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A Maximal Regularity Result with Applications to Parabolic Problems with Nonhomogeneous Boundary Conditions.

DAVIDE GUIDETTI (*)

Introduction.

The aim of this paper is to study mixed problems of parabolic type with nonhomogeneous (possibly nonlinear) boundary conditions.

The main tool is a certain maximal regularity result (theorem 2.1) allowing to study the general problem by simple contraction mapping arguments. The basic idea comes from B. Terreni's [TE1] and starts, from an explicit representation of the solution (formula 12) together with suitable estimates in the linear autonomous case.

However, while Terreni's results are ultimately based on time regularity assumptions, here everything is done starting from a suitable spatial regularity of the data on the lines of Da Prato-Grisvard's work in the case of homogeneous boundary conditions (see [DPG]). Also we remark that this approach allows to avoid regularity assumptions on the initial datum of the kind of Terreni's conditions 5.1, which seems to cause some difficulties, for example in the study of global solutions. Finally, we consider equations and not system. The extension to this more general case does not seem to introduce further difficulties.

The paper is arranged as follows: the first paragraph contains the general properties of little Nikolskii spaces, which are the basic spaces

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we shall use, with a brief study of linear elliptic problems, which is necessary in the following and some variants of the results of section 2 in [TE]. Here the basic results are propositions 1.4 and 1.5. The second paragraph is dedicated to the study of the parabolic autonomous problem and is directed to prove the basic maximal regularity result, theorem 2.1. The third paragraph contains a simple result of perturbation, with the study of linear nonautonomous problems. The fourth and final section is dedicated to quasilinear problems.

Finally, we specify that the expressions C and « const » will be used to indicate constant which may change from time to time.

1. Elliptic problems in little-Nikolskii spaces.

We shall be concerned with a class of subspaces of $L^p(\mathbb{R}^n)$, the so called little-Nikolskii spaces $\lambda_p^{k+\theta}(\mathbb{R}^n)$: if $k \in \mathbb{N} \cup \{0\}$, $\theta \in]0, 1]$, $p \in [1, +\infty]$, it is defined as

$$\left\{u \in W^{k,p}(\mathbb{R}^n) \colon \forall \alpha, |\alpha| = k, |h|^{-\theta p} \int_{\mathbb{R}^n} |\partial^\alpha u(x + h) - \partial^\alpha u(x)|^p \, dx \to 0 (h \to 0)\right\},$$

with the usual modification if $p = +\infty$ and the natural norm. We remark that the space $\lambda_\infty^{k+\theta}(\mathbb{R}^n)$ is exactly the space of function of class C^k which are bounded and little-hölder continuous with all the derivatives of order not exceeding k.

Now, let Ω be an open (nonempty) subset of $\mathbb{R}^n$ with suitably smooth boundary. We indicate with T_0 the trace operator $T_0 f = f|_{\partial\Omega}$. We pose

$$\lambda_p^{k+\theta}(\Omega) = \{u|_\Omega \colon u \in \lambda_p^{k+\theta}(\mathbb{R}^n)\}$$

with the natural norm of quotient space $\|\cdot\|_{k+\theta,p,\Omega}$.

The proof of the following result is routine (see also [ZO] for other results of multiplication in Besov and Nikolskii spaces):

PROPOSITION 1.1. – *Let Ω be an open (nonempty) subset of $\mathbb{R}^n$, $a \in \lambda_\infty^\theta(\Omega)$, $u \in \lambda_p^\theta(\Omega)$. Then $au \in \lambda_p^\theta(\Omega)$ and*

$$\|au\|_{\theta,p,\Omega} \leqslant \|a\|_{\theta,\infty,\Omega} \|u\|_{\theta,p,\Omega} .$$

An easy consequence of proposition 1.1 is the following: consider a differential operator

$$B(x, \partial) = \sum_{|\gamma| \leqslant q} b_\gamma(x)\, \partial^\gamma$$

with $b_\gamma \in \lambda_\infty^{r+\theta}(\Omega)$; if $u \in \lambda_p^{s+\theta}(\Omega)$, $s = q + r$, $B(x, \partial)\; u \in \lambda_p^{r+\theta}(\Omega)$.

Our interest in these spaces lies on the following facts: if A_0, A_1 are a couple of compatible Banach spaces (see [BL] 2.3), and if we denote with $(A_0, A_1)_\sigma$ ($\sigma \in]0, 1[$) the continuous interpolation space determined by A_0, A_1 (see [DPG]), for $j, k \in \mathbf{N} \cup \{0\}$,

$$\left(W^{j,p}(\mathbf{R}^n),\, W^{k,p}(\mathbf{R}^n)\right)_\sigma = \lambda_p^{l+\theta}(\mathbf{R}^n)\,,$$

with

$$l = [(1-\sigma)j + \sigma k]\,, \qquad \theta = (1-\sigma)j + \sigma k - [(1-\sigma)j + \sigma k]\,,$$

(here and in the following, if $\lambda \in \mathbf{R}$, $[\lambda]$ will be the integer part of λ).

The well known Calderon-Stein extension theorem implies (see [ST], ch. VI, th. 5):

PROPOSITION 1.2. *Let Ω be an open subset of $\mathbf{R}^n$ satisfying the assumptions of* [ST], ch. VI, 3.3. *Let $\sigma \in]0, 1[$, $j, k \in \mathbf{N} \cup \{0\}$, $(1-\sigma)j + \sigma k \notin \mathbf{Z}$. Then, $\left(W^{j,p}(\Omega), W^{k,p}(\Omega)\right)_\sigma = \lambda_p^{l+\theta}(\Omega)$, with*

$$l = [(1-\sigma)j + \sigma k]\,, \qquad \theta = (1-\sigma)j + \sigma k - [(1-\sigma)j + \sigma k]\,.$$

Assume A_0, A_1, A are compatible Banach spaces and let $s \in]0, 1[$. We recall that A is of class $K_s(A_0, A_1)$ (see for example [GR] def. 4) if $(A_0, A_1)_{s,1} \subseteq A \subseteq (A_0, A_1)_{s,\infty}$ (continuous inclusions) (here $(\;,\;)_{s,\omega}$ is the real interpolation functor of indexes s, ω).

PROPOSITION 1.3. *Let Ω be an open subset of $\mathbf{R}^n$ satisfying the assumptions of* [ST], ch. VI, 3.3. *For $p \in]1, +\infty[$, $k \in \mathbf{N} \cup \{0\}$, we put $\lambda_p^k(\Omega) = W^{k,p}(\Omega)$. Assume $s_0, s_1, s_2 \in \mathbf{R}$, $0 \leqslant s_0 < st < s_2$.*

Then $\lambda_p^{s_1}(\Omega)$ is of class $K_{(s_1-s_0)/(s_2-s_0)}(\lambda_p^{s_0}(\Omega), \lambda_p^{s_2}(\Omega))$.

PROOF. If is sufficient to consider the case $\Omega = \mathbf{R}^n$.

One has

$$\left(\lambda_p^{s_0}(\mathbf{R}^n),\, \lambda_p^{s_2}(\mathbf{R}^n)\right)_{(s_1-s_0)/(s_2-s_0),1} = B_{p,1}^{s_1}(\mathbf{R}^n)$$

(see [GR] 1.10), as $\lambda_p^{s_0}(\mathbb{R}^n)$ is of class $K_{s_0/([s_2]+1)}(L^p(\mathbb{R}^n), W^{[s_2]+1,p}(\mathbb{R}^n))$, $\lambda_p^{s_2}(\mathbb{R}^n)$ is of class $K_{s_2/([s_2]+1)}(L^p(\mathbb{R}^n), W^{[s_2]+1,p}(\mathbb{R}^n))$ and by the reiteration property. In the same way,

$$(\lambda_p^{s_0}(\mathbb{R}^n), \lambda_p^{s_2}(\mathbb{R}^n))_{(s_1-s_0)/(s_2-s_0),\infty} = B^{s_1}_{p,\infty}(\mathbb{R}^n) .$$

It remains to show that

$$B^{s_1}_{p,1}(\mathbb{R}^n) \subseteq \lambda_p^{s_1}(\mathbb{R}^n) \subseteq B^{s_1}_{p,\infty}(\mathbb{R}^n) .$$

If $s_1 \notin \mathbb{Z}$, $\lambda_p^{s_1}(\mathbb{R}^n)$ is the closure of $C_0^\infty(\mathbb{R}^n)$ in $B^{s_1}_{p,\infty}(\mathbb{R}^n)$. As $B^{s_1}_{p,1}(\mathbb{R}^n) \subseteq$ $\subseteq B^{s_1}_{p,\infty}(\mathbb{R}^n)$ and $C_0^\infty(\mathbb{R}^n)$ is dense in $B^{s_1}_{p,1}(\mathbb{R}^n)$, the result follows. If $s_1 \in \mathbb{Z}$, by theorem 6.4.4 in [BL], for $p \in]1, 2]$,

$$B^{s_1}_{p,1}(\mathbb{R}^n) \subseteq B^{s_1}_{p,p}(\mathbb{R}^n) \subseteq H_{s_1,p}(\mathbb{R}^n) = \lambda_p^{s_1}(\mathbb{R}^n) \subseteq B^{s_1}_{p,\infty}(\mathbb{R}^n)$$

and, for $2 \leqslant p < +\infty$,

$$B^{s_1}_{p,1}(\mathbb{R}^n) \subseteq B^{s_1}_{p,2}(\mathbb{R}^n) \subseteq H_{s_1,p}(\mathbb{R}^n) = \lambda_p^{s_1}(\mathbb{R}^n) \subseteq B^{s_1}_{p,p}(\mathbb{R}^n) \subseteq B^{s_1}_{p,\infty}(\mathbb{R}^n) .$$

So the result is proved.

Now we consider a bounded open subset Ω of $\mathbb{R}^n$ with boundary of class C^{2m+1} and a linear differential operator $A(x, \partial) = \sum_{|\alpha| \leqslant 2m} a_\alpha(x)\, \partial^\alpha$ of order $2m$. We assume that:

($A1$) $\forall \alpha$, $|\alpha| \leqslant 2m$ $a_\alpha \in \lambda^\theta_\infty(\mathbb{R}^n)$, $\theta \in]0, 1[$.

($A2$) $A(x, \partial)$ is properly elliptic.

($A3$) There exists $\phi_0 \in]\pi/2, \pi[$ such that $\forall x \in \bar{\Omega}$, $\forall \xi \in \mathbb{R}^n - \{0\}$,

$$(-1)^m A^0(x, \xi)/|A^0(x, \xi)| \neq e^{i\phi} \quad \forall \phi \in [-\phi_0, \phi_0] (A^0(x, \xi) = \sum_{|\alpha| = 2m} a_\alpha(x) \xi^\alpha) .$$

For $k = 1, \dots, m$ a linear differential operator $B_k(x, \partial) = \sum_{|\beta| \leqslant m_k} b_{k,\beta}(x)\, \partial^\beta$ is given, in such a way that:

($B1$) $0 \leqslant m_k \leqslant 2m - 1$ and $\forall \beta$, $\forall k$, $b_{k,\beta} \in \lambda_\infty^{2m - m_k + \theta}(\mathbb{R}^n)$.

($B2$) For $k \neq l$ $m_k \neq m_l$.

($B3$) For $k = 1, \dots, m$, the boundary $\partial\Omega$ is noncharacteristic in each of its points with respect to $B_k(x, \partial)$.

($B4$) At each point $x \in \partial\Omega$, if $\nu(x)$ is the inward unit vector to $\partial\Omega$ in x and $\xi \neq 0$ any real vector parallel to the boundary in x, we indicate with $t_j^+(\xi, \lambda)$ the j-th root with positive imaginary part of the polynomial in $z(-1)^m A^0(x, \xi + z\nu(x)) - \lambda$. Then, if $|\text{Arg}\, \lambda| \leq \phi_0$, the m polynomials $\{B_k^0(x, \xi + z\nu(x)) : k = 1, \ldots, m\}$ are linearly independent modulo $\prod_{j=1}^{m} (z - t_j^+(\xi, \lambda))$ $(B_k^0(x, \xi) = \sum_{|\beta| = m_k} b_{k,\beta}(x)\, \xi^\beta)$.

We shall consider the problem

$$(1) \qquad A(x, \partial)u = f \quad \text{in } \Omega \qquad B_k(x, \partial)u - g_{k|\partial\Omega} = 0\,, \quad k = 1, \ldots, m\,.$$

We remark that a priori estimates for (1) (case of regular problems) were obtained, for example, in [ADN], [BR1, 2, 3], [AR], [TR]. Here, we need the following

PROPOSITION 1.4. *Consider* (1) *with the assumptions* ($A1$)-($A3$), ($B1$)-($B4$). *If* $f \in \lambda_p^\theta(\Omega)$ $(\theta \in]0, 1[, 1 < p < +\infty)$, $g_k \in \lambda_p^{2m - m_k + \theta}(\Omega)$, $u \in W^{2m,p}(\Omega)$ *and solves* (1), *then* $u \in \lambda_p^{2m+\theta}(\Omega)$ *and*

$$(2) \qquad \|u\|_{2m+\theta,p,\Omega} \leq C\Big(\|f\|_{\theta,p,\Omega} + \sum_{k=1}^{m} \|g_k\|_{2m-m_k+\theta,p,\Omega} + \|u\|_{0,p,\Omega}\Big)\,,$$

with C *independent of* u.

