RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

UGO BESSI

Multiple closed orbits for singular conservative systems via geodesic theory

Rendiconti del Seminario Matematico della Università di Padova, tome 85 (1991), p. 201-215

<http://www.numdam.org/item?id=RSMUP_1991__85__201_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 85 (1991)

Multiple Closed Orbits for Singular Conservative Systems via Geodesic Theory.

UGO BESSI (*)

Introduction.

The aim of this paper is to find a lower bound for the number of closed trajectories of the following problem:

$$\begin{cases} \ddot{q} + V'(q) = 0\,, \\ \dfrac{1}{2}|\dot{q}|^2 + V(q) = h\,, \end{cases} \tag{1}$$

for a fixed $h > 0$ and for a potential V behaving, roughly, like $-(1/|x|^\delta)$ with $\delta > 2$.

We are going to take advantage of a remarkable similarity between the gradient flow of the functional employed in [2] and that of the energy functional on the manifold of closed, H^1 curves on the sphere; indeed our main theorem, Theorem 4.1, is very close to the well-known «Theorem of the three closed geodesics». One of its consequences is the following:

THEOREM 1. *Suppose* $W \in C^2(\boldsymbol{R}^n, \boldsymbol{R})$ *and that*

$$V_\kappa(x) = -\frac{1}{|x|^\delta} + \kappa W(x)$$

(*) Indirizzo dell'A.: Scuola Normale Superiore, Piazza dei Cavalieri 7, Pisa.

with $\delta > 2$. Then, $\forall h > 0$, $\exists \kappa^* > 0$ such that for $|\kappa| \in (0, \kappa^*)$ the problem:

$$\begin{cases} \ddot{q} + V_\kappa'(q) = 0\,, \\ \dfrac{1}{2}|\dot{q}|^2 + V_\kappa(q) = h\,, \end{cases} \tag{2}$$

has at least $2(n-1) - s - 1$ closed solutions, where s is the smallest integer such that $n - 1 = 2^k + s$ for some $k \in \boldsymbol{N}$.

The problem of finding one periodic solution of (1) by variational methods has been tackled, among others, by Tonelli in [8], by Benci and Giannoni in [4] and by A. Ambrosetti and V. Coti Zelati in [2]. A multiplicity result has been proved in [3] with a different method and for a different class of potentials, namely for V even and behaving, roughly, like $-(1/|x|) - (\varepsilon/|x|^2)$.

The paper is organized as follows: section 1 contains the functional setting, while section 2 contains the abstract result, which is wholly analogous to the corresponding result for closed geodesics. Section 3 is devoted to some estimates necessary to prove the minimal period of the orbits we will find, while section 4 contains the applications, including Theorem 1.

1. We begin with some notations. We will denote the euclidean scalar product in $\boldsymbol{R}^n$ by $x \cdot y$ or simply xy, the euclidean norm by $|x|$; $x\hat{0}y$ will be the angle between the two vectors x and y, and $e_1, \ldots, e_n$ will stay for an othonormal basis of $\boldsymbol{R}^n$. By $\operatorname{supp}\sigma$ we will denote the support of a chain σ, that is the union of the images of all its singular simplexes. We define the diameter of an orbit u as:

$$\operatorname{diam}(u) = \sup_{t_1, t_2 \in [0,1]} |u(t_1) - u(t_2)|\,.$$

Given a set $A \subset H^1(S^1, \boldsymbol{R}^n)$, we denote by $N(A, \delta)$ the δ neighbourhood of A. We define $g(n) = 2n - s - 1$, where s is the smallest positive integer such that $n = 2^k + s$.

Our functional setting will be that of [2]; namely we will consider the open set of $H^1(S^1, \boldsymbol{R}^n)$

$$\Lambda = \{u \in H^1(S^1, \boldsymbol{R}^n) | u(t) \neq 0 \;\; \forall t\}$$

with its subset:

$$M = \left\{ u \in \Lambda \middle| \int_0^1 [V(u) + \frac{1}{2} V'(u) u] \, dt - h = 0 \right\}$$

and the functional

$$f(u) = \frac{1}{4} \int_0^1 |\dot{u}|^2 \, dt \int_0^1 V'(u) u \, dt .$$

Now we list our hypotheses on V.

