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The Classification of Groups in which Every Product of Four Elements can be Reordered.

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1. Introduction.

There has been considerable interest in groups G for which, given a fixed integer $n \geq 2$, every product of n elements can be reordered, *i.e.* for all n-tuples $(x_1, x_2, \ldots, x_n)$, $x_i \in G$, there exists a non-trivial element $\sigma \in \Sigma_n$ such that

$$x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)} = x_1 x_2 \ldots x_n \, .$$

The class of such groups G is denoted by P_n and P denotes the union of the classes P_n, $n \geq 2$. Clearly every finite group belongs to P and each class P_n is closed with respect to forming subgroups and factor groups. The concept has a strong connection with PI-rings and semigroup algebras, but our interest here is solely with the groups and in particular the complete description of the class P_4.

Trivially P_2 is the class of abelian groups and in [2] P_3 was shown to be precisely those groups G for which the derived subgroup G' has order ≤ 2. Also the class P is known to coincide with the class of groups G possessing a subgroup N with $|G: N|$ and $|N'|$ both finite [3]. The situation with regard to P_4 seems to be more complicated. Graham Higman [4] considered the problem and obtained two striking results. First:

THEOREM 1. *If G is a group with $G' \cong V_4$ (the 4-group), then $G \in P_4$.*

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Secondly

THEOREM 2. *A finite group G of odd order belongs to P_4 if and only if*

(i) *G is abelian, or*

(ii) $|G'| = 3$, *or*

(iii) $|G'| = 5$ *and* $|G/Z(G)| = 25$.

Next it was shown in [1] that if a finite group G belongs to P_4, then G' is nilpotent. This was improved in [6] where *all* P_4-groups were shown to be metabelian. Then in [9] a complete description of the non-nilpotent groups in P_4 was given:

THEOREM 3. *A group G belongs to P_4 if and only if one of the following holds:*

(i) *G has an abelian subgroup of index 2;*

(ii) *G is nilpotent of class ≤ 4 and $G \in P_4$;*

(iii) $G' \cong V_4$;

(iv) $G = B\langle a, x\rangle$, *where* $B \leq Z(G)$, $|a| = 5$ *and* $a^x = a^2$.

Finally in [7] finite 2-groups of class 2 in P_4 were classified by the following:

THEOREM 4. *Let G be a finite 2-group of class 2 with* $\exp G' = 2$. *Then $G \in P_4$ if and only if*

(i) *G has an abelian subgroup of index 2, or*

(ii) $|G'| \leq 4$, *or*

(iii) $|G'| = 8$ *and $G/Z(G)$ can be generated 3 elements, or*

(iv) $|G'| = 8$, *$G/Z(G)$ can be generated by 4 elements and G is not diabelian.*

Following Philip Hall, a group is said to be *diabelian* if it is a product of 2 abelian subgroups.

These are our starting points in order to characterize all P_4-groups.

As one might intuitively expect, the order of the derived subgroup of a P_4-group turns out to be «nearly always» bounded. The exceptions are due to the fact that any group G with an abelian subgroup A of index 2 always belongs to P_4. This can be seen without difficulty by distinguishing cases according to how many of the 4 factors of a product belong to A. In particular the wreath product G of an arbitrary abelian group A and a group of order 2 always belongs to P_4 and $G' \cong A$. Thus there is no restriction (other than commutativity) on the derived subgroup of a P_4-group. Then we shall see that a P_4-group which does not have an abelian subgroup of index 2 has derived subgroup of order at most 8.

Our classification of P_4-group is given in

THEOREM 5. *A group G belongs to P_4 if and only if one of the following holds:*

(i) *G has an abelian subgroup of index* 2;

(ii) $|G'| \leqslant 3$;

(iii) $G' \cong V_4$;

(iv) *$G' \cong C_4$ and G has a subgroup B of index 2 with* $|B'| = 2$;

(v) *$G' \cong C_5$ and $G/Z(G)$ is isomorphic to the holomorph of* C_5;

(vi) *$G' \cong C_5$ and* $|G/Z(G)| = 25$;

(vii) *$G' \cong C_4 \times C_2$ and, with $C = C_G(G')$, $|G/C| = 2$, $C' = \mho_1(G')$, all subgroups of G' are normal in G and $G/Z(G)$ can be generated by 2 elements*;

(viii) *$G' \cong C_2 \times C_2 \times C_2$, G has class* 2 *and either $G/Z(G)$ can be generated by* 3 *elements or $G/Z(G)$ can be generated by* 4 *elements and G is not diabelian.*

Notation is as follows.

C_n	cyclic group of order n,
V_4	the 4-group,
Σ_n	symmetric group of degree n,
$\Omega_1(G)$	subgroup generated by elements of order p in a p-group G,
$\mho_i(G)$	subgroup generated by the p^i-th powers of the elements of a p-group G,
g^x	$x^{-1}gx$,
$[x, y]$	$x^{-1}y^{-1}xy$,
$\exp G$	exponent of G.

In § 2-4 we consider groups with derived subgroup of order 4, 8 and greater than 8, respectively, and in § 5 we deal with nilpotent groups. We recall again from [6] that

a P_4-group is always metabelian

and this will be assumed from now on without further reference. For convenience we mention some known results (see [7] and [9]) which we shall use.

1.1. *Let $G \in P_4$ and A be an abelian subgroup of G containing G'. Let $a, b \in A$, $x, y \in G$ and suppose that $[a, x], [a, y]$ and $[a, x^{-1}y]$ are all different from 1 and $[b, y] = 1$.*

(i) *If $[b, x]$ has order 2 and commutes with x, then*

$$[b, x] = [a, x], \ [a, y] \ \text{or} \ [a, x][y, a].$$

(ii) *If $[a, y]$ has order 2 and commutes with x, then*

$$[b, x] = 1, \ [a, x], [a, y] \ \text{or} \ [a, x][a, y].$$

1.2. *Let G be a finite 2-group in P_4 and A be an abelian subgroup of G containing G'. If $G = A\langle x \rangle$, then one of the following holds:*

$$(1)\ [A, x^2] = 1; \quad (2)\ G' \cong V_4; \quad (3)\ G' \cong C_4 \ \text{and} \ G' \leqslant Z(G).$$

1.3. *Let $G = A \times B$ be a finite P_4-group with A of odd order and B a 2-group. Then either A or B is abelian.*

1.4. *Let G be a finite nilpotent of class 2 group in P_4 with* $\exp G' = 4$. *Then $G' \cong C_4$.*

1.5. *Let G be a finite nilpotent of class 2 group with $G' \cong C_4$. Then the following are equivalent:*

(i) $G \notin P_4$;

(ii) *there are elements $x_1, x_2, x_3, x_4 \in G$ such that $[x_1, x_2] = [x_2, x_3] = [x_3, x_4]$ of order 4 and $[x_1, x_3] = [x_1, x_4] = [x_2, x_4] = 1$.*

1.6. *Let G be a finite nilpotent of class 2 group with $G' \cong C_4$. Then $G \in P_4$ if and only if G has a subgroup B of index 2 with $|B'| = 2$.*

Sometimes in the following sections we shall only sketch technical proofs, for the sake of brevity. All details can be found in [8].