PROOF. First of all, we look for an appropriate a priori estimate. We start by assuming that $u \in \lambda_p^{2m+\theta}(\Omega)$, $\text{supp}\, u \subseteq B(x^0, r)$ with $x^0 \in \Omega$, $r < \text{dist}\,(x^0, \partial\Omega)$. By the a priori estimates of [BR1, 2, 3] and [ADN], one has

$$\|u\|_{2m,p,\Omega} \leq C_0\big(\|f\|_{0,p,\Omega} + \|u\|_{0,p,\Omega}\big)$$

For $0 < |\xi| < \text{dist}\,(x^0, \partial\Omega) - r$, $x \in B(x^0, r)$,

$$A(x, \partial)[u(x + \xi) - u(x)] =$$
$$= [A(x, \partial) - A(x + \xi, \partial)]u(x + \xi) + f(x + \xi) - f(x)\,,$$

which, by assumption ($A1$), implies

$$\|u(. + \xi) - u\|_{2m,p,\Omega} \leq C|\xi|^\theta\big(\|u\|_{2m,p,\Omega} + \|f\|_{\theta,p,\Omega} + \|u\|_{\theta,p,\Omega}\big)$$

and

$$\|u\|_{2m+\theta,p,\Omega} \leqslant \text{const}\,(x^0, r)\big(\|f\|_{\theta,p,\Omega} + \|u\|_{\theta,p,\Omega}\big)$$

(here const (x^0, r) indicates a constant depending on x^0, r).

Next, let $x^0 \in \partial\Omega$, $r > r' > 0$, $u \in \lambda_u^{2m+\theta}(\Omega)$, $u(x) = 0$ if $x \notin B(x^0, r')$. Then, u solves

$$\text{(3)} \qquad \begin{cases} A(x^0, \partial)u = [A(x^0, \partial) - A(x, \partial)]u + f \text{ in } \Omega\,, \\ B_k(x^0, \partial)u = [B_k(x^0, \partial) - B_k(x, \partial)]u + g_{k|\partial\Omega}\,, \qquad k = 1, \ldots, m\,. \end{cases}$$

If r is sufficiently small, there exist operators $A^\wedge(x, \partial)$, $B_k^\wedge(x, \partial)$ with coefficients in $C^1(\bar{\Omega})$ and $C^{2m-m_k+1}(\bar{\Omega})$ respectively, satisfying $(A1)$-$(A3)$, $(B1)$-$(B4)$ and coinciding with $A(x^0, \partial)$ and $B_k(x^0, \partial)$ in $B(x^0, r) \cap \bar{\Omega}$. By well known results (see [AG]), there exists $\lambda_0 \in \mathbb{C}$ such that

$$\text{(4)} \qquad \begin{cases} A^\wedge(x, \partial)u - \lambda_0 u = f \\ B_k^\wedge(x, \partial)u - g_{k|\partial\Omega} = 0 \end{cases}$$

has a unique solution $u \in W^{2m,p}(\Omega)$ $\forall f \in L^p(\Omega)$, $(g_k)_{k=1}^m \in \prod_{k=1}^m W^{2m-m_k,p}(\Omega)$, which belongs to $W^{2m+1,p}(\Omega)$ if $f \in W^{1,p}(\Omega)$, $(g_k)_{k=1}^m \in \prod_{k=1}^m W^{2m-m_k+1,p}(\Omega)$. By interpolation, if $f \in \lambda_p^\theta(\Omega)$, $(g_k)_{k=1}^m \in \prod_{k=1}^m \lambda_p^{2m-m_k+\theta}(\Omega)$, $u \in \lambda_p^{2m+\theta}(\Omega)$ and

$$\|u\|_{2m+\theta,p,\Omega} \leqslant C\Big(\|f\|_{\theta,p,\Omega} + \sum_{k=1}^m \|g_k\|_{2m-m_k+\theta,p,\Omega}\Big)\,.$$

Applying this estimate to (3) and using the fact that, if r' is small enough

$$\|(A(x^0, \partial) - A(x, \partial))u\|_{\theta,p,\Omega} + \sum_{k=1}^m \|(B_k(x^0, \partial) - B_k(x, \partial))u\|_{2m-m_k+\theta,p,\Omega} \leqslant$$
$$\leqslant \varepsilon\|u\|_{2m+\theta,p,\Omega} + C\|u\|_{2m,p,\Omega}\,,$$

by localization one draws (2), under the assumption $u \in \lambda_p^{2m+\theta}(\Omega)$.

Now assume only that $u \in W^{2m,p}(\Omega)$. Again using Agmon's results, one proves that there exists $\lambda_0 \in \mathbb{C}$ such that

$$\text{(5)} \qquad (A(x, \partial) - \lambda_0)u = f \text{ in } \Omega\,, \qquad B_k(x, \partial)u - g_{k|\partial\Omega} = 0\,,$$

has a unique solution $u \in W^{2m,p}(\Omega)$ $\forall f \in L^p(\Omega)$, $\forall (g_k)_{k=1}^m \in \prod_{k=1}^m W^{2m-m_k,p}(\Omega)$ and

$$\|u\|_{2m,p,\Omega} \leqslant \left(C\|f\|_{0,p,\Omega} + \sum_{k=1}^m \|g_k\|_{2m-m_k,p,\Omega}\right). \tag{6}$$

Consider

$$A^{(r)}(x,\partial) = \sum_{|\alpha| \leqslant 2m} a_\alpha^{(r)}(x)\,\partial^\alpha, \quad B_k^{(r)}(x,\partial) = \sum_{|\beta| \leqslant m_k} b_{k,\beta}^{(r)}(x)\,\partial^\beta,$$

with

$$a_\alpha^{(r)} \in C^\infty(\mathbb{R}^n), \qquad b_{k,\beta}^{(t)} \in C^\infty(\mathbb{R}^n), \qquad \|a_\alpha - a_\alpha^{(r)}\|_{\theta,\infty,\Omega} \to 0 \qquad (r \to +\infty),$$

$$\|b_{k,\beta}^{(r)} - b_{k,\beta}\|_{2m-m_k+\theta,\infty,\bar{\Omega}} \to 0 \qquad (t \to +\infty)$$

(the existence of $a_\alpha^{(r)}$, $b_{k,\beta}^{(r)}$ follows from the density of $C^\infty(\mathbb{R}^n)$ in $\lambda_\infty^\theta(\mathbb{R}^n)$).

For r large enough the problem

$$A^{(r)}(x,\partial)u - \lambda_0 u = f$$

$$B_k^{(r)}(x,\partial)u - g_{k|\partial\Omega} = 0$$

has a unique solution u in

$$W^{2m,p}(\Omega), \qquad \forall f \in L^p(\Omega), \qquad \forall (g_k)_{k=1}^m \in \prod_{k=1}^m W^{2m-m_k,p}(\Omega),$$

which belongs to $W^{2m+1,p}(\Omega)$ if $f \in W^{1,p}(\Omega)$, $(g_k)_{k=1}^m \in \prod W^{2m-m_k+1,p}(\Omega)$. The usual interpolation argument proves that, if $f \in \lambda_p^\theta(\Omega)$, $(g_k)_{k=1}^m \in \prod_{k=1}^m \lambda_p^{2m-m_k+\theta}(\Omega)$, $u \in \lambda_p^{2m+\theta}(\Omega)$.

Now assume $u \in W^{2m,p}(\Omega)$ is a solution of (1), with $f \in \lambda_p^\theta(\Omega)$, $(g_k)_{k=1}^m \in \prod_{k=1}^m \lambda_p^{2m-m_k+\theta}(\Omega)$. Let $u^{(r)}$ be the solution (belonging to $\lambda_p^{2m+\theta}(\Omega)$) of

$$A^{(r)}(x,\partial)u^{(r)} - \lambda_0 u^{(r)} = f - \lambda_0 u$$

$$B_k^{(r)}(x,\partial)u^{(r)} - g_{k|\partial\Omega} = 0.$$

One has

$$A(x,\partial)u^{(r)} = [A(x,\partial) - A^{(r)}(x,\partial)]u^{(r)} - \lambda_0 u^{(r)} + f - \lambda_0 u,$$

$$B_k(x,\partial)u^{(r)} + [B_k^{(r)}(x,\partial) - B_k(x,\partial)]u^{(r)} - g_{k|\partial\Omega} = 0$$

and, using (2), $\|u^{(r)}\|_{2m+\theta,p,\Omega} \leqslant C$, independent of r.

Then, for $r, s \in \mathbf{N}$,

$$A(x, \partial)(u^{(r)} - u^{(s)}) =$$
$$= \lambda_0(u^{(r)} - u^{(s)}) + [A(x, \partial) - A^{(r)}(x, \partial)]u^{(r)} - [A(x, \partial) - A^{(s)}(x, \partial)]u^{(s)},$$
$$B_k(x, \partial)(u^{(r)} - u^{(s)}) =$$
$$= [B_k(x, \partial) - B_k^{(r)}(x, \partial)]u^{(r)} - [B_k(x, \partial) - B_k^{(s)}(x, \partial)]u^{(s)}|_{\partial\Omega} .$$

The a priori estimate gives

$$\|u^{(r)} - u^{(s)}\|_{2m+\theta, p, \Omega} \leqslant C\|u^{(r)} - u^{(s)}\|_{\theta, p, \Omega} + \varepsilon(r, s)$$

with $\varepsilon(r, s) \to 0$ as $r, s \to +\infty$.

It is easily seen that $\|u^{(r)} - u\|_{2m, p, \Omega} \to 0$ $(r \to +\infty)$. From this the result follows.

The following proposition is on the lines of inequality (2.13) in [TE1]:

PROPOSITIOH 1.5. *Assume* $(A1)$-$(A3)$, $(B1)$-$(B3)$ *are satisfied. Let* $\lambda = |\lambda|e^{i\phi}$, *with* $|\phi| \leqslant \phi_0$, $1 < p < +\infty$, $0 < \theta < p^{-1}$. *There exists* $C \geqslant 0$ *such that, if* $u \in W^{2m,p}(\Omega)$ *solves* $A(x, \partial)u - \lambda u = f$ *in* Ω

$$B_k(x, \partial)u - g_k|_{\partial\Omega} = 0 \qquad k = 1, \ldots, m ,$$

with $f \in L^p(\Omega)$, $(g_k)_{k=1}^m \prod_{k=1}^m \lambda_p^{2m-m_k+\theta}(\Omega)$, $|\lambda| \geqslant C$,

$$(7) \quad \sum_{r=0}^{2m} |\lambda|^{(2m-r)/(2m)} \|u\|_{r,p,\Omega} \leqslant$$
$$\leqslant C\Big(\|f\|_{0,p,\Omega} + \sum_{k=1}^m |\lambda|^{(2m-m_k-\theta)/(2m)} \|g_k\|_{\theta,p,\Omega} + \sum_{k=1}^m |\lambda|^{-\theta/(2m)} \|g_k\|_{2m-m_k+\theta,p,\Omega}\Big) .$$

For the proof we need the following lemma:

LEMMA 1.6. *Assume* $p > 1$, $0 < \theta < p^{-1}$. *If* $\{\chi_r\}_{0 \leqslant r \leqslant 1}$ *is a family of* C^1 *functions in* $\mathbb{R}$ *such that*

(*a*) $\chi_r(t) = 0$ *if* $t > r$, $|\chi_r(t)| \leqslant C$ (*independent of* r),

(*b*) $r|\chi_r'(t)| \leqslant C$ $\forall t \in \mathbb{R}$,

we pose, for $g \in \lambda_p^\theta(\mathbb{R}_+^n)$, $g_r(x) = \chi_r(x_n)g(x)$.

Then $\|g_r\|_{\theta,p,\mathbb{R}_+^n} \leqslant A\|g\|_{\theta,p,\mathbb{R}_+^n}$, *with* A *independent of* g *and* r.

PROOF. As $\theta < p^{-1}$, if we define $g^\wedge(x) = g(x)$ if $x_n \geqslant 0$, $g^\wedge(x) = 0$ if $x_n < 0$, we have that $g^\wedge \in \lambda_p^\theta(\mathbf{R}^n)$ and $\|g^\wedge\|_{\theta,p,\mathbf{R}^n} \leqslant C\|g\|_{\theta,p,\mathbf{R}_+^n}$ (see the result of [SH] and interpolate).

One has also

$$\|g_r\|_{\theta,p,\mathbf{R}_+} \leqslant \|g^\wedge\|_{\theta,p,\mathbf{R}^n} = \|g^\wedge \chi_r(x_n)\|_{0,p,\mathbf{R}^n} + \\ + \sup_{h\neq 0}\Big(|h|^{-\theta p}\int_{\mathbf{R}^n}|g^\wedge(x+h)\chi_r(x_n+h_n) - g^\wedge(x)\chi_r(x_n)|^p\,dx\Big)^{1/p}.$$

First of all,

$$\|g^\wedge \chi_r(x_n)\|_{0,p,\mathbf{R}^n} \leqslant C\|g^\wedge\|_{0,p,\mathbf{R}^n}.$$

Also, if $h_n = 0$,

$$\Big(|h|^{-\theta p}\int_{\mathbf{R}^o}|g^\wedge(x+h)\chi_r(x_n+h_n) - g^\wedge(x)\chi_r(x_n)|^p\,dx\Big)^{1/p} \leqslant C\|g^\wedge\|_{\theta,p,\mathbf{R}^n}.$$

Assume $h = (0, \eta)$, with $\eta \geqslant r$. Then

$$\Big(|h|^{-\theta,p}\int_{\mathbf{R}^n}|g^\wedge(x+h)\chi_r(x_n+h_n) - g^\wedge(x)\chi_r(x_n)|^p\,dx\Big)^{1/p} =$$

$$= \eta^{-\theta}\Big(\int_{-\eta\leqslant x_n\leqslant r-\eta}|\chi_r(x_n+\eta)g^\wedge(x+\eta e^n)|^p\,dx + \int_{0\leqslant x_n\leqslant r}|\chi_r(x_n)g^\wedge(x)|^p\,dx\Big)^{1/p} \leqslant$$

$$\leqslant C\eta^{-\theta}\Big(\int_{0\leqslant x_n\leqslant r}|g^\wedge(x)|^p\,dx\Big)^{1/p} = C\eta^{-\theta}\Big(\int_{0\leqslant x_n\leqslant r}|g^\wedge(x) - g^\wedge(x - re^n)|^p\,dx\Big)^{1/p} \leqslant \\ \leqslant C\eta^{-\theta}r^\theta\|g^\wedge\|_{\theta,p,\mathbf{R}^n} \leqslant C\|g^\wedge\|_{\theta,p,\mathbf{R}^n}.$$

Assume $h = (0, \eta)$, $0 < \eta < r$.