(V1) $\quad 3V'(x) + V''(x) x \cdot x < 0 \quad \forall x \in \boldsymbol{R}^n - \{0\}$,

(V2) $\quad V'(x) x > 0 \quad \forall x \in \boldsymbol{R}^n - \{0\}$,

(V3) $\quad \exists \alpha, \beta \in (2, \infty)$: $-\alpha V(x) \leq V'(x) x \leq -\beta V(x) \quad \forall x \in \boldsymbol{R}^n - \{0\}$.

The next lemma, taken, from [2], states that our functional setting makes sense.

Lemma 1.1. *If V satisfies* (V1), (V2), (V3) *and $h > 0$, then*

1) *$M \neq \emptyset$, M is a C^1 manifold in $H^1(S^1, \boldsymbol{R}^n)$ and a strong deformation retract of Λ;*

2) *the sublevels $\{u \in M | f(u) \leq c\}$ are complete for every c and f satisfies the Palais-Smale condition on M;*

3) *$\forall u \in M$ $f(u) \geq 0$, $f(u) = 0$ if and only if u is a constant orbit; the level $M_0 = \{u \in M | f(u) = 0\}$ is radially homeomorphic to the unit sphere of $\boldsymbol{R}^n$; we will call S the image in $\boldsymbol{R}^n$ of M_0;*

4) *let u be a non-constant critical point of $f|_M$, let also*

$$\frac{1}{T^2} = \frac{\int_0^1 V'(u) u \, dt}{\int_0^1 |\dot{u}|^2 \, dt} . \tag{1.2}$$

Then $q(t) = u((1/T)t)$ is a T-periodic solution of (1) *of class C^2 and $q(t) \neq 0$ $\forall t$; viceversa, if q is a C^2 solution of* (1) *of period T, then $u(t) = q(tT)$ is a critical point of $f|_M$.*

The proof of Lemma 2 is contained in [2], Lemmas 2.1, 2.3 and 3.3. Let us remark that (2) depends heavily on the fact that β, $\alpha > 2$. We also note that M is stable for the standard action of $O(2)$ on M: namely, if $u(\cdot)\in M$ then $u(\cdot+\theta)\in M$ and $u(-\cdot)\in M$.

2. We are going to find critical points of $f|_M$ with the method of subordination of homology classes of Lusternik and Schnirelmann; we assume that the reader is familiar with such a theory and refer, whenever possible, to [7] or [1]. We set

$$f^c=\{u\in M|f(u)\leqslant c\},$$

$$\Phi(u)=\int_0^1\left[V(u)+\frac{1}{2}V'(u)u\right]dt.$$

LEMMA 2.1. *If* (V1)-(V3) *hold, then for* κ *small enough,* f^κ *is* $O(2)$-*homotopic to* M_0.

PROOF. Since $u\in M$, by (V3) we have:

$$h=\int_0^1\left[V(u)+\frac{1}{2}V'(u)u\right]dt\leqslant\left(\frac{1}{2}-\frac{1}{\beta}\right)\int_0^1 V'(u)u\,dt$$

from which we deduce

$$\int_0^1 V'(u)u\,dt\geqslant\frac{2\beta h}{\beta-2}. \tag{2.1}$$

Thus $u\in f^\kappa$ implies $\int_0^1|\dot u|^2dt\leqslant C_1\kappa$; it follows readily $\operatorname{diam}(u)\leqslant$ $\leqslant C_2\kappa^{\frac{1}{2}}$.

We consider $\phi(x)=V(x)+\frac{1}{2}V'(x)x$. We have that $\phi|_S=h$; (V1) implies that the restriction of ϕ to any ray through the origin is a strictly monotone function. Thus $\phi>h$ or $\phi<h$ in each of the two connected components of $\boldsymbol{R}^n-S$ and, by the mean principle, neither of them will contain the image of any $u\in M$:

$$\forall u\in M\exists\tilde t\ \text{such that}\ u(\tilde t)\in S.$$

Thus $u\in f^\kappa$ implies:

$$\inf_{\tilde u\in M_0}\|u-\tilde u\|_{L^2}\leqslant\inf_{\tilde u\in M_0}\|u-\tilde u\|_{L^\infty}\leqslant\|u-u(\tilde t)\|_{L^\infty}\leqslant C_2\kappa^{\frac{1}{2}}.$$

From this we deduce

$$\inf_{\tilde{u} \in M_0} \|u - \tilde{u}\|_{H^1} \leqslant C_1 \kappa + C_2 \kappa^{\frac{1}{2}}$$

which implies $f^{\kappa} \subset N(M_0, C_3 \kappa^{\frac{1}{2}})$.