2. Groups with cyclic derived subgroup of order 4.

If $G' \cong C_4$, then G is nilpotent of class 2 or 3. If G has class 2 and is a finite 2-group, then we know already that $G \in P_4$ if and only if G has a subgroup B of index 2 with $|B'| = 2$ (1.6). It turns out that the hypothesis that G is a finite 2-group here is unnecessary (2.2). Also, rather curiously, we find that if G has class 3, then G always belongs to P_4 (2.3). The situation is summed up in 2.4.

First a routine argument (which we omit) establishes

2.1. *For any n, an inverse limit of P_n-groups belongs to P_n.*

Now we can handle the class 2 groups.

2.2. *Let G be nilpotent of class 2 with $G' \cong C_4$. Then $G \in P_4$ if and only if G has a subgroup B of index 2 with $|B'| \leq 2$.*

PROOF. Let $G \in P_4$. We proceed in 3 steps.

(i) *Suppose that G is finite.* Then the 2-complement of G is abelian and so we may assume that G is a 2-group. Then B exists, by 1.6.

(ii) *Suppose that G is finitely generated.* Then G is residually finite and there is a subgroup $N \lhd G$ with G/N finite and $N \cap G' = 1$. By (i), there is a subgroup B of index 2 in G with $B \geq N$ and $|(B/N)'| \leq 2$, *i.e.* $B'N/N \cong B'$ has order at most 2.

(iii) *Suppose that G is arbitrary.* In this case, the argument proceeds by considering a local system of finitely generated subgroups with derived subgroups equal to G' and employing the theory of complete projection sets (see [5], volume 2, p. 168).

Conversely, suppose that G has a subgroup B of index 2 with $|B'| \leq 2$. In order to prove that $G \in P_4$, we may assume that G is finitely generated and therefore residually finite. Thus, by 2.1, it suffices to assume that G is finite. Then $G \in P_4$, by 1.6. ∎

Turning our attention to class 3 groups, we have

2.3. *Let G be nilpotent of class 3 with $G' \cong C_4$. Then $G \in P_4$.*

PROOF. Suppose, for a contradiction, that $G \notin P_4$ and let $x_1 x_2 x_3 x_4$ be a product which cannot be reordered. Let $G' = \langle a \rangle$. Thus $[x_1, x_2] = a^\alpha, [x_1, x_3] = a^\beta, [x_1, x_4] = a^\gamma, [x_2, x_3] = a^\lambda, [x_2, x_4] = a^\mu, [x_3, x_4] = a^\nu$, where $-1 \leq \alpha, \beta, \gamma, \lambda, \mu, \nu \leq 2$. Also $a^{x_i} = a^{\varepsilon_i}$, $\varepsilon_i = \pm 1$, $1 \leq i \leq 4$. Now $G/\langle a^2 \rangle \in P_3$. Let $y_1 = x_1 x_2$, $y_2 = x_3$, $y_3 = x_4$. Then there exists $\sigma (\neq 1)$ in Σ_3 such that

$$y_1 y_2 y_3 = y_{\sigma(1)} y_{\sigma(2)} y_{\sigma(3)} a^2 .$$

It follows that $\alpha \neq 2$ and so $\alpha = \pm 1$. Without loss of generality we may assume that

$$\text{(1)} \qquad \begin{cases} \alpha = 1 ; \\ \text{similarly } \lambda = \pm 1, \nu = \pm 1 . \end{cases}$$

Following the idea of the proof of 1.6, we have

$$x_1 x_2 x_3 x_4 = x_4 x_1 x_2 x_3 t = x_4 x_3 x_2 x_1 u = x_4 x_3 x_1 x_2 v$$

with $t, u, v \in G'$. Clearly $G' = \{1, t, u, v\}$. Let $x_1 x_2 x_3 x_4 = x_4 x_1 x_3 x_2 w$. Then $w = u$ or v. Distinguishing these possibilities in the light of (1), we get the required contradiction. ■

The situation when $G' \cong C_4$ can now be summarised in

2.4. *Let $G' \cong C_4$. Then $G \in P_4$ if and only if G has a subgroup B of index* 2 *with* $|B'| \leqslant 2$.

PROOF. Let $G \in P_4$. If G has class 2, the result follows by 2.2. If G has class 3, then $C = C_G(G')$ has index 2 in G. By the 3-subgroup lemma, $[C', G] = 1$ and so $C' \leqslant Z(G) \cap G' \cong C_2$. Then take $B = C$.

Conversely, let $B \leqslant G$ with $|G:B| = 2$ and $|B'| \leqslant 2$. Again 2.2 takes care of the case when G has class 2 and 2.3 gives the other case. ■

3. Groups with derived subgroup of order 8.

Since P_4-groups are metabelian, we have to consider the case when G' is isomorphic to $C_4 \times C_2$, $C_2 \times C_2 \times C_2$ or C_8. These are discussed in §§ 3.1, 3.2, 3.3 respectively. In the second case, we shall see that the P_4-groups G all have class 2. Then local arguments can be applied to the results of § 3.2 of [7] in our final classification in § 5 below. In the third case, it turns out that the only P_4-groups are those with an abelian subgroup of index 2.

3.1. *The case $G' \cong C_4 \times C_2$.*

The automorhism group of $C_4 \times C_2$ is dihedral of order 8. Thus if $G' \cong C_4 \times C_2$ and $C = C_G(G')$, then G/C is trivial, the 4-group, cyclic of order 4 or cyclic or order 2. We distinguish these cases.

3.1.1. *Let G be nilpotent of class* 2 *with G' a* 2*-group of exponent* $\geqslant 4$ *and order* $\geqslant 8$ *Then* $G \notin P_4$.

PROOF. Suppose, for a contradiction, that $G \in P_4$. We may assume that G is finitely generated, so G' is finite and G is residually finite. Thus there is $N \lhd G$ with G/N finite and $N \cap G' = 1$. Now the derived subgroup of G/N is a 2-group of exponent $\geqslant 4$ and order $\geqslant 8$ and it is the derived subgroup of the Sylow 2-subgroup P of G/N. By 1.2, P' has exponent 4 and therefore $P' \cong C_4$, by 1.4, a contradiction. ■

The next two cases are contained in

3.1.2. *Let* $G' \cong C_4 \times C_2$ *and* $C = C_G(G')$. *If* (i) $G/C \cong C_2 \times C_2$ *or* (ii) $G/C \cong C_4$, *then* $G \notin P_4$.

PROOF. (i) We have $G = CX$ where $X = \langle x, y\rangle$. We distinguish the cases $X' = G'$, $X' \cong C_4$ and $\exp X' \leqslant 2$.

(*a*) *Suppose that* $X' = G'$. Then $[x, y] = a$ (say) has order 4 and $\langle a\rangle \ntriangleleft X$. Therefore we may assume that $a^x \notin \langle a\rangle$. Routine checking shows that, replacing y by xy if necessary, we may further assume that $a^y = a^{-1}$. Let $z = xy$. It is then an exercise to show that $zxay$ cannot be reordered.

(*b*) *Suppose that* $X' \cong C_4$. *Then* $X' \lhd CX = G$. Also

$$(1) \qquad [C, X] \nleqslant X'.$$

For, $C' \leqslant Z(G)$ and if (1) were false, then $G' = C'X'$ implies $G' \leqslant Z(G)X'$, *i.e.* $C = C_G(X')$. But then $|G : C| = 2$, a contradiction. Therefore (1) is true. We may assume that there is an element $c \in C$ such that $[c, x] \notin X'$. Let $[x, y] = a$. Then $[c, x] = a^\alpha b$, where $|b| = 2$ and $G' = \langle a\rangle \times \langle b\rangle$. Without loss of generality, $\alpha = 0$ or 1.

