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Measurable products of modules

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Measurable Products of Modules.

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SUMMARY - In this paper all groups are abelian, rings are associative with identity, and modules are left unitary. We are interested in modules of the form $\prod_I G_i$, a direct product of submodules G_i over an index set I. Many theorems about such modules require that |I| be non-measurable. Here we let I be arbitrary, put mild restrictions on the G_i's, and obtain new results. In Section I we eqtablish some decomposition theorems. We then apply them to homomorphisms of the form $f\colon \prod G_i \to A$ where: in Section 2 A is a slender module and in Section 3 A is an infinite direct sum of submodules.

0. Preliminaries.

Let I be a set and $P(I)$ its power set. Here (as in [3]) $|I|$ is *measurable* if there is a 0,1 countably additive function μ on $P(I)$ such that $\mu(I) = 1$ and $\mu(\{i\}) = 0$ for each $i \in I$. If no such function exists, $|I|$ is *non-measurable*. If β is the least measurable cardinal, it is a regular limit cardinal such that $\alpha < \beta$ implies $2^\alpha < \beta$. If all sets are constructible ($V = L$), measurable cardinals do not exist. A good discussion of these matters may be found in [5].

If $(S, +, \cdot)$ is a Boolean ring, an ideal K is here called a γ-ideal if, whenever $\{s_1, s_2, \ldots\}$ is a countable set of orthogonal elements in S, $\sum_{n \geqslant k} s_n \in K$ for some k in N, the natural numbers,

Let R be a ring and A a R-module. A *filter* F is a set of principal

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right ideals in R such that, for each aR, bR in F, there is a cR in F contained in aR and bR. A is *torsion-free* if, for $r \in R$ and $x \in A$, $rx = 0$ implies $r = 0$ or $x = 0$. A is *divisible* if $rA = A$ for all non-zero r in R. $D(A)$ is the maximal divisible submodule of A and A is *reduced* if $D(A) = 0$.

In general our terminology agrees with that in Fuchs [3].

1. Decomposition theorems.

We begin with a set-theoretic lemma. We omit its proof since its statements are well-known or easily proved (e.g. see page 161 in II of [3] or pages 342-356 in [5]).

LEMMA 1.1. Let I be a set, $S = P(I)$, and K a proper γ-ideal in the Boolean ring $(S, +, \cdot)$.

(*a*) S/K is finite and there are orthogonal elements, say $u_1, \dots, u_e$, in S which map onto the atoms of S/K.

(*b*) If $\{s_j\}$, $j \in J$, is a set of orthogonal elements in S, almost all s_j's are in K. If $|J|$ is non-measurable, then $\sum_{J'} s_j \in K$ for some cofinite subset J' of J.

(*c*) If $|I|$ is non-measurable, then $K = P(I')$ for some cofinite subset I' of I.

REMARK. If K is an ideal of finite index in a complete Boolean ring, it need not be a γ-ideal (consider $S = 2^N$ and let K be a maximal ideal containing $2^{(N)}$). However, if I is a set, then $|I|$ is measurable if and only if $S = P(I)$ has a proper γ-ideal K containing the atoms of S.

We now use Lemma 1.1 to obtain decompositions of specific modules. In the next two theorems the ring R is arbitrary.

THEOREM 1.2. Let $X = \prod_I G_i$ be a R-module where the G_i's are pairwise isomorphic of non-measurable cardinality. If K is a proper γ-ideal in $S = P(I)$ and $H = \langle \prod_s G_i : s \in K \rangle$, then $X = L \oplus H$ where $L \simeq \bigoplus_E G_i$ for some finite subset E of I. If I is infinite, then $X \simeq H$.

