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On a Maximum Principle for Elliptic Systems with Constant Coefficients.

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1. Introduction.

Let $\Omega \subset R^n$ be a bounded open set and let N be a positive integer. Let $(\,|\,)_N$, $\|\;\|_N$ be the scalar product and the norm in R^N (1). We set $D_i = \partial/\partial x_i$, $i = 1, \ldots, n$.

Let $H^1(\Omega, R^N)$ be the usual Sobolev space of vectors $u\colon \Omega \to R^N$ with norm

$$\|u\|_{H^1(\Omega,R^N)} = \left\{\int_\Omega \|u\|^2\,dx + \int_\Omega \sum_{i=1}^n \|D_i u\|^2\,dx\right\}^{\frac12} \tag{1.1}$$

and let $H^1_0(\Omega, R^N)$ be the closure of $C^\infty_0(\Omega, R^N)$ with respect to norm (1.1).

Let $L^{2,\lambda}(\Omega, R^N)$, $0 \leqslant \lambda \leqslant n$, be the Banach space defined as follows

$$L^{2,\lambda}(\Omega, R^N) = \left\{u \in L^2(\Omega, R^N)\colon \|u\|^2_{L^{2,\lambda}(\Omega,R^N)} = \sup_{\substack{\varrho>0\\ x\in\overline{\Omega}}} \varrho^{-\lambda} \int_{B(x,\varrho)\cap\Omega} \|u\|^2\,dy < +\infty\right\}$$

(here $B(x, \varrho) = \{y \in R^n\colon \|x - y\| < \varrho\}$) and

$$H^{1,(\lambda)}(\Omega, R^N) = \{u \in H^1(\Omega, R^N)\colon D_i u \in L^{2,\lambda}(\Omega, R^N),\ 1 \leqslant i \leqslant n\}$$

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(1) We shall often omit the subscript N and write simply $(\,|\,)$, $\|\;\|$.

$H^{1,(\lambda)}(\Omega, R^N)$ is a Banach space with norm

$$\|u\|_{H^{1,(\lambda)}(\Omega,R^N)} = \|u\|_{L^2(\Omega,R^N)} + \sum_{i=1}^{n} \|D_i u\|_{L^{2,\lambda}(\Omega,R^N)} .$$

Let $A_{ij}(x)$ $(i, j = 1, \dots, n)$ be $N \times N$ matrices satisfying the ellipticity condition

$$\sum_{i,j=1}^{n} \xi_i \xi_j (A_{ij}(x)\eta|\eta)_N \geqslant \nu \|\xi\|_n^2 \|\eta\|_N^2 , \qquad \nu > 0 , \qquad \forall x \in \bar{\Omega},\ \forall \xi \in R^n,\ \forall \eta \in R^N . \tag{1.2}$$

The following regularity theorem is proved in [1] (²)

THEOREM 1.I. *Let $\Omega \subset\subset R^n$ be a C^1 (³) open set, $u \in H^{1,(\lambda)}(\Omega, R^N)$, $f_i \in L^{2,\lambda}(\Omega, R^N)$ $(0 \leqslant \lambda < n,\ i = 1, \dots, n)$ and let A_{ij} be continuous in $\bar{\Omega}$ and satisfy* (1.2). *Then, if v is the solution of Dirichlet problem*

$$\begin{cases} v - u \in H_0^1(\Omega, R^N) , \\ \displaystyle\int_\Omega \sum_{i,j=1}^{n} (A_{ij}(x) D_j v | D_i \varphi)\, dx = \int_\Omega \sum_{i=1}^{n} (f_i | D_i \varphi)\, dx \qquad \forall \varphi \in H_0^1(\Omega, R^N) , \end{cases} \tag{1.3}$$

v belongs to $H^{1,(\lambda)}(\Omega, R^N)$ and

$$\|v\|_{H^{1,(\lambda)}(\Omega,R^N)} \leqslant C_1 \Big\{ \sum_{i=1}^{n} \|f_i\|_{L^{2,\lambda}(\Omega,R^N)} + \|u\|_{H^{1,(\lambda)}(\Omega,R^N)} \Big\} . \tag{1.4}$$

In this paper we prove a more specific regularity result which can be summarized as follows

THEOREM 1.II. *Let $\Omega \subset\subset R^n$ be a C^1 convex (⁴) open set, let u belong to $H^{1,(n-2)} \cap L^\infty(\Omega, R^N)$ and let A_{ij}^0 be $N \times N$ constant matrices satisfying*

(²) In [1] the result is stated in the case of only one equation; it is known that it holds unchanged in the case of systems.