If $[a, x] = 1$, then $a^y = a^{-1}$ and replacing y by xy if necessary, we may assume that $b^y = b$. Then we obtain $[c, y, x] = a^{2\alpha}$. Suppose that $\alpha = 0$. Then $[c, y] = a^\beta$, $-1 \leqslant \beta \leqslant 2$. For $\beta = \pm 1$, take c, a, y, x for a, b, x, y in 1.1 (i) to give $G \notin P_4$. For $\beta = 2$, let $z = xy$. Then routine checking shows that $zcyx$ cannot be reordered. The case $\beta = 0$ is covered replacing c by ca and applying the case $\beta = 2$. Therefore suppose that $\alpha = 1$. Then $[c, y] = \alpha^\beta b$, $-1 \leqslant \beta \leqslant 2$ and $\beta = 0$ or 2 is disposed of by taking c, a, y, x again for a, b, x, y in 1.1 (i). If $\beta = -1$, routine checking shows that $xzcy$ cannot be reordered (with $z = xy$ still). And if $\beta = 1$, replace c by ca again and apply the case $\beta = -1$.

Finally suppose that $[a, x] = a^2$. Replacing y by xy if necessary, we may assume that $a^y = a$. If $[c, y] \notin \langle a\rangle$, then we can argue as above with x and y interchanged. On the other hand, if $[c, y] \in \langle a\rangle$, then replacing x by xy if necessary, we may assume $b^y = b$ and another routine check shows that, with c, a, x, y for a, b, x, y in 1.1 (i), $G \notin P_4$.

(*c*) *Suppose that* $\exp X' \leqslant 2$. It is easy to see that $\exp [C, X] = 4$ and so we may assume that $[c, x] = a$ (say) has order 4 for some $c \in C$. Then $[x, yc]$ has order 4 and replacing y by yc, cases (*a*) and (*b*) apply.

(ii) Here $G/C \cong C_4$ and $G = CX$, $X = \langle x\rangle$. It is easy to see that there is an element $c \in C$ such that $[c, x] = a$ (say) has order 4. (In fact

$[C, X] = G'$.) Then $G' = \langle a \rangle \times \langle b \rangle$ with $|b| = 2$ and, replacing c by ca if necessary, we have $a^x = ab$, $b^x = a^2 b$. Taking c, b, x, x^2 for a, b, x, y in 1.1 (i) shows that $G \notin P_4$. ■

Now we begin the analysis of the final case, where $G/C \cong C_2$.

3.1.3. *Let* $G \in P_4$, $G' \cong C_4 \times C_2$, $C = C_G(G')$ *and* $|G/C| = 2$. *Then either*

(i) *G has an abelian subgroup of index 2, or*

(ii) $C' = \mho_1(G')$, *all subgroups of* G' *are normal in* G *and* $G/Z(G)$ *is 2-generator.*

PROOF. Suppose C is not abelian (otherwise (i) is true). Since $C' \leqslant Z(G)$, $C' \neq G'$. Thus $C' \cong C_4$, V_4 or C_2 and we distinguish these cases.

(*a*) *Suppose that* $C' \cong C_4$. Then there are $c_1, c_2 \in C$ such that $[c_1, c_2] = a$ (say) has order 4. Let $G = CX$, where $X = \langle x \rangle$ is cyclic. If $[c_1, x] \notin \langle a \rangle$, then with $G' = \langle a \rangle \times \langle b \rangle$ ($|b| = 2$), we have $[c_1, x] = a^{\lambda} b$, $-1 \leqslant \lambda \leqslant 2$. Since $a \in Z(G)$, $b^x = a^2 b$. Taking c_1, b, x, c_2 for a, b, x, y in 1.1 (i) (and noting that $[c_1, x^{-1}] = a^{-\lambda+2} b$) we find $G \notin P_4$, a contradiction. Thus $[c_1, x] = a^{\lambda} (-1 \leqslant \lambda \leqslant 2)$ and similarly $[c_2, x] = a^{\mu}$ $(-1 \leqslant \mu \leqslant 2)$.

Now there must exist $c_3 \in C$ such that $[c_3, x] = a^{\nu} b$ $(-1 \leqslant \nu \leqslant 2)$. Let $[c_1, c_3] = a^{\alpha}$, $[c_2, c_3] = a^{\beta} (-1 \leqslant \alpha, \beta \leqslant 2)$. If $\alpha = \pm 1$, then the above argument with c_3 for c_2 gives $G \notin P_4$. Therefore $\alpha = 0$ or 2 and similarly $\beta = 0$ or 2. if $\lambda = 1$, take c_1, $c_1^{\nu+1} c_2^{\alpha} c_3$, x, x^2 for a, b, x, y in 1.1. Since $[c_1, x^2] = a^2$ and $c_1^{\nu+1} c_2^{\alpha} c_3$ commutes with c_1 and x^2, we can apply 1.1 (ii) and obtain $[c_1^{\nu+1} c_2^{\alpha} c_3, x] \in \langle a \rangle$, a contradiction. If $\lambda = -1$, replacing c_1 by c_1^{-1} and c_2 by c_2^{-1}, the same argument applies. Finally if $\lambda = 0$ or 2, we obtain a contradiction by choosing $c_1 c_3$, c_1^2, c_2, x for a, b, x, y in 1.1 (i). Thus $C' \not\cong C_4$.

(*b*) *Suppose that* $C' \cong V_4$. Again let $G = CX$, with $X = \langle x \rangle$. Then there exists $c_1 \in C$ such that $[c_1, x] = a$ (say) has order 4. Suppose that $\langle a \rangle \ntriangleleft G$, *i.e.* $a^x \notin \langle a \rangle$, and let $G_1 = \langle c_1, x, G' \rangle$. Then $G_1' = G'$ and $C_1 = \langle c_1, x^2, G' \rangle = C_{G_1}(G_1')$. But $C_1' = \langle [c_1, x^2] \rangle \cong C_2$. Since $\langle a \rangle \ntriangleleft G_1$, we shall see in (*c*) that $G_1 \notin P_4$, a contradiction. Therefore $\langle a \rangle \triangleleft G$ and so $a^x = a^{-1}$, since $\Omega_1(G') = C' \leqslant Z(G)$.

If there is an element $c_2 \in C$ such that $[c_1, c_2] = b \notin \langle a^2 \rangle$, then $G' = \langle a \rangle \times \langle b \rangle$. Taking c_1, a, x, c_2, for a, b, x, y in 1.1 (i), we find $G \notin P_4$, a contradiction. Therefore $[c_1, C] \leqslant \langle a^2 \rangle$. Now there are elements c_2, c_3

in C such that $[c_2, c_3] = b$ (say) has order 2 and $b \neq a^2$. So $G' = \langle a \rangle \times \times \langle b \rangle$. By the above argument, $[c_2, x]$ and $[c_3, x]$ lie in $\langle a^2, b \rangle$. But $[c_1 c_2, x] = a[c_2, x] = a_1$ (say) has order 4 and $[c_1 c_2, c_3] = [c_1, c_3] b$ has order 2 and does not belong to $\langle a^2 \rangle = \langle a_1^2 \rangle$. Then the above argument, with c_1 replaced by $c_1 c_2$ and c_2 replaced by c_3, gives $G \notin P_4$, a contradiction.

