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Some Remarks on Boundary Value Problems for Linear Stochastic Differential Equations.

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SUMMARY - In this paper two negative results dealing with boundary conditions for stochastic differential equations are examined.

1. Introduction.

For years there have been considered some theorems in which the uniqueness of solutions of some boundary value problems implies the existence of solutions of these problems. We mean such theorems for algebraic equations, integral equations (the Fredholm's alternative), deterministic differential equations ([1]). The authors have already made an attempt to obtain similar results for random differential equations ([2]) in which the stochastic integral did not occur. The consequent continuation of the researches originated in the paper mentioned above is an attempt at obtaining the analogous results for the stochastic differential equations.

In the one paper ([3]) known to the authors on boundary value problems with the Itô stochastic integral, this integral appears in the solution formula only, whereas the equation is a random one without stochastic integral and the boundary condition is a deterministic one.

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In the present paper we have attempted to obtain some results concerning the relation between the existence and the uniqueness of the solutions for linear stochastic differential equations with the stochastic boundary conditions. Unfortunately, we have received negative results which may, however, be interesting from the methodological point of view. We have found that the reason of the situation is the fact that the diffusion process can be uniquely determined by means of the boundary condition which is measurable with respect to σ-algebra much poorer than this σ-algebra to which refers the diffusion process.

In this paper the authors have used the topological properties of the Hilbert spaces ([4]).

2. Notations.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, P)$ be a complete probability space, where $\{\mathcal{F}_t\}_{t\geqslant 0}$ is a right continuous filtration.

We consider the linear stochastic differential equation

$$(1) \qquad dx_t = \big(A(t)x_t + a(t)\big)\,dt + \sum_{i=1}^{n} \big(B_i(t)x_t + b_i(t)\big)\,dw_t^{(i)}$$

with a general stochastic boundary condition

$$(2) \qquad \Lambda x = y\,,$$

where $x_t\colon [0, T]\times\Omega \to R^d$ is a d-dimensional stochastic process and $w_t = (w_t^{(1)}, \dots, w_t^{(n)})$ is and n-dimensional, $\{\mathcal{F}_t\}$-adapted Wiener process in Ω.

Let $A(t)$, $B_i(t) \in L^2(\Omega, \mathcal{F}_t; R^{d\times d})$, $a(t)$, $b_i(t) \in L^2(\Omega, \mathcal{F}_t; R^d)$ for $i = 1, \dots, n$. Furthermore we shall denote $L^2(\Omega, \mathcal{F}) = L^2(\Omega, \mathcal{F}; R^d)$, $C([0, T]) = C([0, T], R^d)$.

By $\{\mathcal{F}'_t\}_{t\geqslant 0}$ we define the family of the σ-algebras generated by the process w_t.

We take the mapping $\Lambda\colon L^2\big(\Omega, \mathcal{F}_T; C([0, T])\big) \to L^2(\Omega, \mathcal{F}_T)$ such that

$$\Lambda\big(L^2(\Omega, \mathcal{F}'_t; C([0, T])\big) \subset L^2(\Omega, \mathcal{F}'_t)\,.$$

3. Existence and uniqueness theorems.

We shall prove the following

THEOREM 1. Let the functions $A(t), b_i(t), a(t), b_i(t)$ for $i = 1, \dots, n$ be progressively measurable. Let $\mathcal{F}_t = \mathcal{F}'_t$ for every $t \in [0, T]$.

Then the set of the random variables $y \in L^2(\Omega, \mathcal{F}_T)$ such that problem (1), (2) has a solution is nonwhere dense in $L^2(\Omega, \mathcal{F}_T)$.

PROOF. Let us consider the mapping

$$T: R^d \to L^2(\Omega, \mathcal{F}_T; C([0, T]))$$

which to every $z \in R^d$ applies the solution of equation (1) with condition

$$x_0 = z \,. \tag{3}$$

The composition $\Lambda T: R^d \to L^2(\Omega, \mathcal{F}_T)$ is a linear and continuous mapping. Therefore, the image of this mapping is a nonwhere dense set in the space $L^2(\Omega, \mathcal{F}_T)$. But if x is the solution of equation (1), then it is $\{\mathcal{F}_t\}$-adapted because the family $\{\mathcal{F}_t\}$ is generated by the process w_t. Therefore, x_0 must be such that $x_0 \in R^d$. This means that for every solution x of equation (1) we have

$$\Lambda x \in \operatorname{im} \Lambda T \,. \tag{4}$$

But $\operatorname{im} \Lambda T$ is a finite dimensional subspace of $L^2(\Omega, \mathcal{F}_T)$ and hence the theorem is proved.

Now we may state

THEOREM 2. If $A(t), a(t), B_i(t), b_i(t)$ are stochastic processes progressively measurable with respect to $\{\mathcal{F}'_t\}$, then there exists such a $y \in L^2(\Omega, \mathcal{F}_T)$ that problem (1), (2) has no solution.

PROOF. Let $\hat{x}_t = E(x_t | \mathcal{F}'_T)$. Then the process $\hat{x}_t$ satisfies equation

$$d\hat{x}_t = \big(A(t)\hat{x}_t + a(t)\big)\,dt + \sum_{i=1}^{n} \big(B_i(t)\hat{x}_t + b_i(t)\big)\,dw_t^{(i)} \tag{5}$$

which results from the properties of the conditional expectation. From the factorization condition for the Hilbert spaces there exists such a $\tilde{\Lambda}$: $L^2\big(\Omega, \mathcal{F}'_T; C([0, T])\big) \to L^2(\Omega, \mathcal{F}'_T)$ that

$$E(\Lambda x|\mathcal{F}'_T) = \tilde{\Lambda}E(x|\mathcal{F}'_T)\,,$$

where

$$E(x|\mathcal{F}'_T)_t = E(x_t|\mathcal{F}'_T)\,.$$

This equation (5) with the condition $\tilde{\Lambda}x = E(y|\mathcal{F}'_T)$ fulfills the assumptions of Theorem 1, which completes the proof.

REMARK. Theorem 2 remains true if only $A(t)$, $B_i(t)$ are $\{\mathcal{F}'_t\}$-progressively measurable and $a(t)$, $b_i(t)$ are $\{\mathcal{F}_t\}$-progressively measurable.

4. Conclusions.

The above theorems imply that, in general, it is useless to consider the boundary value problems for the stochastic differential equations.

An exception might be boundary conditions like

$$E(\Lambda x|\mathcal{F}_0) = y \tag{6}$$

as well as some exceptional situations when, for example, we can construct an isomorphism between the spaces $L^2(\Omega, \mathcal{F}_0)$ and $L^2(\Omega, \mathcal{F}_T)$. Such an isomorphism exists when in the space $L^2(\Omega, \mathcal{F}_0)$ is determined the canonical $\{\mathcal{F}_0\}$-measurable Wiener process.

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