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Soluble Products of Two Locally Finite Groups with Min-p for Every Prime p.

BERNHARD AMBERG (*)

Introduction.

It is known that every soluble product of two Černikov groups is a Černikov group; see (1) and (4). Also, in (3), Folgerung 5.3, p. 237, it was shown that every soluble product of two locally nilpotent π-groups with min-p for every prime p in a set of primes π, is a (locally finite) π-group with min-p for every p. In the following these results will be generalized.

If the soluble group $G = AB$ is the product of two π-subgroups A and B, one of which satisfies min-p for every prime p in the set of primes π, then G is a π-group. Thus, in particular, G is locally finite, and the set of primes p, for which there is an element of order p in G, equals the set of primes, for which there is an element of this order in A or B (Corollary 2.2). If even both, A and B, satisfy min-p for every p, and if A or B is almost locally-nilpotent, then G satisfies min-p for every p (Corollary 2.5). Further related results are discussed in section 2.

These results suggest the conjecture that every soluble product of two locally finite groups with min-p must satisfy min-p. To solve this problem one would have to search for additional « factorized » subgroups of the factorized group $G = AB$.

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Notations.

π	= set of primes;
πG	= set of primes p, for which there is an element of order p in the group G;
$\boldsymbol{J}(G)$	= intersection of all subgroups of finite index in G;
$\boldsymbol{O}_p(G)$	= maximal normal p-subgroup of G;
G_π	= π-component of the locally nilpotent group G;
factor	= epimorphic image of a subgroup;
π-group	= group, in which the order of every element contains only primes from the set of primes π.

A group satisfies *min-p* for the prime p, if each of its p-subgroups satisfies the minimum condition on subgroups,

If $\boldsymbol{X}$ is a class of groups, a group is an *almost* $\boldsymbol{X}$*-group* or is *almost* $\boldsymbol{X}$ if it contains a normal subgroup of finite index which is contained in the class $\boldsymbol{X}$,

A group is a *Černikov group* if it is almost abelian and satisfies the minimum condition on subgroups.

1. Lemmas.

The following general reduction lemma is useful for the investigation of factorized soluble groups.

LEMMA 1.1. *Let* $\boldsymbol{X}$ *be a class of groups closed under the forming of subgroups, epimorphic images and extensions. If there exists a soluble group* $G = AB$, *which is the product of two* $\boldsymbol{X}$*-subgroups* A *and* B, *but is itself not an* $\boldsymbol{X}$*-group, then there also exists a soluble group* $F = S_1 S_2 = KS_1 = KS_2$, *which is the product of two* $\boldsymbol{X}$*-subgroups* S_1 *and* S_2 *and is itself not an* $\boldsymbol{X}$*-group and where* K *is an abelian normal subgroup of* F *with* $K \cap S_1 = K \cap S_2 = 1$. *Furthermore, if the soluble group* $T = T_1 T_2$ *is the product of two* $\boldsymbol{X}$*-subgroups* T_1 *and* T_2 *and if the sum of the derived lengths of* T, T_1 *and* T_2 *is smaller than the sum of the derived lengths of* F, S_1 *and* S_2, *then* T *is an* $\boldsymbol{X}$*-group.*

REMARK. $F = S_1 S_2$ is found below as a factor of a suitable « counterexample » $G = AB$, where S_1 is a factor of A and S_2 is a factor of B.

Proof. Among all soluble products of two $\mathbf{X}$-groups, which themselves are no $\mathbf{X}$-groups, choose one $G = AB$ such that the sum of the derived lengths of G, A and B is minimal. Let C be the last nontrivial term of the derived series of G.

The group $G/C = (AC/C)(BC/C)$ is the product of two $\mathbf{X}$-subgroups $AC/C \simeq A/(A \cap C)$ and $BC/C \simeq B/(B \cap C)$. By induction G/C is an $\mathbf{X}$-group. Since $\mathbf{X}$ is extension closed, C is not an $\mathbf{X}$-group.

By Amberg (1), Theorem 1.7, p. 108, the factorizer $X = \mathbf{X}(C)$ of C in G satisfies

$$X = AC \cap BC = C(A \cap BC) = C(B \cap AC) = (A \cap BC)(B \cap AC) .$$

Since C is an abelian normal subgroup of G, $A \cap C$ is a normal subgroup of $X = C(A \cap BC)$ and $B \cap C$ is a normal subgroup of $X = C(B \cap AC)$. Therefore $C^* = (A \cap C)(B \cap C)$ is normal in X. As an abelian product of two $\mathbf{X}$-groups, C^* is an $\mathbf{X}$-group.