(c) *Suppose that* $C' \cong C_2$. We have to establish (ii). As before let $G = CX$ with $X = \langle x \rangle$ and let $c_1 \in C_1$ such that $[c_1, x] = a$ (say) has order 4. Since C is not abelian, it is not difficult to see that, among such elements c_1, there is one with $c_1 \notin Z(C)$. Also the subgroups of order 4 in G' are normal in G. Moreover, $C' = \mho_1(G')(= \langle a^2 \rangle)$. For, if not, then $\Omega_1(G') = \mho_1(G') \times C' \leq Z(G)$ and so $a^x = a^{-1}$. Also there exists $c_2 \in C$ such that $[c_1, c_2] = b$ of order 2 (since $c_1 \notin Z(C)$) and $G' = \langle a \rangle \times \langle b \rangle$. Choosing c_1, a, x, c_2 for a, b, x, y in 1.1 (i), we find $G \notin P_4$. Thus our claim follows.

Let $G' = \langle a \rangle \times \langle b \rangle$. It is straightforward to show there is $c_2 \in C$ with $[c_2, x] = b$ and $[c_1, c_2] = a^2$. Then

(2) *all subgroups of* G' *are normal in* G.

For, if not, changing a if necessary, we may assume that $a^x = a$. But then, with $u_1 = x$, $u_2 = c_1 c_2$, $u_3 = c_1^{-1} x$, $u_4 = c_2$, routine checking shows that $u_1 u_2 u_3 u_4$ cannot be reordered. So (2) holds. It remains to show that $G/Z(G)$ can be generated by 2 elements. Let $C_0 = C_C(x)$. Then $C = = \langle c_1, c_2 \rangle C_0$ and it suffices to show that $C_0 \leq Z(G)$.

Let $d, e, f, g \in G$ and write $[d, e] = r$, $[d, f] = s$, $[d, g] = t$, $[e, f] = = u$, $[e, g] = v$, $[f, g] = w$. Suppose that $d, f, g \in C$, $e \notin C$, $s = t = 1$. Considering the 23 reorderings of *defg*, we find that at least one of the following holds:

(3) $w = 1, u = 1, \quad uv \in \langle a^2 \rangle, \quad v = a^2,$

$$r \in \langle a^2 \rangle, \quad ru = 1, \quad ruv \in \langle a^2 \rangle, \quad rv = a^2 .$$

If $[\langle c_1, c_2 \rangle, C_0] \neq 1$, then there is $c_3 \in C_0$ such that

$[c_1, c_3] = a^2, \quad [c_2, c_3] = 1; \quad$ or $\quad [c_1, c_3] = 1,$

$$[c_2, c_3] = a^2; \quad \text{or} \quad [c_1, c_3] = [c_2, c_3] = a^2 .$$

In each case, take (respectively) $d = c_2 c_3$, $e = x$, $f = c_1 c_3$, $g = c_3$; or $d = c_1 c_3$, $e = c_2 x$, $f = c_3$, $g = c_2$; or $d = c_1 c_2 c_3$, $e = x$, $f = c_2$, $g = c_3$. In each case, none of the relations (3) holds. Thus $[\langle c_1, c_2 \rangle, C_0] = 1$ and it suffices to show that C_0 is abelian. If not, then there are elements

$c_3, c_4 \in C_0$ such that $[c_3, c_4] = a^2$. Take $d = c_2 c_3$, $e = x$, $f = c_1 c_4$, $g = c_3$. Again none of the relations (3) holds. This contradiction completes the proof of 3.1.3. ■

In the opposite direction, we have

3.1.4. *Let $G' \cong C_4 \times C_2$, $C = C_G(G')$, $|G/C| = 2$, $C' = \mho_1(G')$, with all subgroups of G' normal in G and suppose that $G/Z(G)$ is 2-generator. Then $G \in P_4$.*

PROOF. Suppose, for a contradiction, that $G \notin P_4$. By hypothesis there are elements $c_1, c_2 \in C$ such that $C = \langle c_1, c_2 \rangle Z(G)$. Let $G = CX$, $X = \langle x \rangle$. Then $G' = [C, X]$ and we may assume that $[c_1, x] = a$ (say) has order 4. Thus $G' = \langle a \rangle \times \langle b \rangle$ with $|b| = 2$. Replacing c_2 if necessary, we may also assume that $[c_2, x] = b$. Note that $[c_1, c_2] = a^2$.

Now there are elements $d, e, f, g \in G$ such that *defg* cannot be reordered. *Suppose that $e \notin C$ while $d, f, g \in C$.* Without loss of generality, $e = x$. Let $d = c_1^\lambda c_2^\mu$, $f = c_1^\alpha c_2^\beta$, $g = c_1^\xi c_2^\eta$ and let $r, \ldots, w$ be the commutators as in 3.1.3 (c). Since $fg \neq gf$, $w = a^2$. But $w = a^{2(\alpha\eta + \beta\xi)}$ and so

$$(4) \qquad \alpha\eta + \beta\xi \equiv 1(2).$$

Now $r = a^\lambda b^\mu$ and since $de \neq ed$ and $defg \neq edgf$, $r \notin \langle a^2 \rangle$. Thus either $\lambda \equiv 1(2)$ or $\mu \equiv 1(2)$. Suppose that $s = 1$, *i.e.* $\alpha\mu + \beta\lambda \equiv 0(2)$. One checks that $\lambda \equiv \alpha(2)$,

$$(5) \qquad \mu \equiv \beta(2)$$

and $t = a^2$, *i.e.* $\lambda\eta + \mu\xi \equiv 1(2)$. Then, since $def \neq fed$ and $defg \neq fegd$, $ru \notin \langle a^2 \rangle$. But $ru = a^{\lambda - \alpha} b^{\mu + \beta}$ and this contradicts (5). Thus $s = a^2$. Similarly if $t = 1$, we get contradictions to $defg \neq gedf$ or $gefd$, and therefore

$$(6) \qquad t = a^2 .$$

By (4), u, v and $uv \notin \langle a^2 \rangle$ and so $G' = \langle a^2, u, v \rangle$. But $r \notin \langle a^2 \rangle$ and so ru, rv or $ruv \in \langle a^2 \rangle$. The first possibility is ruled out as above, the second contradicts (6), while $defg \neq fged$ or $gfed$ shows that $ruv \notin \langle a^2 \rangle$.

Using the facts that $C \in P_3$ and that $g^{-1} f^{-1} e^{-1} d^{-1}$ cannot be reordered, there are 7 remaining cases depending on which of d, e, f, g belong to C and they are handled in the same fashion. (See [8] for details.) ■

Summarising the results of 3.1, we have proved

3.1.5. *Let $G' \cong C_4 \times C_2$. Then $G \in P_4$ if and only if $C = C_G(G')$ has index 2 in G, $C' = \mho_1(G')$, all subgroups of G' are normal in G and $G/Z(G)$ is 2-generator.*

3.2. *The case $G' \cong C_2 \times C_2 \times C_2$.*

Suppose that G is nilpotent of class $\geqslant 3$ with $G' \cong C_2 \times C_2 \times C_2$. We show that $G \notin P_4$. If $C = C_G(G')$, then $G/C \cong C_2 \times C_2$, C_4 or C_2. We distinguish these cases.