PROOF. Let $\pi_i : G_i \to G$ be an isomorphism for some group G and each i. Let $u_1, \dots, u_e$ be as in Lemma 1.1. For each u_n let $A_n =$

$= \{\sum_{u_n} \pi_i^{-1}(g) : g \in G\}$. Clearly each $A_n \cong G$ and we set $L = \bigoplus_1^e A_n$. We claim $X = L \oplus H$. Let $x = \sum_I x_i$, $x_i \in G_i$, be an element in X. If $s \in S$, we define $x_s = \sum_{i \in s} x_i$. Hence $x = \sum_G x_{s_g}$ where $i \in s_g$ exactly if $\pi_i(x_i) = g \in G$. Since the s_g's partition I and $|G|$ is non-measurable, $\sum_{G'} s_g \in K$ for some cofinite subset G' of G by Lemma 1.1 and so $\sum_{G'} x_{s_g}$ is in H. Now x will be in $L + H$ if we can show x_{s_g} is in it for any fixed g. But $s_g = \sum a_n u_n + v$ where each $a_n = 0$ or 1 and $v \in K$. So $x_{s_g} = \sum a_n x_{u_n} + x_v - 2\sum a_n x_{u_n v}$, which is in $L + H$ ($u_n v$ is in K). Suppose now $y_1 + \ldots + y_e + z = 0$ with y_n in A_n and z in H. For any fixed n and some i in u_n the ith component of z is 0 since u_n is not in K. Hence the ith component of y_n is 0 and $y_n = 0$. Therefore $z = 0$ and $X = L \oplus H$, as desired. If E consists of one element from each u_n, then $L \cong \bigoplus_E G_i$. Since $X = L \oplus \prod_{I \setminus E} G_i$, we have $H \cong \prod_{I \setminus E} G_i \cong X$ if I is infinite.

THEOREM 1.3. Let $X = \prod_I G_i$ be a R-module where each G_i and the set of their isomorphism classes have non-measurable cardinality. If K is a γ-ideal in $S = P(I)$ and $H = \langle \prod_s G_i : s \in K \rangle$, then $X = L \oplus H$ where, for some finite subset E, $L \cong \bigoplus_E G_i$ and $H \cong \prod_{I \setminus E} G_i$.

PROOF. Write $I = \bigcup_J s_j$ where i, i' are in the same s_j exactly if $G_i \cong G_{i'}$. Then $\{s_j\}$ partitions I, $|J|$ is non-measurable, and $\prod_I G_i = \prod_J G_{s_j}$ where $G_{s_j} = \prod_{i \in s_j} G_i$. By Lemma 1.1 there is a J' cofinite in J such that $\sum_{J'} s_j \in K$ and $\prod_{J'} G_{s_j} \in H$. For fixed j G_{s_j} is a product of isomorphic groups and, if $S_j = P(s_j)$, then $K_j = S_j \cap K$ is a γ-ideal in S_j. By the last theorem $G_{s_j} = L_j \oplus H_j$ where $H_j = \langle \prod_s G_i : s \in K_j \rangle$ and, for a finite subset t_j of s_j, $L_j \cong \bigoplus_{t_j} G_i$ and $H_j \cong \prod_{s_j \setminus t_j} G_i$. So $X = \bigoplus_{J \setminus J'} L_j \oplus \left[\bigoplus_{J \setminus J'} H_j \oplus \prod_{J'} G_{s_j}\right]$ Let L be the left sum and let H' be the module in the bracket. If $E = \bigcup_{J \setminus J'} t_j$, we have $L \cong \bigoplus_E G_i$ and $H' \cong \prod_{I \setminus E} G_i$. Since $H' \subseteq H$ and $H \cap L = 0$, $H = H'$ and the proof is complete.

Our next proposition will prove useful later.

PROPOSITION 1.4. Let $\{G_i\}$, $i \in I$, be a set of R-modules and let V be the set of their isomorphism classes.

(*a*) If $|R|$ and each $|G_i|$ is $\leqslant \alpha$, a fixed non-measurable cardinal, then $|V|$ is non-measurable.

(*b*) If each $|G_i|$ and $|V|$ are non-measurable, then $|G_i| \leqslant \alpha$ for some non-measurable α and all i.

