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Finite Groups with a Standard-Component of Type $L_3(4)$, II.

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0. Introduction.

In this paper we finish the investigation of the $L_3(4)$-type standard-subgroup problem. Because of the result of [3] we have to treat here only the case in which the 2-rank of the center of the standard-subgroup is equal to 1, that is, we assume in what follows that the 2-part of the center is cyclic and different from $\langle 1 \rangle$.

The results obtained in [5] will be assumed; we shall retain the notations introduced there. As in [5], we consider a fixed standard-subgroup A of our group G with $A/\boldsymbol{Z}(A) \cong L_3(4)$ and put $K = \boldsymbol{C}(A)$. By X we denote a fixed S_2-subgroup of $\boldsymbol{N}(A)$ and put $X \cap A = S$, $X \cap K = Q$. Thus, X is « contained » in $\{QS, QS\langle \varphi \rangle, QS\langle \varkappa \rangle, QS\langle \varphi\varkappa \rangle, QS\langle \varphi, \varkappa \rangle\}$; here $S = \langle Q \cap A, \pi, \tau, \mu, \lambda, \zeta, \xi \rangle$, where the relations between the generators are those valid in $P \in \mathrm{Syl}_2(L_3(4))$ but modulo $Q \cap A$; of course $P \cong S/Q \cap A$.

The Schur-multiplier of $L_3(4)$ is isomorphic to $Z_4 \times Z_4 \times Z_3$. Thus, we have to handle the cases $Q \cap S \cong Z_2$ and $Q \cap S \cong Z_4$. The case $Q \cap S = \langle 1 \rangle$ has been treated in [3], and there it is proved that then G is isomorphic to the sporadic simple group of Suzuki. Thus, making use of all earlier results we shall have proved the following theorem:

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THEOREM. Let G be a finite, nonabelian simple group which possesses a standard-subgroup A such that $A/\mathbf{Z}(A)$ is isomorphic to $L_3(4)$. Then, G is isomorphic to Sz, He, or O'N.

Here, Sz, He, and O'N denote the sporadic simple groups discovered by Suzuki, Held, and O'Nan, respectively. We remark that by a result of Aschbacher, Q is elementary abelian if the 2-rank of K is greater than 1. In that case we put $Q \cong E_{2^n}$.

1. The case $Q \cap S \cong Z_2$.

(1.1) *Some properties of subgroups of* $\mathbf{N}(A)$.

We have $Q \cap A \cong Z_2$; clearly $|\mathbf{O}_3(A)| \in \{1, 3\}$. Now, A is quasisimple, and so, A is an epimorphic image of the full covering group of $L_3(4)$. Thus, A is an epimorphic image of the perfect central extension of $Z_2 \times Z_2 \times Z_3$ by $L_3(4)$.

Since such an extension possesses an automorphism of order 3 acting fixed—point—free on the 2-part of its center, we see that $A/\mathbf{O}_3(A)$ is uniquely determined up to isomorphism. Using the results of [5] we get the following relations:

$$[\mu, \xi] = \pi\tau , \quad [\lambda, \xi] = \tau , \quad [\mu, \zeta] = q\pi , \quad [\lambda, \zeta] = q\pi\tau ,$$

$$R_1 = \langle q, \pi, \tau, \mu, \lambda \rangle \cong R_2 = \langle q, \pi, \tau, \zeta, \xi \rangle \cong E_{2^5} , \quad \langle q \rangle = Q \cap S .$$

From the results of [5], we get that A possesses the «field»-automorphism φ and the «transpose-inverse»-automorphism $\varkappa$. Thus, $\mathbf{aut}(A)/A$ is a four-group. As in [5], we get

$$\varphi\colon\ q \to q , \quad \pi \to \pi , \quad \tau \to \pi\tau ;$$

$$\varkappa\colon\ q \to q , \quad \pi \to \pi , \quad \tau \to \tau ;$$

$$\varphi\varkappa\colon\ q \to q , \quad \pi \to \pi , \quad \tau \to \pi\tau .$$

Every involution of S lies in R_1 or R_2. Set $S_i = \mathbf{\Omega}_1(QR_i) = \mathbf{\Omega}_1(Q)R_i$. Then, $S_i = R_i$ if $m(Q) = 1$; and $S_i = QR_i \cong E_{2^{n+4}}$ if $m(Q) > 1$. As q has no roots in S—see (1.3)—we get that $\mathbf{\Omega}_1(QS) = S_1S_2$ with

$$(\mathbf{\Omega}_1(QS))' = \langle q, \pi, \tau \rangle .$$

It is clear that $S/\langle q\rangle$ is isomorphic to a S_2-subgroup of $L_3(4)$ and that $\boldsymbol{N}_A(S)/\boldsymbol{Z}(A)$ is isomorphic to a S_2-normalizer of $L_3(4)$. There is an element $g \in \boldsymbol{N}_A(S)\setminus\boldsymbol{Z}(A)S$—defined as in [5]—such that g operates on S mod $\langle q\rangle$ in the following way.

$$g\colon \pi \to \pi\tau \to \tau\,, \quad \mu \to \mu\lambda \to \lambda\,, \quad \zeta \to \zeta\xi \to \xi\,.$$

Further, acting with g on suitable commutators, one obtains

$$g\colon \pi \to q\pi\tau \to q\tau\,.$$

In particular, $\boldsymbol{Z}(S)^{\#} = \langle q, \pi, \tau\rangle^{\#}$ splits into three conjugate classes under $\boldsymbol{N}_A(S)$ with representatives q, π, and $q\pi$.

Obviously, 3 does not divide the order of $\boldsymbol{N}(A)/AK$, since an automorphism of order 3 of the full cover of $L_3(4)$ which is not inner acts fixed-point-free on the 2-part of the Schur-multiplier. Thus, $\boldsymbol{N}(A) = AKX$ and $\langle q\rangle = \boldsymbol{Z}(A)_2$.

(1.2) Lemma. The subgroups S_1 and S_2 are the only elementary abelian subgroups of X of their orders.

Proof. This is a direct consequence of the structures of S, Q, and SQ.

(1.3) Lemma. The involution q has no root in S and $X \in \mathrm{Syl}_2(G)$. If i is an involution in QS, then i is contained in S_1 or S_2. Further, i is conjugate to an involution in $\boldsymbol{\Omega}_1(Q)\langle\pi\rangle$ under A.

Proof. Let x be a root of q in S; set

$$\bar{x} = \langle q\rangle x\,, \qquad \bar{R}_j = R_j/\langle q\rangle\,,$$

$j = 1$ and 2, and $\bar{S} = S/\langle q\rangle$. Then, $\bar{x}$ is an involution of $\bar{S}$. The structure of $\bar{S}$ gives $\bar{x} \in \bar{R}_1 \cup \bar{R}_2$. Hence, $x \in R_1$ or $x \in R_2$. Since R_j is elementary abelian for $j \in \{1, 2\}$, we get $x^2 = 1$. Thus, q has no root in S. In particular, q has no root in $\boldsymbol{\Omega}_1(QS) = S_1S_2$.

Let X_1 be a subgroup of G which contains X as a subgroup of index 2. Then, X_1 normalizes $\langle q, \pi, \tau\rangle = S' = (\boldsymbol{\Omega}_1(Q)S)'$. Under the action of $\boldsymbol{N}_A(S)$ the set $\langle q, \pi, \tau\rangle^{\#}$ splits into three classes with representatives $q, \pi, q\pi$. Clearly, X_1 cannot centralize q, and X_1 normalizes $\boldsymbol{\Omega}_1(Q)S$. Now, π has the root $\mu\lambda\xi$ and $q\pi$ has the root $\mu\zeta$, and both

$\mu\lambda\xi$ and $\mu\zeta$ lie in $S \subseteq \Omega_1(Q)S$. But q has no root in S, and so, q has no root in $\Omega_1(Q)S$. It follows $X \in \mathrm{Syl}_2(G)$.

An involution i of QS has the form $i = us$, $u \in Q$ and $s \in S$. Therefore, $1 = i^2 = u^2 s^2$, so that $u^2 = s^{-2} \in Q \cap S = \langle q \rangle$. Since q has no root in S, we get $u^2 = s^{-2} = 1$. Thus, $u \in \Omega_1(Q)$ and $s \in R_1 \cup R_2$. Thus, i lies in S_1 or S_2, where $S_j = \Omega_1(Q)R_j$ for $j \in \{1, 2\}$. As $A/\boldsymbol{Z}(A)$ possesses exactly one class of involutions and $[Q, A] = \langle 1 \rangle$, one gets that i is conjugate to an element of $\Omega_1(Q)\langle \pi \rangle$.

(1.4) LEMMA. Depending on X, one has:

$$\begin{aligned}
&\boldsymbol{C}_S(\varphi) = \langle q, \pi, \mu\lambda, \xi \rangle \cong \boldsymbol{Z}_2 \times \boldsymbol{D}_8 \quad \text{and} \quad \mho^1(\boldsymbol{C}_S(\varphi)) = \langle \pi \rangle ;\\
&\boldsymbol{C}_S(\varkappa) = \langle q, \pi, \tau \rangle \cong E_{2^3};\\
&\boldsymbol{C}_S(\varphi\varkappa) = \langle q, \mu\lambda\xi\tau \rangle \cong \boldsymbol{Z}_2 \times \boldsymbol{Z}_4 \quad \text{and} \quad \mho^1(\boldsymbol{C}_S(\varphi\varkappa)) = \langle \pi \rangle .
\end{aligned}$$

PROOF. The first two assertions follow immediately from the structure of the automorphism group of $L_3(4)$. Now, $\boldsymbol{C}_S(\varphi\varkappa) \subseteq \langle q, \pi, \mu\lambda\xi\tau, \mu\zeta\tau \rangle$.

We compute $\mu\lambda\xi\tau \xrightarrow{\varkappa} \mu\lambda\xi\pi\tau \xrightarrow{\varkappa} \xi\mu\lambda\pi\tau = \mu\lambda\xi[\xi, \mu\lambda]\pi\tau = \mu\lambda\xi\tau$, and $\mu\zeta\tau \xrightarrow{\varphi k} \zeta\mu\pi\tau = \mu\zeta[\zeta, \mu]\pi\tau = \mu\zeta q\pi\pi\tau = \mu\zeta\tau q$.

Note that $(\mu\lambda\xi\tau)^2 = \pi$. The lemma is proved.