3.2.1. *Let G be nilpotent with $G' \cong C_2 \times C_2 \times C_2$, $C = C_G(G')$ and $G/C \cong C_2 \times C_2$. Then $G \notin P_4$.*

PROOF. Let $G = CX$, where $X = \langle x, y\rangle$. Then $X' \lhd G$ and we consider 4 possibilities for X'.

(i) *Suppose that $X' = G'$.* It is routine to check that there is a basis $\{a, b, c\}$ of G' such that $a^x = ab$, $b^x = b$, $c^x = c$, $a^y = ac$, $b^y = b$, $c^y = c$ and $[x, y] = a$. But then $u_1 u_2 u_3 u_4$ cannot be reordered, where $u_1 = y^2$, $u_2 = x$, $u_3 = y$, $u_4 = x^2$. Therefore $G \notin P_4$.

(ii) *Suppose that $X' \cong C_2 \times C_2$.* We can take $X' = \langle b, c\rangle$, $[x, y] = b$, $b^x = bc$, $b^y = b$, $c^x = c$, $c^y = c$. Clearly $G' = C'[C, X]X'$ and since $C' \leqslant Z(G)$, we have $C = C_G([C, X]X')$. Thus, by hypothesis, $[C, X]X' = G'$ and so there is an element $c_1 \in C$ such that *either* $[c_1, x] \notin X'$ *or* $[c_1, y] \notin X'$. If $[c_1, x] \in X'$, then $[c_1, y] = a \notin X'$ and we can show that $G' = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ with $a^x = a$, $a^y = ac$, $1 \not\equiv [c_1, x] \not\equiv [c_1, y] \not\equiv 1 \bmod \langle c\rangle$. Then the hypotheses of 1.1 (i) are satisfied with c_1, a, y, x for a, b, x, y. Since $[a, y] = c$, it follows that $G \notin P_4$.

Suitable substitutions for x and y reduce the second possibility ($[c_1, x] \notin X'$) to the case just handled.

(iii) *Suppose that $X' \cong C_2$.* Let $[x, y] = c$. Thus $1 \neq c \in Z(G)$. Now $C'X' \leqslant Z(G)$, and so $G' = [C, X]$. Therefore we may assume that there is an element $c_1 \in C$ such that $[c_1, x] = b \notin \langle c\rangle$. Then $[x, c_1 y] = b^y c \notin \langle c\rangle$. But by cases (i) and (ii) we may assume that $\langle x, c_1 y\rangle'$ has order 2 and so $b^y c \in Z(G)$, *i.e.* $b \in Z(G)$. Therefore $|Z(G) \cap G'| = 4$. Choose $c_2 \in X$ and $\varepsilon = 0$ or 1 such that

$$[c_2, xy^{\varepsilon}] = a \notin Z(G).$$

Then

$$[xy^{\varepsilon}, c_2 y] = [xy^{\varepsilon}, y][xy^{\varepsilon}, c_2]^y = ca^y \notin Z(G).$$

Therefore $|\langle xy^{\varepsilon}, c_2 y\rangle'| \geq 4$ and $G \notin P_4$ by cases (i) and (ii).

(iv) *Finally suppose that X is abelian.* Again $G = [C, X]$ and so there is an element $c_1 \in C$ such that, without loss of generality, $[c_1, x] = c \neq 1$. Then $[x, c_1 y] = c^y \neq 1$ and $\langle x, c_1 y\rangle$ is not abelian. Thus $G \notin P_4$ by the previous cases. ■

The next case is much easier.

3.2.2. *Let G be nilpotent with $G' \cong C_2 \times C_2 \times C_2$, $C = C_G(G')$ and $G/C \cong C_4$. Then $G \notin P_4$.*

PROOF. Let $G = CX$ with $X = \langle x\rangle$. Then $G' = C'[C, X]$ and since $C' \leq Z(G)$, it follows that $G' = [C, X]$. One sees easily that G' is indecomposable as an X-module. Choose a basis $\{a, b, c\}$ of G' such that $[a, x] = b, [b, x] = c$, $[c, x] = 1$. Also choose $c_1 \in C$ such that $[c_1, x] = ab^{\lambda} c^{\mu}$, $0 \leq \lambda, \mu \leq 1$. Then $[a^{\lambda} b^{\mu} c_1, x] = a$ and replacing c_1 by $a^{\lambda} b^{\mu} c_1$ we may assume that $[c_1, x] = a$. Taking c_1, b, x, x^2 for a, b, x, y in 1.1 (i) we find $G \notin P_4$. ■

The final case is

3.2.3. *Let G be nilpotent with $G' \cong C_2 \times C_2 \times C_2$, $C = C_G(G')$ and $|G/C| = 2$. Then $G \notin P_4$.*

PROOF. There is a basis $\{a, b, c\}$ of G' such that $a^x = ab$, $b^x = b$, $c^x = c$. Also $G' = C'[C, X]$ and $C' \leq \langle b, c\rangle$ since $C' \leq Z(G)$. We may assume that *if $c_1 \in C$ and $[c_1, x] \notin \langle b, c\rangle$, then $[c_1, C] \leq \langle b\rangle$*. For, if not, then there exists $c_2 \in C$ such that $[c_1, c_2] \notin \langle b\rangle$. Then taking c_1, a, x, c_2 for a, b, x, y in 1.1 (i) we see that $G \notin P_4$. Now it is not hard to see that $[C, X] \nleq \langle a, b\rangle$ and we can find $c_1, c_3 \in C$ such that $[c_1, x] = ab^{\alpha} c^{\gamma}$, $[c_3, x] = b^{\mu} c$ and $[c_1, c_3] = 1$. Taking c_1, c_3, x, x^2 for a, b, x, y in 1.1 (i) shows that $G \notin P_4$. ■

Summarising the previous 3 results, we have

3.2.4. *Let G be nilpotent of class ≥ 3 with $G' \cong C_2 \times C_2 \times C_2$. Then $G \notin P_4$.*

3.3. *The case $G' \cong C_8$.*

Finally for groups G with $|G'| = 8$ we have

3.3.1. *Let $G' \cong C_8$ and suppose that G does not have an abelian subgroup of index 2. Then $G \notin P_4$.*

PROOF. Suppose that G has class 2. Using 2.1, we may assume that G is a finite 2-group. But then $G \notin P_4$, by 1.2.

Now suppose that class $G \geqslant 3$. Let $C = C_G(G')$. Then G/C is either the 4-group or of order 2 and we distinguish these cases.

Case (i). Suppose that $G/C \cong C_2 \times C_2$. Let $G = CX$, where $X = \langle x, y\rangle$. Thus $G' = C'[C, X]X'$ and since $C' \leqslant Z(G)$, $C' < G'$ and we have $G' = [C, X]X'$. Let $G' = \langle a\rangle$. If $X' \leqslant \langle a^2\rangle$, then there are elements $c \in C$, $z \in X\backslash C$ such that $[c, z] = a$. Choose $t \in X$ such that $G = C\langle z, t\rangle$. Then $[ct, z] = a^t a^{2i}$ (some integer i) and so $[ct, z] \notin \langle a^2\rangle$. Therefore replacing X by $\langle ct, z\rangle$, we may assume that $X' = G' = \langle a\rangle$. Thus, without loss of generality, $[x, y] = a$, $a^x = a^{-1}$, $a^y = a^3$. Let $z = xy$. Then it is routine to check that $xayz$ cannot be reordered and so $G \notin P_4$.