We will show that M_0 is a compact, non-degenerate critical submanifold of M and that this implies that $\exists \delta^* > 0$ such that, for $0 < \delta \leqslant \delta^*$, M_0 is a deformation retract of $N(M_0, \delta)$.

It is a standard fact that the map

$$\psi(v) = v - \tilde{u} - (\Phi'(\tilde{u}), \tilde{u} - v)_{H^1} \frac{\Phi'(\tilde{u})}{\|\Phi'(\tilde{u})\|^2} + \Phi(v) \frac{\Phi'(\tilde{u})}{\|\Phi'(\tilde{u})\|^2}$$

maps diffeomorphically a neighbourhood of $\tilde{u} \in M_0$ in the tangent space of M at $\tilde{u}$. Trivially $(f \circ \psi^{-1}(\tilde{u}))' = 0$ $\forall \tilde{u} \in M_0$. We have to show that

$$\forall \tilde{u} \in M_0 \qquad T_{\tilde{u}} M_0 = \ker (f \circ \psi^{-1}(\tilde{u}))''$$

and that $(f \circ \psi^{-1}(\tilde{u}))''$ is a Fredholm operator of index 0. An easy calculation yields that, for $h, k \in T_{\tilde{u}}(M)$,

$$(f \circ \psi^{-1}(\tilde{u}))''(h, k) = \frac{1}{2} \left[\int_0^1 V'(\tilde{u}) \tilde{u} \, dt \right]^2 \int_0^1 \dot{h} \cdot \dot{k} \, dt$$

from which it readily follows that the second derivative is Fredholm.

Using Lagrange multiplers, we have that $k \in \ker (f \circ \psi^{-1}(\tilde{u}))''$ if and only if

$$0 = \int_0^1 \left[\frac{3}{2} V'(\tilde{u}) k + V''(\tilde{u})(\tilde{u}, k) \right] dt \tag{2.2}$$

and

$$\forall h \in H \qquad 0 = -\frac{1}{2} \left[\int_0^1 V'(\tilde{u}) \tilde{u} \, dt \right]^2 \int_0^1 (\ddot{k} \cdot h) \, dt. \tag{2.3}$$

From (2.3) we deduce $\ddot{k} = 0$, which implies that $k \in \boldsymbol{R}^n$. Equation (2.2) implies that k is tangent to S in $\tilde{u}$, which shows $T_{\tilde{u}} M_0 = \ker f''|_M (\tilde{u})$. Thus for u in the domain of a chart at $\tilde{u}$, there exists a unique pair, a, b

depending continuously and equivariantly on $\tilde{u}$, such that $\psi(\tilde{u}) = a(\tilde{u}) + b(\tilde{u})$ with $a(\tilde{u}) \in T_{\tilde{u}} M_0$ and $b(\tilde{u}) \in \operatorname{Im} f''|_M(\tilde{u})$. The homotopy

$$H(\lambda, u) = \psi^{-1}(a(u) + (1-\lambda)\, b(u))$$

retracts a neighbourhood of $\tilde{u}$ onto M_0. Glueing the local charts together, we get the homotopy we want. Q.E.D.

LEMMA 2.2. *Let u be a non-constant critical point of $f|_M$ of minimal period $1/l$, $l \geqslant 2$ and set $w(t) = u(t/l)$. Then we have that w is a critical point of $f|_M$, and*

$$f(u) = l^2 f(w) \geqslant 4f(w). \tag{2.4}$$

PROOF. We take T as in (1.2). We know that $q(t) = u(t/T)$ is a solution of (1) of minimal period T/l. Setting:

$$\tilde{q}(t) = q|_{[0, T/l]}$$

we have that $\tilde{q}$ is a solution of (2) and thus, because of Lemma 2.2, $w(t) = \tilde{q}((T/l)\, t)$ is a critical point of $f|_M$; (2.4) is then verified easily. Q.E.D.

In [2] it is shown that it is possible to find a continuous function $a\colon \Lambda \to \boldsymbol{R}$ such that, given any $u \in \Lambda$, $a(u)\, u \in M$. Thus the homotopy

$$h\colon [0,1] \times \Lambda \to \Lambda, \qquad h\colon \lambda, u \to (1-\lambda)\, u + \lambda a(u)\, u \tag{2.5}$$

retracts Λ on M and is $O(2)$-equivariant.