(1.5) LEMMA. Let $y \in X \setminus QS$ with $y \in \{\varphi, \varkappa, \varphi\varkappa\}$. Let z be an involution from QSy. Then, Sz contains at most two classes of involutions under G with representatives z and qz. If $y = \varphi$, then $\mho^1(\boldsymbol{C}_S(z)) = \langle \pi \rangle$ and $z \underset{S}{\sim} \pi z$. If $y = \varphi\varkappa$, then $\Omega^1(\boldsymbol{C}_S(z)) = \langle \pi \rangle$ and $z \sim \pi z$. If $y = \varkappa$, then $\boldsymbol{C}_S(z) = \langle q, \pi, \tau \rangle$ and $z \sim \pi z \underset{S}{\sim} q\tau z \sim q\pi\tau z$ under S.

PROOF. As in [5], one shows that $\boldsymbol{C}_S(z) \underset{S}{\sim} \boldsymbol{C}_S(y)$. Let $y \in \{\varphi, \varphi\varkappa\}$. Then, we have $\mho^1(\boldsymbol{C}_S(y)) = \langle \pi \rangle$. Since $\pi \in \boldsymbol{Z}(S)$, it follows $\mho^1(\boldsymbol{C}_S(z)) = \langle \pi \rangle$. We have $\tau^y = \pi\tau$. Because of $\tau \in \boldsymbol{Z}(S)$, we get $\tau^z = \pi\tau$. Thus, $z^\tau = \pi z$ and $z \sim \pi z$ in S.

Let $y = \varkappa$. We know that $\boldsymbol{C}_S(\varkappa) = \langle q, \pi, \tau \rangle = \boldsymbol{Z}(S)$. Hence, $\boldsymbol{C}_S(z) = \langle q, \pi, \tau \rangle$. Put $z = us\varkappa$, where $u \in Q$ and $s \in S$. As in [5], one shows that $ss^\varkappa \in Q \cap S$; it follows $s^{-2}s\varkappa^{-1}s\varkappa = [s, \varkappa] \in (Q \cap S)\langle s^2 \rangle \subseteq S' = \langle q, \pi, \tau \rangle$. Hence, $s \in \boldsymbol{C}_S(\varkappa \bmod S') = \langle q, \pi, \tau, \mu\lambda\xi, \lambda\zeta \rangle$. Denote the latter group by E. We have $E' = \langle 1 \rangle$. Compute: $[\varkappa, \lambda\zeta] = \varkappa^{-1}\zeta\lambda\varkappa\lambda\zeta = \lambda\zeta\lambda\zeta = q\pi\tau$, $[\varkappa, \mu\lambda\xi] = \pi$, and $[\varkappa, \mu\zeta\xi] = q\tau$.
It follows that

$$\varkappa \sim \pi\varkappa \sim q\tau\varkappa \sim q\pi\tau\varkappa$$

holds under E. Since $E' = \langle 1 \rangle$ and $[u, E] = \langle 1 \rangle$, we get

$$z \sim \pi z \sim q\tau z \sim q\pi\tau z$$

under E. The Lemma is proved.

(1.6) LEMMA. Two involutions of $\boldsymbol{Z}(\boldsymbol{\Omega}_1(Q)S)$ are conjugate in G if, and only if, they are conjugate in $\boldsymbol{N}(\boldsymbol{\Omega}_1(Q)S) \subseteq \boldsymbol{N}(A)$.

PROOF. Note that $\boldsymbol{\Omega}_1(QS) = \boldsymbol{\Omega}_1(Q)S$ is the subgroup of X which is generated by all subgroups of X which are isomorphic to S_1. Let x and y be two involutions of $\boldsymbol{Z}(\boldsymbol{\Omega}_1(QS))$. Then, $\boldsymbol{\Omega}_1(QS)$ lies in $\boldsymbol{C}(x) \cap \cap \boldsymbol{C}(y)$. Assume that there is $g \in G$ such that $x^g = y$. Denote by X_x a S_2-subgroup of $\boldsymbol{C}(x)$ containing $\boldsymbol{\Omega}_1(QS)$ and by X_y a S_2-subgroup of $\boldsymbol{C}(y)$ containing $\boldsymbol{\Omega}_1(QS)$. Then, $X_x^{gh} = X_y$ for some $h \in \boldsymbol{C}(y)$. Clearly, $gh \in \boldsymbol{N}(\boldsymbol{\Omega}_1(QS))$ and $x^{gh} = y^h = y$. Since $(\Omega_1(QS))' = S' = \langle q, \pi, \tau \rangle$, and since q is the only element of S' which has no root in $\boldsymbol{\Omega}_1(QS)$, the assertion follows.

(1.7) LEMMA. (i) Let $m(Q) = 1$, and let $\langle q, s \rangle$ be a four-group contained in QS. Then, $q \nsim qs \nsim s \nsim q$ in G. (ii) Let $m(Q) > 1$. Then, $\langle q \rangle$ is strongly closed in QS with respect to G. If i is an involution of S and $i^g \in QS$ for some $g \in G$, then $i^g \in S$. Further, $\pi \nsim q\pi$. In particular, $QS \subset X$.

PROOF. Assume first that $m(Q) = 1$. Then $\langle q, s \rangle$ and $\langle q, \pi \rangle$ are conjugate via an element of A. We have $\langle q, \pi \rangle \subseteq \boldsymbol{Z}(\boldsymbol{\Omega}_1(Q)S)$, and by assumption $\boldsymbol{\Omega}_1(Q)S = S$. Application of (1.6) gives that G-conjugates in $\langle q, \pi \rangle$ are conjugate under the action of $\boldsymbol{N}(S)$ which lies in $\boldsymbol{N}(A) = AKX$. Clearly, $KX \subseteq \boldsymbol{N}(S)$ and $[\langle q, \pi \rangle, KX] = \langle 1 \rangle$. So, a conjugation of two elements should be performed by an element of $A \cap \boldsymbol{N}(S)$. But q, π, and $q\pi$ are representatives of $\boldsymbol{N}_A(S)$-classes. Assume now that $m(Q) > 1$. If q is conjugate to an element q' of QS, then—by the structure of A—we may assume that q' lies in $Q\langle \pi \rangle$. We have $Q\langle \pi \rangle \subseteq \boldsymbol{Z}(QS)$; note that $\boldsymbol{\Omega}_1(Q)S = QS$. Application of (1.6) yields that $q \sim q'$ holds in $\boldsymbol{N}(A)$. But $\boldsymbol{N}(A) = AKX$, and so, we must have $q = q'$.

Let i be an involution of S and let $i^g \in QS$ for some $g \in G$. We may assume $m(Q) > 1$. There are elements $a, b \in A$ such that i^{ga}, i^b lie in $Q\langle \pi \rangle \subseteq \boldsymbol{Z}(QS)$. Application of (1.6) yields that i^{ga} and i^b are conjugate in $\boldsymbol{N}(A)$; let c be the conjugating element of $\boldsymbol{N}(A)$ with

$i^{ga} = i^{bc}$. Obviously, i^{bc} lies in A, and so, $i^{ga} \in A$. It follows $i^g \in$ $\in A \cap QS = S$. Assume that $\pi \sim q\pi$. By (1.6) this conjugation is performed by an element of $N(A)$. But $N(A) = AKX$, and so, the conjugation $\pi \sim q\pi$ is done by an element of A. Since π, $q\pi$ lie in $Z(S)$, the conjugation is done by an element of $N(S) \cap A$. But this is not the case. The element q is not conjugate to any element different from q in QS. Application of a well-known result of Glauberman yields $QS \subset X$.

(1.8) LEMMA. Let $y \in X \setminus QS$ with $y \in \{\varphi, \varphi\varkappa\}$. Then, q is not conjugate to an element of QSy.

PROOF. Assume that $q \sim z$ for $z \in QSy$. Let $y = \varphi$. From (1.5) we get $\mho^1(C_S(z)) = \langle\pi\rangle$ and $z \sim \pi z$ under S. Let $\tilde{X} \in \mathrm{Syl}_2(C_S(z))$ with $\tilde{X} \supseteq C_X(z)$. Let $\tilde{A}$ be the unique standard-subgroup of $C(z)$; note that $\tilde{A} \sim A$ in G. Set $\tilde{Q} = \tilde{X} \cap C(\tilde{A})$ and $\tilde{S} = \tilde{X} \cap \tilde{A}$. Then, $\tilde{Q} \sim Q$, $\tilde{S} \sim S$, and $\langle z\rangle = \tilde{Q} \cap \tilde{S}$. Further, $\mho^1(\tilde{X}/\tilde{Q}\tilde{S}) = \langle 1\rangle$. Since $\langle\pi\rangle =$ $= \mho^1(C_S(z))$, we get $\pi \in \tilde{Q}\tilde{S}$. Since $z \underset{S}{\sim} \pi z$ and $\pi z \in \tilde{Q}\tilde{S} \setminus \langle z\rangle$, we get a contradiction to (1.7). In the case $y = \varphi\varkappa$ one arrives at a contradiction in the same way. The lemma is proved.

(1.9) LEMMA. The case $X = QS\langle\varkappa\rangle$ does not occur.

PROOF. Assume by way of contradiction that $X = QS\langle\varkappa\rangle$. Since $X \in \mathrm{Syl}_2(G)$, we get from (1.7) and a result of Glauberman that q is conjugate to an involution z of $QS\varkappa$. We know that $C_S(\varkappa) = \langle q, \pi, \tau\rangle$ and that

$$z \sim \pi z \sim q\tau z \sim q\pi\tau z$$

holds under S.

Let $\tilde{X} \in \mathrm{Syl}_2(C(z))$ with $C_X(z) \subseteq \tilde{X}$. Define $\tilde{Q}$, $\tilde{S}$, and $\tilde{A}$ as in (1.8). Then, $|\tilde{X} : \tilde{Q}\tilde{S}| = 2$, and so, $\langle\pi, q\tau\rangle \cap \tilde{Q}\tilde{S} \neq \langle 1\rangle$. Assume that π lies in $\tilde{Q}\tilde{S}$. Then, we get $\pi \in \tilde{S}$ from (1.7), and we know that $z \sim z\pi$. However, this contradicts (1.7) as $\langle z\rangle = \tilde{Q} \cap \tilde{S}$. If $q\tau$ or $q\pi\tau$ is in $\tilde{Q}\tilde{S}$, then we get the same contradiction, since $\langle q\tau, q\pi\tau\rangle \subseteq S$ and by (1.7).

(1.10) LEMMA. Under the assumptions of the theorem the case $Q \cap S \cong Z_2$ does not occur.