Case (ii). Suppose that $|G/C| = 2$. Now $G = CX$ with $X = \langle x\rangle$ and $G' = [C, X]$. So there is an element $c \in C$ such that $[c, x] = a$ (say) generates G'. If $a^x = a^{2i+1}$ ($i = 1$ or 2), then $[a, x] = a^{2i}$ and $[c, x^2] = a^{2(i+1)}$. Take c, a, x, x^2 for a, b, x, y in 1.1. If $i = 1$, then $[c, x^2] = a^4$ and 1.1 (ii) fails; while if $i = 2$, then $[a, x] = a^4$ and 1.1 (i) fails. Thus $G \notin P_4$.

Therefore we may suppose that $a^x = a^{-1}$. Since $C' \leqslant Z(G)$, $C' \leqslant \langle a^4\rangle$. By hypothesis $C' \neq 1$ and so $C' = \langle a^4\rangle$. Then there are elements $c_1, c_2 \in C$ such that

$$[c_1, c_2] = a^4, \quad \langle [c_1, x]\rangle = G' = \langle a\rangle. \tag{7}$$

For, certainly there are elements $c_1, c_2 \in C$ such that $[c_1, c_2] = a^4$. Thus suppose that $[c_1, x] \in \langle a^2\rangle$. If $[c_1 c, c_2] = 1$, then $[c, c_2] = a^4$ and we can take $c_1 = c$. On the other hand, if $[c_1 c, c_2] = a^4$, then since $[c_1 c, x] \notin \langle a^2\rangle$, we can take $c_1 c$ for c_1. Therefore we have (7) and so we may assume that $[c_1, x] = a$. Then $[c_1, x^{-1}] = a$. Take c_1, a, x, c_2 for a, b, x, y in 1.1 (ii). We find that $G \notin P_4$. ■

4. Derived subgroups of order exceeding 8.

We shall prove that if $G \in P_4$ and G does not have an abelian subgroup of index 2, then $|G'| \leqslant 8$. Finite groups are considered in 4.2

and infinite groups in 4.3. In the finite case we argue by induction on order and in anticipation of this we derive some technical results in 4.1.

4.1. *Groups with a metabelian subgroup of index* 2.

Throughout this subsection we assume that

$G \notin P_4$, *G does not have an abelian subgroup of index* 2, $N \lhd G$, $N \leqslant G'$, G/N *has an abelian subgroup* B/N *of index 2 and* $G = \langle B, h \rangle$.

By *rank* we always understand Prüfer rank.

4.1.1. G'/N *has rank* $\leqslant 2$.

PROOF. Suppose, for a contradiction, that $\operatorname{rank} G'/N \geqslant 3$. Clearly we may assume that $B' = N$ and it is not difficult to show that there are elements $w, x, y \in B$ such that

$$[w, x] \neq 1, \ [w, h] = q \notin N, \ [x, h] = r \notin \langle q, N \rangle, \ [y, h] = s \notin \langle q, r, N \rangle.$$

Then we claim that *wxhy cannot be reordered.* For, $wxhy = wxyhs^{-1}$ and modulo N every reordering of $wxhy$ is uniquely expressible as

$$wxyhq^{\alpha} r^{\beta} s^{\gamma} \tag{8}$$

with $\alpha, \beta, \gamma \in \{0, -1\}$. Let $\lambda = \alpha + \beta + \gamma$ and $a = q^{\alpha} r^{\beta} s^{\gamma}$. Thus with $\dots$ representing factors in a reordering (8) of $wxhy$, $h\dots$ has $\lambda = -3$, $.h..$ has $\lambda = -2$ and $\dots h$ had $\lambda = 0$. Also $..hw$ has $a = q^{-1}$, $..hx$ has $a = r^{-1}$ and $xwhy \neq wxhy$. Thus our claim follows, contradicting $G \in P_4$. ■

Next we consider some cases when G' has rank 2.

4.1.2. *Let* $G' = \langle a \rangle \times \langle b \rangle$ *with* $|a| = 2^m$, $|b| = 2^n$, $m \geqslant n$, *and let* $N = \Omega_1 \langle a \rangle (\lhd G)$. *Then* $m \leqslant 2$.

PROOF. Suppose, for a contradiction, that $m \geqslant 3$. Since B is not abelian, $B' = N$. Then $G' = N[B, h] = [B, h]$. Again it is not difficult to find elements $x, y \in B$ such that

$$[x, y] \neq 1, \quad [x, h] \notin \langle a^2, b \rangle, \quad [y, h] \in \langle a^2, b \rangle.$$

Thus $[x, h] = a^i b^j$ (i odd) and $[x^2, h] \equiv a^{2i} b^{2j} \bmod N$. Let $z = x^2$. Then $[y, z] = 1$ and $yxhz \equiv x^3 yha^{-2i} b^{-2j} \bmod N$. By considering the position of h in reorderings of $yxhz$ (as in 4.1.1 and again working modulo N

when convenient), we find that $yxhz$ cannot be reordered, contradicting $G \in P_4$. ■

4.1.3. *If* $|N| = 2$, *then* $G' \ncong C_4 \times C_4$.

PROOF. Suppose, for a contradiction, that $G' \cong C_4 \times C_4$ and let $G' = \langle a \rangle \times \langle b \rangle$ with $N = \langle a^2 \rangle$. Then $B' = N$ and $G' = [B, h]$. As in 4.1.2 there are elements $x, y \in B$ such that $[x, y] = a^2$, $[x, h] = ab^j$ (replacing a by a^{-1} if necessary). We may assume that $j = 0$ or 1. In the former case, we may assume (replacing y by a suitable element if necessary) that $[y, h] = b$. Thus $[y^2, h] \equiv b^2 \bmod N$. Put $z = y^2$. Then easy checking shows $xyhz$ cannot be reordered. In the second case, when $[x, h] = ab$, we may assume that $[y, h] \notin \langle a^2, b \rangle$. Thus $[y, h] = a^i b^k$ where either (i) i is odd and k is even or (ii) i is even and k is odd. If (i) holds, then the previous case applies with x and y interchanged. Thus (ii) holds. Replacing a by a^{-1} and b by $a^2 b$ gives $[y, h] = b^{\pm 1}$. Then the previous case applies with xy and x for x and y. ■

4.2. *Bounding* $|G'|$: *the finite case.*

Suppose that $G \in P_4$ and G does not have an abelian subgroup of index 2. If $|G'| > 5$, then G is nilpotent, by Theorem 3. Thus if G is finite, then $G = A \times B$ with A a 2-group and $|B|$ odd. If $B' \neq 1$, then 1.3 shows that A is abelian; and $|B'| = 3$ or 5, by Theorem 2. Thus B must be abelian and in order to prove that $|G'| \leqslant 8$, we may assume that G is a 2-group.

4.2.1 *Let G be a finite group in P_4. Then either G has an abelian subgroup of index 2 or* $|G'| \leqslant 8$.

PROOF. Suppose that G does not have an abelian subgroup of index 2. Thus we may assume that G is a 2-group (see above).

Assume, for a contradiction, that $|G'| \geqslant 16$ and let $\exp G' = 2^e$.

Choose $N \lhd G$, $|N| = 2$ and with $N \leqslant \mho_{e-1}(G')$. Then

(9) *G/N does not have an abelian subgroup of index* 2.