$$\Big(|h|^{-\theta p}\int_{\mathbf{R}^n}|g^\wedge(x+h)\chi_r(x_n+h_n) - g^\wedge(x)\chi_r(x_n)|^p\,dx\Big)^{1/p} =$$

$$= \eta^{-\theta}\Big(\int_{-\eta\leqslant x_n\leqslant 0}|\chi_r(x_n+\eta)g^\wedge(x+\eta e^n)|^p\,dx + \\ + \int_{0\leqslant x_n\leqslant r-\eta}|\chi_r(x_n+\eta)g^\wedge(x+\eta e^n) - \chi_r(x_n)g(x)|^p\,dx + \\ + \int_{r-\eta\leqslant x_n\leqslant r}|\chi_r(x_n)g^\wedge(x)|^p\,dx\Big)^{1/p}.$$

One has

$$\int\limits_{-\eta\leqslant x_n\leqslant 0} |\chi_r(x_n+\eta)g^\wedge(x+\eta e^n)|^p\,dx \leqslant C \int\limits_{0\leqslant x_n\leqslant \eta} |g(x)|^p\,dx \leqslant (\eta^\theta \|g^\wedge\|_{\theta,p,\mathbf{R}^n})^p \cdot$$

$$\cdot\Big(\int\limits_{0\leqslant x_n\leqslant r-\eta} |\chi_r(x_n+\eta)g^\wedge(x+\eta e^n) - \chi_r(x_n)g(x)|^p\,dx\Big)^{1/p} \leqslant$$

$$\leqslant \Big(\int\limits_{0\leqslant x_n\leqslant r-\eta} |\chi_r(x_n+\eta)-\chi_r(x_n)|^p\,|g^\wedge(x+\eta e^n)|^p\,dx\Big)^{1/p} +$$

$$+\Big(\int\limits_{0\leqslant x_n\leqslant r-\eta} |\chi_r(x_B)|^p\,|g^\wedge(x+\eta e^n) - g^\wedge(x)|^p\,dx\Big)^{1/p} \leqslant$$

$$\leqslant \text{const}\Big[\Big(\int\limits_{\eta\leqslant x_n\leqslant r} \eta^p r^{-p} |g^\wedge(x)|^p\,dx\Big)^{1/p} +$$

$$+\Big(\int\limits_{0\leqslant x_n\leqslant r-\eta} |g^\wedge(x+\eta e^n) - g^\wedge(x)|^p\,dx\Big)^{1/p}\Big] =$$

$$= \text{const}\Big[\Big(\int\limits_{\eta\leqslant x_n\leqslant r} \eta^p r^{-p} |g^\wedge(x) - g^\wedge(x-re^n)|^p\,dx\Big)^{1/p} + \eta^\theta \|g^\wedge\|_{\theta,p,\mathbf{R}^n}\Big] \leqslant$$

$$\leqslant \text{const}\,\eta^\theta((\eta/r)^{1-\theta}+1)\,\|g^\wedge\|_{\theta,p,\mathbf{R}^n} \leqslant \text{const}\,\eta^\theta \|g^\wedge\|_{\theta,p,\mathbf{R}^n} \cdot$$

$$\cdot \int\limits_{r-\eta\leqslant x_n\leqslant r} |\chi_r(x_n)g^\wedge(x)|^p\,dx = \int\limits_{r-\eta\leqslant x_n\leqslant r} |\chi_r(x_n) - \chi_r(x_n+\eta)|^p\,|g^\wedge(x)|^p\,dx \leqslant$$

$$\leqslant \text{const}\,r^{-p}\eta^p \int\limits_{r-\eta\leqslant x_n\leqslant r} |g^\wedge(x)|^p\,dx \leqslant \text{const}(r^{(\theta-1)}\eta \|g^\wedge\|_{\theta,p,\mathbf{R}^n})^p\,.$$

The case $h=(0,\eta)$, with $\eta<0$, can be treated in the same way. So,

$$\|g_r\|_{\theta,p,\mathbf{R}^n_+} \leqslant C\|g^\wedge\chi_r(x_n)\|_{\theta,p,\mathbf{R}^n} \leqslant C\|g^\wedge\|_{\theta,p,\mathbf{R}^n} \leqslant C\|g\|_{\theta,p,\mathbf{R}^n_+}$$

and the result is proved.

Proof of proposition 1.5. From [AG1] one draws the existence of $C>0$ such that, if $|\lambda|\geqslant C$, $|\text{Arg}\,\lambda|\leqslant\phi_0$,

$$(8)\quad \sum_{r=0}^{2m} |\lambda|^{(2m-r)/(2m)}\|u\|_{r,p,\Omega} \leqslant$$

$$\leqslant C\Big(\|f\|_{0,p,\Omega} + \sum_{k=1}^{m} |\lambda|^{(2m-m_k)/(2m)}\|g_k\|_{0,p,\Omega} + \sum_{k=1}^{m} \|g_k\|_{2m-m_k,p,\Omega}\Big)\,.$$

For every $x^0 \in \partial\Omega$ there exists an open neighbourhood U of x^0, $\Phi: U \to \mathbb{R}^n$ of class C^{2m+1}, such that

$$\Phi(x^0) = 0\,, \qquad \Phi(U \cap \partial\Omega) = \Phi(U) \cap \{(y', 0): y' \in \mathbb{R}^{n-1}\}\,,$$
$$\Phi(U \cap \Omega) = \Phi(U) \cap \mathbb{R}^n_+\,.$$

As Ω is bounded, there exist $x_1^0, \ldots, x_N^0$, with corresponding neighbourhoods $U_1, \ldots, U_N$ and diffeomorphisms $\Phi_1, \ldots, \Phi_N$, such that $\partial\Omega \subseteq \bigcup_{j=1}^{N} U_j$: Let $\{\psi_1, \ldots, \psi_N\}$ be a partition of unity in a neighbourhood of $\partial\Omega$, such the supp $\psi_j \subseteq U_j$ $(j = 1, \ldots, N)$. Let $(\chi_r)_{0<r\leqslant 1}$ be a family of C^∞ functions in $\mathbb{R}$, such that $\chi_r(t) = 1$ if $|t| \leqslant r/2$, $\chi_r(t) = 0$ if $t \geqslant r$, $|\chi_r^{(j)}(t)| \leqslant C_j r^{-j}$ and put $g_{k,r}(x) = \sum_{j=1}^{N} g_{k,r,j}(x)$ with

$$g_{k,r,j}(x) = \psi_j(x) g_k(x) \chi_r\big(\Phi_j(x)_n\big)\,.$$

Then, $\forall r > 0$, $B_k(x, \partial)u - g_{k,r|\partial\Omega} = 0$, so that $\forall r > 0$

$$\sum_{j=0}^{2m} |\lambda|^{(2m-j)/(2m)} \|u\|_{j,p,\Omega} \leqslant$$
$$\leqslant C\Big(\|f\|_{0,p,\Omega} + \sum_{k=1}^{m} |\lambda|^{(2m-m_k)/(2m)} \|g_{k,r}\|_{0,p,\Omega} + \sum_{k=1}^{m} \|g_{k,r}\|_{2m-m_k,p,\Omega}\Big)\,.$$

Now we remark that, if $g \in \lambda_p^\theta(\mathbb{R}^n_+)$, $1 < p < +\infty$, $0 < \theta < p^{-1}$ and $\operatorname{supp} g \subseteq \{x \in \mathbb{R}^n: 0 \leqslant x_n \leqslant r\}$, $\|g\|_{\theta,p,\mathbb{R}^n_+} \leqslant r^\theta \|g\|_{\cdot\theta,p,\mathbb{R}^n_+}$ This implies that

$$\|g_{k,r}\|_{0,p,\Omega} \leqslant C \sum_{j=1}^{N} \|g_{k,r,j} \circ \Phi_j^{-1}\|_{0,p,\mathbb{R}^u_+} \leqslant C r^\theta \sum_{j=1}^{N} \|g_{k,r,j} \circ \Phi_j^{-1}\|_{0,p,\mathbb{R}_+} \leqslant$$
$$\leqslant (\text{by lemma 1.6})\ C r^\theta \sum_{j=1}^{N} \|(\psi_j \circ \Phi_j^{-1})(g_k \circ \Phi_j^{-1})\|_{\theta,p,\mathbb{R}^n_+} \leqslant C r^\theta \|g_k\|_{\theta,p,\Omega}\,.$$

With the same method,

$$\|g_{k,r}\|_{2m-m_k,p,\Omega} \leqslant C \sum_{i=0}^{2m-m_k} r^{\theta-i} \|g_k\|_{2m-m_k-i+\theta,p,\Omega}\,,$$

so that

$$|\lambda|^{(2m-m_k)/(2m)}\|g_{k,r}\|_{0,p,\Omega}+\sum_{k=1}^{m}\|g_{k,r}\|_{2m-m_k,p,\Omega}\leqslant$$
$$\leqslant C\Big(|\lambda|^{(2m-m_k)/(2m)}r^{\theta}|g_k|_{\theta,p,\Omega}+\sum_{i=0}^{2m-m_k}r^{\theta-i}\|g_k\|_{2m-m_k-i+\theta,p,\Omega}\Big)\,.$$

Choosing $r=|\lambda^{-1/(2,m)}$ and using the moment inequality implied by prop. 1.3, the result follows.

2. Linear autonomous parabolic problems.

Now we shall use the results of paragraph 1 to obtain a maxima regularity result for linear autonomous parabolic problems like

$$(9)\quad\begin{cases}\dfrac{\partial u}{\partial t}(t,x)=A(x,\partial)u(t,x)+f(t,x)\,, & t\in[0,T]\,,\quad x\in\Omega\\ B_k(x,\partial)u(t,x)=g_k(t,x)\,, \quad t\in[0,T]\,, & x\in\partial\Omega\,,\quad 1\leqslant k\leqslant m\,.\\ u(0,x)=u_0(x)\,,\quad x\in\Omega\,. & \end{cases}$$

As already declared, the main tool will be an explicit representation of the solution, following a method developed in [TE1]. We remark that theorems of maximal regularity with nonautonomous boundary conditions were also obtained, for example, by Solonnikov in a series of papers (see [SO1, 2]) in the more general case of systems, by Agranovich-Vishik [AV] (anisotropic Sobolev spaces), Lunardi [LU] and [TE2] (spaces of continuous and hölder continuous functions). However the result of theorem 2.1 seems to be new.

Now, assume that $A(x,\partial)=\sum_{|\alpha|\leqslant 2m}a_\alpha(x)\,\partial^\alpha$, $B_k(x,\partial)=\sum_{|\beta|\leqslant m_k}b_{k,\beta}(x)\,\partial^\beta$ satisfy the assumptions $(A1)$-$(A3)$, $(B1)$-$(B4)$. Fix $p\in\,]1,+\infty[$. Define:

$$(10)\quad\begin{cases}D(A)=\{u\in W^{2m,p}(\Omega)\colon B_k(x,\partial)u|_{\partial\Omega}=0,\ k=1,\dots,m\}\,.\\ Au(x)=A(x,\partial)u(x)\,.\end{cases}$$

A is the infinitesimal generator of an analytic semigroup $\{\exp(tA)$:

$t \geqslant 0\}$ in $L^p(\Omega)$. For simplicity, we shall assume (this is not restrictive at all) that $\sigma(A) \subseteq \{z \in \mathbb{C}: \operatorname{Re} z < 0\}$. Fix $\theta \in]0, p^{-1}[$ and let u be a solution of (9), $u \in C^1([0, T]; \lambda_p^\theta(\Omega)) \cap C([0, T]; \lambda_k^{2m+\theta}(\Omega))$ (here and in the following we identify mappings of (t, x) with corresponding mappings of t with values in a space of functions of domain Ω).

Then necessarily $f \in C([0, T]; \lambda_p^\theta(\Omega))$, $u_0 \in \lambda_p^{2m+\theta}(\Omega)$.

Further, $B_k(., \partial) u \in C([0, T]; \lambda_p^{2m-m_k+\theta}(\Omega))$ $(k = 1, \ldots, m)$. By the moment inequality, which is a consequence of proposition 1.3, $u \in \lambda^{1-m_k/(2m)}([0, T]; \lambda_p^{m_k+\theta}(\Omega))$ (if E is a Banach space, $\alpha \in]0, 1[$, $\lambda^\alpha([0, T]; E) = \{u \in C([0, T]; E): \lim_{\delta \to 0} \sup \{(t-s)^{-\alpha} \|u(t) - u(s)\|_E: s, t \in \in [0, T], 0 < |t-s| < \delta\} = 0\}$, $\lambda^0([0, T]; E) = C([0, T]; E)$, $\lambda^1([0, T]; E) = = C^1([0, T]; E)$). This implies

$$B_k(\cdot, \partial) u \in \lambda^{1-m_k/(2m)}([0, T]; \lambda_p^\theta(\Omega)) .$$

Therefore, we can assume

$$g_k \in C([0, T]; \lambda_p^{2m-m_k+\theta}(\Omega)) \cap \lambda^{1-m_k/(2m)}([0, T]; \lambda_p^\theta(\Omega)) .$$

Finally, necessarily, $B_k(\cdot, \partial) u_0 - g_k(0)|_{\partial\Omega} = 0$, for $k = 1, .., m$.

Now, suppose $\lambda \in \varrho(A)$. For $k = 1, \ldots, m$ the problem

$$\begin{aligned} \lambda u - A(\cdot, \partial) u &= 0 \text{ in } \Omega \\ B_r(\cdot, \partial) u|_{\partial\Omega} &= 0 \quad \text{ if } r \neq k \\ B_k(\cdot, \partial) u - g|_{\partial\Omega} &= 0 \end{aligned}$$

has a unique solution $u = N_k(\lambda) g$, $\forall g \in W^{2m-m_k, p}(\Omega)$. By proposition 1.4, if $g \in \lambda_p^{2m-m_k+\theta}(\Omega)$, $u \in \lambda_p^{2m+\theta}(\Omega)$ and

$$\text{(11)} \quad \sum_{r=0}^{2m} |\lambda|^{(2m-r)/(2m)} \|N_k(\lambda) g\|_{r,p,\Omega} \leqslant \\ \leqslant C\left((1 + |\lambda|)^{(2m-m_k-\theta)/(2m)} \|g\|_{\theta,p,\Omega} + (1 + |\lambda|)^{-\theta/(2m)} \|g\|^{2m-m_k+\theta,p,\Omega}\right) .$$

Put

$$\gamma = \{r \exp(-i\phi_0): r \in [0, +\infty[\} \cup \{r \exp(i\phi_0): r \in [0, +\infty[\} ,$$

oriented from $-\infty \exp(-i\phi_0)$ to $+\infty \exp(i\phi_0)$.