PROOF. Application of (1.7), (1.8), (1.9) and a result of Glauberman yields that $X = QS\langle\varphi, \varkappa\rangle$, and that q is conjugate to an involution z

of $QS\varkappa$. We know that q is not conjugate to an involution of $QS\varphi \cup QS\varphi\varkappa$. We have $\tau^{\varphi} = \tau^{\varphi\varkappa} = \pi\tau$, $[\varphi, \varkappa] \in Q$.

Let $\tilde{Q}$, $\tilde{S}$, $\tilde{X}$, and $\tilde{A}$ be the subgroups of $C(z)$ defined as in (1.8). Then, $\langle z \rangle = \tilde{Q} \cap \tilde{S}$ and $\tilde{X}/\tilde{Q}\tilde{S}$ is a four-group. We know that $C_S(z) = \langle q, \pi, \tau \rangle$ and that $z \sim \pi z \sim q\tau z \sim q\pi\tau z$ holds under the action of S. As z is isolated in $\tilde{Q}\tilde{S}$, we see as above that $\pi, q\tau, q\pi\tau, q \notin \tilde{Q}\tilde{S}$.

If $\tau \notin \tilde{Q}\tilde{S}\langle q \rangle$, then $\tilde{Q}\tilde{S}\langle q, \tau \rangle = \tilde{X}$ and $\pi \in \tilde{Q}\tilde{S}q \cup \tilde{Q}\tilde{S}\tau$, since $q\pi\tau \notin \tilde{Q}\tilde{S}$. If $\tau \in \tilde{Q}\tilde{S}\langle q \rangle$, then, as $q\tau \notin \tilde{Q}\tilde{S}$, we must have $\tau \in \tilde{Q}\tilde{S}$. If in addition $\pi \in \tilde{Q}\tilde{S}q$, then we would obtain $q\pi\tau \in \tilde{Q}\tilde{S}$ which is not the case. Hence we have to handle the following two possibilities:

(*a*) $\tilde{X} = \tilde{Q}\tilde{S}\langle q, \tau \rangle$ and $\pi \in \tilde{Q}\tilde{S}q \cup \tilde{Q}\tilde{S}\tau$; and

(*b*) $\tilde{X} = \tilde{Q}\tilde{S}\langle q, \pi \rangle$ and $\tau \in \tilde{Q}\tilde{S}$.

Suppose that $\tilde{X} = \tilde{Q}\tilde{S}\langle q, \tau \rangle$ and $\pi \in \tilde{Q}\tilde{S}q$. Then, $q\pi \in \tilde{Q}\tilde{S}$. Since $q\pi \in S$ and $S \sim \tilde{S}$, we get $q\pi \in \tilde{S}$. The G-fusion of the involutions of QS yields $q\pi\tau \nsim q \nsim \tau$ by (1.7). Consider $\langle q\pi, z \rangle\tau$ in $\tilde{S}\tau$. We know that $q \sim z \sim q\pi\tau z$ holds in G. It follows $q\pi\tau \nsim q\pi\tau z \nsim \tau$.

Since Sy with $y \in \{\varphi, \varkappa, \varphi\varkappa\}$ contains at most two G-classes of involutions, we get $\tau \sim q\pi\tau$ under $\tilde{S}$. Using the structure of $N_A(S)$, we get $q\pi \sim \pi\tau \sim \tau$ and $\pi \sim q\pi\tau \sim q\tau$. It follows $\pi \sim q\pi$ in G, against (1.7).

Suppose now that $\tilde{X} = \tilde{Q}\tilde{S}\langle q, \tau \rangle$ and $\pi \in \tilde{Q}\tilde{S}\tau$. Then, $\pi\tau \in \tilde{Q}\tilde{S}$, and so, $\pi\tau \in \tilde{S}$. Consider the set $\langle z, \pi\tau \rangle q\tau$ in $\tilde{S}q\tau$. We know that $q\tau z \sim z \sim q \nsim q\pi$ and $q \nsim q\tau$. Hence, $q\tau \nsim q\tau z \nsim q\pi$. Since in $\langle z, \pi\tau \rangle q\tau$ there are at most two G-classes of involutions, we derive $q\tau \sim q\pi$. However, π is conjugate to $q\tau$ via a 3-element in $N_A(S)$, and this gives a contradiction.

Finally, we handle the case (*b*). Here, we have $\tilde{X} = \tilde{Q}\tilde{S}\langle q, \pi \rangle$ and $\tau \in \tilde{Q}\tilde{S}$. Thus, $\tau \in \tilde{S}$. Consider the set $\langle z, \tau \rangle q\pi$ which lies in $\tilde{S}q\pi$. We know that $q\pi\tau z \sim q$, $q\pi\tau \sim \pi \nsim q$, and $q\pi \nsim q$ and $q\pi \nsim \pi$. Hence, in $\langle z, \tau \rangle q\pi$ we have three G-classes of involutions against the fact that in $\tilde{S}q\pi$ there are at most two G-classes of involutions. This final contradiction proves the lemma.

2. The case $Q \cap S \cong Z_4$.

(2.1) *Some properties of subgroups of* $N(A)$.

We are interested in the possible structures for S. Set $Q \cap S = \langle t \rangle$ with $t^2 = q$ and $S = \langle t, \pi, \tau, \mu, \lambda, \zeta, \xi \rangle$.

We put

$$[\mu, \xi] = q^{\gamma}\pi\tau\,, \quad [\lambda, \xi] = q^{\delta}\tau\,, \quad [\mu, \zeta] = tq^{\beta}\pi\,, \quad [\lambda, \zeta] = tq^{\alpha}\pi\tau\,,$$

where $\alpha, \beta, \gamma, \delta \in \{0, 1\}$ and $t^{-1} = tq = t^3$. If $s \in S$, then $o(s) = o(sq)$ if $s \neq q$. We replace $q^{\gamma}\pi\tau$ by $\pi\tau$ and $q^{\delta}\tau$ by τ without changing the defining relations of S. Interchanging t and t^{-1} if necessary, we may put $[\mu, \zeta] = t\pi$. Thus, we get:

$$[\mu, \xi] = \pi\tau\,, \quad [\lambda, \xi] = \tau\,, \quad [\mu, \zeta] = t\pi\,, \quad [\lambda, \zeta] = tq^{\alpha}\pi\tau\,,$$

where $\alpha \in \{0, 1\}$. Furthermore, we have the freedom to choose μ, λ, ζ, and ξ to be involutions, since for each $x \in \{\mu, \lambda, \zeta, \xi\}$ either $o(x) = 2$ or $o(tx) = 2$, and the commutator relations given above remain unchanged with tx in place of x.

There is an element g in $\boldsymbol{N}_A(S)\diagdown\boldsymbol{Z}(A)S$ which acts fixed-point-free on S modulo $\langle t\rangle$ in the following way:

$$g\colon \pi \to \pi\tau \to \tau\,, \quad \mu \to \mu\lambda \to \lambda\,, \quad \zeta \to \zeta\xi \to \xi\,.$$

In fact, $\boldsymbol{N}_A(S) = \boldsymbol{Z}(A)S\langle g\rangle$. We have $t\pi = [\mu, \zeta] \xrightarrow{g^2} [\lambda, \xi] = \tau$, and so, $q\pi^2 = \tau^2 \in \langle q\rangle$; this means that either $o(\pi) = 4$ and $o(\tau) = 2$, or $o(\pi) = 2$ and $o(\tau) = 4$. We compute:

$$\pi\tau = [\mu, \xi] \xrightarrow{g} [\mu\lambda, \zeta] = [\mu, \zeta]^{\lambda}[\lambda, \zeta] = t\pi^{\lambda}\cdot tq^{\alpha}\pi\tau = qq^{\alpha+i(\lambda)}\pi^2\tau;$$

thus $\pi\tau\pi\tau = \pi^2\tau\pi^2\tau = \tau^2$, and so, $\pi\tau\pi = \tau$. One obtains two cases:

a) $o(\pi) = 4$ and $o(\tau) = 2$; then $\tau\pi\tau = \pi^{-1}$, $\langle\pi, \tau\rangle \cong \boldsymbol{D}_8$, and $\langle\pi, t\tau\rangle \cong Q_8$.

b) $o(\pi) = 2$ and $o(\tau) = 4$; then $[\pi, \tau] = 1$, and $\langle t, \pi, \tau\rangle = \langle t, \pi, t\tau\rangle \cong \boldsymbol{Z}_4 \times \boldsymbol{Z}_2 \times \boldsymbol{Z}_2$.

Put $R_1 = \langle t, \pi, \tau, \mu, \lambda\rangle$ and $R_2 = \langle t, \pi, \tau, \zeta, \xi\rangle$. Then, $S = R_1R_2$. Clearly, $\boldsymbol{D}(S) = S' = \langle t, \pi, \tau\rangle$. If $S' = \boldsymbol{D}(S) = \boldsymbol{Z}(S)$, then S would be special, and hence, S' would be elementary abelian, namely: Let $x, y \in S$; then $[x, y] \in \boldsymbol{Z}(S)$ and $[x, y]^2 = x^{-2}y^{-1}x^2y = 1$, since $\mho^1(S) \subseteq \boldsymbol{D}(S)$; hence every commutator of S has order 2 or 1 which implies that S' is elementary abelian. This is, however, not the case. Thus,

$\boldsymbol{Z}(S) \subset \langle t, \pi, \tau \rangle$, and since g acts fixed-point-free on $\langle \pi, \tau, t \rangle / \langle t \rangle$, we get $\boldsymbol{Z}\langle S \rangle = \langle t \rangle$.

It follows that not both R_1 and R_2 are abelian. We know that $\mho^1(S) \subseteq \langle t, \pi, \tau \rangle = R_1 \cap R_2$. Assume that R_1 was abelian. Put $R_1^* = \boldsymbol{\Omega}_1(R_1)$. We have $R_1/\boldsymbol{D}(R_1) \simeq E_{2^5}$, and so, $\boldsymbol{\Omega}_1(R_1)$ is elementary abelian of order 2^5. From the Jordan-canonical-form of ζ and ξ on R_1^*, we get $|\boldsymbol{C}_{R_1^*}(\langle \zeta, \xi \rangle)| \geqslant 4$. Since $S = \boldsymbol{C}_S(R_1)\langle \zeta, \xi \rangle$, we see that $\boldsymbol{C}_{R_1^*}(\langle \zeta, \xi \rangle)$ lies in $\boldsymbol{Z}(S)$. But $\boldsymbol{Z}(S) = \langle t \rangle$, and we have derived a contradiction. Thus, $R_1' \neq \langle 1 \rangle$. Similarly, we get $R_2' \neq \langle 1 \rangle$. It follows $R_1' = R_2' = \langle q \rangle$.