For, suppose that this is not the case. Then, by 4.1.1, $\operatorname{rank} G'/N \leqslant 2$ and therefore $\operatorname{rank} G' \leqslant 3$. If $\operatorname{rank} G' = 3$, then $e \geqslant 2$ and since $N \leqslant \Phi(G')$, $\operatorname{rank} G'/N = 3$. Thus $\operatorname{rank} G' \leqslant 2$ and so, by 4.1.2, $e \leqslant 2$. Since $|G'| \geqslant 16$, we would have $G' \cong C_4 \times C_4$, contradicting 4.1.3. Then (9) follows.

By induction on $|G|$, we may assume that $|G'/N| = 8$ and so $|G'| = 16$. We distinguish the cases in which G' is isomorphic to

$C_8 \times C_2$, $C_4 \times C_4$ and $G'/N \cong C_2 \times C_2 \times C_2$. (Recall that $G' \not\cong C_{16}$, otherwise $G'/N \cong C_8$, contradicting 3.3.1.)

Case (i). Suppose that $G' \cong \langle a \rangle \times \langle b \rangle$, $|a| = 8$, $|b| = 2$. Here $N = \langle a^4 \rangle$ and $G'/N \cong C_4 \times C_2$. Let $C = C_G(G'/N)$ By 3.1.3

$$(10) \qquad |G : C| = 2, \qquad (C/N)' = C'N/N = \langle a^2 \rangle / N$$

$$\text{and} \quad N \leqslant X \leqslant G' \quad \text{implies} \quad X \lhd G.$$

Therefore $C' = \langle a^2 \rangle$. Let $G = \langle C, h \rangle$. Thus $G' = C'[C, h] = [C, h]$ since $a^2 \in \Phi(G')$. Also $[a, C] \leqslant N$ and therefore $[a^2, C] = 1$. Then we easily find elements $x, y \in C$ such that $[x, y] = a^4$ and $|[x, h]| = 8$. Now $[b, h] \in N$ and $h \notin C$. Therefore $[a, h] \notin N$ and so

$$(11) \qquad [a, h] = a^{\pm 2}.$$

Let $H = \langle x, y, h, G' \rangle$, $K = \langle x, y, h^2, G' \rangle$. Thein either $H' = G'$ or $H' \cong C_8$. Also $N \leqslant K'$ and $K \leqslant C$. Thus $|H : K| = 2$ and $K = C \cap H$. Let $C^* = C_H(H'/N)$. Then $K \leqslant C^*$ and so $C^* = K$, since $h \notin C^*$ (by (11)). But K is not abelian and therefore H does not have an abelian subgroup of index 2. However, by (10) and (11), $[C, h^2] \leqslant N$ and thus K/N is abelian. Then H/N has an abelian subgroup of index 2, contradicting 4.1.2.

Case (ii). Now suppose that $G' = \langle a \rangle \times \langle b \rangle$, $|a| = |b| = 4$. We can take $N = \langle a^2 \rangle$. As in *(i)*, $C = C_G(G'/N)$ has index 2 in G and $(C/N)' = \langle b^2 \rangle N/N$, *i.e.* $C' \leqslant \langle a^2, b^2 \rangle$. All subgroups of G'/N are normal in G/N and so $\langle a \rangle \lhd G$. By assumption, $C' \neq 1$. Also $|C'| \neq 2$ (by 4.1.3). Thus $C' = \langle a^2, b^2 \rangle = \Phi(G')$. Then with $G = \langle C, h \rangle$, we have $G' = [C, h]$. Since $\langle a \rangle \lhd G, [b, h] \notin N$. Also $\langle b, N \rangle \lhd G$ and so $\langle b^2 \rangle \lhd G$, *i.e.* $C' \leqslant Z(G)$. By 4.1.3, $G/\langle b^2 \rangle$ cannot have an abelian subgroup of index 2 and thus, by 3.1.3, $\langle b \rangle \lhd G$. Similarly $\langle ab \rangle \lhd G$. Therefore, by conjugation, h inverts a and b and thus every element outside C does the same. Hence $C = C_G(G')$.

In a routine way we find elements $x, y \in C$ such that (replacing a and b by suitable elements if necessary) $[x, y] \neq 1$, $[x, h] = a$, $[y, h] = b$. Let $H = \langle x, y, h, G' \rangle$, $K = \langle x, y, h^2, G' \rangle$. So $H' = G'$ and clearly $|H/K| = 2$. Since $[C, h^2] = 1$, $K/\langle [x, y] \rangle$ is abelian and hence $H/\langle [x, y] \rangle$ has an abelian subgroup of index 2. If H has an abelian subgroup A of index 2, then $H' \leqslant A \leqslant C_H(G') = K$ and so $A = K$, contradicting $K' \neq 1$. Thus H does not have an abelian subgroup of index 2, contradicting 4.1.3.

For the remaining case, we need the following two results.

4.2.2. *Let* $G = \langle x_1, x_2, x_3 \rangle$ *be nilpotent of class 2 with* G' *of exponent 2. Suppose that there is an element* $\sigma \in \Sigma_3, \sigma \neq 1$, *such that* $x_1 x_2 x_3 = x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)}$. *Then* $|G'| \leqslant 4$.

The proof can safely be left as an easy exercise.

4.2.3. *Let* $G \in P_4$ *and* $N \leqslant Z(G)$ *with* G/N *nilpotent of class 2. Let* $x_1, x_2, x_3 \in G$ *with the property that the only element* $\sigma \in \Sigma_3$ *such that* $x_1 x_2 x_3 \equiv x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} \bmod N$ *is* $\sigma = 1$. *Then for all* $a \in G'$, $[a, x_1] = 1$ *if and only if* $[a, x_3] = 1$.

Again the proof is an exercise—assume $[a, x_1] = 1$ and consider a reordering of $x_1 x_2 x_3 a$.

PROOF OF 4.2.1 CONTINUED. *Case (iii). Finally suppose that* $G'/N \cong C_2 \times C_2 \times C_2$. Let bars denote subgroups and elements of G modulo N. By 3.2.4, $\overline{G}$ has class 2 and by Theorem 4 *either* (a) $\overline{G}/Z(\overline{G})$ is 3-generator *or* (b) $\overline{G}/Z(\overline{G})$ is 4-generator and $\overline{G}$ is not diabelian.

Consider case (a) where $\overline{G} = \langle Z(\overline{G}), \overline{x}_1, \overline{x}_2, \overline{x}_3 \rangle$. Then $\langle \overline{x}_1, \overline{x}_2, \overline{x}_3 \rangle'$ is elementary of rank 3. Let $a \in G'$. Since $|G \colon C_G\langle a, N \rangle| \leqslant 2$, we may assume that $[a, x_1] = 1$. By 4.2.2, neither $\overline{x}_1 \overline{x}_2 \overline{x}_3$ nor $\overline{x}_2 \overline{x}_1 \overline{x}_3$ can be reordered. Therefore, by 4.2.3, $[a, x_3] = 1$ and hence $[a, x_2] = 1$. Let $\overline{c} \in Z(\overline{C})$ Replacing $\overline{x}_2$ in the above argument by $\overline{cx}_2$, we obtain similarly $[a, cx_2] = 1$, *i.e.* $[a, c] = 1$. Thus $a \in Z(G)$ and therefore $G' \leqslant Z(G)$. But this contradicts Theorem 4.