Let $K = C/C^*$, $S_1 = (A \cap BC)\,C^*/C^*$ and $S_2 = (B \cap AC)\,C^*/C^*$. Then $C \cap (A \cap BC) \subseteq (A \cap C) \subseteq C^*$ and $C \cap (B \cap AC) \subseteq (B \cap C) \subseteq C^*$. Hence $F = S_1 S_2 = KS_1 = KS_2$ where $K \cap S_1 = K \cap S_2 = 1$. Assume that F is an $\mathbf{X}$-group. Then also the subgroup K of F is an $\mathbf{X}$-group. Since C^* is an $\mathbf{X}$-group, also C is an $\mathbf{X}$-group. This contradiction shows that F is not an $\mathbf{X}$-group. The other assertions are clear.

For any class of groups $\mathbf{X}$, let $\boldsymbol{R}_{\mathbf{X}}(Y)$ be the product of all normal $\mathbf{X}$-subgroups of the group Y, the so-called *$\mathbf{X}$-radical* of Y. If A is a subgroup and N a normal subgroup of the group G, then for any epimorphism closed class of groups $\mathbf{X}$, obviously the following holds: $\boldsymbol{R}_{\mathbf{X}}(A)\,N/N \subseteq \boldsymbol{R}_{\mathbf{X}}(AN/N)$. For, if K is a normal $\mathbf{X}$-subgroup of A, then $KN/N \simeq K/(K \cap N)$ is a normal $\mathbf{X}$-subgroup of AN/N. Some other statements of this kind are contained in the following lemma.

Lemma 1.2. *If G is a group, A a subgroup and N a normal subgroup of G with $A \cap N = 1$, then for any class of groups $\mathbf{X}$ the following holds*:

(*a*) $\boldsymbol{R}_{\mathbf{X}}(A)\,N/N \subseteq \boldsymbol{R}_{\mathbf{X}}(AN/N)$,

(*b*) *If for every group Y the $\mathbf{X}$-radical $\boldsymbol{R}_{\mathbf{X}}(Y)$ is an $\mathbf{X}$-group, then $\boldsymbol{R}_{\mathbf{X}}(A)\,N/N = \boldsymbol{R}_{\mathbf{X}}(AN/N)$; in particular $\boldsymbol{R}_{\mathbf{X}}(A)\,N$ is a normal subgroup of AN,*

(*c*) *If B is another subgroup of G such that $AN = BN$ and if for every group Y the $\mathbf{X}$-radical $\boldsymbol{R}_{\mathbf{X}}(Y)$ is an $\mathbf{X}$-group, then $\boldsymbol{R}_{\mathbf{X}}(A)\,N = \boldsymbol{R}_{\mathbf{X}}(B)\,N$ and this group is normalized by AB.*

Proof. (*a*) If K is a normal $\mathbf{X}$-subgroup of A, then $KN/N \simeq K/(K \cap N) \simeq K$ is a normal $\mathbf{X}$-subgroup of AN/N. This shows (*a*).

(*b*) Since $\boldsymbol{R}_{\mathbf{X}}(A)N/N \simeq \boldsymbol{R}_{\mathbf{X}}(A)/(\boldsymbol{R}_{\mathbf{X}}(A) \cap N) \simeq \boldsymbol{R}_{\mathbf{X}}(A)$, the group $\boldsymbol{R}_{\mathbf{X}}(A)N/N$ is a normal $\mathbf{X}$-subgroup of AN/N. Therefore (or directly by (*a*))

$$\boldsymbol{R}_{\mathbf{X}}(A)N/N \subseteq \boldsymbol{R}_{\mathbf{X}}(AN/N) .$$

Let S be the uniquely determined normal subgroup of AN such that $N \subseteq S$ and $S/N = \boldsymbol{R}_{\mathbf{X}}(AN/N)$. By the modular law $S = S \cap AN = (A \cap S)N$. This implies

$$\boldsymbol{R}_{\mathbf{X}}(AN/N) = S/N = (A \cap S)N/N \simeq (A \cap S)/(A \cap S) \cap N \simeq (A \cap S) .$$

Thus $A \cap S$ is a normal $\mathbf{X}$-subgroup of A. Therefore $(A \cap S) \subseteq \boldsymbol{R}_{\mathbf{X}}(A)$ and thus

$$\boldsymbol{R}_{\mathbf{X}}(AN/N) = (A \cap S)N/N \subseteq \boldsymbol{R}_{\mathbf{X}}(A)N/N .$$

This proves (*b*).