A subgroup of A involving A_5 acts transitively on $R_i/\langle t \rangle$. This implies $\boldsymbol{Z}(R_1) = \boldsymbol{Z}(R_2) = \langle t \rangle$.

Clearly, $\boldsymbol{Z}(X) \subseteq QS$, and so, we have $\boldsymbol{Z}(X) \subseteq Q$; note that Q is cyclic by a result of Aschbacher. It follows that $X \in \mathrm{Syl}_2(G)$ as $\boldsymbol{C}(q) \subseteq \boldsymbol{N}(A)$.

Since an element of order 5 of A acts fixed-point-free on $R_i/\langle t \rangle$, and since $R_i/\langle t^2 \rangle$ is elementary abelian, we deduce that $R_1 = \langle t \rangle \curlywedge E_1$ and $R_2 = \langle t \rangle \curlywedge E_2$, where E_i is extraspecial of order 2^5 and of type $D_8 \curlywedge Q_8$; here $\curlywedge$ denotes the central product with amalgamated center of at least one factor. Clearly, E_i possesses 10 off-central involution and 20 elements of order 4. Thus, R_i possesses 30 off-central involutions and 30 off-central elements of order 4. The 2-rank of E_i is 2 as the maximal abelian subgrups of E_i are of type (2, 4). Thus, the 2-rank of R_1 and of R_2 is equal to 3.

We know that $t\langle t^2 \rangle$, $t^2 = q$, does not possess a root in $S/\langle q \rangle$. Hence, t has no root in S. Let i be an involution in QS. Then, $i = us$, $u \in Q$ and $s \in S$. We have $1 = i^2 = u^2 s^2$, and so, $u^{-2} = s^2 \in Q \cap S = \langle t \rangle$. Since t has no root in S, we get $u^{-2} = s^2 \in \langle q \rangle$. Since Q is cyclic and $t \in Q$, we have $u \in \langle t \rangle$. It follows $i = us \in S$. From the structure of S follows $i \in R_1 \cup R_2$.

Assume by way of contradiction that S had an elementary abelian subgroup E of order 16. From the structure of $L_3(4)$ we get that if x is an element of $R_i \setminus \langle t, \pi, \tau \rangle$, then $\boldsymbol{C}_S(x) \subseteq R_i$ for $i \in \{1, 2\}$. Since the 2-rank of R_i is 3, we get $|R_i E| \geqslant 2^7$ for $i \in \{1, 2\}$. Assume that $|R_1 E| = 2^7$. Then, $R_1 E \in \{R_1\langle \zeta \rangle, R_1\langle \xi \rangle, R_1\langle \zeta\xi \rangle\}$. There is an involution $e \in E$ such that $R_1 E = R_1\langle e \rangle$. We know that $e \in R_2$. Since $\boldsymbol{C}_S(e) \subseteq R_2$, we get $E \subseteq R_2$, and this is a contradiction. Similarly, one sees that $|R_2 E| = 2^7$ does not happen. Assume now that $R_i E = S$ for $i = 1$ or $i = 2$. Then, there is an involution $e \in E \setminus R_i$, and so, $E \subseteq \boldsymbol{C}_S(e) \subseteq R_j$, $j \neq i$; again we arrived at a contradiction. We have shown that the 2-rank of QS is precisely 3.

(2.2) LEMMA. The S_2-subgroup Q of K is cyclic, $\boldsymbol{Z}(S) = \langle t \rangle = \boldsymbol{Z}(R_i)$ for $i \in \{1, 2\}$. If i is an involution of $QS \setminus \langle q \rangle$, then i is conjugate to an involution of $\langle t \rangle \pi$ under A. The involutions of $\langle t \rangle \pi$ are conjugate under S.

PROOF. We have to prove only the last assertion. If $o(\pi) = 4$, then $o(t\pi) = 2$. Clearly, $\pi \sim \pi q$ under R_i, since E_i is extraspecial.

(2.3) LEMMA. The case (a) of (2.1) does not occur. Thus, we have $o(\pi) = 2$, $o(\tau) = 4$, and $\langle t, \pi, t\tau \rangle$ is of type $(4, 2, 2)$.

PROOF. Put $V = \langle t, \pi, t\tau \rangle$, and assume that we are in case (a). Since $V = \boldsymbol{Z}_2(S)$ and $\boldsymbol{Z}(S) = \langle t \rangle$, we get $|S : \boldsymbol{C}_S(V)| = 2^2$; note that $\langle t \rangle \langle \pi \rangle$ and $\langle t \rangle \langle \tau \rangle$ are both normal in S and that $\langle t \rangle \pi$ contains precisely two involutions; the last assertion is also true for $\langle t \rangle \tau$. Since $V = \langle t \rangle \curlywedge \langle \pi, t\tau \rangle$ with $\langle \pi, t\tau \rangle \cong Q_8$, we get $V \cap \boldsymbol{C}_S(V) = \langle t \rangle$, and so, $V\boldsymbol{C}_S(V) = S$. But S/V is elementary abelian of order 16, and $\boldsymbol{C}_S(V)/\langle t \rangle \cong S/V$. Hence, $(V/\langle t \rangle)(\boldsymbol{C}_S(V)/\langle t \rangle) = S/\langle t \rangle$ would be elementary abelian against the structure of a S_2-subgroup of $L_3(4)$.

(2.4) LEMMA. The involutions of $A \setminus \boldsymbol{Z}(A)$ form a single conjugate class. Further, $\boldsymbol{C}_S(V) \setminus \langle q, \pi, t\tau \rangle$ does not contain involutions; here and in what follows, we put $V = \langle q, \pi, t\tau \rangle$. Clearly, $|\boldsymbol{C}_S(V)| = 2^6$.

PROOF. The first assertion follows from the fact that $A/\boldsymbol{Z}(A) \cong$ $\cong L_3(4)$ and that $\pi \sim q\pi$ under R_1 and R_2. Let x be an involution of $\boldsymbol{C}_S(V) \setminus V$. Then, $V \times \langle x \rangle$ is elementary abelian of order 16 against the fact that the 2-rank of S is 3.

(2.5) LEMMA. We have

$$\boldsymbol{\Omega}_1\big(\mho^1(\boldsymbol{C}_s(V))\big) = \boldsymbol{\Omega}_1(\boldsymbol{C}_S(V)) = V = \boldsymbol{\Omega}_1(\mho^1(QS)) .$$

Further,

$$\boldsymbol{N}_{\boldsymbol{N}(A)}(V)/\boldsymbol{C}(V) \simeq \begin{cases} A_4 , & \text{if } X \in \{QS, QS\langle \varkappa \rangle\} , \\ \Sigma_4 , & \text{if } X \in \{QS\langle \varphi \rangle, QS\langle \varphi\varkappa \rangle, QS\langle \varphi, \varkappa \rangle\} . \end{cases}$$

PROOF. We know that $S/\langle q \rangle$ has exponent 4. The first assertion follows from the fact that $\boldsymbol{C}_S(V) \setminus V$ does not contain involutions and that every involution of QS lies in $R_1 \cup R_2$; note that $\mho^1(S) \subseteq (QS)' =$ $= \langle t, \pi, \tau \rangle$. Clearly, $S/\boldsymbol{C}_S(V)$ is elementary of order 4, and S and

$C_S(V)$ are g-invariant. The element g acts fixed-point-free on $S/\langle t\rangle$, and so, A induces an automorphism group isomorphic to A_4 of V. Clearly, X normalizes V, $C_S(V)$, S, and $Z_2(S) = \langle t, \pi, \tau\rangle$. If $\varkappa \in X$, then $[q, \varkappa] = 1$; and also $[\varkappa, V] = 1$, since the centralizer of $\varkappa$ involves a section of A isomorphic to A_5, and we know that $C_S(\varkappa) \subseteq \langle t, \pi, t\tau\rangle$. If $y \in \{\varphi, \varphi\varkappa\}$, then we get from the last section that $y \in N_{N(A)}(V)\setminus C(V)$. Clearly, $C(V) \subseteq C(q) = N(A) = KAX$ with $K \subseteq C(V)$. Since $N_{N(A)}(V)/C(V)$ is a subgroup of $L_3(2)$ which has no element of order 7, the assertion of the lemma follows.

We want to get more information on the multiplication table of S. Clearly, $\mu\lambda$ or $\mu\lambda t$ is an involution. Compute $(\mu\lambda\xi)^2 \equiv \pi \bmod \langle q\rangle$; thus $o(\mu\lambda\xi) = 4$. It follows that $\langle \mu\lambda, \xi\rangle$ or $\langle \mu\lambda t, \xi\rangle$ is dihedral of order 8 with center in $\langle q, \pi\rangle\setminus\langle q\rangle$. We have shown that $\langle t, \pi, \mu\lambda, \xi\rangle = F \cong Z_4\times D_8$ and $Z(F) = \langle t, \pi\rangle$. Hence, $C_{R_1}(\pi) = \langle t, \pi, t\tau, \mu\lambda\rangle$ and $C_{R_2}(\pi) = \langle t, \pi, t\tau, \xi\rangle$, and $|C_{R_i}(\pi)| = 2^5$ for $i \in \{1, 2\}$. It follows $\pi^\mu = \pi^\lambda = q\pi$, $\pi^\zeta = \pi^{\zeta\xi} = q\pi$. Further, since the 2-rank $m(S)$ is equal to 3, we get $(t\tau)^{\mu\lambda} = qt\tau = (t\tau)^\xi$, $(t\tau)^{\mu\lambda\xi} = t\tau$.

Compute: $q = [\pi, \mu] \xrightarrow{g^2} [t\tau, \lambda] = q$, hence $[\tau, \lambda] = q$; also $1 = [\pi, \mu\lambda] \xrightarrow{g^2} [t\tau, \mu] = 1$, hence $[\tau, \mu] = 1$. Further, we have $1 = [\pi, \xi] \xrightarrow{g^2} [\tau, \zeta\xi] = 1$, and so, $\tau^\zeta = q\tau$. It follows $C_S(\pi) = \langle t, \pi, t\tau, \mu\lambda, \mu\zeta, \xi\rangle$ has order 2^7. Thus, $C_S(\langle t, \pi, t\tau\rangle) = \langle t, \pi, t\tau, \mu\lambda\xi, \mu\zeta\xi\rangle = C_S(V)$, where $V = \Omega_1(Z_2(S)) = \langle q, \pi, t\tau\rangle$.