We denote by Z the set of non-constant critical points of $f|_M$, by A the set of all circles on the unit sphere and by $\tilde{A}$ the set $h(1, A)$. We set

$$k = \min_Z f, \qquad K = \max_{\tilde{A}} f.$$

The following is the translation of a classical theorem in the theory of closed geodesics. All the homology modules we will consider are with Z_2 coefficients.

THEOREM 2.3. *Let* (V1)-(V3) *hold and $K < 4k$; then problem* (1) *has at least $g(n-1)$ closed solutions, all of which are prime and none of which can be brought into the other by the $O(2)$ action.*

PROOF. We set $\Lambda(S^{n-1}) = H^1(S^1, S^{n-1})$.

By (2.5) the identity on $(\Lambda, \boldsymbol{R}^n - \{0\})$ is $O(2)$-homotopic to the retraction on (M, S); it is a standard fact that it is also $O(2)$-homotopic to the retraction on $(\Lambda(S^{n-1}), S^{n-1})$. Because of the $O(2)$-equivariance we can pass to the quotient and state the isomorphism

$$H_*(\Lambda(S^{n-1})/O(2), S^{n-1}) \simeq H_*(M/O(2), S). \tag{2.6}$$

As a consequence of Lemma 2.1 we can now define a cap-product:

$$\frown: H_*(M/O(2), S) \otimes H^*(M/O(2) - S) \to H_*(M/O(2), S)$$

see for instance [7], Lemma 2.1.9 for the proof. From the isomorphism (2.6) it now follows from [7], Theorems 2.3.4 and 2.3.5, that we have $g(n-1)$ subordinated homology classes $\{\sigma_1\}, \ldots, \{\sigma_{g(n-1)}\}$ for the cap product defined above, each with a representative in $H_*(\tilde{A}/O(2), S)$. We thus have $g(n-1)$ minimax levels

$$c_i = \inf_{\sigma \in \{\sigma_i\}} \max_{u \in \operatorname{supp}(\sigma)} f(u)$$

with $c_i \leqslant c_{i+1}$ because of subordination. Since M is locally $O(2)$-contractible and $f|_M$ satisfies the Palais-Smale condition, it follows from a standard minimax principle that we can find $u_1, \ldots, u_{g(n)}$ geometrically distinct non-constant critical points.

We now notice that

$$c_{g(n-1)} = \inf_{\sigma \in \{\sigma_{g(n-1)}\}} \max_{u \in \operatorname{supp}(\sigma)} f(u) \leqslant \max_{\tilde{A}} f = K.$$

Thus, $u_i \in f^K$ $\forall i$. This and Lemma 2.2 imply that each u_i is of minimal period, from which follows that none of them will cover one of the others n times. Q.E.D.

3. In this section we will look for conditions assuring $K < 4k$. We will always assume that V satisfies (V1)-(V3), and that $h > 0$; it is easy to see that in these hypotheses the set $\{\frac{1}{2}V'(x)x > h\}$ is bounded; we fix a positive R such that

$$\{V'(x)x > h\} \subset B_R.$$

LEMMA 3.1. *Let u be a non-constant critical point of $f|_M$. Then $\forall t$ $u(t) \in B_R$.*

Proof. We set $q(t) = u(t/T)$ with T defined as in Lemma 1.1, and take $\bar{t}$ such that $q(\bar{t})$ is the aphelion. Then in $q(\bar{t})$ q is tangent from the inside to the sphere of radius $|q(\bar{t})|$; going to second derivatives, this translates into

$$-\frac{1}{2}\ddot{q}(\bar{t})\cdot\frac{q(\bar{t})}{|q(\bar{t})|} \geqslant \frac{1}{2}\frac{|\dot{q}(\bar{t})|^2}{|q(\bar{t})|}. \tag{3.1}$$

Since $\ddot{q}(\bar{t}) = -V'(q(\bar{t}))$ from (3.1) we get

$$\frac{1}{2}V'(q(\bar{t}))\,q(\bar{t}) \geqslant \frac{1}{2}|\dot{q}(\bar{t})|^2 > h.$$

From this the thesis follows. Q.E.D.

