)]Z(A') = U_i$ and $[Z_2(A'), (A \cap Q)] \subseteq Z(A')$. So there are no $A \cap Q$-modules of U_i/U_i' which are operator isomorphic to those of $Z_2(A')/Z(A')$. This shows that $[x, y]$ is contained in $Z(A')$ and by symmetry in $Z(B')$. In all the other cases of U_i and V_j it is obvious that $[x, y]$ is contained in $Z(B') \cap Z(A')$. We deduce that $A'B'/Z(A')Z(B')$ is a direct product of the form prescribed, and from $Z(A') \subset A''$; $Z(B') \subseteq B''$ and the direct product description of $Z(A')$, $Z(B')$ we deduce that $[A', Z(B')] = [B', Z(A')] = = 1$; $Z(A'B') = Z(A')Z(B')$. This completes the proof of (4); the theorem has been proved.

REMARK. It is obvious by the construction that the initial object is contained in the key section (and in the corresponding Fitting class). Whether the key section is minimal with respect to the initial object is doubtful and probably depends on the particular initial object. For this aspect we consider an initial object constructed as in example 1 with $q = 5$ and $p = 11$, and we remember $3^5 = 1 + 2 \cdot (11^2)$. The group $H = = \langle a, b, c \rangle$ with the relations

$$a^5 = b^{11} = [b, [b, c^r]] = 1 \quad \text{for } 1 \leqslant r \leqslant 3;$$

$$[b, {}_3 c]c^{-11} = a^{-1}cac^{-3} = a_{-1}bab^{-27} = 1;$$

is one of these possibilities. We will construct a group T belonging to the key section as would be found by the theorem, which does not obviously belong to the smallest Fitting class containing H:

$$T = \langle a, b_1, c_1, b_2, c_2 \rangle$$

with the following relations:

$$a^5 = b_1^{11} = b_2^{11} = c_1^{-11}[b_{1,3}c_1] = c_2^{-11}[b_{2,3}c_2] = 1,$$

$$[b_1,[b_1, c_1^r]] = [b_2,[b_2, c_2^r]] = 1 \quad \text{for } 1 \leqslant r \leqslant 3,$$

$$a^{-1}c_1 ac_1^{-3} = a^{-1}c_2 ac_2^{-81} = 1,$$

$$a^{-1}b_1 ab_1^{-27} = a^{-1}b_2 ab_2^{-9} = 1,$$

$$[b_1, c_2] = [b_2, c_1] = 1,$$

$$[[b_1, b_2], x] = 1 \quad \text{for all } x \in T,$$

$$[b_1, b_2] = [c_1, c_2].$$

On the other hand it can be seen rather easily that the smallest Fitting class will indeed contain the group T^* with the same relations except the last, which is now $[c_1, c_2] = 1$. More detailed analysis will be needed to work out the Fitting classes *generated* by the initial objects.

4. – Consequences.

The constructions just given allow to find Fitting classes such that

(1) the quotient groups $G/F(G)$ of all members G are elementary abelian q-groups for some q,

(2) the nilpotent residuals $G^{\mathfrak{N}}$ of all members G are nilpotent of given class (not smaller than 3) and of exponent p or p^2,

(3) p-chief factors of all members are of ranks m (depending on q) or 1.

Of course this includes many supersoluble cases; the nilpotency class of the nilpotent residual is no longer restricted to 3 as in Menth [7]. In the examples of Traustason [8] the nilpotency class mentioned in (2) and the exponent mentioned in (1) were still the same but different from 3. The examples of Menth [7] and Traustason [8] were restricted also by the fact that all minimal normal subgroups of their members

were central, and that there were no central chief factors in $G^{\pi}/Z(G^{\pi})$. In the constructions of this paper, these restrictions are no longer valid, however there is another restriction: the quotient $Z_2(G^{\pi})/Z(G^{\pi})$ is hypercentral in G. For the argument in section 3 this fact is crucial, and the description of the key sections will probably become much more complicated if this condition is weakened further. As the constructions in [4] show, key sections with nilpotent residual of nilpotency class two can lead to classes with Fitting quotient arbitrary out of the class all q-groups, and chief factor ranks $1, m, mq, mq^2, \ldots$, where m minimal such that q divides $p^m - 1$.

The examples of this note show in particular, that question 12.91 of the Kourovskaya Notebook [6], which was posed by the author, has a negative answer.

REFERENCES

[1] R. DARK, *Some examples in the theory of injectors of finite soluble groups*, Math. Z., **127**(1956), pp. 145-156.
[2] K. DOERK, T. HAWKES, *Finite Soluble Groups*, Berlin, New York (1992).
[3] T. HAWKES, *On metanilpotent Fitting classes*, J. Alg., **63** (1980), pp. 459-483.
[4] H. HEINEKEN, *Fitting classes of certain metanilpotent groups*, Glasgow Math. J., **36** (1994), pp. 185-195.
[5] B. HUPPERT, *Endliche Gruppen I*, Berlin, Heidelberg, New York (1967).
[6] *Unsolved problems of group theory, Kourovskaya Tetrad*, Russian Academy of Sciences, Siberian section, Mathematical Institute, Novosibirsk 1992 (in Russian).
[7] M. MENTH, *A family of Fitting classes of supersoluble groups*, Math. Proc. Cambridge Phil. Soc., **118** (1995), pp. 49-57.
[8] G. TRAUSTASON, *Supersoluble Fitting classes* (to be published).