Put $W = C_S(V)$. We summarize:

(2.6) Lemma. We have the following relations for the generators $t, \pi, \tau, \mu, \lambda, \zeta, \xi$ of S:

$$t^4 = \pi^2 = \tau^4 = \mu^2 = \lambda^2 = \zeta^2 = \xi^2 = 1\,, \quad t^2 = \tau^2 = q\,, \quad [\pi, \tau] = 1\,,$$

$$\pi^\mu = \pi^\lambda = \pi^\zeta = q\pi\,, \quad \pi^{\mu\lambda} = \pi^\xi = \pi\,, \quad \tau^\xi = \tau^\lambda = \tau^\zeta = q\tau\,,$$

$$[\tau, \mu] = 1\,, \quad [\mu, \lambda] \in \langle q\rangle\,, \quad [\zeta, \xi] \in \langle q\rangle; \quad C_S(\pi) = \langle t, \pi, \tau, \mu\lambda, \mu\zeta, \xi\rangle\,,$$

$$C_S(\langle t, \pi, t\tau\rangle) = \langle t, \pi, t\tau, \mu\lambda\xi, \mu\zeta\xi\rangle; \quad [\mu, \xi] = \pi\tau\,, \quad [\mu, \zeta] = t\pi\,,$$

$$[\lambda, \xi] = \tau\,, \quad [\lambda, \zeta] = tq^\alpha\pi\tau; \quad g\colon \pi \to q^{1+\alpha}t\pi\tau \to qt\tau\,.$$

From the action of the outer automorphism group of the full cover A^* of $L_3(4)$ on $O_2(A^*)$ one gets that our standard-subgroup A possesses the « automorphism $\varphi\varkappa$ ». Put $q^\iota = [\mu, \lambda]$ and compute $q^\iota = [\mu, \lambda] \xrightarrow{\varphi\varkappa} [\zeta, \xi] = q^\iota$. We want to determine under what conditions the elements $\mu\lambda\xi$ and $\mu\zeta\xi$ commute. Compute: $\mu\lambda\xi \xrightarrow{\mu} \mu q^l \lambda\pi\tau\xi \xrightarrow{\zeta} \mu t\pi q^l \lambda t q^\alpha \pi\tau\pi\tau q^l \xi = q^\alpha \mu\pi\lambda\xi \xrightarrow{\xi} q^\alpha \mu\pi\tau\pi\lambda\tau\xi = q^\alpha \mu\lambda\xi$.

We get:

(2.7) Lemma. $[\mu\lambda\xi, \mu\zeta\xi] = 1$ if and only if, $\alpha = 0$. Here, $[\mu, \lambda] = [\zeta, \xi] = q^l$, $l \in \{1, 2\}$. Further,

$$(\mu\lambda\xi)^2 = q^{1+l}\pi\,, \qquad (\mu\zeta\xi)^2 = q^l t\tau\,, \qquad (\mu\lambda\xi\mu\zeta\xi)^2 = q^{1+\alpha}\pi t\tau\,.$$

Thus, W is abelian of type $(4, 4, 4)$ if $\alpha = 0$, and $W' = \langle q\rangle$ if $\alpha = 1$.

(2.8) Lemma. Let $y \in X\setminus QS$ with $y \in \{\varphi, \varkappa, \varphi\varkappa\}$. If QSy contains an involution y^*, then there is an involution z in QSy conjugate to y^* under S which acts on S in the same way as y does.

Proof. The assertion follows from the proof of [5; Lemma 3.1].

(2.9) Here, we shall study the situation of a subgroup $\tilde W$ of X with $\tilde W \cong W$.

If W is abelian, then W is of type $(4, 4, 4)$; if $W' \neq \langle 1\rangle$, then $W' = \langle q\rangle$ and $\boldsymbol{Z}(W) = \langle t, \pi, t\tau\rangle$ and $\boldsymbol{\Omega}_1(W) = \langle q, \pi, t\tau\rangle$. Note that $\exp(W) = 4$. We denote by $\tilde W$ a subgroup of X isomorphic to W.

We assume first that $\tilde W$ lies in QS. We know that $\boldsymbol{\Omega}_1(\tilde W) \subseteq S$, and since QS/S is cyclic, we get $|\tilde W \cap S| \geqslant 2^5$. Put $\hat W = \tilde W \cap S$. We assume that $\tilde W \not\subseteq S$. Then, there is an element us of order 4 of $\tilde W\setminus S$, $u \in Q^\#$ and $s \in S^\#$. We compute: $u^4 = s^{-4} \in S \cap Q = \langle t\rangle$, and hence, $u^4 = s^{-4} = q$ as t has no root in S, since otherwise $u^4 = 1$ and $u \in S$. Thus, $o(u) = o(s) = 8$. Since $|\hat W| = 2^5$ and $\exp(W) = 4$, we get $|\hat W\langle s\rangle| \geqslant 2^6$; clearly, s centralizes $\boldsymbol{Z}(\tilde W)$ and operates on $\hat W$ in the same way as us does. If $|\hat W\langle s\rangle| = 2^8$, then $\hat W\langle s\rangle = S$, and $\boldsymbol{Z}(S)$ would contain $\boldsymbol{\Omega}_1(\hat W)$ which is not cyclic. If $|\hat W\langle s\rangle| = 2^7$, then, as $\boldsymbol{\Omega}_1(\tilde W)$ lies in $\boldsymbol{Z}(\hat W\langle s\rangle)$, we get a contradiction to $\boldsymbol{Z}(S) = \langle t\rangle$ by the Jordan-canonical-form. Thus, we have $|\hat W\langle s\rangle| = 2^6$. If $\hat W\langle s\rangle W = S$, then $\hat W\langle s\rangle \cap W$ has order 2^4, and from the structure of W, we see that the intersection contains a four-group, which lies in the center of $\hat W\langle s\rangle$ and of W; note that the 2-rank of QS is 3 and that $\tilde W \cong W$. We get a contradiction to $\boldsymbol{Z}(S) = \langle t\rangle$. If $|\hat W\langle s\rangle W| = 2^7$, then $|\hat W\langle s\rangle \cap W| = 2^5$ and $\hat W\langle s\rangle \cap W$ contains an elementary abelian subgroup of order 8. Thus, $\boldsymbol{\Omega}_1(W)$ lies in $\boldsymbol{Z}(\hat W\langle s\rangle W)$, and again we get a contradiction by the Jordan-canonical-form. The case $|\hat W\langle s\rangle W| = 2^6$ is not possible as $\exp(W) = 4$ and $o(s) = 8$. We have shown that $\tilde W$ must lie in S. But then $|\tilde W \cap W| \geqslant 2^4$, and hence, the center of S would not be cyclic. It follows that if $\tilde W \subseteq QS$ then $\tilde W = W$.

Finally, we have to consider the case that $\tilde{W}$ lies in X but not in QS. Remember that $|X:QS|\leqslant 4$. Thus, $QS\cap\tilde{W}$ contains a subgroup of type $(2,2,4)$. Note that $\mathbf{\Omega}_1(\tilde{W})\subseteq \mathbf{Z}(\tilde{W})$ and that $\mathbf{\Omega}_1(\tilde{W})=$ $=\mho^1(\tilde{W})$. We know that $\mathbf{\Omega}_1(\tilde{W})$ must lie in S as X/QS is elementary and S contains the involutions of QS. Put $X^*=X/Q$, $S^*=SQ/Q$, and $\tilde{W}^*=\tilde{W}Q/Q$. Then, $|\tilde{W}^*|\geqslant 2^4$, since $\exp(\tilde{W})=4$, and also $|S^*\cap$ $\cap\tilde{W}^*|\geqslant 2^2$. As $\mathbf{\Omega}_1(\tilde{W})\subseteq S$, we see that there is a four-group in $S^*\cap\tilde{W}^*$ which is centralized by $\tilde{W}^*$. Let us assume first that $X^*=S^*\tilde{W}^*$ and $|X^*:S^*|=4$. Then, we get a contradiction, because—computing in $P\langle\varphi,\varkappa\rangle\in \mathrm{Syl}_2\big(\mathbf{aut}\,(L_3(4))\big)$—we see that $\mathbf{C}_P(s_1\varphi)\cap \mathbf{C}(s_2\varkappa)$ is cyclic for $s_1, s_2\in P$; note that in $\mathbf{aut}\,(L_3(4))$ we have $\mathbf{C}_P(s\varphi)\subseteq\langle\pi,\tau,\mu\lambda,\xi\rangle\setminus\{\tau\}$, $\mathbf{C}_P(s\varkappa)\subseteq\langle\lambda\zeta,\mu\lambda\xi\rangle$ which is abelian of type $(4,4)$, and

$$\mathbf{C}_P(s\varphi\varkappa)\subseteq\langle\pi,\tau,\mu\lambda\xi\tau,\mu\zeta\tau\rangle\setminus\{\tau\}\quad\text{for } s\in P;$$

note also that $\langle\pi,\tau,\mu\lambda,\xi\rangle\cong Z_2\times D_8$ and that $\langle\pi,\tau,\mu\lambda\xi\tau,\mu\zeta\tau\rangle\cong$ $\cong Z_2\times Q_8$.

Now, we consider the case that $|QS\tilde{W}:QS|=2$. From the structure of $\mathbf{aut}\,(L_3(4))$ we get that $QS\tilde{W}=QS\langle\varphi\varkappa\rangle$ is impossible, since $\mathbf{C}_P(s\varphi\varkappa)$ does not contain a four-group, but $\mathbf{Z}(\tilde{W}^*)$ contains a four-group in $\tilde{W}^*\cap S^*$. Thus, either $QS\tilde{W}=QS\langle\varphi\rangle$ or $QS\tilde{W}=QS\langle\varkappa\rangle$.

Assume that $QS\tilde{W}=QS\langle\varkappa\rangle$. We know that $\mathbf{C}(s\varkappa)\subseteq\langle t,\pi,t\tau,\lambda\zeta,$ $\mu\lambda\xi\rangle=W$ for $s\in S$. Since $\exp(X/QS)=2$, we see that $\mho^1(\tilde{W})\subseteq QS$, and so $\mho^1(\tilde{W})=\mathbf{\Omega}_1(\tilde{W})\subseteq S$. There is $u\in Q$, $s\in S$ such that $us\varkappa\in\tilde{W}$, and so, $s\varkappa$ centralizes $\mathbf{\Omega}_1(\mathbf{Z}(\tilde{W}))=\mathbf{\Omega}_1(\tilde{W})$, an elementary abelian group of order 8. It follows $\mathbf{\Omega}_1(\tilde{W})\subseteq W$, and so, $\mathbf{\Omega}_1(\tilde{W})=\mathbf{\Omega}_1(W)=\langle q,\pi,t\tau\rangle$.