(*c*) By (*b*) we have

$$\boldsymbol{R}_{\mathbf{X}}(A)N/N = \boldsymbol{R}_{\mathbf{X}}(AN/N) = \boldsymbol{R}_{\mathbf{X}}(BN/N) = \boldsymbol{R}_{\mathbf{X}}(B)N/N .$$

If $g = ab$ where $a \in A$ and $b \in B$, then

$$(\boldsymbol{R}_{\mathbf{X}}(A)N)^g = (\boldsymbol{R}_{\mathbf{X}}(A))^b N = (\boldsymbol{R}_{\mathbf{X}}(A)N)^b = (\boldsymbol{R}_{\mathbf{X}}(B)N)^b = \boldsymbol{R}_{\mathbf{X}}(B)N = \boldsymbol{R}_{\mathbf{X}}(A)N .$$

Thus $\boldsymbol{R}_{\mathbf{X}}(A)N$ is normalized by AB.

For any group Y denote by $\boldsymbol{J}(Y)$ the intersection of all subgroups of Y with finite index. If Y is a Černikov group, then $Y/\boldsymbol{J}(Y)$ is finite and $\boldsymbol{J}(Y)$ is a characteristic radicable abelian p-group of finite rank.

Lemma 1.3. *If the locally soluble group $G = AB$ is the product of two torsion subgroups A and B with min-p for the prime p, then the following holds:*

(*a*) $\langle \boldsymbol{J}(\boldsymbol{O}_p(A)), \boldsymbol{J}(\boldsymbol{O}_p(B)) \rangle$ *is a p-subgroup of G and therefore locally finite,* (*b*) *If* $\boldsymbol{O}_p(A)$ *and* $\boldsymbol{O}_p(B)$ *are finite, then* $\langle \boldsymbol{O}_p(A), \boldsymbol{O}_p(B)\rangle$ *is a finite* p*-subgroup of* G.

PROOF. As a p-subgroup of A, the group $\boldsymbol{O}_p(A)$ satisfies the minimum condition on subgroups. As a locally soluble group with minimum condition on subgroups $\boldsymbol{O}_p(A)$ is a Černikov group; see Kegel and Wehrfritz (6), Theorem 1.E.6, p. 31. Thus $\boldsymbol{J}(\boldsymbol{O}_p(A))$ is a characteristic radicable abelian p-subgroup of finite rank and with finite index $|\boldsymbol{O}_p(A):\boldsymbol{J}(\boldsymbol{O}_p(A))|$. As a characteristic subgroup of the characteristic subgroup of $\boldsymbol{O}_p(A)$ *of* A also $\boldsymbol{J}(\boldsymbol{O}_p(A))$ is a characteristic subgroup A.—Analoguously $\boldsymbol{J}(\boldsymbol{O}_p(B))$ is a characteristic radicable abelian p-subgroup of B with finite rank. Every finite subset of $\boldsymbol{J}(\boldsymbol{O}_p(A))$ is contained in a finite normal subgroup A_1 of A with $A_1 \subseteq \boldsymbol{J}(\boldsymbol{O}_p(A))$. Also, every finite subset of $\boldsymbol{J}(\boldsymbol{O}_p(B))$ is contained in a finite normal subgroup B_1 of B with $B_1 \subseteq \boldsymbol{J}(\boldsymbol{O}_p(B))$.

Let $S_1 = \langle A_1, B_1\rangle$. Since A and B are torsion groups, by Kegel (5), Lemma 1.2, p. 536, the normalizer $\boldsymbol{N}(S_1)$ is factorized. Since G is locally soluble, by Amberg (1), Proposition 5.3, p. 117, S_1 is finite and $\pi S_1 = \pi A_1 \cup \pi B_1 = \{p\}$.

