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Existence Results for Infinite Dimensional Differential Equations without Compactness.

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ABSTRACT. - Let E be a Banach space and $f: E \to E$ be limit of a sequence of bounded Lipschitz functions, uniformly on compact sets. We show that there exists a continuous extension $F: \mathbb{R} \times E \to \mathbb{R} \times E$ of f such that the Cauchy problem $X' = F(X)$ admits solutions, in the future, for all initial conditions in $[0, +\infty) \times E$. We also prove an existence result valid for the map $f(x) = x/\sqrt{\|x\|}$ for $x \neq 0$, $f(0) = 0$, hence admitting a non-compact set of solutions.

1. Introduction.

Let E be a Banach space and $f: E \to E$ a function. A result of Godunov [6] states that E is finite dimensional if and only if the Cauchy problem

$$x' = f(x), \qquad x(0) = x_0$$

admits solutions for all $x_0 \in E$ and all continuous f (a detailed version of Godunov's original proof is presented in [11]; different proofs are given in [12], [5]; see also [1], [7]). The earliest example of nonexistence, due to Dieudonné [4], is constructed in the space c_0 of all real sequences converging to zero: having set $f = (f_n)_n$ to be $(\sqrt{|x_n|} + 1/n)_n$, the Cau-

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chy problem $x' = f(x)$, $x(0) = 0$ does not admit solutions in $\boldsymbol{c}_0$. However, it admits solutions for every initial point, if considered in the (1-dimension) larger space $\boldsymbol{c}$ of all converging sequences. According to this example, we consider the following problem:

Let E be a Banach space and $f: E \to E$ a continuous function; does there exist a continuous extension F of f to the Banach space $\mathbb{R} \times E$ such that the Cauchy problem

$$X' = F(X), \qquad X(0) = X_0$$

($X = (\lambda, x)$) admits solutions for all $X_0 \in \mathbb{R} \times E$?

A partial positive answer is given in Section 2.

We consider also a second problem, originated by the following example, due to A. Cellina. Let $f(x) = x/\sqrt{\|x\|}$ for $x \neq 0$, $f(0) = 0$. The Cauchy problem $x' = f(x)$, $x(0) = 0$ has a nonempty set of solutions, which is not compact in the space $C(I; E)$ for any interval I, provided E is infinite dimensional. On the other hand, all the existence criteria known by the authors (see [2], [3], [8], [10], [13], [14] and [15]) use a combination of compactness and uniqueness assumptions on f (α-Lipschitzeanity and α-dissipativity), which as a by product provide a compact set of solutions. According to this remark, Cellina stated the problem of finding a condition for existence without obtaining a compact set of solutions.

Section 3 contains a result, valid in a Hilbert space, with the above considered features.

Both the existence results obtained here, unlikely from the classical techniques, are based on conditions providing a lack of uniqueness for an auxiliary scalar differential equation. We prove existence by assuming it everywhere except in a region, which, due to the assumption on the auxiliary problem, can be reached by a solution in a finite backward time.

2. An extension ensuring existence.

Let $f: E \to E$ be continuous, bounded and such that

(A) there exists a sequence (f_n) of bounded Lipschitzean functions such that f_n converges to f uniformly on the compact subsets of E.

We remark that the right hand side of Dieudonné's counterexample satisfies (A) (actually, it is the uniform limit of a sequence of Lipschitzean functions). We have

PROPOSITION 2.1. *Let f be continuous, bounded and satisfying* (A). *Then there exists a continuous map F from $\mathbb{R} \times E$ into itself such that $F(0, x) = f(x)$ for all $x \in E$ and, for any $X_0 = (\lambda_0, x_0) \in [0, +\infty) \times E$, the Cauchy problem $X' = F(X)$, $X(0) = X_0$ admits solutions on* $[0, +\infty)$.

PROOF. Let L_n be the Lipschitz constant of f_n (assume $(L_n)_n$ increasing) and let $0 < \lambda_n \leq 1/n$ be decreasing, $\lambda_1 = 1$, and such that

$$\sum_{n \geq 1} L_{n+1}(\sqrt{\lambda_n} - \sqrt{\lambda_{n+1}}) < +\infty .$$

Set, for $\lambda \in [\lambda_{n+1}, \lambda_n)$, $F_0(\lambda, x) = (\lambda - \lambda_{n+1})(\lambda_n - \lambda_{n+1})^{-1} f_n(x) + (\lambda_n - \lambda)(\lambda_n - \lambda_{n+1})^{-1} f_{n+1}(x)$; for $\lambda \geq 1$ set $F_0(\lambda, x) = f_1(x)$ and for $\lambda \leq 0$ set $F_0(\lambda, x) \equiv f(x)$. We claim that the map F_0 is a continuous extension of f to $\mathbb{R} \times E$ and there exists a function $L(\lambda)$ such that, for all $\lambda > 0$ and $x, y \in E$,