Assume that $QS\tilde{W}=QS\langle\varphi\rangle$. Clearly, $|\tilde{W}\cap QS|=2^5$. Since $S(\tilde{W}\cap$ $\cap QS)/S$ is contained in the cyclic group QS/S, and since $\mho^1(\tilde{W})=$ $=\mathbf{\Omega}_1(\tilde{W})\subseteq S$, we get $|\tilde{W}\cap S|\geqslant 2^4$; note that $S(\tilde{W}\cap QS)/S\cong(\tilde{W}\cap$ $\cap QS)/(\tilde{W}\cap S)$. Now, $\mathbf{\Omega}_1(\tilde{W})$ is centralized by an element $vs\varphi$, where $v\in Q$, $s\in S$; hence $\mathbf{\Omega}_1(\tilde{W})$ is centralized by $s\varphi$. We know that $\mathbf{C}_S(s\varphi)$ is contained in $\langle t,\pi,t\tau,\mu\lambda,\xi\rangle$ which group we denote by B. Also we know that $\mathbf{\Omega}_1(\tilde{W})$ lies in R_1 or R_2. Note that $(t\tau)^{s\varphi}\neq t\tau$ for any $s\in S$. Hence, $\mathbf{\Omega}_1(\tilde{W})\neq\mathbf{\Omega}_1(W)$. We have $|B\cap R_i|=2^5$, and so, $|\langle t,$ $\pi,t\tau\rangle\cap\mathbf{\Omega}_1(\tilde{W})|\geqslant 2^2$; it follows $|\mathbf{\Omega}_1(W)\cap\mathbf{\Omega}_1(\tilde{W})|=4$. From the structure of $L_3(4)$ follows that $\mathbf{C}_S(\mathbf{\Omega}_1(\tilde{W}))$ lies in R_1 or R_2. From the symplectic structure of R_i follows $|\mathbf{C}_S(\mathbf{\Omega}_1(\tilde{W}))|=2^4$; note that $q\in$ $\in\mathbf{\Omega}_1(\tilde{W})$ as the 2-rank of S is 3. We have derived $|S\cap\tilde{W}|=2^4$. But $|SQ\cap\tilde{W}|=2^5$. Thus, there exists $u\in Q$, $s\in S$ such that $us\in\tilde{W}\setminus S$. This implies $u\notin\langle t\rangle$. Since $\mathbf{\Omega}_1(\tilde{W})\subseteq S$, we have $o(us)=4$, and we know that $s\in R_i$, $i=1$ or 2, as $[s,\mathbf{\Omega}_1(\tilde{W})]=\langle 1\rangle$. This implies $s^4=1$.

But then $u^4 s^4 = 1$ implies $u^4 = 1$ and $u \in \langle t \rangle$ which is not possible. We have shown that in $QS\langle \varphi \rangle$ there is only one subgroup isomorphic to W, namely W itself. We summerize:

(2.10) LEMMA. Let $\tilde{W}$ be a subgroup of X isomorphic to W and assume $\tilde{W} \neq W$. Then $\tilde{W}$ is not contained in QS, $QS\langle \varphi \rangle$, or $QS\langle \varphi \varkappa \rangle$. The case $QS\tilde{W} = QS\langle \varphi, \varkappa \rangle$ is not possible. If $QS\tilde{W} = QS\langle \varkappa \rangle$, then $\mathbf{\Omega}_1(W) = \mathbf{\Omega}_1(\tilde{W}) = \langle q, \pi, t\tau \rangle$.

(2.11) LEMMA. If $\pi \sim q$ holds in G, then $\pi \sim q$ holds in $\mathbf{N}(\mathbf{\Omega}_1(W))$.

PROOF. Denote by J the intersection of all subgroups $\tilde{W}$ of X which are isomorphic to W. Then, $\mathbf{\Omega}_1(W) = \mathbf{\Omega}_1(\mathbf{Z}(J))$.

Assume that $q \sim \pi$ holds in G. Denote by X_π a S_2-subgroup of $\mathbf{C}_G(\pi)$ which contains $X \cap \mathbf{C}(\pi)$. We have $W \subseteq X \cap X_\pi$. Thus, $\mathbf{\Omega}_1(W)$ is normalized by X and X_π, and so, as $\mathbf{Z}(X)$ is cyclic, we get $q \sim \pi$ in $\langle X, X_\pi \rangle \subseteq \mathbf{N}(\mathbf{\Omega}_1(W))$.

(2.12) LEMMA. The case $QS = X$ does not occur.

PROOF. Note that in QS there are only two $\mathbf{N}(A)$-classes of involutions with representatives q and π. By a result of Glauberman we have $q \sim \pi$ in G. From (2.11) we get that q and π are conjugate under the action of $\mathbf{N}(\mathbf{\Omega}_1(W))$. Since π has 6 conjugates under $\mathbf{N}(S)$, we see that an element of order 7 of $\mathbf{N}(V)/\mathbf{C}(V)$ acts fixed-point-free on V. Thus, G induces $L_3(2)$ on V, against $|QS : \mathbf{C}_{QS}(V)| = 4$. The lemma is proved.

(2.13) LEMMA. If $\varphi, \varkappa$, or $\varphi\varkappa$ are present in X, then $\mathbf{C}_S(\varphi) \subseteq \subseteq \langle t, \pi, \mu\lambda, \xi \rangle$, $\mathbf{C}_S(\varkappa) \subseteq \langle t, \pi, t\tau \rangle$, $\mathbf{C}_S(\varphi\varkappa) \subseteq \langle t, \mu\lambda\xi\tau \rangle$; further $(\mu\lambda\xi\tau)^2 = q^l \pi$.

PROOF. The assertion is a consequence of (1.4).

(2.14) LEMMA. Let q be conjugate to an involution z in $X \setminus QS$. If $[t, z] = 1$, then $q \sim \pi$ holds in G.

PROOF. Let $q \sim z \in X \setminus QS$ and $[t, z] = 1$. Denote by $\tilde{X}$ a S_2-subgroup of $\mathbf{C}_G(z)$ with $\mathbf{C}_X(z) \subseteq \tilde{X}$ and by $\tilde{A}$ the unique standard-subgroup of $\mathbf{C}(z)$. Put $\tilde{X} \cap \tilde{A} = \tilde{S}$ and $\tilde{X} \cap \mathbf{C}(\tilde{A}) = \tilde{Q}$. We have $z \in \in \tilde{Q} \cap \tilde{S}$. Since $\tilde{X}/\tilde{Q}\tilde{S}$ is elementary abelian and $t \in \tilde{X}$, we get $t^2 = q \in \in \tilde{Q}\tilde{S}$, and so $q \in \tilde{S}$ as $o(q) = 2$. Clearly, $q \neq z$. It follows that q is conjugate to π in G, since all involutions of $\tilde{A} \setminus \langle z \rangle$ are conjugate to π; note that $q \in \tilde{S} \subseteq \tilde{A} \sim A$.

(2.15) LEMMA. We have $[\pi, \varphi] = [\pi, \varkappa] = [t\tau, \varkappa] = 1$. Also $\alpha = 0$ if, and only if $[t, \varphi] = 1$; further $\alpha = 1$ if, and only if $[t, \varkappa] = 1$; $t^{\varphi\varkappa} = t^{-1}$ always.

PROOF. Since the centralizers of φ and $\varkappa$ involve $L_3(2)$ and A_5, respectively, we see easily that $\pi \in \boldsymbol{C}_S(\varphi)$ and $\langle t^2, \pi, t\tau\rangle \subseteq \boldsymbol{C}_S(\varkappa)$. Compute $t\pi = [\mu, \zeta] \xrightarrow{\varphi} [\lambda, \zeta\xi] = [\lambda, \xi][\lambda, \zeta]^{\xi} = \tau(tq^{\alpha}\pi\tau)^{\xi} = tq^{\alpha}\pi$; thus $\alpha = 0$ if, and only if $[t, \varphi] = 1$. Compute further $t\pi = [\mu, \zeta] \xrightarrow{\varkappa} [\zeta\xi, \lambda] = [\zeta, \lambda]^{\xi}[\xi, \lambda] = t\pi q^{1+\alpha}$; thus $\alpha = 1$ if, and only if $[t, \varkappa] = 1$. Finally, we have $t\pi = [\mu, \zeta] \xrightarrow{\varphi\varkappa} [\zeta, \mu] = [\mu, \zeta]^{-1} = (t\pi)^{-1} = t^{-1}\pi$, and so, $t^{\varphi\varkappa} = t^{-1}$, since obviously $[\pi, \varphi\varkappa] = 1$ as $|\boldsymbol{C}_V(\varphi\varkappa)| = 2^2$.

(2.16) LEMMA. Let z be an involution of $QS\varkappa$ which operates on S in the same way as $\varkappa$ does. If $t^z = t^{-1}$, then all elements of $\langle q, \pi, t\tau\rangle z$ are conjugate.

PROOF. We prove the assertion by a series of computations:

$$(\mu\lambda\xi)^z = \xi t^{\beta}\mu\lambda t^{\beta} = \xi\mu\lambda q^{\beta} = \xi\lambda\mu q^{1+\beta};$$

thus

$$(z\mu\lambda\xi z)\xi\lambda\mu = q^{1+\beta}\pi\,, \qquad \text{and so}\,, \qquad z \sim q^{1+\beta}\pi z \underset{t}{\sim} q^{\beta}z\,.$$

Hence,

$$z \underset{t}{\sim} qz \sim q\pi z \sim \pi z\,.$$

Also,

$$(\mu\zeta\xi)^z = (\zeta\xi)t^{\gamma}\mu t^{\gamma} = \zeta\xi\mu q^{\gamma} = \xi\zeta\mu q^{1+\gamma};$$

thus $(z\mu\zeta\xi z)\xi\zeta\mu = q^{\gamma}t\tau$. Hence,

$$z \sim q^{\gamma}t\tau z \underset{t}{\sim} q^{1+\gamma}t\tau z\,.$$

Finally,

$$(\mu\lambda\xi\mu\zeta\xi)^z = \xi\lambda\mu q^{1+\beta}\xi\zeta\mu q^{1+\gamma}\,,$$

and it follows

$$(z\mu\lambda\xi\mu\zeta\xi z)\xi\zeta\mu\xi\lambda\mu = q^{1+\beta+\gamma}\pi t\tau;$$

thus

$$z \sim q^{1+\beta+\gamma}\pi t\tau z \underset{t}{\sim} q^{\beta+\gamma}\pi t\tau z\,.$$

Here, β and γ are suitable exponents; the proof can also be done by looking at the structure of $S\langle g, z\rangle$.