Now suppose that case (b) holds. By 3.2.6 of [7], $\overline{G} = \langle \overline{A}, \overline{x}, \overline{y} \rangle$ with $\overline{A} \lhd \overline{G}$, $\overline{A}$ abelian; and $\overline{G}$ is not diabelian. By choosing $\overline{A}$ as large as possible, we may assume that $N \leqslant A$ and so $G = \langle A, x, y \rangle$. Again *we claim that class* $G = 2$. For, let $a \in G'$. As before, we may assume that $[a, x] = 1$. Since $\overline{G}/\overline{A}$ is elementary, it follows from (9) that $\overline{G}/\overline{A} \cong C_2 \times \times C_2$ and $[\overline{A}, \overline{xy}] \neq 1$. Thus we can find $c \in A$ such that $[\overline{c}, \overline{xy}] \neq 1$. Let $z = = cx$. Since $\overline{G}$ is not diabelian, it follows easily that $\overline{xyz}$ cannot be reordered. Therefore, by 4.2.3, $[a, z] = 1$ and so $[a, c] = 1$. Since $\overline{A}$ is generated by such elements $\overline{c}$, it follows that $[a, A] = 1$. Also $[\overline{A}, \overline{x}] \neq 1 \neq [\overline{A}, \overline{yx}]$ and therefore $\overline{A} \neq C_{\overline{A}}(\overline{x}) \cup C_{\overline{A}}(\overline{yx})$. Thus there is an element $d \in A$ such that $[\overline{d}, \overline{x}] \neq 1 \neq [\overline{d}, \overline{yx}]$. As before, it follows easily that $\overline{y}\overline{x}\overline{d}$ cannot be reordered. (If $\overline{y}\overline{x}\overline{d} = \overline{d}\overline{x}\overline{y}$, then $[\overline{d}\overline{x}, \overline{y}\overline{x}] = 1$ and $\overline{G} = \overline{A}\langle \overline{d}\overline{x}, \overline{y}\overline{x} \rangle$ would be diabelian.) Since $[a, d] = 1$, 4.2.3 gives $[a, y] = 1$. Thus $a \in Z(G)$ and so $G' \leqslant Z(G)$, as claimed. As in case (a), this contradicts Theorem 4. ■

4.3. *Bounding* $|G'|$: *the general case.*

First, a straightforward inverse limit argument (see, for example [5], vol. 2, p. 167) gives

4.3.1. *Suppose that every finitely generated subgroup of G is abelian or has an abelian subgroup of index* 2. *Their either G is abelian or G has an abelian subgroup of index* 2.

Now we can establish

4.3.2. *Let $G \in P_4$. Then either G has an abelian subgroup of index 2 or* $|G'| \leqslant 8$.

PROOF. As we saw at the beginning of 4.2, we may assume that G is nilpotent.

(i) *Suppose that G is finitely generated.* Then G' is finitely generated and has finite exponent ([3], 2.2). Thus G' is finite. Also G is residually finite and so there is $N \lhd G$ with G/N finite and $G' \cap N = 1$. If G/N has an abelian subgroup A/N of index 2, then A is abelian and has index 2 in G. On the other hand, if G/N does *not* have an abelian subgroup of index 2, then $|G'| = |(G/N)'| \leqslant 8$, by 4.2.1.

(ii) *Now suppose that G is arbitrary.* Suppose that G is not abelian and does not have an abelian subgroup of index 2. By 4.3.1, there is a finitely generated subgroup X of G with the same properties. If $|G'| > 8$, then, by enlarging X if necessary, we may assume that $|X'| > 8$. But this would contradict (i). Thus $|G'| \leqslant 8$. ∎

5. Nilpotent P_4-groups.

The classification of the nilpotent P_4-groups is as follows.

5.1. *Let G be a nilpotent group. Then $G \in P_4$ if and only if one of the following holds:*

(i) *G has an abelian subgroup of index* 2;

(ii) $|G'| \leqslant 3$;

(iii) $G' \cong V_4$;

(iv) $G' \cong C_4$ *and G has a subgroup B of index 2 with* $|B'| = 2$;

(v) $G' \cong C_5$ *and* $|G/Z(G)| = 25$;

(vi) $G' \cong C_4 \times C_2$ *and, with* $C = C_G(G')$, $|G/C| = 2$, $C' = \mho_1(G')$, *all subgroups of* G' *are normal in* G *and* $G/Z(G)$ *is* 2*-generator*;

(vii) $G' \cong C_2 \times C_2 \times C_2$, G *has class* 2 *and either* $G/Z(G)$ *is* 3*-generator or* $G/Z(G)$ *is* 4*-generator and* G *is not diabelian.*

PROOF. Let $G \in P_4$ and suppose that (i) does not hold. By 4.3.2, $|G'| \leqslant 8$. Routine arguments allow us to assume that G is finitely generated. Then, as in 4.3.2 (i), there is a finite quotient G/N with $(G/N)' \cong G'$. Therefore, by 1.3 and Theorem 2, either $|G'| \leqslant 5$ or $|G'| = 8$. If $|G'| \leqslant 4$, then G satisfies (ii), (iii) or (iv) (by 2.4). Suppose that $|G'| = 8$. By 3.3.1, $G' \not\cong C_8$; and if $G' \cong C_4 \times C_2$, then G satisfies (vi), by 3.1.3. Thus we may assume that $G' \cong C_2 \times C_2 \times C_2$. By 3.2.4, class $G = 2$ and so $G/Z(G)$ is a finite elementary abelian 2-group. Let $L = Z(G) \cap N$. Then G/L is finite and $Z(G/L) = Z(G)/L$. By Theorem 4, G/L modulo its centre has rank $\leqslant 4$ and hence rank $G/Z(G) \leqslant 4$. Also if rank $G/Z(G) = 4$ and G is diabelian, then G/L has structure contradicting Theorem 4. Thus G satisfies (vii). Finally, if $|G'| = 5$, then, by Theorem 2, $G/Z(G) \cap N$ has order 25 modulo its centre, namely $Z(G)/Z(G) \cap N$. Thus $|G/Z(G)| = 25$ and (v) holds.

Conversely, if G satisfies (i), then $G \in P_4$, by Theorem 3. If $|G'| = 2$, then $G \in P_3$, by [2]. Suppose that $|G'| = 3$. To show that $G \in P_4$, we may assume that G is finitely generated and, by 2.1, even finite and hence a 3-group. Then $G \in P_4$, by Theorem 2. If (iii) holds, then $G \in P_4$, by Theorem 1. If G satisfies (iv), then $G \in P_4$, by 2.4.

Suppose that (v) holds. To show that $G \in P_4$, again we may assume that G is finite and hence a 5-group and the result follows from Theorem 2. If (vi) holds, 3.1.4 shows that $G \in P_4$. Finally, suppose that G satisfies (vii). We may assume that G is finitely generated and so $Z(G)$ is finitely generated. If B is a complement in $Z(G)$ of the 2-component of $Z(G)$, then it suffices to show that $G/B \in P_4$, since G embeds in $G/B \times G/G'$. But G/B is a finite 2-group and satisfies (vii) and so $G/B \in P_4$, by Theorem 4. ■

Taking Theorem 3 and 5.1 together, we have a complete description of P_4-groups and their structure is as described in Theorem 5 of § 1.

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