(³) We say that a bounded open set $\Omega \subset R^n$ is of class C^1 if for every point $x_0 \in \partial\Omega$ we can find an open neighbourhood $\Omega(x^0)$ and a C^1 homeomorfism $x \to \phi(x)$ which maps $\overline{\Omega(x^0)}$ onto $\overline{B(0, 1)}$, $\Omega(x^0) \cap \Omega$ onto the set $\{x \in B(0, 1) : x_n > 0\}$ and $\Omega(x^0) \cap \partial\Omega$ onto $\{x \in B(0, 1) : x_n = 0\}$.

(⁴) The hypothesis that Ω is convex may be eliminated.

(1.2). *Then, if v is the solution of Dirichlet problem*

$$(1.5)\qquad \begin{cases} v-u\in H_0^1(\Omega, R^N)\,, \\ \int\limits_\Omega \sum\limits_{i,j=1}^n (A_{ij}^0 D_j v|D_i\varphi)\,dx = 0 \qquad \forall\varphi\in H_0^1(\Omega, R^N)\,, \end{cases}$$

v belongs to $H^{1,(n-2)}\cap L^\infty(\Omega, R^N)$ and

$$(1.6)\qquad \sup_\Omega \|v\| + \sum_{i=1}^n \|D_i v\|_{L^{2,n-2}(\Omega,R^N)} \leqslant C_2\Big\{\sup_\Omega \|u\| + \sum_{i=1}^n \|D_i u\|_{L^{2,n-2}(\Omega,R^N)}\Big\}\,.$$

A trivial consequence of theorem 1.II is the following maximum principle

THEOREM 1.III. *Let $\Omega\subset\Lambda\subset\subset R^n$, be two open sets and let Ω be convex* [4] *and of class C^1. Let $u\in H^1\cap L^\infty(\Lambda, R^N)$ be such that*

$$(1.7)\qquad \begin{aligned} &D_i u\in L^{2,n-2}(\Omega, R^N)\,, \qquad 1\leqslant i\leqslant n\,, \\ &\sum_{i=1}^n \|D_i u\|_{L^{2,n-2}(\Omega,R^N)} \leqslant C_3 \sup_\Lambda \|u\|\,. \end{aligned}$$

Then, if v is the solution of Dirichlet problem (1.5), *v verifies the following inequality*

$$(1.8)\qquad \sup_\Omega \|v\| \leqslant C_4 \sup_\Lambda \|u\|\,.$$

Property (1.7) is quite usual in the study of nonlinear elliptic systems; consider, for example, the following problem.

Let $A_{ij}(x, w)$ $(i, j = 1, \ldots, n)$ be $N\times N$ bounded continuous matrices defined in $\bar\Lambda\times R^N$, satisfying the following ellipticity condition

$$(1.9)\qquad \begin{aligned} &\sum_{i,j=1}^n (A_{ij}(x, w)\xi^j|\xi^i) \geqslant \nu(K)\sum_{i=1}^n \|\xi^i\|^2\,, \qquad \nu>0\,, \\ &\forall(x, w)\in\bar\Lambda\times\{\|w\|\leqslant K\},\ \forall\xi^1, \ldots, \xi^n\in R^N\,. \end{aligned}$$

Let $f\colon \Lambda\times R^N\times R^{nN}\to R^N$ be measurable in $x\in\Lambda$ and continuous in (w, p); suppose also that f has quadratic growth

$$\|f(x, w, p)\|_N \leqslant a(K)\|p\|^2_{nN} + b(K)\,, \qquad \forall (x, w, p) \in \Lambda \times \{\|w\| \leqslant K\} \times R^{nN}\,. \tag{1.10}$$

Finally, let Dw denote the vector $(D_1 w|\ldots|D_n w)$ of R^{nN}.