Lemma 3.2. *If u is a non-constant critical point of $f|_M$, if T is defined as in* (1.1) *and $q(t) = u(t/T)$, then the following equalities hold:*

$$[4f(u)]^{\frac{1}{2}} = \int_0^T \left[\frac{1}{2}|\dot{q}|^2 + \frac{1}{2}V'(q)\,q\right]dt, \tag{3.2}$$

$$[4f(u)]^{\frac{1}{2}} = T\int_0^1 V'(u)\,u\,dt. \tag{3.3}$$

Proof. It is an easy verification.

We need a further hypothesis on our potential:

(V4) $\exists\gamma > 0$ such that $\forall\eta \in \boldsymbol{R}^n - \{0\}$, $\forall K \subset \boldsymbol{R}^n - \{0\}$, K compact,

$$\min_{\widehat{x0\eta} \leqslant \gamma,\, x \in K} V'(x)\,\eta > 0.$$

We introduce the following set

$$\Theta = \{u \in \Lambda \,|\, \forall\eta \in \boldsymbol{R}^n \;\; \exists t \text{ with } \widehat{\eta 0 u(t)} \geqslant \gamma\}.$$

Lemma 3.3. *If u is a non-constant critical point of $f|_M$ and* (V4) *holds, then*

$$u \in \Theta.$$

PROOF. From Lemma 1.2 we know that $q(t) = u(t/T)$ is a C^2 solution of:

$$\ddot{q}(t) = -V'(q(t)) \qquad \forall t.$$

If $u \notin \Theta$, then we could find a vector η such that the image of q is contained in the half-cone

$$x \in \boldsymbol{R}^n \text{ such that } x\hat{0}\eta \leqslant \gamma.$$

From this and the periodicity of q we would deduce:

$$0 = \int_0^1 \eta \cdot \ddot{q}\, dt = -\int_0^1 \eta \cdot V'(q)\, dt < 0,$$

a contradiction. Q.E.D.

REMARK 3.4. We can write $u(t) = \rho(t)\, w(t)$ with $|w(t)| = 1$. If we denote by d the geodesic distance on the sphere S^{n-1}, the condition $u \in \Theta$ is then equivalent to this one

$$\nexists \eta \in S^{n-1} \text{ such that } d(\eta, w(t)) \geqslant \gamma \qquad \forall t.$$

It is easy to see that this condition implies

$$\int_0^1 |\dot{w}(t)|\, dt \geqslant 4\gamma.$$

We set:

$$c_1 = R^\beta \max_{|x| = R}(-V(x)), \qquad c_2 = R^\alpha \min_{|x| = R}(-V(x))$$

and note that (V3) implies

$$-\frac{c_1}{|x|^\beta} \leqslant V(x) \leqslant -\frac{c_2}{|x|^\alpha} \quad \text{for } |x| \leqslant R. \tag{3.4}$$

We are now ready to state the estimate on the minimum of the functional.

LEMMA 3.5. *If u is a non-constant critical point of $f|_M$ and $V(x)$ satisfies* (V1)-(V4), *then the following holds*:

$$f(u) \geqslant \frac{1}{4}\left(\frac{2\beta h}{\beta - 2}\right)^2 \left(\frac{2\alpha h}{\alpha - 2}\right)^{-(\alpha+2)/\alpha} \left(\frac{\alpha c_2}{2}\right)^{2/\alpha} (4\gamma)^2 \alpha^{2/\alpha} \left(\frac{1}{2} + \frac{1}{\alpha}\right)^{(\alpha+2)/\alpha}.$$

PROOF. For any solution u (3.2) and (V3) yield:

$$[4f(u)]^{\frac{1}{2}} = \int_0^T \left[\frac{1}{2}|\dot{q}|^2 + \frac{1}{2}V'(q)\,q\right]dt \geq \int_0^T \left[\frac{1}{2}|\dot{q}|^2 - \frac{\alpha}{2}V(q)\right]dt\,.$$

We set $\varphi(s) = c_2 s^{\alpha}$; (3.4) and Lemma 3.1 imply $-V(q) \geq \varphi(1/|q|)$.
From this and Lemma 3.3 we deduce

$$[4f(u)]^{\frac{1}{2}} \geq \inf_{q\in\Theta} \int_0^T \left[\frac{1}{2}|\dot{q}|^2 + \frac{\alpha}{2}\varphi\left(\frac{1}{|q(t)|}\right)\right]dt\,.$$