PROOF. (*a*) Let Y be a free R-module of rank α. Then $G_i \simeq Y/H_i$ for some H_i and each i. The number of distinct submodules of Y is $\leqslant 2^{|Y|} \leqslant 2^{\alpha}$ which is non-measurable since α is. So $|V|$ is non-measurable since α is. So $|V|$ is non-mesaurable. (*b*) Let β be the least non-measurable cardinal (it's an ordinal). Let J be a subset of I such that the ordinals $|G_j|$, $j \in J$, are distinct with least upper bound $\alpha \leqslant \beta$. Since β is a regular cardinal if $\alpha = \beta$, we have $\beta = |J| \leqslant |V| < \beta$, a contradiction. So $\alpha < \beta$.

2. Slender modules.

In this section we apply the theorems of the last section to mappings of the form $f\colon \prod_I G_i \to A$ where A is a slender R-module. In the literature there appear various definitions of a «slender» module (or group). Actually these definitions are essentially the same as we shall show. Therefore, we define: a R-module A is *slender* if it satisfies any (hence all) of the four conditions of Proposition 2.1.

PROPOSITION 2.1. Let A be a R-module. The following are equivalent.

(1) If $f\colon R^N \to A$ is a homomorphism, almost all components in R^N map to 0.

(2) If $f\colon R^N \to A$ is a homomorphism, there is a cofinite subset C in N such that $f(R^C) = 0$.

(3) If $\{G_n\}, n \in N$, is a countable set of R-modules and $f\colon \prod_N G_n \to A$ is a homomorphism, then $f\left(\prod_{n \geqslant k} G_n\right) = 0$ for some k in N.

(4) If $\{G_n\}, n \in N$, is a countable set of R-modules and $f\colon \prod_N G_n \to A$ is a homomorphism, then $f(G_n) = 0$ for almost all n in N.

PROOF. (1) $\Rightarrow$ (2). We write $R^N = \prod_N Re_n$ where e_n is a N-tuple with 1 in the nth position and 0 elsewhere. It suffices to assume $f(e_n) = 0$ for all n but $f(x) \neq 0$ for some $x \in R^N$ and to derive a contradiction. Write $x = \sum r_n e_n$, $r_n \in R$, and, for each n in N, set $a_n = x - (r_1 e_1 + \ldots + r_n e_n)$. For each k the kth component in R^N of a_n is 0 for almost all n. Hence, if $B = \prod Ra_n$, there is a natural imbedding $\varphi: B \to R^N$ with $\varphi(a_n) = a_n$ for each n. Consider the map $f: B \to A$. Since $f\varphi(a_n) = f(x) \neq 0$ for each n, we have a contradiction of (1) with respect to $f\varphi$. Therefore (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3). Suppose (3) is false. For each k in N choose $x_k \in \prod_{n \geqslant k} G_n$ so that $f(x_k) \neq 0$. There is a natural map $\varphi: R^N \to \prod G_n$ carrying e_n to x_n. Then $f\varphi: R^N \to A$ is a homomorphism and by (2) $f\varphi\left(\prod_{k \geqslant m} Rx_k\right) = 0$ for some m. So $0 = f\varphi(x_k) = f(x_k)$ for $k \geqslant m$, a contradiction. Thus (2) $\Rightarrow$ (3). Clearly (3) $\Rightarrow$ (4) $\Rightarrow$ (1) and the proposition is true.

REMARK. (1) is the definition of slender used by Fuchs [3, vol. II, pg. 159] in the case $R = Z$ and A is torsion-free (a slender R-module is torsion-free if $R = Z$ but not in general. See Example 3 on p. 399 of [4]). (2) and (3) are the definitions of slender used in [2] and [4].

We now apply Theorem 1.3 to a map from a direct product of modules to a slender module.