(2.17) Lemma. The case $X = QS\langle\varkappa\rangle$ is not possible.

Proof. By way of contradiction we assume $X = QS\langle\varkappa\rangle$. As always put $V = \langle q, \pi, t\tau\rangle$. We know that $\langle Q, \varkappa\rangle$ centralizes V. Thus, $X/C_X(V)$ is a four-group and this implies that G does not induce $L_3(2)$ on V. Hence, $\pi \nsim q$ in G.

We know that all involutions of $QS\setminus\langle q\rangle$ are conjugate to π. Hence, by a result of Glauberman, there is $z \in QS\varkappa$ such that $z \sim q$ in G and such that z operates in the same way as $\varkappa$ does on S. Application of (2.14) yields that $[z, t] \neq 1$ as $\pi \nsim q$. Application of (2.15) gives $\alpha = 0$ as $[t, \varkappa] \neq 1$.

Let $\tilde{X}, \tilde{Q}, \tilde{S}$, and $\tilde{A}$ as in the proof of (2.14). We have $z \in \tilde{Q} \cap \tilde{S}$. Obviously, all involutions of $\tilde{Q}\tilde{S}\setminus\langle z\rangle$ are conjugate to π in G. We have $C_S(z) = C_S(\varkappa) \supseteq \langle q, \pi, t\tau\rangle$. Thus, $\langle q, \pi\rangle \subseteq \tilde{X}$, and hence $\langle q, \pi\rangle \cap \cap \tilde{Q}\tilde{S} \neq \langle 1\rangle$. Clearly, $q \notin \tilde{Q}\tilde{S}$, since $q \neq z$ and $q \nsim \pi$ in G. It follows that π or $q\pi$ lies in $\tilde{Q}\tilde{S}$. Application of (2.16) yields that $z \sim z\pi \sim zq\pi$. But $z\pi$ or $zq\pi$ is in $\tilde{Q}\tilde{S}\setminus\langle z\rangle$. This would give $z \sim q \sim \pi$ which is not possible. The lemma is proved.

(2.18) Lemma. The case $X = QS\langle\varphi\rangle$ is not possible.

Proof. We have $C_X(\pi) = QC_S(\pi)\langle\varphi\rangle$ and $|X : C_X(\pi)| = 2$; clearly, $C_S(\pi) = \langle t, \pi, \tau, \mu\lambda, \mu\zeta, \xi\rangle$, $S' = Z_2(S) = \langle t, \pi, t\tau\rangle$, $W = C_S(S') = \langle t, \mu\lambda\xi, \mu\zeta\xi\rangle \subseteq C_X(\pi)$. We know that W is the only subgroup of X isomorphic to W.

Case 1. The subgroup W is nonabelian. In that case, we have $W' = \langle q\rangle$ and $\alpha = 1$. Lemma (2.15) implies $[t, \varphi] \neq 1$.

Assume that $q \underset{G}{\sim} \pi$. Consider $C_G(\pi)$, and let $\tilde{X}$ be in $\mathrm{Syl}_2(C_G(\pi))$ such that $C_X(\pi) \subseteq \tilde{X}$. Since $W \subseteq \tilde{X}$ and since $\tilde{X} \sim X$, we see that W is the unique subgroup of $\tilde{X}$ isomorphic to W. It follows that q and π are conjugate inside $N(W)$. But—as $W' = \langle q\rangle$—this is not possible. Hence, $\pi \nsim q$ in G.

By a result of Glauberman there is an involution z in $X\setminus QS$ such that $z \sim q$ in G. We choose z so that z operates on S in the same way as φ does. Denote by $\tilde{X}, \tilde{Q}, \tilde{S}$, and $\tilde{A}$ subgroups of $C_G(z)$ as in the proof of (2.14). Clearly, all involutions of $\tilde{Q}\tilde{S}\setminus\langle z\rangle$ are conjugate to π in G. We have $\langle q, \pi\rangle \subseteq C_X(z) \subseteq \tilde{X}$. But $q \notin \tilde{Q}\tilde{S}$. Since $|X : QS| = 2$, we get that π or $q\pi$ lies in $\tilde{Q}\tilde{S}$. Thus, πz or $q\pi z$ lies in $\tilde{Q}\tilde{S}\setminus\langle z\rangle$, and this implies that πz or $q\pi z$ is conjugate to π in G. Compute $\tau^z = [\lambda, \xi]^z = [\mu, \xi] = \pi\tau$. It follows $z^\tau = \pi z$; but $z^t = zq$, and so,

$$z \sim \pi z \sim \pi q z .$$

This is not possible as $z \nsim \pi$ holds in G.

Case 2. The subgroup W is abelian. In that case we have $\alpha = 0$. Lemma (2.15) gives $[t, \varphi] = 1$.

We show that $C_X(\pi)$ is normal in $N_G(X)$. Let $x \in N(X)$. Then, $x \in N(A)$, and hence, x normalizes $X \cap C(A) = Q$. But $Z(X/Q) = \langle \pi Q \rangle$, and so, $\pi^x = \pi$ or $\pi^x = q\pi$. Clearly, $C_X(\pi) = C_X(q\pi)$, and this implies $x \in N_G(C_X(\pi))$. We show further that $\langle q \rangle$ char $C_X(\pi)$. Put $C = C_X(\pi)$; note that $X = QS\langle\varphi\rangle$ and $C_S(\varphi) = \langle t, \pi, \mu\lambda, \xi\rangle$ and that $[t, \varphi] = 1$ as $\alpha = 0$. Obviously, $\langle t, \pi\rangle \subseteq Z(C)$, and $Z(C) \subseteq Q\langle\pi\rangle$. Hence, $\langle q \rangle$ char C.

We assume that $q \sim \pi$ holds in G. Let $\tilde{A}$ be the unique standard-subgroup of type $L_3(4)$ in $C_G(\pi)$ and let $\tilde{X}$ be in $\mathrm{Syl}_2(C(\pi))$ such that $C_X(\pi) \subseteq \tilde{X}$. There is g' in G such that $q^{g'} = \pi$ and $X^{g'} = \tilde{X}$. We have $C_X(\pi)^{g'^{-1}}$ as a subgroup of index 2 in X. Since $X \in \mathrm{Syl}_2(G)$, we may apply a theorem of Burnside, and get $C_X(\pi)^{g'^{-1}} = C_X(\pi)^y$ for some $y \in N(X)$. This implies $g' \in N(C_X(\pi))$. It follows $[g', q] = 1$ against $q^{g'} = \pi$. We have shown that $\pi \nsim q$ holds in G. A result of Glauberman yields the existence of an element $z \in X \setminus QS$ with $q \sim z$ in G. Application of (2.14) yields $\pi \sim q$ in G which is a contradiction. The lemma is proved.

(2.19) Lemma. Let z be an involution in $QS\varphi\varkappa$ which acts in the same way as $\varphi\varkappa$ on S. Then, $C_S(z) = \langle q, \mu\lambda\xi\tau\rangle$ or $\langle q, \mu\lambda\xi\tau t\rangle$ and all involutions of Sz are conjugate to z under S. Further, $\mho^1(C_S(z)) = \langle q^\varepsilon \pi\rangle$ for some $\varepsilon \in \{0, 1\}$.

Proof. From (2.13) we get $C_S(z) \subseteq \langle t, \mu\lambda\xi\tau\rangle$. Note that $t^z = t^{-1}$ by (2.15). The coset $\langle t \rangle z$ consists of four involutions. Computing in $\mathbf{aut}(L_3(4))$ we have $C_P(\varphi\varkappa) \simeq Q_8$, and so, $\varphi\varkappa$ has precisely $2^6 : 2^3 = 2^3$ conjugates under the action of P in $P\varphi\varkappa$. Thus, the number of conjugates of z under the action of S is at most $8 \cdot 4 = 32$. This forces $|S : C_S(z)| \leqslant 2^5$ which implies $|C_S(z)| = 2^3$. The lemma is proved.

(2.20) Lemma. If $X = QS\langle\varphi\varkappa\rangle$, then W is abelian and G induces an automorphism group isomorphic to $L_3(2)$ on W.

Proof. Assume that $q \nsim \pi$ in G. Then, $q \sim z \in X \setminus QS$; we assume that z acts on S in the same way as $\varphi\varkappa$ does. Let $\tilde{X}$, $\tilde{A}$, $\tilde{S}$, and $\tilde{Q}$ be subgroups of $C_G(z)$ as in the proof of (2.14). Then, π or $q\pi$ lies in $\tilde{S}$, and so, πz or $q\pi z$ is conjugate to π in G. Application of (2.19) gives that $z \sim \pi z \sim q\pi z$. But $q \sim z$, and we have got a contradiction. Hence, $q \sim \pi$ holds in G. Now, W is the only subgroup of X isomorphic to W, and $\Omega_1(Z(W)) = \langle q, \pi, t\tau\rangle$. Hence, $q \sim \pi$ holds in $N_G(W)$. It follows $N(W)/C(W) \cong L_3(2)$ and the lemma is proved, since $W' = \langle q\rangle$ cannot happen.

(2.21) Lemma. Let $X = QS\langle\varphi, \varkappa\rangle$ and $\alpha = 0$. If $\pi \nsim q$ in G, then q is not conjugate to an involution of $QS\varkappa$.

Proof. Assume that $q \sim z \in QS\varkappa$; we choose z so that it operates on S as $\varkappa$ does. Application of (2.15) yields that $[t, z] \neq 1$. We have $C_S(z) = \langle q, \pi, t\tau\rangle$. Lemma (2.16) says that all involutions of Vz are conjugate to z. Denote by $\tilde{X}, \tilde{S}, \tilde{Q}$, and $\tilde{A}$ subgroups of $C_G(z)$ as in the proof of (2.14). Since $|X:QS| = 4$, we have $\langle q, \pi, t\tau\rangle \cap \tilde{Q}\tilde{S} = V \cap \tilde{S} \neq \langle 1\rangle$. Let x be a nontrivial element of that intersection. Then, $z \sim zx \sim q$. Since $\tilde{Q} \cap \tilde{S}$ contains z, we have $zx \sim \pi$. But this is against the assumption of the lemma.

(2.22) Lemma. Let $X = QS\langle\varphi, \varkappa\rangle$ and $\alpha = 0$. If q is conjugate to an involution z of $QS\varphi$, then $q \sim \pi$ holds in G.