It follows that $\langle \boldsymbol{J}(\boldsymbol{O}_p(A)), \boldsymbol{J}(\boldsymbol{O}_p(B))\rangle$ is a p-group. Since G is locally soluble, p-subgroups of G are locally finite.

The proof of (*b*) is similar.

The following criterion for a factorized almost locally-nilpotent group to satisfy min-p is needed.

LEMMA 1.4. *If the almost locally-nilpotent group* $G = AB$ *is the product of two* π*-subgroups* A *and* B *with min-*p *for the prime* p *in the set of primes* π*, then* G *is a* π*-group with min-*p.

PROOF. The Hirsch-Plotkin radical $R = \boldsymbol{R}(G)$ has finite index $|G:R|$ in G. Then also the indices

$$|A:(A\cap R)| \leqslant |G:R| \quad \text{and} \quad |B:(B\cap R)| \leqslant |G:R|$$

are finite. By Amberg (1), Lemma 5.1, p. 116, also $V = \langle A\cap R, B\cap R\rangle$ has finite index in G and is contained in the π-component of R. Hence there exists a normal π-subgroup N of G contained in V with finite factor group G/N. As a finite product of two π-subgroups $G/N = (AN/N)(BN/N)$ is a π-group. Hence also G is a π-group.

As a locally finite-nilpotent group, $R = R_p \otimes R_{p'}$ is the direct product of its p-component R_p and its p'-component $R_{p'}$. The factor group $G/R_{p'} = (AR_{p'}/R_{p'})(BR_{p'}/R_{p'})$ contains a normal p-subgroup $R/R_{p'}$ of finite index and is therefore an almost p-group.

The groups $AR_{p'}/R_{p'}$ and $BR_{p'}/R_{p'}$ are isomorphic to almost p-factors of A resp. B and therefore satisfy the minimum condition on subgroups. By Amberg (1), Corollary 3.3, p. 112, the normal subgroups of $G/R_{p'}$ satisfy the minimum condition. Since $R/R_{p'}$ has finite index in $G/R_{p'}$, also the normal subgroups of $R/R_{p'}$ satisfy the minimum condition; see Robinson (7), Theorem 5.21, p. 147.

As a locally nilpotent group with minimum condition on normal subgroups $R/R_{p'}$ is a Černikov group; see Kegel and Wehrfritz (6), Theorem 1.H.4, p. 44. Thus also $G/R_{p'}$ is a Černikov group. This implies that G satisfies min-p.

The basis of the subsequent considerations is the following proposition.

PROPOSITION 1.5. *If the locally soluble group $G = AB = AK = BK$ is the product of two π-subgroups A and B with min-p for the prime p in the set of primes π and an abelian normal subgroup K of G with $A \cap K = B \cap K = 1$, then the following holds:*

(*a*) *$K_p \boldsymbol{O}_p(A) = K_p \boldsymbol{O}_p(B)$, and this group is a normal p-subgroup of G,*

(*b*) *If $A \simeq B$ satisfies min-p for every p, then*

$$H = \prod_{p \in \pi} K_p \boldsymbol{O}_p(A) = K_\pi \boldsymbol{R}(A)$$

is a normal π-subgroup of G contained in $\boldsymbol{R}(G)$; and if, in addition, $A \simeq B$ is almost locally-nilpotent, then G is an almost locally-nilpotent π-group with min-p for every p.

PROOF. By Lemma 1.2 (*c*) $K\boldsymbol{O}_p(A) = K\boldsymbol{O}_p(B)$, and this group is a normal subgroup of $AB = G$. By hypothesis $\boldsymbol{O}_p(A)$ and $\boldsymbol{O}_p(B)$ are Černikov groups, so that $J_1 = \boldsymbol{J}(\boldsymbol{O}_p(A))$ and $J_2 = \boldsymbol{J}(\boldsymbol{O}_p(B))$ are radicable abelian p-subgroups of finite rank and with finite index in $\boldsymbol{O}_p(A)$ resp. $\boldsymbol{O}_p(B)$. By Lemma 1.2 (*c*) $KJ_1 = KJ_2$, and this group is a normal subgroup of $AB = G$.

