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Nearly Normal Projectivities.

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1. Introduction.

For two groups G and H a projectivity f from G to H, that is, a subgroup lattice isomorphism from the subgroup lattice $L(G)$ onto $L(H)$, need not map a normal subgroup of G to a normal subgroup of H. This can happen even if f is nearly normal, by which we mean that $A \lhd B$ implies $f(A) \lhd f(B)$ for any proper subgroup B of G. In 1927 Rottländer [3] produced the first such f by showing that the direct product $Q = Z_{p^k} \times Z_p$ is lattice isomorphic to the non-abelian semi-direct product $R = Z_{p^k} * Z_p$ where p is an odd prime and k is 2. This result holds for all p and $k \geqslant 2$ if R has the presentation

$$R = \langle a, b \colon a^{p^k} = 1 = b^p, b^{-1}ab = a^{1+p^{k-1}} \rangle$$

except for the case $p = 2 = k$. All claims hold since the non-central subgroups of order p are not normal in R, and every maximal subgroup of R is abelian. Only groups with the given presentation can be nonabelian projective images of Q [5, p. 13].

We say that a projectivity f from G to G' is normality preserving, or more sumply, f is normal if and only if $A \lhd B \leqslant G$ implies $f(A) \lhd f(B)$. Nearly normal projectivities are significant, because if f is a nonnormal projectivity of a finite group G then there must be a (not necessarily proper) subgroup H of G so that the restriction of f to H

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is nearly normal but nonnormal. We study such projectivities in the next section and in the last section develop some useful conditions sufficient to show that f is normal.

2. Nonnormal, nearly normal projectivities.

In this section f is a nearly normal projectivity from a finite group G to a group G' and N is a normal subgroup of G such that $N'=f(N)$ is not normal in G'. G/N must be cyclic of prime power order, p^k, and the interval of subgroups containing N is a chain of normal subgroups $G > N_1 > N_2 > \dots > N_k = N$. We write H' for $f(H)$ where $H \leqslant G$ and let K' be the core of N', that is, K' is the intersection of all conjugates of N'. By a result of R. Schmidt [4] K is normal in G, and G/K is a P-group or N'/K' is a quasi-normal (permutable) subgroup of G'/K'. Clearly any maximal subgroup of G contains K so that minimal generating sets of G and G/K have the same cardinality.

RESULT 1. The following are equivalent. *a*) f is index preserving. *b*) $N_1' \lhd G'$ (so N is not maximal in G). *c*) G'/K' is a p-group. *d*) G/K and G'/K' are isomorphic to the groups Q and R, respectively, from the introduction. *e*) G and G' are both (doubly generated) p-groups.

PROOF. *a*) implies *b*) is a result of Suzuki [5, p. 42]. To show *b*) implies *c*) it suffices to show that $[G':N'] = p^k$, since then the conjugates of N' are all in the normal subgroup N_1' so that the index of K' in N' is a power of p. If $[G':N'] \neq p^k$, then the restriction of f to any sylow p-subgroup S of G is not index preserving, $S(S')$ is cyclic of order p^n (q^n) for $n \geqslant k$, $G = C * S$ for some unique subgroup C in G, $G' = C' * S'$, and $C' \leqslant N'$ so that $N' \lhd G'$ [5, pp. 43-44].

c) implies *d*). Any projectivity of a nonabelian p-group must be index preserving, and so f^{-1} restricted to the interval above K' must be index preserving. Let M be a maximal subgroup of G other than N_1. Then $M \geqslant K$, $M' \lhd G'$, and by Suzuki's result $M \lhd G$. From the fact that $M \cap N$ is normal in N and also in M we have that $M' \cap N'$ is normal in G'. Since $M' \cap N'$ has index p in N', we have that $K' = M' \cap N'$ and G/K is isomorphic to $Z_{p^k} \times Z_p = Q$.

d) implies *e*). For a sylow p-group C in G the factor group $C/C \cap K$ is abelian while $C'/C' \cap K'$ is nonabelian. If C were properly contained in G, then f restricted to C would not preserve normality. Finally, e) implies a), since any projectivity between p-groups is index preserving. □

The next two results settle special cases.