Proof. From (2.15) we get $[t, \varphi] = 1$. Application of (2.14) yields the assertion.

(2.23) Lemma. Let $X = QS\langle\varphi, \varkappa\rangle$ and $\alpha = 0$. Then, $\pi \sim q$ in $N_G(V)$.

Proof. By way of contradiction assume that $q \nsim \pi$ in G. By a result of Glauberman, $q \sim z$ for z in $QS\varphi, QS\varkappa$, or $QS\varphi\varkappa$. Application of (2.21) and (2.22) yields that $z \in QS\varphi\varkappa$. We get from (2.19) that $z \sim z\pi \sim z\pi q$. Let $\tilde{A}, \tilde{X}, \tilde{Q}$, and $\tilde{S}$ be subgroups of $C(z)$ as in the proof of (2.14). We get $\mho^1(C_S(z)) = \langle q^e\pi\rangle \subseteq \tilde{S}$. Thus, $zq^e\pi \sim \pi$ in G. This implies $z \sim \pi \sim q$ which is against the assumption. The assertion is now a consequence of (2.11).

(2.24) Lemma. Let $X = QS\langle\varphi, \varkappa\rangle$ and $\alpha = 0$. Then, $Q = \langle t\rangle$ and $|X| = 2^{10}$.

Proof. We know that $q \sim \pi$ holds in $N(V)$. Thus, $N(V)/C(V) \cong L_3(2)$. Since $C(V) \subseteq C(q)$, we get that $QW\langle\varkappa\rangle$ is a S_2-subgroup of $C(V)$. Clearly, $QW\langle\varkappa\rangle$ is nonabelian, and since QW is abelian, we get $Z(QW\langle\varkappa\rangle) \subset QW$. Now, V lies in $Z(QW\langle\varkappa\rangle)$, and since the 2-rank of QS is 3, we get $V = \Omega_1(Z(QW\langle\varkappa\rangle))$. Denote by uw an element of $C_{WQ}(\varkappa)$ with $u \in Q$, $w \in W$, and $o(uw) = 4$. Then, $u^4w^4 = 1$ which implies $u^4 = 1$, and this means $u \in \langle t\rangle$. Thus, $C_{QW}(\varkappa) = V = \langle q, \pi, t\tau\rangle = Z(WQ\langle\varkappa\rangle)$. We have $|QW| = 2^n2^4$. Assume there were a subgroup Q^*W^* in $QW\langle\varkappa\rangle$ isomorphic to QW and different from QW. Then, $(QW)(Q^*W^*) = QW\langle\varkappa\rangle$ and $QW \cap Q^*W^*$ has order 2^n2^3 and would be contained in the center of $QW\langle\varkappa\rangle$; it would follow $n = 0$ which is not the case. Thus, QW is unique in $QW\langle\varkappa\rangle$. By the Frattini-argument, $N(V)$ induces an automorphism σ of order 7 of $QW\langle\varkappa\rangle$ which acts fixed-point-free on V, thus σ has no fixed-points on QW as $\Omega_1(QW) = \langle q, \pi, t\tau\rangle$. This implies that QW is homocyclic, and so, $Q \subseteq W$. The lemma is proved.

(2.25) LEMMA. If $\alpha = 0$, then the case $X = QS\langle\varphi, \varkappa\rangle$ is not possible.

PROOF. We have $C_X(V) = W\langle\varkappa\rangle$. Since $\pi \sim q$ holds in $N(V)$, we have $N(V)/C(V) \cong L_3(2)$. Clearly, $W\langle\varkappa\rangle \in \mathrm{Syl}_2\,(C(V))$. Since $C_W(\varkappa) = V$, we see that W is the only subgroup of $W\langle\varkappa\rangle$ of its type. We have $N(W) \subseteq N(V)$. Now, $C(V) = (O \times W)\langle\varkappa\rangle$. Denote by $\tilde{W}$ a subgroup of $C(V)$ isomorphic to W and assume $\tilde{W} \neq W$. Then, $(O \times W)\tilde{W} = C(V)$. Since $W \trianglelefteq C(V)$, we get that $W\tilde{W}$ is a group of order 2^7, and so, $|W \cap \tilde{W}| = 2^5$. But then, a S_2-subgroup of $C(V)$ would have a center of order greater than 8 which is not the case. Hence, in $C(V)$ the subroup W is unique. It follows that $N(V) \subseteq N(W)$, and hence, $N(V) = N(W)$. By Frattini's argument there is an automorphism σ of order 7 of $W\langle\varkappa\rangle$ induced by an element of $N(V)$ which acts fixed-point-free on V. Hence, $C(\sigma) \cap W\langle\varkappa\rangle = \langle z\rangle$ has order 2. Since $C(\sigma) \cap W = \langle 1\rangle$, we get $W\langle\varkappa\rangle = W\langle z\rangle$. It follows $z \in S\varkappa$. Clearly, all involutions of W are conjugate in $N(V)$. From the structure of X we get that X/W is a direct product of $\langle W\varkappa\rangle$ and a dihedral group of order 8. Thus, $N(W)/C(W)$ is isomorphic to $L_3(2) \times Z_2$.

Denote by N^* the subgroup of index 2 of $N(W)$ which contains $C(W)$ such that $N^*/C(W) \cong L_3(2)$. Put $X^* = X \cap N^*$. Then, $X^* \cap S \supset W$. Note that the involutions of $N^*/C(W)$ are all conjugate in that factor group. Let s be an involution of $(S \cap X^*)\setminus W$. If x is any involution of $X^*\setminus W$, then $sC(W) \sim xC(W)$ in $N^*/C(W)$. We have $sC(W) = s(W \times O) \subseteq S \times O$; so all involutions of $sC(W)$ are conjugate as $\pi \sim q$ in G. It follows that all involutions of X^* are conjugate to q in G; as a matter of fact, S lies in X^* as S is normalized by g. Note that X^* is a maximal subgroup of X. A transfer lemma of J. G. Thompson gives $z \sim q$ in G. The last statement produces a contradiction in the following way. In the normalizer of V in G there is an element σ' which centralizes z and conjugates all the elements of $V^{\#}$. It follows that $\langle z\rangle \times \boldsymbol{\Omega}_1(W)$ lies in the unique standard-subgroup A_z of $\boldsymbol{C}(z)$. Thus, the 2-rank of S would be 4 which is a contradiction. The lemma is proved.

(2.26) Lemma. The case $X = QS\langle\varphi, \varkappa\rangle$ and $\alpha = 1$ is not possible.

Proof. Assume by way of contradiction that $q \sim \pi$ in G. Then, $q \sim \pi$ holds in $\boldsymbol{N}(V)$. We have $\boldsymbol{C}(V) = (QW\langle\varkappa\rangle)O$, where $O = \boldsymbol{O}(N(A))$. From Frattini's argument we get $\boldsymbol{N}(V) = O(\boldsymbol{N}(QW\langle\varkappa\rangle) \cap \boldsymbol{N}(V))$. Since $[O, V] = \langle 1\rangle$, we see that $q \sim \pi$ happens in $\boldsymbol{N}(QW\langle\varkappa\rangle)$. However, $t \in \boldsymbol{Z}(QW\langle\varkappa\rangle) \subseteq \langle Q, \pi, t\tau\rangle$, and therefore $\langle q\rangle$ char $QW\langle\varkappa\rangle$. It follows that $q \nsim \pi$ holds in G.

From Glauberman's result we get that q is conjugate to an involution z in $X \setminus QS$. From (2.14) we get that $z \notin QS\varkappa$, since $[t, \varkappa] = 1$. Assume that $z \in QS\varphi\varkappa$. We assume also that z acts in the same way on S as $\varphi\varkappa$ does. Application of (2.19) yields that all involutions of Sz are conjugate to z. We have $\mho^1(\boldsymbol{C}_S(z)) = \langle q^\varepsilon \pi\rangle$ for some $\varepsilon \in \{0, 1\}$. Clearly, $\pi \underset{G}{\nsim} q^\varepsilon\pi$. In $\boldsymbol{C}(z)$ we choose $\tilde{A}, \tilde{Q}, \tilde{S}$, and $\tilde{X}$ as usual. Then, $z \in \tilde{Q} \cap \tilde{S}$. We have $\langle q^\varepsilon\pi\rangle \subseteq \tilde{S}$, and so, $z\pi q^\varepsilon \sim \pi$ in G. However, $z\pi q^\varepsilon$ lies in Sz and is an involution. Thus, $z \sim z\pi q^\varepsilon \sim \pi$, against $q \sim z$ and $q \nsim \pi$.

We have still to treat the case that $q \sim z \in QS\varphi$. Denote again by $\tilde{A}, \tilde{S}, \tilde{X}, \tilde{Q}$ the usual subgroups of $\boldsymbol{C}(z)$. We have $\mho^1(\boldsymbol{C}_S(\varphi)) = \langle q^\varepsilon\pi\rangle$ as $t^\varphi = t^{-1}$. Thus, $\langle q^\varepsilon\pi\rangle \in \tilde{S}$. It follows $zq^\varepsilon\pi \sim q^\varepsilon\pi \sim \pi \nsim z \sim q$ in G. Now, z and $q^\varepsilon\pi z$ are involutions of Sz. We may assume that z acts in the same way on S as φ does. Compute: $\tau^z = [\mu, \xi]^z = [\mu, \xi] = \pi\tau$. It follows $z^\tau = \pi z$; but $z^t = zq$, and so,

$$z \sim \pi z \sim \pi q z\,.$$

This is not possible as $z \nsim \pi$ in G.

We are left with the situation of (2.20). We have $W' = \langle 1\rangle$, and W is the only subgroup of its type in $X = QS\langle\varphi\varkappa\rangle$. Now, $\boldsymbol{N}(W)/\boldsymbol{C}(W) = L_3(2)$ and $\boldsymbol{C}(W) = QWO$, where $O = \boldsymbol{O}(N(A))$. Since $\boldsymbol{N}(W)$ operates transitively on $\boldsymbol{\Omega}_1(W) = V$, we see that $|Q| \leqslant 4$ as $\langle q\rangle$ is not characteristic in WQ. It follows $X = S\langle\varphi\varkappa\rangle$. Clearly, $\boldsymbol{N}(W)/O$ is a nonsplitting extension of an abelian group of type (4, 4, 4) by $L_3(2)$. By a result of Alperin, we see that X is isomorphic to a S_2-subgroup of

O'Nan's simple group. This is enough to get $G \cong$ O'N; but we may invoke a result of G. Stroth [6] to identify G with O'N.

The theorem is proved.

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