$$\|F_0(\lambda, x) - F_0(\lambda, y)\| \leq L(\lambda)\|x - y\| \tag{2.1}$$

and

$$\int_0^1 L(\lambda^2)\, d\lambda < +\infty . \tag{2.2}$$

Indeed, to check the continuity on $\{0\} \times E$, let $\mu_k \to 0$, $x_k \to x$. Then, for a suitable sequence of integers (n_k), there holds $\mu_k \in [\lambda_{n_k+1}, \lambda_{n_k})$ and

$$\begin{aligned}
&\|F_0(\mu_k, x_k) - F_0(0, x)\| \leq \\
&\quad \leq \|F_0(\mu_k, x_k) - F_0(0, x_k)\| + \|F_0(0, x_k) - F_0(0, x)\| = \\
&\quad = \|(\mu_k - \lambda_{n_k+1})(\lambda_{n_k} - \lambda_{n_k+1})^{-1}[f_{n_k}(x_k) - f(x_k)] + \\
&\quad + (\lambda_{n_k} - \mu_k)(\lambda_{n_k} - \lambda_{n_k+1})^{-1}[f_{n_k+1}(x_k) - f(x_k)]\| + \|f(x_k) - f(x)\| .
\end{aligned}$$

Both terms of the last expression tend to zero as k tends to ∞: the first one because the set $\{x_k, x\}$ is compact, while the second one by continuity. The properties (2.1) and (2.2) hold by construction, with $L(\lambda) = L_{n+1}$ for $\lambda \in [\lambda_{n+1}, \lambda_n)$.

Now define $F\colon \mathbb{R} \times E \to \mathbb{R} \times E$ by setting

$$F(\lambda, x) = (2\sqrt{|\lambda|}, F_0(\lambda, x)) . \tag{2.3}$$

We claim that the Cauchy problem $X' = F(X)$, $X(0) = X_0 := (\lambda_0, x_0) \in [0, +\infty) \times E$ admits solutions on $[0, +\infty)$. In fact, for

$\lambda_o \geqslant 0$ this system splits into $\lambda'(t) = 2\sqrt{\lambda(t)}$, $\lambda(0) = \lambda_0$ and $x'(t) = F_0(\lambda(t), x(t)), x(0) = x_0$. If the solution $\lambda(t) = (\lambda_0 + t)^2$ to the first equation is chosen, the Picard operator of the second one is Lipschitzean, by (2.1), (2.2); hence there exists a solution on $[0, T)$ for some $T > 0$. Since F_o is bounded, $T = +\infty$. ■

The approximation property (A) plays a central role in the above result. The question whether every continuous map $f: E \to E$ satisfies it (in every Banach space or in particular types of spaces) seems to be open. However, it is not hard to prove that the set of bounded continuous functions from E into itself which can be approximated uniformly by Lipschitzean functions is nowhere dense in the space of all bounded continuous functions. (In fact, choose a sequence $\{e_k\}$ in E with $\|e_k - e_h\| > 2$ $(k \neq h)$ and consider a sequence of continuous functions $\varphi_m: E \to [0, 1/m]$ satisfying $\varphi_m(e_k) = 1/m$, $k = 1, 2, \dots$ and $\varphi_m(e) = 0$ whenever $e \notin \bigcup_{k=1}^{\infty} \{d \in E \mid \|d - e_k\| \leqslant 1/k\}$, $m = 5, 6, \dots$. Finally, fix $e_0 \in E$ with $\|e_0\| = 1$. Given an arbitrary function $f: E \to E$, define $V_m = \{g: E \to E \mid \sup_{e \in E} \|g(e) - f(e) - e_0 \varphi_m(e)\| < 1/m^2\}$, $W_m = \{g: E \to E \mid \sup_{e \in E} \|g(e) - f(e) + e_o \varphi_m(e)\| < 1/m^2\}$, $m = 5, 6, \dots$. Then, foreach m separately, either V_m or W_m consists entirely of functions that cannot be uniformly approximated by Lipschitzean maps. Indeed, let $g \in V_m$ and ψ_1 be a function with Lipschitz constant L_1 such that $\sup_{e \in E} \|\psi_1(e) - g(e)\| \leqslant 1/m^2$, and for each $k \geqslant 1$ let $d_k \in E$ with $\|d_k - e_k\| = 1/k$. Since $\varphi_m(d_k) = 0$, $\varphi_m(e_k) = 1/m$, there holds

$$\begin{aligned}\|f(d_k) - f(e_k) - e_0/m\| &\leqslant \|f(d_k) + e_0 \varphi_m(d_k) - g(d_k)\| + \\ &+ \|g(d_k) - \psi_1(d_k)\| + \|\psi_1(d_k) - \psi_1(e_k)\| + \\ &+ \|\psi_1(e_k) - g(e_k)\| + \|g(e_k) - f(e_k) - e_0 \varphi_m(e_k)\| \leqslant \\ &\leqslant \frac{1}{m^2} + \frac{1}{m^2} + \frac{L_1}{k} + \frac{1}{m^2} + \frac{1}{m^2}.\end{aligned}$$

If also some $g \in W_m$ can be approximated within a distance $1/m^2$ by a function ψ_2 with Lipschitz constant L_2, an analogous argument shows that $\|f(d_k) - f(e_k) + e_0/m\| \leqslant 4/m^2 + L_2/k$. Applying the triangle inequality again, it follows that $2/m = \|2e_0/m\| \leqslant 8/m^2 + (L_1 + L_2)/k$. But this is impossible if k is large. On the other hand, an arbitrary continuous map $f: E \to E$ can be uniformly approximated by a sequence of *locally* Lipschitzean maps (see [9]). This provides an existence result, for the extended problem, for a dense set of initial conditions.