By Lemma 1.3 (*a*) $\langle J_1, J_2 \rangle$ is a p-group. If $a \in J_1$, then $a = sb$ with $b \in J_2$ and $s \in K$. Then $s = s_1 s_2$ where s_1 is a p'-element and s_2 is a p-element. Since the p-component K_p of the abelian normal subgroup K of G is a normal subgroup of G, $\langle J_1, J_2 \rangle K_p$ is a p-subgroup of G. Thus $s_1 = ab^{-1}s_2^{-1}$ is a p'-element and a p-element, and therefore equals 1. This shows $J_1 \subseteq K_p J_2$. Similarly $J_2 \subseteq K_p J_1$. If $P = K_p J_1 = K_p J_2$, then by Lemma 1.2 (c) P is a normal subgroup of G.

The group $G/P = (AP/P)(BP/P)$ is the product of two π-subgroups $AP/P \simeq A/(A \cap P)$ and $BP/P \simeq B/(B \cap P)$, whose maximal normal p-subgroups are finite, since by the modular law

$$J_1 \subseteq J_1(A \cap K_p) = A \cap K_p J_1 = (A \cap P) \subseteq \boldsymbol{O}_p(A) ,$$

and the index $|\boldsymbol{O}_p(A) : J_1|$ is finite.

Therefore $\boldsymbol{O}_p(AP/P)$ and $\boldsymbol{O}_p(BP/P)$ are finite p-subgroups of G/P. By Lemma 1.3 (b) the group

$$\langle \boldsymbol{O}_p(AP/P), \boldsymbol{O}_p(BP/P)\rangle \supseteq \langle \boldsymbol{O}_p(A)P/P, \boldsymbol{O}_p(B)P/P\rangle = \langle O_p(A), \boldsymbol{O}_p(B)\rangle P/P$$

is a finite p-group. Then also the group isomorphic to the last group

$$\langle \boldsymbol{O}_p(A), \boldsymbol{O}_p(B)\rangle / \big(P \cap \langle \boldsymbol{O}_p(A), \boldsymbol{O}_p(B)\rangle\big)$$

is a p-group. Hence also $\langle \boldsymbol{O}_p(A), \boldsymbol{O}_p(B)\rangle$ is a p-group.

As above it is shown that $K_p\boldsymbol{O}_p(A) = K_p\boldsymbol{O}_p(B)$, and that this group is a normal subgroup of G. This proves (a).

If $A \simeq B$ is a π-group with min-p for every prime p, then by (a) every group $K_p\boldsymbol{O}_p(A) = K_p\boldsymbol{O}_p(B)$ is a normal p-subgroup of G, which is locally nilpotent and therefore contained in the Hirsch-Plotkin radical $\boldsymbol{R}(G)$. Since A is locally finite, $\boldsymbol{R}(A)$ is the direct product of the maximal normal p-subgroups $\boldsymbol{O}_p(A)$. Also, $\boldsymbol{R}(B)$ is the direct product of the maximal normal p-subgroups $\boldsymbol{O}_p(B)$. Then

$$H = \prod_{p\in\pi} K_p\boldsymbol{O}_p(A) = \prod_{p\in\pi} K_p \prod_{p\in\pi} \boldsymbol{O}_p(A) = K\boldsymbol{R}(A)$$

$$= \prod_{p\in\pi} K_p\boldsymbol{O}_p(B) = \prod_{p\in\pi} K_p \prod_{p\in\pi} \boldsymbol{O}_p(B) = K\boldsymbol{R}(B)$$

is a normal π-subgroup of G which is contained in $\boldsymbol{R}(G)$. In particular $\langle \boldsymbol{R}(A), \boldsymbol{R}(B)\rangle \subseteq H$.

If $A \simeq B$ is almost locally-nilpotent, then $\boldsymbol{R}(A)$ has finite index in A and $\boldsymbol{R}(B)$ has finite index in B. By Amberg (1), Lemma 5.1,

p. 116, also $\langle \boldsymbol{R}(A), \boldsymbol{R}(B) \rangle$ and therefore H has finite index in $G = AB$. Thus G is almost locally-nilpotent. By Lemma 1.4 G is a π-group with min-p for every prime p. This proves (*b*).

2. Results.

The first theorem is formulated for a rather general class of groups $\boldsymbol{X}$.