THEOREM 2.2. Let $X = \prod_I G_i$ be a module where each $|G_i|$ and the set of their isomorphism classes are non-measurable. If $f: X \to A$ is a homomorphism and A is slender, then $X = L \oplus H$ where $f(H) = 0$ and, for some finite subset E, $L \simeq \bigoplus_E G_i$ and $H \simeq \prod_{I \setminus E} G_i$.

PROOF. For $S = P(I)$ let $K = \left\{s \in S: f\left(\prod_s G_i\right) = 0\right\}$. Clearly K is an ideal and it is a γ-ideal by (3) of Proposition 2.1. Theorem 1.3 completes the proof.

COROLLARY 2.3. Let R be a commutative integral domain not a field and let A be a countable torsion-free reduced R-module. If R-module X equals $\prod_I G_i$ where $|G_i| \leqslant \alpha$ for some non-measurable α and all i and if $f: X \to A$ is a homomorphism, then $X = L \oplus H$ where $f(H) = 0$ and, for some finite subset E, $L \simeq \bigoplus_E G_i$ and $H \simeq \prod_{I \setminus E} G_i$.

PROOF. We assume A is non-zero and hence $|R|$ is countable. Thus the set of isomorphism classes of the G_i's is non-measurable by Proposition 1.4. The result now follows from Theorem 2.2.

COROLLARY 2.4. Let A be a torsion-free abelian group which does not contain a copy of Q, Z^N, or the p-adic integers for a prime p. Let $X = \prod_I G_i$ be an abelian group where $|G_i| \leqslant \alpha$ for some non-measurable α abd all i. If $f: X \to A$ is a homomorphism, then $X = L \oplus H$ where $f(H) = 0$ and, for some finite E, $L \simeq \bigoplus_E G_i$ and $H \simeq \prod_{I \setminus E} G_i$.

PROOF. A is a slender Z-module by Theorem 95.3 in [3]. Proposition 1.4 and Theorem 2.2 complete the proof.

NOTE. If $\{G_i\}$ is a set of indecomposable groups of non-measurable cardinality, the least upper bound of $|G_i|$ may not be non-measurable. There exists arbitrarily large indecomposable groups (Theorem 2.1 in [7]),

3. Direct products and sums.

Suppose $X = \prod_I G_i$, $A = \bigoplus_J A_j$ are modules and $f: X \to A$ is a homomorphism. Most known theorems dealing with this situation require that $|I|$ be non-measurable (see [6] for references). In this section we let I be arbitrary, put some restrictions on the G_i's and obtain new results. Other results with I arbitrary may be found in part 2 of [6]. By f_j we will mean the map f followed by the projection to A_j.

THEOREM 3.1. Let $X = \prod_I G_i$, $A = \bigoplus_J A_j$ be two R-modules and let $f: X \to A$ be a homomorphism. Suppose each G_i and the set of their isomorphism classes have non-measurable cardinality. If (a) F is a filter of non-zero principal right ideals in R or (b) R is a commutative integral domain, then $X = L \oplus H$ where $L \simeq \bigoplus_E G_i$ and $H \simeq \prod_{I \setminus E} G_i$ for some finite E such that, for some non-zero b in R, $bf_j(H)$ is contained in (a) $\bigcap rA$ or (b) $D(A)$ for almost all j in J.

PROOF. Let $S = P(I)$. (*a*) Let $K = \left\{s \in S\colon \text{for some } r_s R \text{ in } F \text{ we have } r_s f_j\left(\prod_s G_i\right) \subseteq \bigcap_{rR \in F} rA \text{ for almost all } j \text{ in } J\right\}$.

It is easy to see that K is an ideal in S and it is a γ-ideal by Chase's Theorem (Theorem 2.1 in [1] or Theorem 1.1 in [6]). Conclusion (*a*) follows immediately from Theorem 1.3 above and from Theorem 1.3 in [6]. (*b*) Let $K = \left\{s \in S\colon \text{for some } r_s \neq 0 \text{ in } R \text{ we have } r_s f_j\left(\prod_s G_i\right) \subseteq \subseteq D(A) \text{ for almost all } j\right\}$. Then K is a γ-ideal in S by Theorem 1.5 in [6] and, from that theorem and Theorem 1.3 above, we have conclusion (*b*).