RESULT 2. The following are equivalent. a) f is not index preserving. b) N_1' is not normal in G'. c) G and G' are P-groups and either: 1) $G = Z_p \times Z_p$, $G' = Z_p * Z_q$ or 2) $G = Z_q * Z_p$, $G' = Z_q * Z_r$. d) N is maximal in G.

PROOF. By Result 1 a) implies b). To simplify notation in b) we suppose that $N = N_1$. Since every maximal quasi-normal subgroup of a group is normal [5, p. 7], G/K is a P-group. By [4] N/K has prime order q and N'/K' has different prime order r. Again since f is nearly normal the order of G/K is p^2 ($q = p$) or pq, and the number of subgroups of index q in G is q or more. Now the restriction of f to some Sylow q-subgroup S of G is not index preserving and S must be elementary abelian or cyclic. By [5] there is a normal Hall subgroup C of G such that either $G = C * S$ or $G = C \times P$ where S is a normal subgroup of P. If $G = C * S$ and S is cyclic, then the number of subgroups of index q in G is one, a contradiction. In the remaining cases the restriction of f to S or P is not index preserving and thus not normality preserving. Since f is nearly normal, C must be trivial, G is S or P, and the cases are as in c). Clearly c) implies d) and d) implies a). □

RESULT 3. f is index preserving and G is abelian if and only if $G' = \langle a, b\colon a^{p^r} = b^{p^s} = 1,\ bab^{-1} = a^{1+p^{r-1}} \rangle$ where $r > 1$ or $r > 2$, if $p = 2$.

PROOF. Consider the if part first. G' has a modular subgroup lattice by the result in [5, p. 13] on finite modular groups and is clearly non-Hamiltonian. Baer has shown [5, p. 39] that every non-Hamiltonian modular group has a projectivity induced by a bijection mapping the group to an abelian group. Such a projectivity must be index preserving. Clearly the semi-direct product is carried by this projectivity to a direct product in the abelian group. We take f to be the inverse of this projectivity. Then f carries a subgroup of the abelian group onto the nonnormal subgroup generated by b, and every maximal subgroup of G' is abelian so that f is also nearly normal.

Now consider the only if part. We know that G is doubly generated finite abelian p-group so that $G = Z_{p^r} \times Z_{p^s}$. Designate the Z_{p^r}

factor by A and the Z_{p^s} factor by B. If A' and B' were both normal in G', then G' would be abelian and every subgroup would be normal. So one of A' and B' must be nonnormal and possibly both. In the rest of this proof we drop the prime notation from the elements of G' in order to make the notation less cumbersome. In particular suppose that $A' = \langle a \rangle$ and $B' = \langle b \rangle$. From the fact that f is nearly normal we have that $\langle a^p, b \rangle = \langle a^p \rangle \times B'$ and $\langle a, be \rangle = A' \times \langle b^p \rangle$ so that $\langle a^p, b^p \rangle$ is the center of G'.

Even if A' and B' are nonnormal the subgroups immediately above them, i.e. $A'\langle b^{p^{s-1}} \rangle$ and $B'\langle a^{p^{r-1}} \rangle$ are still normal by Result 1. The commutator (a, b) is in the intersection of these two subgroups which is $\langle a^{p^{r-1}}, b^{p^{s-1}} \rangle = E'$. If D' denotes the commutator subgroup of G', then $D' \leqslant E' \leqslant Z(G')$ so that the third term in the descending central series of G' must be the identity subgroup. By a well known result D' must be a cyclic subgroup of E' so that $D' = \langle (a, b) \rangle$ has order p in the elementary abelian group E'.

If A' and B' are nonnormal then D' is not in either one of them. Let D be the preimage of D' with respect to f. The crucial thing to see is that any cyclic subgroup of maximal order among the cyclic subgroups of G containing D is a direct summand of G. Relabel if necessary and call one of these subgroups A and let any one of the complementary subgroups be B. Then A' is normal in G' because it contains D' and B' is nonnormal in G' because it does not contain D'. Let $A' = \langle a \rangle$ and $B' = \langle b \rangle$ as before. Then G' has the given presentation, because D' has order p in A'. □

COROLLARY. Let f be an autoprojectivity of G', the nonabelian group in Result 3, and take all notation from there so that D' is the commutator subgroups of G'. Then 1) f is nearly normal, 2) f is nonnormal if and only if $f(D') \neq D'$, 3) $f(D')$ can be chosen to differ from D' only if $r \leqslant s$.