Theorem 2.1. *Let* $\boldsymbol{X}$ *be a class of groups closed under the forming of subgroups, epimorphic images and extensions such that for a set of primes* π *every soluble* π*-group is an* $\boldsymbol{X}$*-group. If the soluble group* $G = AB$ *is the product of an* $\boldsymbol{X}$*-subgroup* A *and a* π*-subgroup* B *one of which satisfies min-*p *for every prime* p*, then* G *is an* $\boldsymbol{X}$*-group.*

Proof. Assume that the theorem is false. Then there is a soluble group $G = AB$, which is the product of an $\boldsymbol{X}$-subgroup A and a π-subgroup B, one of which satisfies min-p for every p, but itself is not an $\boldsymbol{X}$-group. By Lemma 1.1 it may be assumed that G contains an abelian normal subgroup K such that $G = AB = AK = BK$ and $A \cap K = B \cap K = 1$. Also, it may be assumed that every soluble group $T = T_1 T_2$, which is the product of an $\boldsymbol{X}$-subgroup A and a π-subgroup B, one of which satisfies min-p for every p, is an $\boldsymbol{X}$-group, if the sum of the derived lengths of T, T_1 and T_2 is smaller than the sum of the derived lengths of G, A and B.

Then $A \simeq B$ is a π-group with min-p for every p. By Proposition 1.5 $H = \prod_{p \in \pi} K_p \boldsymbol{O}_p(A)$ is a normal π-subgroup of G, which contains $\boldsymbol{R}(A)$ and $\boldsymbol{R}(B)$. The group $G/H = (AH/H)(BH/H)$ is the product of two π-subgroups $AH/H \simeq A/(A \cap H)$ and $BH/H \simeq B/(B \cap H)$ with min-p for every p. By induction G/H is an $\boldsymbol{X}$-group. As a soluble π-group also H is an $\boldsymbol{X}$-group. Hence G is an $\boldsymbol{X}$-group.

If $\boldsymbol{X}$ is the class of all soluble π-groups, the following corollary mentioned in the introduction, is obtained.

Corollary 2.2. *If the soluble group* $G = AB$ *is the product of two* π*-subgroups* A *and* B *for the set of primes* π*, one of which satisfies min-*p *for every prime* p*, then* G *is a* π*-group; in particular* G *is locally finite and* $\pi G = \pi A \cup \pi B$.

Does in the situation of Corollary 2.2 G satisfy min-p for every p, if A and B do? In the following this is proved under an additional requirement for A or B. The following theorem generalizes Theorem B of Amberg (2), p. 2.

THEOREM 2.3. *Let* $\mathbf{X}$ *be a class of groups closed under the forming of subgroups, epimorphic images and extensions such that for a set of primes* π *every almost locally-nilpotent* π*-group with min-*p *for every prime* p *is an* $\mathbf{X}$*-group. If the soluble group* $G = AB$ *is the product of an* $\mathbf{X}$*-subgroup* A *and an almost locally-nilpotent* π*-subgroup* B *with min-*p *for every* p*, then* G *is an* $\mathbf{X}$*-group.*

PROOF. Assume that the theorem is false. Then there exists a soluble group $G = AB$, which is the product of an $\mathbf{X}$-subgroup A and an almost locally-nilpotent π-subgroup B with min-p for every p, but which is not an $\mathbf{X}$-group. By Lemma 1.1 it may be assumed that G contains an abelian normal subgroup K such that $G = AB = AK = BK$ and $A \cap K = B \cap K = 1$.

Then $A \simeq B$ is an almost locally-nilpotent π-group with min-p for every p. By Proposition 1.5 (*b*) G is an almost locally-nilpotent π-group with min-p for every p. By hypothesis G is therefore an $\mathbf{X}$-group. This contradiction proves the theorem.

A soluble group G has *finite abelian section rank*, if every abelian factor (section) of G has finite p-rank, where $p = 0$ or a prime. The class of soluble groups with finite abelian section rank is closed under the forming of subgroups, epimorphic images and extensions, and a soluble torsion group has finite abelian section rank, if and only if it satisfies min-p for every prime p; see Robinson (8), chapter 9. Thus Theorem 2.3 has the following consequence.

COROLLARY 2.4. *If the soluble group* $G = AB$ *is the product of two subgroups* A *and* B *with finite abelian section rank, one of which is almost locally finite-nilpotent, then* G *has finite abelian section rank.*

If in Theorem 2.3 $\mathbf{X}$ is the class of soluble π-groups with min-p for every prime p, the following is obtained.