PROPOSITION 2.2. *Let $f\colon E \to E$ be continuous and bounded. Then there exist a set D dense in E and a continuous map F from $\mathbb{R} \times E$ into itself such that $F(0, x) = f(x)$ for all $x \in E$ and, for any $X_0 = (0, x_0) \in \mathbb{R} \times D$, the Cauchy problem $X' = F(X)$, $X(0) = X_0$ admits solutions on $[0, +\infty)$.*

PROOF. Let (f_n) be a sequence of bounded, locally Lipschitzean maps uniformly converging to f. Define $F_0\colon \mathbb{R} \times E \to E$ as done in the proof of the previous result and fix $x_0 \in E$, $\varepsilon > 0$. Assume F_0 is bounded by $M > 0$. Fix $0 < \tau < \varepsilon/M$ and consider the Cauchy problem $X' = F(X)$, $X(\tau) = (\tau^2, x_0)$, where F is given by (2.3). This system admits a unique solution $X := (\lambda, x) = (t^2, x(t))$, by the local Lipschitzeanity, which is prolongable down to $t = 0$. Clearly, $|x_0 - x(0)| \leqslant M\tau < \varepsilon$. Hence in $B(x_0, \varepsilon)$ falls a point $x^* = x(0)$ such that the problem $X' = F(X)$, $X(0) = (0, x^*)$ admits a solution. ■

3. Existence without compactness of the solution set.

In what follows, X is a Hilbert space with unit sphere $S = \{x \in X \mid \|x\| = 1\}$. The Bouligand tangent cone to a set Γ at $x \in \Gamma$, i.e. the set $\{v \in E \mid \liminf_{h \to 0^+} d(x + hv, \Gamma)/h = 0\}$, is denoted by $T_\Gamma(x)$.

PROPOSITION 3.1. *Let $f\colon X \to X$ continuous, bounded on the unit ball and such that*

1) *there exists $\Omega \subseteq S$, closed, such that for every $\xi \in \Gamma := (0, +\infty)\Omega$ the Cauchy problem*

$$x' = f(x), \qquad x(0) = \xi$$

has local existence;

2) *for every $x \in \Gamma$, $-f(x) \in T_\Gamma(x)$;*

3) *there exists $k\colon (0, +\infty) \to (0, +\infty)$, continuous, such that for every $x \in \Gamma$*

$$(f(x), x) \geqslant \frac{\|x\|^2}{k(\|x\|)};$$

4) $$\frac{1}{2}\int_{0^+}^{1} \frac{k(\sqrt{u})}{u}\, du = T < +\infty .$$

Then, for every $\xi \in \Omega$ *there exists a solution* x_ξ *of*

$$x' = f(x), \qquad x(0) = 0$$

on some interval $[0, \tau]$, $0 < \tau \leqslant T$ *such that* $x_\xi(\tau) = \xi$.

PROOF. Let $\xi \in \Omega$ and let x_ξ be a solution of $x' = f(x)$, $x(T) = \xi$ defined on some interval $I = (\sigma, T]$, $0 \leqslant \sigma < T$, and such that $x(t) \in \Gamma$ for all $\sigma \leqslant t \leqslant T$. Such a solution exists by 1), 2), see § VI.2 in [10]. We may assume that x_ξ is nonextendable to the left.

From 3), 4) we obtain that r, the nonextendable left maximal solution to the Cauchy problem $u' = 2u/k(\sqrt{u})$, $u(T) = 1$ is defined at least on $J = (0, T] \supset I$ and

$$\|x_\xi(t)\|^2 \leqslant r(t) \qquad \text{for all } t \in I.$$

In virtue of 4), $\lim_{t \to 0^+} r(t) = 0$. If $\sigma = 0$, then $\lim_{t \to 0^+} x_\xi(t) = 0$, hence x_ξ can be extended as a solution down to $x_\xi(0) = 0$. If $\sigma > 0$, then, being $x_\xi(\cdot)$ bounded on I and σ finite, $\lim_{t \to \sigma^+} x_\xi(t) = \bar{x}$ exists. By 1), 2) $\bar{x}$ must be zero, otherwise the maximality of I is contradicted. A straightforward reparametrization of $x_\xi(\cdot)$ then concludes the proof. ■

REMARK. The map $f(x) = x/\sqrt{\|x\|}$ for $x \neq 0$ and 0 for $x = 0$ satisfies the assumptions of Proposition 3.1, with $\Omega = S$, $k(\cdot) = \sqrt{\cdot}$ and $T = 2$. As it is shown by the trivial example $f(x) = v$, $v \in S$ fixed, $\Omega = \{v\}$, $k(\cdot) = \cdot$ and $T = 1$, the assumptions of Proposition 3.1 do not imply that $f(0) = 0$.

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