It is then known ([3]) that every solution $u \in H^1 \cap L^\infty(\Lambda, R^N)$ of system

$$\int_\Lambda \sum_{i,j=1}^n (A_{ij}(x, u) D_j u|D_i\varphi)_N\, dx = \int_\Lambda (f(x, u, Du)|\varphi)_N\, dx \qquad \forall \varphi \in H^1_0 \cap L^\infty(\Lambda, R^N) \tag{1.11}$$

which satisfies the following inequality (with $M = \sup_\Lambda \|u\|$)

$$2Ma(M) < \nu(M) \tag{1.12}$$

is Hölder continuous in $\Lambda \setminus \Lambda_0$, where Λ_0 is closed in Λ and such that $H_{n-q}(\Lambda_0) = 0$ (5) for a certain $q > 2$.

The proof given in [3] needs a boundedness result of the following kind:

let $u \in H^1 \cap L^\infty(\Lambda, R^N)$ *be a solution of system* (1.11) *verifying* (1.12);

let A^0_{ij} $(i, j = 1, \ldots, n)$ *be constant* $N \times N$ *matrices satisfying* (1.2);

let v *be the solution of Dirichlet problem*

$$\begin{cases} v - u \in H^1_0(B(x_0, r), R^N) \text{ with } B(x_0, 2r) \subset\subset \Lambda \text{ and } 0 < r \leqslant 1 \\ \displaystyle\int_{B(x_0,r)} \sum_{i,j=1}^n (A^0_{ij} D_j v|D_i\varphi)\, dx = 0 \qquad \forall \varphi \in H^1_0(B(x_0, r), R^N)\,. \end{cases}$$

Then

$$v \in L^\infty(B(x_0, r), R^N) \quad \text{and} \quad \sup_{B(x_0,r)} \|v\| \leqslant C_5 \sup_\Lambda \|u\| \tag{1.13}$$

where C_5 *does not depend on* x_0 *and* r.

(5) H_α, $\alpha \geqslant 0$, is the α-dimensional Hausdorff measure.

In order to get (1.13), the proof of [3] recalls the maximum principle proved in [2], which depends on the possibility of representing v by adequate potentials.

In section 3 we prove that (1.13) may be obtained in a simpler way, showing that u verifies the hypotheses of Theorem 1.III.

This method can be extended to more general situations, such as elliptic systems of order $2m \geqslant 2$, even with continuous coefficients, and C^1 bounded open sets Ω not necessarily convex.

I would like to thank S. Campanato for the useful discussions we had on this subject.

2. Proof of Theorem 1.II.

Having fixed $y \in \Omega$, we define

$$d = \operatorname{dist}(y, \partial\Omega) = \|y - z\| \quad \text{with } z \in \partial\Omega .$$

As v solves problem (1.5) and A^0_{ij} are constant, the following inequality holds ([1], Lemma 7.I)

$$\int_{B(y,\varrho)} \|v\|^2\, dx \leqslant C(\nu) \left(\frac{\varrho}{d}\right)^n \int_{B(y,d)} \|v\|^2\, dx \qquad \forall 0 < \varrho \leqslant d . \tag{2.1}$$

On the other hand

$$\int_{B(y,d)} \|v\|^2\, dx \leqslant \int_{B(z,2d)\cap\Omega} \|v\|^2\, dx \leqslant \leqslant C(n)\Big[d^n \sup_\Omega \|u\|^2 + \int_{B(z,2d)\cap\Omega} \|v-u\|^2\, dx\Big] . \tag{2.2}$$

As $v - u \in H^1_0(\Omega, R^N)$, Poincaré inequality is valid ([4]):

$$\int_{B(z,2d)\cap\Omega} \|v-u\|^2\, dx \leqslant C(n)\, d^2 \int_{B(z,2d)\cap\Omega} \sum_{i=1}^n \|D_i(v-u)\|^2\, dx \;(^6) . \tag{2.3}$$

(6) As Ω is convex the constant $C(n)$ does not depend on y (in general we shall write $C(n, \nu, \ldots)$ to mean a constant that depends on the algebraic data $n, \nu, \ldots$).