We have the following

THEOREM 2.1. *Assume* (*A*1)-(*A*3), (*B*1)-(*B*4) *are satisfied.*
Let $p \in]1, +\infty[$, $0 < \theta < p^{-1}$, $f \in C([0, T]; \lambda_p^\theta(\Omega))$ *and, for* $k = 1, \dots, m$

$$g_k \in C([0, T]; \lambda_p^{2m-m_k+\theta}(\Omega)) \cap \lambda^{1-m_k/(2m)}([0, T]; \lambda_p^\theta(\Omega)) ,$$

$$B_k(\cdot, \partial)u_0 - g_k(0)|_{\partial\Omega} = 0 .$$

Then, (9) *has a unique solution*

$$u \in C^1([0, T]; \lambda_p^\theta(\Omega)) \cap C([0, T]; \lambda_p^{2m+\theta}(\Omega)) ,$$

$$u(t) = \exp(tA)u_0 + \int_0^t \exp((t-s)A) f(s)\, ds + \sum_{k=1}^m \int_0^t K_k(t-s) g_k(s)\, ds , \tag{12}$$

with $K_k(t) = (2\pi i)^{-1}\int_\gamma \exp(\lambda t) N_k(\lambda)\, d\lambda$ $(t \in]0, +\infty[)$.

We start by remarking that, owing to (11),

$$\|K_k(t)g\|_{0,p,\Omega} \leqslant C\big(t^{(m_k+\theta)/(2m)-1}\|g\|_{\theta,p,\Omega} + t^{\theta/(2m)-1}\|g\|_{2m-m_k+\theta,p,\Omega}\big) ,$$

$\forall g \in \lambda_p^{2m-m_k+\theta}(\Omega)$, so that the last integral in (12) converges, at least in $L^p(\Omega)$. To prove theorem 2.1, we start by considering the first integral in (12):

LEMMA 2.2. *Let*

$$0 < \theta < p^{-1} , \qquad f \in C([0, T]; \lambda_p^\theta(\Omega)) , \qquad u(t) = \int_0^t \exp((t-s)A) f(s)\, ds .$$

Then, $u \in C^1([0, T]; \lambda_p^\theta(\Omega)) \cap C([0, T]; \lambda_p^{2m+\theta}(\Omega))$ *and solves*

$$\frac{\partial u}{\partial t}(t, x) = A(x, \partial)u(t, x) + f(t, x) , \qquad t \in [0, T] , \quad x \in \Omega$$

$$B_k(x, \partial)u(t, x) = 0 , \qquad t \in [0, T] , \quad x \in \partial\Omega , \quad 1 \leqslant k \leqslant m .$$

$$u(0, x) = 0 , \qquad x \in \Omega .$$

PROOF. As $0 < \theta < p^{-1}$, $\lambda_p^\theta(\Omega) = (L^p(\Omega), D(A))_{\theta/(2m)}$ (see [DPG], th. 6.10). So, by [DPG] th. 3.1, we conclude that $u \in C([0, T]; W^{2m,p}(\Omega))$ and $A(\cdot, \partial)u \in C([0, T]; \lambda_p^\theta(\Omega))$. Therefore the result follows from proposition 1.4.

LEMMA 2.3. *Let*

$$1 \leqslant k \leqslant m\,, \quad g \in C([0, T]; \lambda_p^{2m-m_k+\theta}(\Omega)) \cap \lambda^{1-m_k/(2m)}([0, T]; \lambda_p^\theta(\Omega))\,,$$

with $g(0) = 0$. *Put*

$$u(t) = \int_0^t K_k(t-s)g(s)\,ds\,.$$

Then u is the only solution in $C^1([0, T]; L^p(\Omega)) \cap C([0, T]; W^{2m,p}(\Omega))$ *of*

$$\begin{aligned}
&\frac{\partial u}{\partial t}(t, x) = A(x, \partial)u(t, x)\,, \qquad t \in [0, T]\,, \quad x \in \Omega\\
&B_r(x, \partial)u(t, x) = 0\,, \qquad t \in [0, T]\,, \quad x \in \partial\Omega\,, \quad r \neq k\\
&B_k(x, \partial)u(t, x) = g(t, x)\,, \qquad t \in [0, T]\,, \quad x \in \partial\Omega\\
&u(0, x) = 0\,.
\end{aligned}$$

PROOF. One has $u(t) = u_0(t) + u_1(t)$, with

$$u_0(t) = \int_0^t K_k(t-s)g(t)\,ds\,, \qquad u_1(t) = \int_0^t K_k(t-s)[g(s) - g(t)]\,ds\,.$$

We indicate with γ_t the positively oriented boundary of

$$\{\lambda \in \mathbf{C}\colon |\mathrm{Arg}\,\lambda| \leqslant \varphi_0\} \cap \{\lambda \in \mathbf{C}\colon |\lambda| \geqslant t^{-1}\} \qquad (t > 0)\,.$$

Then

$$u_0(t) = (2\pi i)^{-1}\int_{\gamma_t} \lambda^{-1} \exp[\lambda t) N_k(\lambda) g(t)\,d\lambda\,.$$

$$\begin{aligned}
\|u_0(t)\|_{2m,p,\Omega} \leqslant\; & C\int_{\gamma_t} \exp(t\,\mathrm{Re}\,\lambda)\big(|\lambda|^{-(m_k+\theta)/(2m)}\|g(t)\|_{\theta,p,\Omega} +\\
&+ |\lambda|^{-1-\theta/(2m)}\|g(t)\|_{2m-m_k+\theta.p,\Omega}|d\lambda| \leqslant\\
&\leqslant Ct^{\theta/(2m)}\int_{\gamma_1} \exp(\mathrm{Re}\,\lambda)\big(|\lambda|^{-(m_k+\theta)/(2m)}\|g\|_{\lambda^{1-m_k(2m)}([0,T];\lambda_p^\theta(\Omega))} +\\
&\qquad + |\lambda|^{-1-\theta/(2m)}\|g\|_{C([0,T];\lambda_p^{2m-m_k+\theta}(\Omega))}\big)|d\lambda|\,.
\end{aligned}$$

This proves that

$$u_0(t) \in W^{2m,p}(\Omega)\,, \qquad \forall t \in\,]0, T]\,, \qquad \|u_0(t)\|_{2m,p,\Omega} \to 0 \qquad (t \to 0)\,.$$

Further, $\forall t \in [0, T]$,

$$A(\cdot, \partial)\big(\lambda^{-1} \exp(\lambda t) N_k(\lambda) g(t)\big) = \exp(\lambda t) N_k(\lambda) g(t)$$

and

$$\|A(\cdot, \partial)(\lambda^{-1} \exp(\lambda t) N_k(\lambda) g(t)\|_{0,p,\Omega} \leqslant C \|\lambda^{-1} \exp(\lambda t) N_k(\lambda) g(t)\|_{2m,p,\Omega}$$

so that we have

$$A(\cdot, \partial) u_0(t) = \\ = (2\pi i)^{-1} \int_\gamma \exp(\lambda t) N_k(\lambda) g(t)\, d\lambda \qquad \text{if } t \in\,]0, T]\,, \quad A(\cdot, \partial) u_0(0) = 0\,.$$

If $r \neq k$,

$$T_0 B_r(\cdot, \partial)\big(\lambda^{-1} \exp(\lambda t) N_k(\lambda) g(t)\big) = 0$$

and this implies

$$T_0 B_r(\cdot, \partial) u_0(t) = 0 \qquad \forall t \in [0, T]\,, \qquad r \neq k\,.$$

Further,

$$T_0 B_k(\cdot, \partial)\big(\lambda^{-1} \exp(\lambda t) N_k(\lambda) g(t)\big) = \lambda^{-1} \exp(\lambda t) T_0 g(t)$$

and so

$$T_0 B_k(\cdot, \partial) u_0(t) = T_0\Big((2\pi i)^{-1} \int_{\gamma_1} \lambda^{-1} \exp(\lambda t) g(t)\, d\lambda\Big) = T_0 g(t)$$

by the residue theorem, as the integral is absolutely convergent in $W^{2m-m_k,p}(\Omega)$. So

$$u_0 \in C\big([0, T]; W^{2m,p}(\Omega)\big)\,,$$
$$A(\cdot, \partial) u_0(t) = (2\pi i)^{-1} \int_\gamma \exp(\lambda t) N_k(\lambda) g(t)\, d\lambda\,,$$
$$u_1(t) = (2\pi i)^{-1} \int_0^t \Big(\int_\gamma \exp\big(\lambda(t-s)\big) N_k(\lambda)[g(s) - g(t)]\, d\lambda\Big)\, ds\,.$$

Analogous considerations take to $u_1 \in C([0, T]; W^{2m,p}(\Omega))$,

$$A(\cdot, \partial)u_1(t) = (2\pi i)^{-1}\int_0^t \left(\int_\gamma \lambda \exp(\lambda(t-s)) N_k(\lambda)[g(s) - g(t)]\, d\lambda\right) ds ,$$

$$T_0 B_r(\cdot, \partial)u_1(t) = 0 \quad \text{if } r \neq k ,$$

$$T_0 B_k(\cdot, \partial)u_1(t) = T_0\left((2\pi i)^{-1}\int_0^t \left(\int_\gamma \exp(\lambda(t-s))[g(s) - g(t)]\, d\lambda\right) ds\right) = 0$$

(as $(\lambda, s) \to \exp(\lambda(t-s))[g_k(s) - g_k(t)] \in L^1([0, t]\times\gamma; W^{2m-m_k,p}(\Omega))$, applying Fubini and Cauchy theorems). So we can say that $u \in C([0, T]; W^{2m,p}(\Omega))$,

$$A(\cdot, \partial)u(t) = (2\pi i)^{-1}\int_\gamma \exp(\lambda t) N_k(\lambda) g(t)\, d\lambda +$$

$$+ (2\pi i)^{-1}\int_0^t \left(\int_\gamma \lambda \exp(\lambda(t-s)) N_k(\lambda)[g(s) - g(t)]\, d\lambda\right) ds ,$$

$$T_0 B_r(\cdot, \partial)u(t) = 0 \quad \text{if } t \neq k ,$$

$$T_0 B_k(\cdot, \partial)u(t) = T_0 g(t) .$$

Now fix $\varepsilon \in]0, T[$ and define, for $t \in [\varepsilon, T]$,

$$u_\varepsilon(t) = (2\pi i)^{-1}\int_0^{t-\varepsilon} \left(\int_\gamma \exp(\lambda(t-s)) N_k(\lambda) g(s)\, d\lambda\right) ds .$$

Thinking of u_ε as a function with values in $L^p(\Omega)$, one can differentiate and obtain

$$u'_\varepsilon(t) = (2\pi i)^{-1}\int_\gamma \exp(\varepsilon\lambda) N_k(\lambda) g(t-\varepsilon)\, d\lambda +$$

$$+ (2\pi i)^{-1}\int^{t-\varepsilon} \left(\int_\gamma \lambda \exp(\lambda(t-s)) N_k(\lambda) g(s)\, d\lambda\right) ds =$$

$$= (2\pi i)^{-1}\int_\gamma \exp(\varepsilon\lambda) N_k(\lambda)[g(t-\varepsilon) - g(t)]\, d\lambda +$$

$$+ (2\pi i)^{-1}\int_0^{t-\varepsilon}\Big(\int_\gamma \lambda \exp\big(\lambda(t-s)\big) N_k(\lambda)[g(s)-g(t)]\, d\lambda\Big)\, ds +$$

$$+ (2\pi i)^{-1}\int_\gamma \exp(\lambda t) N_k(\lambda) g(t)\, d\lambda .$$

$$\Big\|(2\pi i)^{-1}\int_\gamma \exp(\varepsilon\lambda) N_k(\lambda)[g(t-\varepsilon)-g(t)]\, d\lambda\Big\|_{0,p,\Omega} \leqslant$$

$$\leqslant \text{const}\int_{\gamma_1} \exp(\operatorname{Re}\lambda)\big[|\lambda|^{-(m_k+\theta)/(2m)} + |\lambda|^{-1-\theta/(2m)}\big]\varepsilon^{\theta/(2m)}|d\lambda| \to 0 \qquad (\varepsilon\to 0) .$$

So we have proved that, for every $t \in]0, T]$, $u'(t)$ exists in $L^p(\Omega)$ and equals $A(\cdot, \partial)u(t)$. A passage to the limit as $t \to 0$ gives the final result.