We next apply Theorem 3.1 to the case where f is the identity map. For best results we let A be torsion free.

THEOREM 3.2. *Let A be a torsion-free R-module with decompositions $A = \prod_I G_i = \bigoplus_J A_j$ where $|G_i| \leqslant \alpha$ for some non-measurable α and all i. If F is a filter of non-zero principal right ideals in R such that $\bigcap_{rR \in F} rA = 0$ or if R is a commutative integral domain and $D(A) = 0$, then there are finite subsets I_1 in I and J_1 in such that $\bigoplus_{I_1} G_i \cong B \oplus \oplus \left(\bigoplus_{J \setminus J_1} A_j\right)$ with $B \subseteq \bigoplus_{J_1} A_j$.*

PROOF. Since A is torsion-free, we may assume $|R| \leqslant \alpha$. By Proposition 1.4 the set of isomorphism classes of the G_i's has non-measurable cardinality. Let f be the identity map in Theorem 3.1. By that theorem and the fact that A is torsion-free we have, for some finite subsets I_1 in I and J_1 in J, $A = L \oplus H$ with $L \cong \bigoplus_{I_1} G_i$ and $H \subseteq \bigoplus_{J_1} A_j$. The conclusion of the theorem now follows.

If $R = Z$ and A in the last theorem is just an abelian group, torsion-freeness is not required to obtain a meaningful decomposition theorem.

THEOREM 3.3. *Let $A = \prod_I G_i = \bigoplus_J A_j$ be a reduced abelian group where $|G_i| \leqslant \alpha$ for some non-measurable α and all i. There are decompositions $G_i = B_i \oplus C_i$, $A_j = T_j \oplus U_j$ and finite subsets I_1 in I, J_1 in J such that*

(*a*) $\prod_I B_i \cong \bigoplus_J T_j$ *and is bounded*

(*b*) $\prod_I C_i \cong \bigoplus_J U_j$

(c) $\bigoplus_{J_1} U_j = P \oplus Q$ such that

(i) $\bigoplus_{I_1} C_i \cong Q \oplus \left(\bigoplus_{J \setminus J_1} U_j\right)$

(ii) $\prod_{I \setminus I_1} C_i \cong P$.

PROOF. By Proposition 1.4 the set of isomorphism classes to which the G_i's belong has non-measurable cardinality. By Theorem 3.1 (with f the identity map and $D(A) = 0$) there is a decomposition $A = L \oplus H$ and finite subsets I_1 and J_1 such that $L \cong \bigoplus_{I_1} G_i$, $H \cong \cong \prod_{I \setminus I_1} G_i$, and $nH \subseteq \bigoplus_{J_1} A_j$ for some n in N. Since our goals are isomorphisms not identities, we may assume $n\left(\prod_{I \setminus I_1} G_i\right) \subseteq \bigoplus_{J_1} A_j$ for n in N. The rest of the proof is exactly like that of Corollary 1.9 in [6].

COROLLARY 3.4. Suppose $A = \prod_I G_i$ is a reduced abelian group, $|G_i| \leqslant \alpha$ for some non-measurable α and all i, and let β be an infinite cardinal. Then A is a direct sum of β non-zero subgroups if and only if (a) some finite sum of G_i's is or (b), for some n in N, each G_i has a n-bounded direct summand B_i such that $\left|\prod_I B_i\right| \geqslant \beta$.

PROOF. Sufficiency is clear. To show necessity set $A = \bigoplus_J A_j$, $A_j \neq 0$ and $|J| = \beta$, and apply Theorem 3.3. Assume the decompositions there have been made and $\left|\prod B_i\right| < \beta$. By ($c$) then $\bigoplus_{I_1} C_1$ is a direct sum of β non-zero subgroups.

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