To give an estimate on the last term, we employ a method of Giannoni - De Giovanni (see for instance [6] or [5]). By reparametrization and the Jensen inequality, we have:

$$\inf_{q\in\Theta} \int_0^T \left[\frac{1}{2}|\dot{q}|^2 + \frac{\alpha}{2}\varphi\left(\frac{1}{|q(t)|}\right)\right]dt \geq$$

$$\geq \frac{1}{T} \inf_{u\in\Theta} \int_0^1 \left[\frac{1}{2}|\dot{u}|^2\right]dt + \frac{T^2\alpha}{2}\varphi\left(\int_0^1 \frac{1}{|u(t)|}\,dt\right).$$

Hence, setting

$$g(u) = \int_0^1 \left[\frac{1}{2}|\dot{u}|^2\right]dt + \frac{T^2\alpha}{2}\varphi\left(\int_0^1 \frac{1}{|u(t)|}\,dt\right),$$

there results

$$[4f(u)]^{\frac{1}{2}} \geq \frac{1}{T}g(u)\,. \tag{3.5}$$

It is easy to check that g is a weakly sequentially lower semicontinuous functional. In the set Θ g is coercive. Indeed, we take t_1 such that $|u(t_1)| = \|u\|_\infty$. Then, as $u \in \Theta$, it is possible to find a time t_2 such that

$$u(t_2)\hat{0}u(t_1) = \gamma$$

from which follows:

$$|u(t_1)| \sin(\gamma) \leqslant |u(t_1) - u(t_2)| \leqslant \int_0^1 |\dot{u}|\, dt \leqslant \|\dot{u}\|_{L^2}$$

which implies coercitivity. We take a minimizing sequence $u_n \in \Theta$ for g, and its weak limit u. There are two cases: either $u \in \Lambda \cap \Theta$, or $u \in \partial\Lambda$. In the first case, the following argument applies.

We will show that the minimum occurs on a planar orbit; indeed we will show that, taken $u \in \Theta$, there exists $\tilde{u} \in \Theta$ planar and with $g(\tilde{u}) \leqslant g(u)$. As in Remark 3.2, we set $u(t) = \rho(t)\, w(t)$ with $|w(t)| = 1$. From the same remark we know that

$$\int_0^1 |\dot{w}(t)|\, dt = \lambda \geqslant 4\gamma\,.$$

We define

$$\lambda(s) = \int_0^s |\dot{w}(t)|\, dt$$

and t^* by $\lambda(t^*) = \lambda/2$. We set

$$\tilde{w}(t) = e_1 \cos(\lambda(t)) + e_2 \sin(\lambda(t)) \qquad 0 \leqslant t \leqslant t^*\,,$$

$$\tilde{w}(t) = e_1 \cos(\lambda - \lambda(t)) + e_2 \sin(\lambda - \lambda(t)) \qquad t^* \leqslant t \leqslant 1\,.$$

The orbit $\tilde{w}(t)$ will be planar and with length 4γ and thus it will not be contained in any geodesic ball of radius less than γ. It follows then readily that $\tilde{u}(t) = \rho(t)\, \tilde{w}(t)$ is an orbit of Θ and $f(u) = f(\tilde{u})$.

We denote by $\tilde{u}$ the minimum of g on Θ; there exist t_1, t_2 such that $\tilde{u}(t_1)\hat{0}\tilde{u}(t_2) \geqslant 2\gamma$. As the problem is invariant for translation of the time, we set $t_1 = 0$.

Let us suppose $4\gamma = (m/k)\, 2\pi$, with m, $k \in N$; the general case will follow by approximation. We set ourselves on the plane on which $\tilde{u}$ lies; here we will denote by $R(\theta)$ the rotation of angle θ and by R the symmetry which leaves $u(t_2)$ fixed.

Set

$$w(t) = \tilde{u}(t) \qquad 0 \leqslant t \leqslant t_2\,,$$

$$w(t) = R\tilde{u}(t) \qquad t_2 \leqslant t \leqslant 1\,.$$

We define a new orbit z in the following way:

$$z(t) = R\left(i\frac{2\pi}{k}\right) w(t-1) \qquad i \leqslant t \leqslant i+1\,, \qquad i = 0, \ldots, k-1\,.$$

We have that z has degree m and period k, and thus

$$\min_{\theta} g \geqslant \frac{1}{k} \min_{\deg z = m} \int_0^k \frac{1}{2} |\dot z(t)|^2 \, dt + \frac{T^2 \alpha k^{1-\alpha}}{2} \phi\left(\int_0^k \frac{1}{|z|}\, dt\right).$$