Lemma 2.4. *Let*

$$1 \leqslant k \leqslant m, \quad g \in C\big([0, T]; \lambda_p^{2m-m_k+\theta}(\Omega)\big) \cap \lambda^{1-m_k/(2m)}\big([0, T]; \lambda_p^\theta(\Omega)\big),$$

with $g(0) = 0$ *and* $1 \leqslant m_k \leqslant 2m-1$. *If* $u(t) = \int_0^t K_k(t-s) g(s)\, ds$,

$$u \in C^1\big([0, T]; \lambda_p^\theta(\Omega)\big) \cap C\big([0, T]; \lambda_p^{2m+\theta}(\Omega)\big) .$$

Proof. By Lemma 2.4,

$$u \in C^1\big([0, T]; L^p(\Omega)\big) \cap C\big([0, T]; W^{2m,p}(\Omega)\big),$$

$$A(\cdot, \partial) u(t) = (2\pi i)^{-1}\int_\gamma \exp(\lambda t) N_k(\lambda) g(t)\, d\lambda +$$

$$+ (2\pi i)^{-1}\int_0^t\Big(\int_\gamma \lambda \exp\big(\lambda(t-s)\big) N_k(\lambda)[g(s)-g(t)]\, d\lambda\Big)\, ds .$$

As already mentioned, $\lambda_p^\theta(\Omega) = \big(L^p(\Omega), D(A)\big)_{\theta/(2m)}$; so, by theorem 2.5 in [DPG], if $f \in L^p(\Omega)$, then $f \in \lambda_p^\theta(\Omega)$ if and only if

$$\lim_{\xi\to\infty} \xi^{\theta/(2m)} A(\xi - A)^{-1} f = 0.$$

Further, an equivalent norm in $\lambda_p^\theta(\Omega)$ is

$$\|f\|_{\theta,p,\Omega} + \sup_{\xi\geqslant 1} \|\xi^{\theta/(2m)}A(\xi - A)^{-1}f\|_{0,p,\Omega}\,.$$

We pose, for

$$0<\delta\leqslant T\,,\qquad \chi(\delta) = \sup\{(t-s)^{m_k/(2m)-1}\|g(t)-g(s)\|_{\theta,p,\Omega} + \\ + \|g(t)-g(s)\|_{2m-m_k+\theta,p,\Omega}\colon 0\leqslant s<t\leqslant T,\, t-s\leqslant\delta\}\,.$$

We recall that $\chi(\delta)\to 0$ $(\delta\to 0)$.
One has

$$\xi^{\theta/(2m)}A(\xi-A)^{-1}A(\cdot,\partial)u(t) = \\ = (2\pi i)^{-1}\xi^{\theta/(2m)}\int_\gamma \lambda(\xi-\lambda)^{-1}\exp(\lambda t)N_k(\lambda)g(t)\,d\lambda + \\ + (2\pi i)^{-1}\xi^{\theta/(2m)}\int_0^t\Big(\int_\gamma\lambda(\xi-\lambda)^{-1}\exp\big(\lambda(t-s)\big)N_k(\lambda)[g(s)-g(t)]\,d\lambda\Big)\,ds\,.$$

$$\Big\|(2\pi i)^{-1}\xi^{\theta/(2m)}\int_\gamma\lambda(\xi-\lambda)^{-1}\exp(\lambda t)N_k(\lambda)g(t)\,d\lambda\Big\|_{0,p,\Omega}\leqslant$$

(owing to proposition 1.5)

$$\leqslant \text{const}\,\chi(t)\Big(t^{1-m_k/(2m)}\xi^{\theta/(2m)}\int_0^{+\infty} r^{1-(m_k+\theta)/(2m)}|r\exp(i\phi_0) - \\ -\xi|^{-1}\exp\big(tr\cos(\phi_0)\big)\,dr\Big) + \xi^{\theta/(2m)}\int_0^{+\infty} r^{-\theta/(2m)}|r\exp(i\phi_0)-\xi|^{-1}\cdot \\ \cdot\exp\big(tr\cos(\phi_0)\big)\,dr = \text{const}\,\chi(t)\big(I_1(\xi,t)+I_2(\xi,t)\big)\,.$$

If $\tau=\xi t$,

$$I_1(\xi,t) = \tau^{\theta/(2m)-1}\int_0^{+\infty} r^{1-(m_k+\theta)/(2m)}|r\tau^{-1}\exp(i\phi_0)-1|^{-1}\exp\big(r\cos(\phi_0)\big)\,dr\to 0\,.$$

On the other hand,

$$I_1(\xi,t) = \\ = \tau^{\theta/(2m)}\int_0^{+\infty} r^{1-(m_k+\theta)/(2m)}|r\exp(i\phi_0)-\tau|^{-1}\exp\big(r\cos(\phi_0)\big)\,dr\to 0(\tau\to 0)\,.$$

Further,

$$I_2(\xi, t) = \tau^{\theta/(2m)}\int_0^{+\infty} r^{-\theta/(2m)}|r\exp(i\phi_0) - \tau|^{-1}\exp\big(r\cos(\phi_0)\big)\,dr =$$

$$= \int_0^{+\infty} r^{-\theta/(2m)}|r\exp(i\phi_0) - 1|^{-1}\exp\big(\tau\, r\cos(\phi_0)\big)\,dr \leqslant$$

$$\leqslant \text{const (independent of } \tau),$$

so that

$$\sup_{\xi\geqslant 1}\|\xi^{\theta/(2m)}A(\xi - A)^{-1}A(\cdot, \partial)u_0(t)\|_{0,p,\Omega} \leqslant \text{const}\,\chi(t)\,. \tag{13}$$

It is also easily seen that, for a fixed t, $I_1(\xi, t) + I_2(\xi, t) \to 0$ $(\xi \to +\infty)$.

So $A(\cdot, \partial)u_0(t) \in \lambda_p^\theta(\Omega)$ $\forall t \in [0, T]$, $\|A(\cdot, \partial)u_0(t)\|_{\theta,p,\Omega} \to 0$ $(t \to 0)$ which implies $u_0(t) \in \lambda_p^{2m+\theta}(\Omega)$, $\|u_0(t)\|_{2m+\theta,p,\Omega} \to 0$ $(t \to 0)$ by proposition 1.4. Analogous estimates prove that $u_0(t) \in C\big([0, T]; \lambda_p^{m+\theta}(\Omega)\big)$.

Next, we consider $A(\cdot, \partial)u_1(t)$.

$$\xi^{\theta/(2m)}A(\xi - A)^{-1}A(\cdot, \partial)u_1(t) =$$

$$= (2\pi i)^{-1}\xi^{\theta/(2m)}\int_0^t\Big(\int_\gamma \lambda^2(\xi - \lambda)^{-1}\exp\big(\lambda(t - s)\big)N_k(\lambda)[g(s) - g(t)]\,d\lambda\Big)\,ds\,.$$

$$\|\xi^{\theta/(2m)}A(\xi - A)^{-1}A(\cdot, \partial)u_1(t)\|_{0,p,\Omega} \leqslant \Big[\text{const}\,\xi^{\theta/(2m)}\int_0^t\Big(\int_0^{+\infty} r^{2-(m_k+\theta)/(2m)}\cdot$$

$$\cdot|r\exp(\phi_0) - \xi|^{-1}\exp\big((t - s)\,r\cos\phi_0\big)\chi(t - s)(t - s)\,dr\Big)\,ds + \xi^{\theta/(2m)}\cdot$$

$$\cdot\int_0^t\Big(\int_0^{+\infty} r^{1-\theta/(2m)}|r\exp(i\phi_0) - \xi|^{-1}\exp\big((t - s)\,r\cos\phi_0\big)\chi(t - s)\,dr\Big)\,ds\Big] =$$

$$= \text{const}\big(I_1(\xi, t) + I_2(\xi, t)\big)\,.$$

$$I_1(\xi, t) = \xi^{0/(2m}\int_0^{+\infty} r^{-\theta/(2m)}|r\exp(i\phi_0) -$$

$$-\xi|^{-1}\Big(\int_0^{rt}\chi(s/r)\exp(s\cos\phi_0)\,s^{(2m-m_k)/(2m)}\,ds\Big)\,dr = \int_0^{+\infty} r^{-\theta/(2m)}\cdot$$

$$\cdot|1 - r\exp(i\phi_0)|^{-1}\Big(\int_0^{\xi r t}\chi\big(s/(\xi r)\big)\exp(s\cos\phi_0)\,s^{(2m-m_k)/(2=m)}\,ds\Big)\,dr \to 0$$

$(\xi \to +\infty)$ and is uniformly majorized by

$$\mathrm{const}\int_0^{+\infty} r^{-\theta/(2m)}|1 - r\exp(i\phi_0)|^{-1}\,dr\,.$$

Also,

$$I_2(\xi, t) \leqslant \mathrm{const}\int_0^{+\infty} r^{-\theta/(2m)}|1 - r\exp(i\phi_0)|^{-1}\left(\int^{\xi r t} \chi(s/(\xi r))\exp(s\cos\phi_0)\,ds\right)dr$$

which can be treated as the foregoing integral.

To prove that $u_1 \in C([0, T]; \lambda_p^{2m+\theta}(\Omega))$ it is sufficient to remark that, for

$$0 \leqslant t' \leqslant t \leqslant T\,, \qquad \xi^{\theta/(2m)}A(\xi - A)^{-1}A(\cdot, \partial)(u_1(t) - u_1(t')) =$$

$$= (2\pi i)^{-1}\xi^{\theta/(2m)}\int_{t'}^{t}\left(\int_{\gamma}\lambda^2(\xi - \lambda)^{-1}\exp(\lambda s)N_k(\lambda)[g(t - s) - g(t)]\,d\lambda\right)ds +$$

$$+ (2\pi i)^{-1}\xi^{\theta/(2m)}\int^{t'}\left(\int_{\gamma}\lambda^2(\xi - \lambda)^{-1}\exp(\lambda s)N_k(\lambda)\cdot\right.$$

$$\left.\cdot[g(t - s) - g(t) - g(t' - s) + g(t')]\,d\lambda\right)ds\,,$$

and to observe that

$$\|g(t - s) - g(t) - g(t' - s) + g(t')\|_{\theta,p,\Omega} \leqslant \chi(t - t')\,s^{1 - m_k(2m)}\,,$$

$$\|g(t - s) - g(t) - g(t' - s) + g(t')\|_{2m+\theta,p,\Omega} \leqslant \chi(t - t')$$

and apply methods analogous to the foregoing estimates.

So we have proved that $u \in C([0, T]; \lambda_p^{2m+\theta}(\Omega))$. As $(du/dt)(t) = A(\cdot, \partial)u(t)$, $u \in C^1([0, T]; \lambda_p^{\theta}(\Omega))$ and the lemma is completely established.

Proof of theorem 2.1. First of all, we remark that it is not restrictive to assume $B_k(x, \partial)u = u$ if $m_k = 0$, so that $B_k(\cdot, \partial)u - g_{k|\partial\Omega}$ becomes $u - g_{k|\partial\Omega} = 0$, with

$$g_k \in C^1([0, T]; \lambda_p^{\theta}(\Omega)) \cap C([0, T]; \lambda_p^{2m+\theta}(\Omega))\,.$$

Substituting $v = u - g_k$ to u, the problem is reduced to the case $g_k = 0$ if $m_k = 0$. Now, $u(t) = v_0(t) + v_1(t)$, with

$$v_0(t) = \sum_{k=1}^{m} \int^{t} K_k(t-s)[g_k(s) - g_k(0)]\,ds + \int_0^t \exp((t-s)A)\,f(s)\,ds\,,$$

$$v_1(t) = \exp(tA)\,u_0 + \sum_{k=1}^{m} \int_0^t K_k(t-s)\,g_k(0)\,\mathrm{d}s\,.$$

By lemmata 2.2 and 2.4, $v_0 \in C^1([0, T]; \lambda_p^\theta(\Omega)) \cap C([0, T]; \lambda_p^{2m+\theta}(\Omega))$.

$$v_1(t) = \exp(tA)u_0 + 2\pi i)^{-1}\int_{\gamma_1} \lambda^{-1} \exp(\lambda t) \sum_{k=1}^{m} N_k(\lambda) g_k(0)\,d\lambda\,,$$

so that, $\forall t \in]0, T]$, $v_1(t) \in W^{2m,p}(\Omega)$ and

$$A(\cdot, \partial) v_1(t) = A \exp(tA) u_0 + (2\pi i)^{-1}\int_{\gamma_1} \exp(\lambda t) \sum_{k=1}^{m} N_k(\lambda) g_k(0)\,d\lambda\,.$$

One has

$$\sum_{k=1}^{m} N_k(\lambda) g_k(0) = u_0 - (\lambda - A)^{-1}(\lambda - A(\cdot, \partial))\,u_0$$

and so,

$$\begin{aligned} A(\cdot, \partial) v_1(t) &= \\ &= A \exp(tA) u_0 - (2\pi i)^{-1}\int_{\gamma_1} \exp(\lambda t)(\lambda - A)^{-1}(\lambda - A(\cdot, \partial))\,u_0\,d\lambda = \\ &= \exp(tA) A(\cdot, \partial) u_0\,. \end{aligned}$$

Again for $t \in]0, T]$, $1 \leqslant r \leqslant m$, $1 \leqslant k \leqslant m$,

$$\begin{aligned} T_0 B_r(\cdot, \partial) \int_0^t K_k(t-s) g_k(0)\,ds &= \\ &= T_0 \Big((2\pi i)^{-1} \int_{\gamma_1} \lambda^{-1} \exp(\lambda t) B_r(\cdot, \partial) N_k(\lambda) g_k(0)\,d\lambda \Big) = 0 \quad \text{if } r \neq k\,, \end{aligned}$$

$$T_0 B_r(\cdot, \partial) \int_0^t K_k(t-s) g_k(0)\,ds = T_0 g_k(0) \quad \text{if } r = k\,,$$

so that

$$T_0 B_k(\cdot, \partial) v_1(t) = T_0 g_k(0) \qquad \text{if } t \in]0, T] .$$

From this

$$A(\cdot, \partial) v_1(t) = \exp(tA) A(\cdot, \partial) u_0 , \qquad \text{if } t \in]0, T] .$$
$$T_0 B_k(\cdot, \partial) v_1(t) = T_0 g_k(0)$$

As $A(\cdot, \partial) u_0 \in \lambda_p^\theta(\Omega)$, by [DPG] th. 2.6,

$$t \to \exp(tA) A(\cdot, \partial) u_0 \in C([0, T]; \lambda_p^\theta(\Omega)) .$$

Therefore the result follows from proposition 1.4.