COROLLARY 2.5. *If the soluble group* $G = AB$ *is the product of two* π*-subgroups* A *and* B *with min-*p *for every prime* p *in the set of primes* π*, and if* A *or* B *is almost locally-nilpotent, then* G *is a* π*-group with min-*p *for every* p*.*

Corollary 2.5 may be further specialized. First the minimum condition on subgroups of finite index is considered.

LEMMA 2.6. *The following conditions of the group G are equivalent:*

(*a*) *The subgroups of finite index in G satisfy the minimum condition,*

(*b*) *The normal subgroups of finite index in G satisfy the minimum condition,*

(*c*) *$G/\boldsymbol{J}(G)$ is finite,*

(*d*) *There is only a finite number of subgroups of finite index in G.*

PROOF. Clearly (*a*) implies (*b*). It follows for instance from Kegel and Wehrfritz (6), Proposition 1.E.4, p. 31, that (*c*) is a consequence of (*b*). If (*c*) holds, G has only finitely many subgroups of finite index, so that also (*d*) holds. Finally (*a*) is an obvious consequence of (*d*).

The next lemma shows that the minimum condition on subgroups of finite index is inherited by products of groups.

LEMMA 2.7. *If the group $G = AB$ is the product of two subgroups A and B with finite $A/\boldsymbol{J}(A)$ and $B/\boldsymbol{J}(B)$, then $G/\boldsymbol{J}(G)$ is finite, and*

$$\boldsymbol{J}(G) = \langle \boldsymbol{J}(A), \boldsymbol{J}(B)\rangle \,.$$

PROOF. Since $A/\boldsymbol{J}(A)$ and $B/\boldsymbol{J}(B)$ are finite, by Amberg (1), Lemma 5.1, p. 116, also $\langle \boldsymbol{J}(A), \boldsymbol{J}(B)\rangle$ has finite index in G. Therefore $\boldsymbol{J}(G) \subseteq \langle \boldsymbol{J}(A), \boldsymbol{J}(B)\rangle$.

If S is any subgroup of finite index in G, then $|A:(A \cap S)| \leqslant |G:S|$ and $|B:(B \cap S)| \leqslant |G:S|$. Therefore $\boldsymbol{J}(A) \subseteq (A \cap S)$ and $\boldsymbol{J}(B) \subseteq (B \cap S)$. Hence $\langle \boldsymbol{J}(A), \boldsymbol{J}(B)\rangle \subseteq \langle A \cap S, B \cap S\rangle \subseteq S$, so that $\langle \boldsymbol{J}(A), \boldsymbol{J}(B)\rangle \subseteq \boldsymbol{J}(G)$. This proves the lemma.

Now Corollary 2.5 is specialized in the following way.

COROLLARY 2.8. *If the soluble group $G = AB$ is the product of two almost locally-nilpotent π-subgroups A and B with min-p for every prime p in the set of primes π and if $A/\boldsymbol{J}(A)$ and $B/\boldsymbol{J}(B)$ are finite, then G is an almost abelian π-group with min-p for every prime p, $G/\boldsymbol{J}(G)$ is finite and*

$$\boldsymbol{J}(G) = \boldsymbol{J}(A)\boldsymbol{J}(B) \,.$$

PROOF. By Corollary 2.5 G is π-group with min-p for every prime p. By Lemma 2.7 $G/\boldsymbol{J}(G)$ is finite, and $\boldsymbol{J}(G) = \langle \boldsymbol{J}(A), \boldsymbol{J}(B)\rangle$. Since G is a soluble torsion group with min-p for every p, by Kegel and Wehrfritz (6), Corollary 3.18, p. 95, $\boldsymbol{J}(G)$ is a radicable abelian group. Thus also $\boldsymbol{J}(G) = \boldsymbol{J}(A)\boldsymbol{J}(B)$.

Finally our considerations give a new proof of the following known result; see (1) and (4).

COROLLARY 2.9. *If the soluble group $G = AB$ is the product of two Černikov π-subgroups A and B for the set of primes π, then G is a Černikov π-group, and*

$$\boldsymbol{J}(G) = \boldsymbol{J}(A)\boldsymbol{J}(B) .$$

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