PROOF. 1) As observed above, every projectivity onto G' is nearly normal. 2) Since every noncyclic subgroup of G' contains D' and is normal, the image of any of these subgroups under f must be normal in G'. Thus f is nonnormal if and only if $f\langle a \rangle$ is not normal in G' if and only if $f\langle a \rangle$ does not contain D' if and only if $f(D') \neq D'$. 3) The only possible images of D' under f are subgroups of order p in G'. If $r > s$, then D' is the only subgroup of order p contained in any cyclic subgroup of order p^r so that $f\langle a \rangle$ must be normal in G'.

By a result in [2] every inclusion preserving mapping defined on the cyclic subgroups of the abelian group G in Result 3 extends to an autoprojectivity of G. Since the group G' is a projective image of G, the same is true for G'. Thus in the Corollary we can strengthen 3) to be an equivalence. In particular if $r \leqslant s$, then we can choose an inclusion preserving mapping from the cyclic subgroups of G' to the cyclic subgroups of G' which maps D' to one of the other $p-1$ subgroups of order p which is contained in a maximal cyclic subgroup of order p^r. In this way we can produce many nonnormal, nearly normal autoprojectivities of G'.

3. Normality preserving projectivities.

If f is a nonnormal projectivity from a finite group G to G', then G has a subgroup H so that f restricted to H is nearly normal but nonnormal. This fact and our first two results yield the following theorem.

THEOREM 1. A projectivity f from a finite group G to G' is normal if and only if f is normal when restricted to any doubly generated subgroup H of G such that H is a p-group or P-group.

If f happens to be index preserving, then one need not check P-subgroups of G. There remains the problem of which doubly generated p-groups admit a nonnormal, nearly normal projectivity f. Result 3 and its Corollary take first steps in this direction. An obstacle to progress has been the somewhat curious fact that among p-groups it is the doubly generated ones that have the least understood subgroup lattices, e.g. [5, p. 35]. The paper [2] overcomes some of these difficulties particularly for the autoprojectivities of groups G which are projective images of abelian groups, as in the remarks after the Corollary to Result 3. In particular these methods work well when G is a metacyclic p-group, but we leave that fairly extensive discussion to a subsequent paper as well as the fact that there are groups other than metacyclic p-groups having nonnormal nearly normal projectivities.

It is also possible to make some further progress when extra restrictions are placed on the projectivity f. For example, suppose that f is nearly isomorphism preserving in the sense that $f(H)$ is isomorphic to H for any proper subgroup H of G. If we avoid the case where G has prime order, then f must be index preserving [5, p. 42-44], but G may not be isomorphic to G' as in the case of the groups discovered by Rottländer [3] in the introduction. Our results come together in the following theorem.

THEOREM 2. Let G be a finite group that is not a p-group, and suppose any sylow subgroup of G is abelian. If f is a nearly isomorphism preserving projectivity from G to G', then f is normalizer preserving, i.e. $f(N_B A) = N_{B'} A'$, for all $A \leqslant B \leqslant G$.

PROOF. If f were not normal, then by our previous theorem there would be a prime p and a doubly generated p-subgroup K of G so that the restriction of f to K is a non-normal, nearly normal projectivity. By the given, K is an abelian proper subgroup of G so that K' is abelian too. Thus f must be normal on K and G too. In a similar way f^{-1} can be shown to be normal. Thus f is normalizer preserving. □

It is fairly common in the literature to show that certain nearly isomorphism preserving projectivities are normalizer preserving. The Author cites only one article [1], because among the known examples for G not a p-group this seems to be one of the rare instances where Theorem 2 does not suffice to show that a nearly isomorphism preserving projectivity is normalizer preserving. Theorem 1 simplifies all such arguments.

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