3. Perturbations and linear nonautonomous parabolic problems.

In this section we shall consider certain perturbations of problem (9) of the kind

$$(14) \qquad \begin{cases} \dfrac{\partial u}{\partial t}(t, x) - A(x, \partial) u = Mu(t) + f(t, x) , \\ B_k(x, \partial) u(t, x) - N_k u(t, x) - g_k(t, x)|_{|\partial\Omega} = 0 , \\ u(0, x) = u_0(x) . \end{cases}$$

For $\delta > 0$ we define

$$E_\delta = \{u \in C([0, \delta]; \lambda_p^{2m+\theta}(\Omega)) \cap C^1([0, \delta]; \lambda_p^\theta(\Omega)) : u(0) = 0\} .$$

We have:

LEMMA 3.1. *Let X_α be a Banach space of type α ($\alpha \in [0, 1[$) between $\lambda_p^\theta(\Omega)$ and $\lambda_p^{2m+\theta}(\Omega)$, $M \in \mathcal{L}(X_\alpha; \lambda_p^\theta(\Omega))$. For $\delta > 0$, define*

$$M_\delta : E_\delta \to E_\delta \qquad M_\delta u(t) = \int_0^t \exp((t-s)A) Mu(s)\, ds .$$

Then, $\|M_\delta\|_{\mathcal{L}(E_\delta)} \to 0$ $(\delta \to 0)$.

Further, assume that, for $1 \leqslant m_k \leqslant 2m-1$, $Y_{k,\alpha}$ *is a space of type* α *between* $\lambda_p^\theta(\Omega)$ *and* $\lambda_p^{m_k+\theta}(\Omega)$, $Z_{k,\alpha}$ *a space of type* α *between* $\lambda_p^{2m-m_k+\theta}(\Omega)$ *and* $\lambda_p^{2m+\theta}(\Omega)$, $N_k \in \mathcal{L}(Y_{k,\alpha}, \lambda_p^\theta(\Omega)) \cap \mathcal{L}(Z_{k,\alpha}, \lambda_p^{2m-m_k+\theta}(\Omega))$. *Define, for* $\delta > 0$,

$$N_{k,\delta}\colon E_\delta \to E_\delta\,, \qquad N_{k,\delta}u(t) = \int_0^t K_k(t-s) N_k u(s)\, ds\,.$$

Then, $\|N_{k,\delta}\|_{\mathcal{L}(E_\delta)} \to 0 \ (\delta \to 0)$.

PROOF. In force of lemmata 2.2 and 2.4, M_δ and $N_{k,\delta} \in \mathcal{L}(E_\delta)$ $\forall \delta > 0$. Moreover, by lemma 2.2,

$$\|M_\delta u\|_{E_\delta} \leqslant \text{const}\, \|Mu\|_{C([0,\delta];\,\lambda_p^\theta(\Omega))} \qquad (\text{const independent of } \delta \in\,]0, T])\,.$$

For $t \in [0, \delta]$,

$$\|Mu(t)\|_{\theta,p,\Omega} \leqslant \|M\|_{\mathcal{L}(X_\alpha, \lambda_p^\theta(\Omega))} \|u(t)\|_{X_\alpha} \leqslant$$
$$\leqslant \text{const}\, (\|u(t)\|_{\theta,p,\Omega})^{1-\alpha} (\|u(t)\|_{2m+\theta,p,\Omega})^\alpha \leqslant \text{const}\, \delta^{1-\alpha} \|u\|_{E_\delta}\,.$$

This proves the result for M_δ. An analogous proof is applicable to $N_{k,\delta}$.

THEOREM 3.2. *Consider the problem* (14), *with* M, N_k *satisfying the assumptions of lemma* 3.1, $N_k = 0$ *if* $m_k = 0$. *Then* (14) *has a unique solution* $u \in C([0,T]; \lambda_p^{2m+\theta}(\Omega)) \cap C^1([0,T]; \lambda_p^\theta(\Omega))$ *for every*

$$f \in C([0,T]; \lambda_p^\theta(\Omega))\,,$$
$$(g_k)_{k=1}^m \in \prod_{k=1}^m \left\{C([0,T]; \lambda_p^{2m-m_k+\theta}(\Omega)) \cap \lambda^{1-m_k/(2m)}([0,T]; \lambda_p^\theta(\Omega))\right\},$$
$$u_0 \in \lambda_p^{2m+\theta}(\Omega),\ B_k(\cdot, \partial)u_0 - N_k u_0 - g_k(0,\cdot)|_{\partial\Omega} = 0 \ (1 \leqslant k \leqslant m)\,.$$

PROOF. Exchanging u with $u - u_0$, it is seen that it is not restrictive to assume $u_0 = 0$, $g_k(0,\cdot)|_{\partial\Omega} = 0$ $(1 \leqslant k \leqslant m)$ and look for a solution in E_T. So we can consider the equation

$$(15) \qquad u(t) = \int_0^t \exp((t-s)A) f(s)\, ds + \sum_{k=1}^m \int_0^t K_k(t-s) g_k(s)\, ds +$$
$$+ \int_0^t \exp((t-s)A) Mu(s)\, ds + \sum_{k=1}^m \int_0^t K_k(t-s) N_k u(s)\, ds\,.$$

By lemma 3.1, if δ is suitably small, by the contraction mapping principle, (15) has a unique solution in E_δ. Then one can repeat the process taking as initial value $u(\delta)$. As the length of definition of the local solution does not depend on the starting point, the solution is defined on $[0, T]$.

To give an example of operators and spaces satisfying the assumptions of lemma 3.1, consider the space $W^{2m,p}(\Omega)$ which is of type $\alpha = 1 - \theta/(2m)$ between $\lambda_p^\theta(\Omega)$ and $\lambda_p^{2m+\theta}(\Omega)$ (see proposition 1.3) and define

$$Mu(x) = \sum_{|\alpha|\leqslant 2m} \int_\Omega K_\alpha(x, y)\,\partial^\alpha u(y)\,dy\,.$$

Assuming $K_\alpha\colon \bar{\Omega}\times\bar{\Omega} \to \mathbb{C}$, $K_\alpha \in L^\infty(\Omega\times\Omega)$,

$$|K_\alpha(x + h, y) - K_\alpha(x, y)| \leqslant \omega(|h|)|h|^\theta$$

with $\omega(\delta) \to 0$ $(\delta \to 0)$, one has that $M \in \mathcal{L}(W^{2m,p}(\Omega), \lambda_p^\theta(\Omega))$.

If $m_k \geqslant 1$,

$$N_k u(x) = \sum_{|\beta|\leqslant m_k - 1} \int_{\partial\Omega} K_\beta(x, y)\,\partial^\beta u(y)\,d\sigma(y)\,,$$

with $K_\beta \in C^{2m-m_k}(\bar{\Omega}\times\partial\Omega)$,

$$|\partial_x^\gamma K_\beta(x + h, y) - \partial_x^\gamma K_\beta(x, y)| \leqslant \omega(|h|)|h|^\theta \quad \forall\gamma\,, \qquad |\gamma| = 2m - m_k\,,$$
$$N_k \in \mathcal{L}(W^{m_k,p}(\Omega), \lambda_p^\theta(\Omega)) \cap \mathcal{L}(W^{2m,p}(\Omega), \lambda_p^{2m-m_k+\theta}(\Omega))\,.$$

Now we consider the problem

$$(16)\quad \left|\begin{array}{l} \dfrac{\partial u}{\partial t}(t, x) - A(t, x, \partial)u = f(t, x)\,, \qquad t \in [0, T]\,, \quad x \in \Omega\,, \\[2ex] B_k(t, x, \partial)u(t, x) - g_k(t, x)|_{x\in\partial\Omega} = 0\,, \qquad t \in [0, T]\,, \quad k = 1, \ldots, m\,, \\[2ex] u(0, x) = u_0(x)\,, \qquad x \in \Omega\,. \end{array}\right.$$

We have:

THEOREM 3.3. *Assume the assumptions (A1)-(A3), (B1)-(B4) are satisfied by Ω and the operators $A(t, x, \partial)$, $(B_k(t, x, \partial))_{k=1}^m$, $\forall t \in [0, T]$).*

Further:

(*k*1) $A(t, x, \partial) = \sum_{|\alpha| \leqslant 2m} a_\alpha(t, x)\partial^\alpha$ *with* $t \to a_\alpha(t, \cdot) \in C([0, T]; \lambda_\infty^\theta(\Omega))$.

(*k*2) *If* $m_k \geqslant 1$, $B_k(t, x, \partial) = \sum_{|\beta| \leqslant m_k} b_{k\beta}(t, x)\partial^\beta$,

with $t \to b_{k\beta}(t, \cdot) \in C([0, T]; \lambda_\infty^{2m-m_k+\theta}(\Omega)) \cap \lambda^{1-m_k/(2m)}([0, T]; \lambda_\infty^\theta(\Omega))$.

Then problem (16) *has a unique solution u in the space*

$$C^1([0, T]; \lambda_p^{2m+\theta}(\Omega)) \cap C^1([0, T]; \lambda_p^\theta(\Omega)) \qquad \forall f \in C([0, T]; \lambda_p^\theta(\Omega)),$$

$$(g_k)_{k=1}^m \in \prod_{k=1}^m \{C([0, T]; \lambda_p^{2m-m_k+\theta}(\Omega)) \cap \lambda^{1-m_k/(2m)}([0, T]; \lambda_p^\theta(\Omega))\},$$

$$u_0 \in \lambda_p^{2m+\theta}(\Omega),$$

such that $B_k(0, \ldots, \partial)u_0 - g_k(0, \cdot)|_{\partial\Omega} = 0$.

PROOF. For $m_k = 0$, we can divide by $b(t, x)$ and assume $b(t, x) = 1$. Now, we put $v(t) = u(t) - u_0$. Then, we have

$$\frac{dv}{dt}(t) - A(t)v(t) = f^\wedge(t), \qquad t \in [0, T],$$

$$B_k(t)v(t) - g_k^\wedge(t)|_{\partial\Omega} = 0,$$

$$v(0) = 0,$$

with

$$A(t) = A(t, \cdot, \partial), \qquad B_k(t) = B_k(t, \cdot, \partial),$$

$$f^\wedge(t) = A(t)u_0 + f(t), \qquad g_k^\wedge(t) = -B_k(t)u_0 + g_k(t).$$

Owing to the assumptions,

$$f^\wedge \in C([0, T]; \lambda_p^\theta(\Omega)),$$

$$g_k^\wedge \in C([0, T]; \lambda_p^{2m-m_k+\theta}(\Omega)) \cap \lambda^{1-m_k/(2m)}([0, T]; \lambda_p^\theta(\Omega)).$$

This shows that we can assume $u_0 = 0$, $g_k(0) = 0$, for $k = 1, \ldots, m$.

Under these (more restrictive) conditions, (16) can be written as

$$(17)\quad \begin{cases} \dfrac{du}{dt}(t) = A(0)u(t) + \big(A(t) - A(0)\big)u(t) + f(t)\,, \qquad t \in [0, T]\,, \\ B_k(0)u(t) + [B_k(t) - B_k(0)]u(t) - g_k t)|_{\partial\Omega} = 0\,, \\ \qquad\qquad\qquad\qquad t \in [0, T]\,, \quad k = 1, \dots, m\,, \\ u(0) = 0\,. \end{cases}$$

Define $R\big(f, (g_k)_m^{k=1}\big)$ as the solution of

$$\frac{dv}{dt}(t) - A(0)v(t) = f(t)\,,$$

$$B_k(0)v(t) - g_k(t)|_{\partial\Omega} = 0\,,$$

$$v(0) = 0\,,$$

in the space $C\big([0, T];\, \lambda_p^{2m+\theta}(\Omega)\big) \cap C^1\big([0, T];\, \lambda_p^\theta(\Omega)\big)$. Then,

$$u = R\Big(\big(A(\cdot) - A(0)\big)u + f,\, \big(g_k + [B_k(0) - B_k(\cdot)]u\big)_{k=1}^m\Big)\,.$$

For $u \in E_\delta$ $(0 < \delta \leqslant T)$, put

$$\tau u = R\Big(\big(A(\cdot) - A(0)\big)u + f,\, \big(g_k + [B_k(0) - B_k(\cdot)]u\big)_{k=1}^m\Big)\,.$$

The assumptions ($k1$)-($k2$) assure that, if δ is sufficiently small, τ is a contraction in E_δ, so that (16) has a solution in $[0, \delta]$ with $0 < \delta \leqslant T$. Starting from the initial value in δ one can prolounge the solution until 2δ (if $2\delta \leqslant T$), as the length of the interval of definition of the local solution does not depend on the starting point and the data.

THEOREM 3.4. *Consider the problem*

$$(18)\quad \begin{cases} \dfrac{du}{dt}(t) - A(t)u(t) - M(t)u(t) = f(t)\,, \\ B_k(t)u(t) + N_k(t)u(t) - g_k(t)|_{\partial\Omega} = 0\,, \\ u(0) = u_0\,, \end{cases}$$

with $A(t) = A(t, x, \partial)$, $B_k(t) = B_k(t, x, \partial)$ *satisfying the assumptions of theorem* 3.3. *Further assume that* $M \in C\big([0, T]; \mathcal{L}(X_\alpha, \lambda_p^\theta(\Omega))\big)$, *with* X_α *intermediate space of type* $\alpha(\in [0, 1[)$ *between* $\lambda_p^\theta(\Omega)$ *and* $\lambda_p^{2m+\theta}(\Omega)$, $N_k \in \lambda^{1-m_k/(2m)}\big([0, T]; \mathcal{L}(Y_{k,\beta}, \lambda_p^\theta(\Omega))\big) \cap C\big(([0, T]; \mathcal{L}(Z_{k,\gamma}, \lambda_p^{2m-m_k+\theta}(\Omega))\big)$, with $Y_{k,\beta}, Z_{k,\gamma}$ intermediate spaces of type β, γ between, respectively,

$$\lambda_k^\theta(\Omega) \text{ and } \lambda_p^{m_k+\theta}(\Omega)\,, \quad \lambda_p^{2m-m_k+\theta}(\Omega) \text{ and } \lambda_p^{2m+\theta}(\Omega)\,, \quad N_k = 0 \text{ if } m_k = 0\,.$$

Then (18) has a unique solution

$$u \in C^1([0, T]; \lambda_p^\theta(\Omega)) \cap C([0, T]; \lambda_p^{2m+\theta}(\Omega))\,, \qquad \forall f \in C([0, T]; \lambda_p^\theta(\Omega))\,,$$

$$(g_k)_{k=1} \in \prod_{k=1}^{m} \{C([0, T]; \lambda_p^{2m-m_k+\theta}(\Omega)) \cap \lambda^{1-m_k/(2m)}([0, T]; \lambda_p^\theta(\Omega))\}$$

$u_0 \in \lambda_p^{2m+\theta}(\Omega)$, *such that* $B_k(0, \ldots, \partial)u_0 - g_k(0, \cdot)|_{\partial\Omega} = 0$.