Setting $v(t) = z(kt)$, we get

$$\min_{\theta} g \geqslant \frac{1}{k^2} \min_{\deg v = m} l(v)$$

where l is defined for $v \in H^1(S^1, \boldsymbol{R}^2)$ by

$$l(v) = \int_0^1 \frac{1}{2} |\dot v(t)|^2 \, dt + \frac{T^2 \alpha k^2}{2} \phi\left(\int_0^1 \frac{1}{|v|}\, dt\right).$$

Using the fact that the Euler equation satisfied by the minimum of l on planar orbits of degree m is a Kepler problem, for which the integrals of the kinetic energy and of the potentials are known, in [5], Lemma 2.4 it has been shown that:

$$\min_{\deg v = m} l(v) \geqslant \min_{r > 0} \left\{ \frac{1}{2} (2m\pi)^2 r^2 + \frac{T^2 \alpha k^2 c_2}{2} \frac{1}{r^\alpha} \right\} =$$

$$= \left(\frac{T^2 \alpha k^2 c_2}{2} \right)^{2/(\alpha+2)} (2m\pi)^{2\alpha/(\alpha+2)} \beta^{2/(\alpha+2)} \left(\frac{1}{2} + \frac{1}{\alpha} \right).$$

This and (3.5) implies

$$(3.6) \qquad [4f(u)]^{\frac{1}{2}} \geqslant \frac{1}{Tk^2} \left(\frac{T^2 \alpha k^2 c_2}{2} \right)^{2/(\alpha+2)} (2m\pi)^{2\alpha/(\alpha+2)} \beta^{2/(\alpha+2)} \left(\frac{1}{2} + \frac{1}{\alpha} \right).$$

In the second case, $u \in \partial \Lambda$, in [6], Proposition 2.4, has been shown that

$$\min_{\partial \Lambda} g \geqslant \min_{r > 0} \left\{ \frac{1}{2} (2\pi)^2 r^2 + \frac{T^2 \alpha k^2 c_2}{2} \frac{1}{r^\alpha} \right\}$$

which yields (3.6). As in (2.1) it follows easily

$$(3.7) \qquad T\int_0^1 V'(u)u\,dt \leqslant \frac{2T\alpha h}{\alpha-2}\,.$$

Plugging (3.7) and (3.6) into (3.2)-(3.3) we get

$$(3.8) \quad (T^2)^{\alpha/(\alpha+2)}\frac{2\alpha h}{\alpha-2} \geqslant \frac{1}{k^2}\left(\frac{\alpha k^2 c_2}{2}\right)^{2/(\alpha+2)}(2m\pi)^{2\alpha/(\alpha+2)}\alpha^{2/(\alpha+2)}\left(\frac{1}{2}+\frac{1}{\alpha}\right).$$

From (3.3) we know that

$$[4f(u)]^{\frac{1}{2}} \geqslant T\frac{2\beta h}{\beta-2}$$

plugging this into (3.8) we get

$$f(u) \geqslant \frac{1}{4}\left(\frac{2\beta h}{\beta-2}\right)^2\left(\frac{2\alpha h}{\alpha-2}\right)^{-(\alpha+2)/\alpha}\left(\frac{\alpha c_2}{2}\right)^{2/\alpha}(4\gamma)^2\alpha^{2/\alpha}\left(\frac{1}{2}+\frac{1}{\alpha}\right)^{(\alpha+2)/\alpha}$$

which is the result for $\gamma = (m/k)\pi$; by approximation it also holds in the general case. Q.E.D.

We are now going to derive an estimate on the minimax levels. We recall that each of the subordinated homology classes has a representative in $\tilde{A}/O(2)$, and that each element of $\tilde{A}$ is a circle u of S^{n-1} multiplied by the unique $a(u)$ such that $a(u)u \in M$. Thus:

$$c = \inf_{\sigma\in\{\sigma_{g(n)}\}}\ \max_{u\in\operatorname{supp}\sigma} f(u) \leqslant \max_{\tilde{A}} f(u) = K\,.$$

By (V3) and (3.4) we have

$$h = \int_0^1\left[Va(u)u + \frac{1}{2}V'a(u)u\,a(u)u\right]dt = \int_0^1[\phi a(u)u]\,dt\,.$$

By the same argument as in Lemma 2.1 we have that the image of $a(u)u$ must intersect S: thus

$$(3.9) \qquad \forall u\in A \qquad a(u) \leqslant \max_{x\in S}|x| := B\,.$$

We can now state the following lemma.