PROOF. Analogous to the proof of theorem 3.3, using $A(t) + M(t)$ and $B_k(t) + N_k(t)$ instead of $A(t)$ and $B_k(t)$.

4. Quasilinear problems.

Now, we shall consider quasilinear parabolic problems like

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x) - \sum\limits_{|\alpha|=} A_\alpha(t, x, u, \ldots, \partial^{2m-1}u)\,\partial^\alpha u = f(t, x, u, \ldots, \partial^{2m-1}u)\,, \\ \sum\limits_{|\beta|=m_k} B_{k,\beta}(t, x, \ldots, \partial^{m_k-1}u)\,\partial^\beta u - g_k(t, x, \ldots, \partial^{m_k-1}u)|_{\partial\Omega} = 0 \\ \hfill (1 \leqslant k \leqslant m)\,, \\ u(0, x) = u_0(x) \qquad (x \in \Omega, t \in [0, T])\,. \end{cases}$$

($\partial^i u$ is the set of the derivatives of order j of u).

We mention that quasilinear problems with nonlinear boundary were considered for example in [FR] (very particular equations of second order), [AM] (systems in variational form), [AT] (applying the linear results of [TE1] and finding, by the linearization method, solutions in $C^{1+\alpha}([0, \tau]; L^p(\Omega)^N) \cap C^\alpha([0, \tau]; W^{2,p}(\Omega)^N)$, [GM] (existence of classical solutions for equations with smooth coefficients and data), [LU] (applying the results of the linear part).

Here we shall apply the linearization techniques to (19) using the results of sections 2 and 3 and finding solutions in $C([0,\tau];\lambda_p^{2m+\theta}(\Omega))\cap$ $\cap\, C^1([0,\tau];\lambda_p^{\theta}(\Omega))$, for some $\tau\in\,]0,T]$. We shall employ the following assumptions: first of all Ω is a bounded open subset of $\mathbb{R}^n$, with boundary of class C^∞. If $N(j)$ is the number of multiindexes of length not overcoming j, we assume that:

($L1$) $\forall\alpha$, $|\alpha|=2m$, $A_\alpha\colon[0,T]\times\bar\Omega\times\mathbb{R}^{N(2m-1)\to\mathbb{R}}$ of class C^∞.

($L2$) $f\colon[0,T]\times\bar\Omega\times\mathbb{R}^{N(m_k-1)\to\mathbb{R}}$ of class C^∞.

($L3$) For $1\leqslant k\leqslant m$, $g_k\colon[0,T]\times\bar\Omega\times\mathbb{R}^{N(m_k-1)\to\mathbb{R}}$ of class C^∞.

($L4$) For $1\leqslant k\leqslant m$, $|\beta|=m_k$, $B_{k,\beta}\colon[0,T]\times\bar\Omega\times\mathbb{R}^{N(m_k-1)\to\mathbb{R}}$ of class C^∞.

($L5$) $\forall(t,x,p)\in[0,T]\times\bar\Omega\times\mathbb{R}^{N(2m-1)}$, $\sum_{|\alpha|=2m}A_\alpha(t,x,p)\partial^\alpha$ is properly elliptic, there exists $\phi_0\in\,]\pi/2,\pi[$ such that

$$(-1)\frac{\sum_{|\alpha|=2m}A_\alpha(t,x,p)\xi^\alpha}{\left|\sum_{|\alpha|=2m}A_\alpha(t,x,p)\xi^\alpha\right|}\neq\exp(i\phi)\,,\qquad\forall\phi\in[-\phi_0,\phi_0]\,,$$

$$\forall\xi\in\mathbb{R}^n\setminus\{0\}\,,\quad\forall x\in\bar\Omega\,,\quad\forall p\in\mathbb{R}^{N(2m-1)}\,.$$

($L6$) For $k=1,\dots,m$, $m_k\leqslant 2m-1$ and the system of operators $\Big\{\sum_{|\beta|=m_k}B_{k,\beta}(t,x,q_k)\Big\}_{k=1}^m$ is normal in $\partial\Omega$ $\forall t\in[0,T]$, $\forall q_k\in\mathbb{R}^{N(m_k-1)}$ such that, for $m_k<m_j$, the coordinates of q_j coincide with the corresponding coordinates of q_k.

($L7$) $\forall t\in[0,T]$, $x\in\partial\Omega$, $q\in\mathbb{R}^{N(2m-1)}$, let $\nu(x)$ be the inward unit vector to $\partial\Omega$ in x and $\xi\neq0$ any vector parallel to the boundary at x. Denote by $t_j^+(\xi;\lambda)$ the j-th root with positive imaginary part of the polynomial in $z(-1)^mA^0(t,x,q;\xi+z\nu(x))-\lambda$. Then, if $|\mathrm{Arg}\,\lambda|\leqslant\phi_0$, the m polynomials in $z\{B_k^0(t,x,q_k;\xi+z\nu(x))\}_{k=1}^m$ are linearly independent modulo $\prod_{j=1}^m(z-t_j^+(\xi;\lambda))$ (with the coordinates of q coinciding with the corresponding coordinates of q_k).

($L8$) $p>n$, $0<\theta<p^{-1}$, $u_0\in\lambda_p^{2m+\theta}(\Omega)$.

($L9$) $$\sum_{|\beta|=m_k}B_{k,\beta}(0,x,\dots,\partial^{m_k-1}u_0)\partial^\beta u_0=g_k(0,x,\dots,\partial^{m_k-1}u_0)|_{\partial\Omega}\qquad(1\leqslant k\leqslant m).$$

For $t \in [0, T]$, $x \in \bar{\Omega}$, $p, q \in \mathbb{R}^{N(2m-1)}$, we pose

$$R_\alpha(t, x, p; q) = A_\alpha(t, x, p + q) - A_\alpha(t, x, p) - \sum_{|\gamma| \leqslant 2m-1} \partial_{p_\gamma} A_\alpha(t, x, p) q_\gamma ,$$

$$r(t, x, p; q) = f(t, x, p + q) - f(t, x, p) - \sum_{|\gamma| \leqslant 2m-1} \frac{\partial f}{\partial p_\gamma} (t, x, p) q_\gamma .$$

$\forall t \in [0, T]$, $x \in \bar{\Omega}, r, s \in \mathbb{R}^{N(m_k-1)}$, we pose, for $1 \leqslant k \leqslant m$, $|\beta| = m_k$,

$$S_{k,\beta}(t, x, p; q) = B_{k,\beta}(t, x, p + q) - B_{k,\beta}(t, x, p) - \sum_{|\gamma| \leqslant m_k - 1} \partial_{p_\gamma} B_{k,\beta}(t, x, p) q ,$$

$$s_k(t, x, p; q) = g_k(t, x, p + q) - g_k(t, x, p) - \sum_{|\gamma| \leqslant m_k - 1} \frac{\partial g_k}{\partial p_\gamma} (t, x, p) q .$$

We remark that, if $m_k = 0$, $S_{k,\beta} = 0$, $s_k = 0$.
We have:

Lemma 4.1. *Let*

$$A \in C^\infty([0, T] \times \bar{\Omega} \times \mathbb{R}^{N(2m-1)}), \qquad p > n , \quad \eta > 2m + 1 + np^{-1} + \theta .$$

Define $\mathcal{A}(u)(t)(x) = A(t, x, \ldots, \partial^{2m-1} u(t, x))$.
Then $\mathcal{A} \in C^1\big(C([0, T]; \lambda_p^\eta(\Omega)); C([0, T]; \lambda_\infty^\theta(\Omega))\big)$ *and*

$$\mathcal{A}'(u)(v)(t)(x) = \sum_{|\gamma| \leqslant 2m-1} \partial_{p_\gamma} A_\alpha(t, x, \ldots, \partial^{2m-1} u(t, x))\, \partial_\gamma v(t, x) .$$

Proof. As $\lambda_p^\eta(\Omega) \subseteq \lambda_\infty^{2m-1+\theta}(\Omega)$, $\forall t \in [0, T]$, $\mathcal{A}(u)(t) \in \lambda_\infty^\theta(\Omega)$. From the inequality

$$|A(t, x, p) - A(s, x, p) - A(t, y, q) + A(s, y, q)| \leqslant$$
$$\leqslant \omega(p, q)|t - s|[|x - y| + |p - q|]$$

(ω bounded on bounded subsets of $\mathbb{R}^{N(2m-1)} \times \mathbb{R}^{N(2m-1)}$) the belonging of $\mathcal{A}(u)$ to $C([0, T]; \lambda_\infty^\theta(\Omega))$ follows. The continuity of $\mathcal{A}$ is a consequence of the estimate

$$|A(t, x, p) - A(t, x, q) - A(t, y, p) + A(t, y, q)| \leqslant \omega(p, q)|x - y|\,|p - q|$$

(ω bounded on bounded subsets of $\mathbb{R}^{N(2m-1)} \times \mathbb{R}^{N(2m-1)}$).

We define

$$A_\gamma(u)(t,x) = \partial_{p_\gamma} A_\alpha(t, x, \ldots, \partial^{2m-1}u(t,x)) .$$

Then $A_\gamma \in C\big(C([0,T]; \lambda_p^\eta(\Omega)); C([0,T]; \lambda_\infty^\theta(\Omega)\big)$. From this the complete result follows easily.

LEMMA 4.2. *Let* $B \in C^\infty([0,T] \times \bar{\Omega} \times \mathbb{R}^{N(m_k-1)})$ $(1 \leqslant m_k \leqslant 2m-1)$. *Define* $B(u)(t)(x) = B(t, x, \ldots, \partial^{m_k-1}u(t,x))$. *If*

$$p > n\,, \qquad \xi > np^{-1} + \theta + m_k - 1\,,$$
$$B \in C^1\big(\lambda^{1-m_k/(2m)}([0,T]; \lambda_p^\xi(\Omega)); \lambda^{1-m_k/(2m)}([0,T]; \lambda_\infty^\theta(\Omega)\big) .$$

If $\xi > 2m - 1 + np^{-1} + \theta$,

$$B \in C^1\big(C([0,T]; \lambda_p^\xi(\Omega)); C([0,T]; \lambda_\infty^{2m-m_k+\theta}(\Omega))\big) .$$

In each case,

$$B'(u)(v)(t)(x) = \sum_{|\gamma| \leqslant m_k - 1} \partial_{p_\gamma} B(t, x, \ldots, \partial^{m_k-1}u(t,x))\, \partial^\gamma v(t,x) .$$

PROOF. The proof does not differ essentially from the proof of lemma 4.1, taking into account that, if $\xi > 2m - 1 + np^{-1} + \theta$, $\lambda_p^\xi(\Omega) \subseteq \lambda_\infty^{2m-1+\theta}(\Omega)$ and, if $\xi > np^{-1} + \theta + m_k - 1$, $\lambda_p^\xi(\Omega) \subseteq \lambda_\infty^{m_k-1+\theta}(\Omega)$.

LEMMA 4.3. *Let*

$$w \in C^1([0,T]; \lambda_p^\theta(\Omega)) \cap C([0,T]; \lambda_p^{2m+\theta}(\Omega))\,, \qquad w(0) = 0\,.$$

Then $\forall \varrho \in [0,1[$, $w \in \lambda^\varrho([0,T]; \lambda_p^{(1-\varrho)2m+\theta}(\Omega))$ *and*

$$\lim_{\delta \to 0} \|w\|_{\lambda^\varrho([0,\delta]; \lambda_p^{(1-\varrho)2m+\theta}(\Omega))}(\Omega)) = 0\,.$$

PROOF. A standard consequence of proposition 1.3.

THEOREM 4.4. *Assume* (L1)-(L9) *are satisfied. then there exists* $\delta \in\,]0, T]$ *such that* (19) *has a unique solution in the space*

$$C^1([0,\delta]; \lambda_p^\theta(\Omega)) \cap C([0,\delta]; \lambda_p^{2m+\theta}(\Omega))\,.$$

PROOF. Defining $v = u - u_0$, we can write (19) in the form

$$
(20)\quad \begin{cases}
\dfrac{\partial v}{\partial t}(t,x) = \sum_{|\alpha|=2m} A_\alpha(t,x,u_0,\dots,\partial^{2m-1}u_0)\partial^\alpha v + \\
+ \sum_{|\gamma|\leqslant 2m-1} \dfrac{\partial f}{\partial p_\gamma}(t,x,u_0,\dots,\partial^{2m-1}u_0)\partial^\alpha u_0\,\partial^\gamma v + \\
+ \sum_{|\alpha|=2m}\ \sum_{|\gamma|\leqslant 2m-1} \partial_{p_\gamma}A_\alpha(t,x,u_0,\dots,\partial^{2m-1}u_0)\,\partial^\alpha u_0 + \\
+ \sum_{|\alpha|=2m} A_\alpha(t,x,u_0,\dots,\partial^{2m-1}u_0)\partial^\alpha u_0 + f(t,x,u_0,\dots,\partial^{2m-1}u_0) + \\
+ \sum_{|\alpha|=2m} R_\alpha(t,x,u_0,\dots,\partial^{2m-1}u_0; v,\dots,\partial^{2m-1}v)\,\partial^\alpha v + \\
+ r(t,x,u_0,\dots,\partial^{2m-1}u_0; v,\dots,\partial^{2m-1}v) + \\
+ \sum_{|\alpha|=2m} R_\alpha(t,x,u_0,\dots,\partial^{2m-1}u_0; v,\dots,\partial^{2m-1}v)\,\partial^\alpha u_0 + \\
\qquad + \sum_{|\alpha|=2m}\ \sum_{|\gamma|\leqslant 2m=1} \partial_{p_\gamma}A_\alpha(t,x,u_0,\dots,\partial^{2m-1}u_0)\,\partial^\gamma v\,\partial^\alpha v\,, \\
\sum_{|\beta|=m_k} B_{k,\beta}(t,x,\dots,\partial^{m_k-1}u_0)\,\partial^\beta v - \sum_{|\gamma|\leqslant m_k-1} \dfrac{\partial g_k}{\partial p_\gamma}(t,x,\dots,\partial^{m_k-1}u_0)\,\partial^\gamma v + \\
+ \sum_{|\beta|=m_k}\ \sum_{|\gamma|\leqslant m_k-1} \partial_{p_\gamma}B_{k,\beta}(t,x,\dots,\partial^{m_k-1}u_0)(\partial^\beta u_0)\,\partial^\gamma v + \\
+ \sum_{|\beta|=m_k} B_{k,\beta}(t,x,\dots,\partial^{m_k-1}u_0)\,\partial^\beta u_0 - g_k(t,x,u_0,\dots,\partial^{m_k-1}u_0) + \\
+ \sum_{|\beta|=m_k} S_{k,\beta}(t,x,\dots,\partial^{m_k-1}u_0; v,\dots,\partial^{m_k-1}v)\,\partial^\beta v - \\
- s_k(t,x,\dots,\partial^{m_k-1}u_0; v,\dots,\partial^{m_k-1}v) + \\
+ \sum_{|\beta|=m_k} S_{k,\beta}(t,x,\dots,\partial^{m_k-1}u_0; v,\dots,\partial^{m_k-1}v)\,\partial^\beta u_0 + \\
+ \sum_{|\beta|=m_k}\ \sum_{|\gamma|\leqslant m_k-1} \partial_{p_\gamma} C_{k,\beta}(t,x,\dots,\partial^{m_k-1}u_0)\,\partial^\beta v\,\partial^\gamma v = 0_{|\partial\Omega} \\
v(0,x) = 0\,.
\end{cases}
$$

Put

$$
A(t,x,\partial) = \sum_{|\alpha|=2m} A_\alpha(t,x,u_0,\dots,\partial^{2m-1}u_0)\,\partial^\alpha + \\
+ \sum_{|\gamma|\leqslant m_k-1} \frac{\partial f}{\partial p_\gamma}(t,x,u_0,\dots,\partial^{2m-1}u_0)\,\partial^\gamma\,,
$$

$$
M(t)v(x) = \sum_{|\alpha|=2m}\ \sum_{|\gamma|\leqslant 2m-1} \partial_{p_\gamma}A_\alpha(t,x,u_0,\dots,\partial^{2m-1}u_0)\,\partial^\gamma v\,\partial^\alpha u_0\,,
$$

$$f(t, x) = \sum_{|\alpha|=2m} A_\alpha(t, x, u_0, \dots, \partial^{2m-1}u_0)\, \partial^\alpha u_0 + f(t, x, u_0, \dots, \partial^{2m-1}u_0) \,,$$

$$B_k(t, x, \partial) =$$

$$= \sum_{|\beta|=m_k} B_{k,\beta}(t, x, \dots, \partial^{m_k-1}u_0)\, \partial^\beta - \sum_{|\gamma|\leqslant m_k-1} \frac{\partial g_k}{\partial p_\gamma}(t, x, \dots, \partial^{m_k-1}u_0)\, \partial^\gamma \,,$$

$$N_k(t)w(x) = \sum_{|\beta|=m_k} \sum_{|\gamma|\leqslant m_k-1} \partial_{p_\gamma} B_{k,\beta}(t, x, \dots, \partial^{m_k-1}u_0)(\partial^\beta u_0)\, \partial^\gamma w(x) \,,$$

$$g_k(t, x) = \sum_{|\beta|=m_k} B_{k,\beta}(t, x, \dots, \partial^{m_k-1}u_0)\, \partial^\beta u_0 - g_k(t, x, u_0, \dots, \partial^{m_k-1}u_0)$$

and consider the problem

$$(21)\quad \begin{cases} \dfrac{\partial w}{\partial t}(t, x) = A(t, x, \partial)w(t, x) + M(t)w(t, x) + f(t, x) \,, \\ B_k(t, x, \partial)w + N_k(t)w(t, x) + g_k(t, x) = 0|_{\partial\Omega} \,, \qquad k = 1, \dots, m \,, \\ w(0, x) = 0 \,. \end{cases}$$

The operators $A(t, x, \partial)$ and $B_k(t, x, \partial)$ $(1 \leqslant k \leqslant m)$ satisfy the assumptions of theorem 3.3, while the operators $M(t)$ verify those of theorem 3.4, whith $X_\alpha = \lambda_p^\xi(\Omega)$, for $2m - 1 + np^{-1} + \theta < \xi < 2m + \theta$, owing to lemma 4.1. Further, by lemma 4.2, the operators $N_k(t)$ are conformal to the conditions of theorem 3.4, with $Y_{k,\alpha} = \lambda_p^\xi(\Omega)$, $m_k - 1 + np^{-1} + \theta < \xi < m_k + \theta$, $Z_{k,\alpha} = \lambda_p^\eta(\Omega)$ with $2m - 1 + np^{-1} + \theta < \eta < 2m + \theta$.

Therefore, (21) has a unique solution $w \in C^1([0, T]; \lambda_p^\theta(\Omega)) \cap C([0, T]; \lambda_p^{2m+\theta}(\Omega))$. Put $z = v - w$. Then,

$$(22)\quad \begin{cases} \dfrac{\partial z}{\partial t}(t, x) = A(t, x, \partial)z(t, x) + M(t)z(t, x) + R(t, x, z + w) \,, \\ B_k(t, x, \partial)z(t, x) + N_k(t)z(t, x) + s_k(t, x, w + z)|_{\partial\Omega} = 0 \,, \\ z(0, x) = 0 \,, \end{cases}$$

with

$$R(t, x, z + w) = \sum_{|\alpha|=2m} R_\alpha\big(t, x, u_0, \dots, \partial^{2m-1}u_0; w + z, \dots, \partial^{2m-1}(w + z)\big) \cdot$$

$$\cdot\, \partial^\alpha(z + w) + r\big(t, x, \dots, \partial^{2m-1}u_0; \dots, \partial^{2m-1}(z + w)\big) +$$

$$+\sum_{|\alpha|=2m} R_\alpha(t, x, u_0, \ldots, \partial^{2m-1}u_0; x+z, \ldots, \partial^{2m-1}(w+z))\,\partial^\alpha u_0 +$$

$$+\sum_{|\alpha|=2m}\ \sum_{|\gamma|\leqslant 2m-1} \partial_{p_\gamma} A_\alpha(t, x, u_0, \ldots, \partial^{2m-1}u_0)\,\partial^\gamma(z+w)\,\partial^\alpha(z+w)\,,$$

$$s_k(t, x, w+z) = \sum_{|\beta|=m_k} S_{k,\beta}(t, x, \ldots, \partial^{m_k-1}u_0; v, \ldots, \partial^{m_k-1}(z+w))\,\partial^\beta(z+w) -$$

$$- s_k(t, x, \ldots, \partial^{m_k-1}u_0; \ldots, \partial^{m_k-1}(z+w)) +$$

$$+\sum_{|\beta|=m_k} S_{k,\beta}(t, x, \ldots, \partial^{m_k-1}u_0; v, \ldots, \partial^{m_k-1}(z+w))\,\partial^\beta u_0 +$$

$$+\sum_{|\beta|=m_k}\ \sum_{|\gamma|\leqslant m_k-1} \partial_{p_\gamma} B_{k,\beta}(t, x, \ldots, \partial^{m_k-1}u_0)\,\partial^\gamma(z+w)\,\partial^\beta(z+w)\,.$$

$\forall\delta\in\,]0, T]\,,\quad \forall f\in C([0,\delta];\, \lambda_p^\theta(\Omega))\,,$

$$\forall(g_k)_{k=1}^m \in \prod_{k=1}^m \{\lambda^{1-m_k/(2m)}([0,\delta];\, \lambda_p^\theta(\Omega)) \cap C([0,\delta];\, \lambda_p^{2m-m_k+\theta}(\Omega))\}$$

(21) has a unique solution $R_\delta(f; (g_k)_{k=1}^m)$ in $C^1([0,\delta];\, \lambda_p^\theta(\Omega)) \cap$ $\cap\, C([0,\delta];\, \lambda_p^{2m+\theta}(\Omega))$. So (22) is equivalent to

$$z = R_\delta\big(R(\cdot,\cdot, z+w), (s_k(\cdot,\cdot, z+w))_{k=1}^m\big)\,. \tag{23}$$

Owing to proposition 1.1 and lemmata 4.1, 4.2, 4.3, $\forall\varepsilon>0$ there exist $\delta_0\in\,]0,T]$, $M(\varepsilon)>0$, such that $\forall z_1, z_2\in E_\delta$, with $0<\delta<\delta_0$, $\|z_1\|_{E_\delta}\leqslant$ $\leqslant M(\varepsilon)$, $\|z_2\|_{E_\delta}\leqslant M(\varepsilon)$,

$$\|R(\cdot,\cdot, z_1+w)\|_{C([0,\delta];\,\lambda_p^\theta(\Omega))} \leqslant \varepsilon(\phi(\delta) + \|z_1\|_{E_\delta})\,,$$

$$\|R(\cdot,\cdot, z_1+w) - R(\cdot,\cdot, z_2+w)\|_{C([0,\delta];\,\lambda_p^\theta(\Omega))} \leqslant \varepsilon\|z_1-z_2\|_{E_\delta}\,,$$

$$\|s_k(\cdot,\cdot, z_1+w)\|_{Z_{k,\delta}} \leqslant \varepsilon(\phi(\delta) + \|z_1\|_{E_\delta})\cdot$$

$$\cdot\big(Z_{k,\delta} = \lambda^{1-m_k/(2m)}([0,\delta];\, \lambda_p^\theta(\Omega)) \cap C([0,\delta];\, \lambda_p^{2m-m_k+\theta}(\Omega))\big)\,,$$

$$\|s_k(\cdot,\cdot, z_1+w) - s_k(\cdot,\cdot, z_2+w)\|_{Z_{k,\delta}} \leqslant \varepsilon\|z_1-z_2\|_{E_\delta}\,,$$

with $\phi(\delta)\to 0$ $(\delta\to 0)$, so that, for δ suitably small, by the contraction mapping principle, there exists $M>0$ such that (23) has a unique fixed point in the set $\{z\in E_\delta\colon \|z\|_{E_\delta}\leqslant M\}$.

We conclude with a simple result of unicity.

THEOREM 4.5. *Under the assumptions* (*L*1)-(*L*9) *ere satisfied, if*

$$u \in C^1([0, \delta]; L^p(\Omega)) \cap C([0, \delta]; W^{2m,p}(\Omega))$$

is a solution of (19) *in* $[0, \delta]$, $u \in C^1([0, \delta]; \lambda_p^\theta(\Omega)) \cap C([0, \delta]; \lambda_p^{2m+\theta}(\Omega))$ (*so the local solution is unique in* $C^1([0, \delta]; L^p(\Omega)) \cap C([0, \delta]; W^{2m,p}(\Omega))$).

PROOF. Put

$$f(t)(x) = f(t, x, \ldots, \partial^{2m-1}u), \qquad g_k(t)(x) = g_k(t, x, \ldots, \partial^{m_k-,p1}u).$$

Then

$$f \in C([0, \delta]; W^{1,p}(\Omega)),$$

$$g_k \in C([0, \delta]; W^{2m-m_k+1,p}(\Omega)) \cap \lambda([0, \delta]; W^{1-m_k/(2m),p}(\Omega)),$$

$$a_\alpha(t, x, \ldots, \partial^{2m-1}u) \in C([0, \delta]; \lambda_\infty^{\theta'}(\Omega)) \ \forall \theta' \in]0, 1 - np^{-1}[,$$

$$b_{k,\beta}(t, x, \ldots, \partial^{m_k-1}u) \in C([0, \delta]; \lambda_\infty^{2m-m_k+\theta'}(\Omega)) \cap$$
$$\cap \lambda^{1-m_k/(2m)}([0, \delta]; \lambda_\infty^{\theta'}(\Omega)) \ \forall \theta' \in]0, 1 - np^{-1}[.$$

Fix $\theta' \in]0, 1 - np^{-1}[$. By theorem 3.3, the problem

$$(24) \quad \begin{cases} \dfrac{\partial v}{\partial t}(t, x) = \sum\limits_{|\alpha|=2m} A_\alpha(t, x, \ldots, \partial^{2m-1}u)\, \partial^\alpha v = f(t, x, u, \ldots, \partial^{2m-1}u), \\ \sum\limits_{|\beta|=m_k} B_{k,\beta}(t, x, \ldots, \partial^{m_k-1}u)\, \partial^\beta v - g_k(t, x, \ldots, \partial^{m_k-1}u)|_{\partial\Omega} = 0, \\ v(0, x) = u_0(x), \end{cases}$$

has a unique solution $v \in C^1([0, \delta]; \lambda_p^{\theta'}(\Omega)) \cap C([0, \delta]; \lambda_p^{2m+\theta'}(\Omega))$.

By theorem 5.4 in [SO1], the solution v is unique in

$$W^{1,p}([0, \delta]; L^p(\Omega)) \cap L^p([0, \delta]; W^{2m,p}(\Omega)).$$

As u solves (24) in this space, $u = v$ and so $u \in C^1([0, \delta]; \lambda_p^{\theta'}(\Omega)) \cap C([0, \delta]; \lambda_p^{2m+\theta'}(\Omega))$. Therefore the result is proved if $\theta < 1 - np^{-1}$. Viceversa, one can iterate the method, starting from the assumption $u \in C^1([0, \delta]; \lambda_p^{\theta'}(\Omega)) \cap C([0, \delta]; \lambda^{2m+\theta'}(\Omega))$.

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