LEMMA 3.6. *If $V(x)$ satisfies* (V1)-(V3), *we have*

$$K \leqslant \pi^2 B^2 \frac{-2\alpha h}{2-\alpha}.$$

PROOF. It suffices to plug (3.9) in the definition of f.

4. The following theorem is an immediate consequence of Theorem 2.3 and of estimates of the previous section.

THEOREM 4.1. *If V satisfies* (V1)-(V4), *if $h > 0$ and*

$$\left(\frac{2\beta h}{\beta-2}\right)^2 \left(\frac{2\alpha h}{\alpha-2}\right)^{-(\alpha+2)/\alpha} \left(\frac{\alpha c_2}{2}\right)^{2/\alpha} (2\gamma)^2 \cdot \left(\frac{1}{2}\alpha^{2/(2+\alpha)} + \frac{1}{\alpha^{\alpha/(2+\alpha)}}\right)^{(\alpha+2)/\alpha} > \pi^2 B^2 \frac{2\beta h}{\beta-2} \tag{3.10}$$

then system (1) *has at least $g(n)$ distinct closed trajectories.*

PROOF. We are indeed in the hypotheses of Theorem 2.3; $K < 4k$ is nothing but (3.10).

PROOF OF THEOREM 1. By Lemma 3.3 we can take R big enough such that B_R contains in its interior every possible solution of (2) with $\kappa = 0$. We also take a smooth function ψ, such that $\psi = 1$ in $[-R, R]$ and $\psi = 0$ in $[-\infty, -R-1] \cup [R+1, \infty]$. We set

$$\tilde{V}_\kappa(x) = -\frac{1}{|x|^\delta} + \kappa W(x)\, \psi(|x|).$$

It is now easy to check (see for instance [3]) that for κ small enough $\tilde{V}_\kappa$ will verify hypotheses (V1)-(V3) with α, β tending to δ and c_1, c_2 tending to 1 as κ tends to 0. Analogously,

$$B \to \left(\frac{\delta-2}{2h}\right)^{1/\delta}$$

and $2\gamma \to \pi$ for $\kappa \to 0$. Moreover for κ small enough by Lemma 3.3 we have that all orbits of $\tilde{V}_\kappa(x)$ will stay in the ball B_R, and thus they will coincide with the solutions of (2).

Thus to show Theorem 1 we have only to check that (3.9) holds for $\kappa = 0$; this is indeed the case as, for $\beta = \alpha = \delta$, $c_1 = c_2 = 1$ and $\gamma = \pi$ (3.10) reduces to

$$4\left(\frac{\delta}{2}\right)^{2/\delta}\left(\frac{1}{2}+\frac{1}{\delta}\right)^{(\delta+2)/\delta}>1$$

which holds trivially. Q.E.D.

Acknowledgement. I wish to thank Professor A. Ambrosetti for his precious assistance during this work.

BIBLIOGRAPHY

[1] S. I. ALBER, *On periodicity problems in the calculus of variations in the large,* A.M.S. Translations, **14** (1960), pp. 107-172.

[2] A. AMBROSETTI - V. COTI ZELATI, *Closed orbits of fixed energy for singular hamiltonian systems,* Archive Rat. Mech. and Anal., to appear.

[3] A. AMBROSETTI - U. BESSI, *Multiple periodic trajectories in a relativistic gravitational field,* preprint S.N.S., **77** (1990).

[4] V. BENCI - F. GIANNONI, *Periodic solutions of prescribed energy for a class of hamiltonian systems with singular potentials,* J. Differential Equations, **82** (1989), pp. 60-70.

[5] V. COTI ZELATI - U. BESSI, *Symmetries and non-collision closed orbits for planar, N-body type problems,* J. Nonlinear. Analysis, to appear.

[6] F. GIANNONI - M. DEGIOVANNI, *Dynamical systems with newtonian type potentials,* Ann. Scuola Norm. Sup. Pisa, **15** (1988), pp. 467-494.

[7] W. KLINGENBERG, *Lectures on Closed Geodesic,* Grundlehren der Math. Wiss., 235.

[8] L. TONELLI, *Sulle orbite periodiche,* Rendiconti R. Accad. dei Lincei, **21-1** (1912), pp. 251-258.

Manoscritto pervenuto in redazione il 28 luglio 1990.