From (2.1), (2.2) and (2.3) we get

$$(2.4)\qquad \frac{1}{\varrho^n}\int_{B(y,\varrho)} \|v\|^2\,dx \leqslant$$

$$\leqslant C(n,\nu)\left[\sup_{\Omega}\|u\|^2 + d^{2-n}\int_{B(z,2d)\cap\Omega}\sum_{i=1}^{n}\|D_i(v-u)\|^2\,dx\right].$$

Theorem 1.I implies that

$$v\in H^{1,(n-2)}(\Omega, R^N)$$

and

$$(2.5)\qquad \|v\|_{H^{1,(n-2)}(\Omega,R^N)}\leqslant C_1\|u\|_{H^{1,(n-2)}(\Omega,R^N)}\,.$$

Combining (2.4) and (2.5) we prove (1.6) and the theorem.

3. Application to quasilinear systems.

Let $\Lambda\subset R^n$ be a bounded open set. Let $A_{ij}(x,u)$ $(1\leqslant i, j\leqslant n)$ be $N\times N$ bounded continuous matrices defined in $\bar\Lambda\times R^N$, satisfying the ellipticity condition

$$(3.1)\qquad \sum_{i,j=1}^{n}(A_{ij}(x,u)\xi^j|\xi^i)\geqslant\nu(K)\sum_{i=1}^{n}\|\xi^i\|^2\,,\qquad \nu>0\,,$$

$$\forall(x,u)\in\bar\Lambda\times\{\|u\|\leqslant K\},\ \forall\xi^1,\dots,\xi^n\in R^N\,.$$

Let $f\colon\Lambda\times R^N\times R^{nN}\to R^N$ be measurable in $x\in\Lambda$, continuous in (u,p) and with quadratic growth

$$(3.2)\qquad \|f(x,u,p)\|_N\leqslant a(K)\|p\|^2_{nN}+b(K)\,,$$

$$\forall(x,u,p)\in\Lambda\times\{\|u\|\leqslant K\}\times R^{nN}\,.$$

Let us consider the quasilinear system in divergence form

$$(3.3)\qquad -\sum_{i,j=1}^{n}D_i\big(A_{ij}(x,u)\,D_ju\big)=f(x,u,Du)\,,\quad \text{in }\Lambda\,.$$

The following lemma can be deduced from a « Caccioppoli inequality » proved in [3].

LEMMA 3.I. *Let* $u \in H^1 \cap L^\infty(\Lambda, R^N)$ *be a weak solution of system* (3.3) *satisfying the following inequality* $\left(\text{with } M = \sup_\Lambda \|u\|\right)$

$$(3.4) \qquad Ma(M) < \nu(M) .$$

Then $u \in H^{1,(n-2)}_{\text{loc}}(\Lambda, R^N)$ *and for every ball* $B(x^0, r) \subset B(x^0, 2r) \subset \Lambda$

$$(3.5) \qquad \sum_{i=1}^{n} \|D_i u\|_{L^{2,n-2}(B(x^0 r), R^N)} \leqslant C' \sup_\Lambda \|u\|$$

where C' *depends on* M, *but neither on* r *nor on* x^0.

PROOF. As $u \in H^1 \cap L^\infty(\Lambda, R^N)$ is a weak solution of (3.3)

$$(3.6) \qquad \int_\Lambda \sum_{i,j=1}^{n} \left(A_{ij}(x, u) D_j u | D_i \varphi\right) dx = \int_\Lambda \left(f(x, u, Du) | \varphi\right) dx$$

$$\forall \varphi \in H^1_0 \cap L^\infty(\Lambda, R^N) .$$

Having fixed $y \in \overline{B(x^0, r)}$ and $0 < \sigma < r/2$, we choose $\theta \in C_0^\infty(B(y, 2\sigma))$ with $0 \leqslant \theta \leqslant 1$, $\theta = 1$ in $B(y, \sigma)$ and $\|D\theta\| \leqslant 2/\sigma$.

If we substitute $\varphi = \theta^2 u$ in (3.6), we get as in [3] the following « Caccioppoli inequality »:

$$\int_{B(y,\sigma)} \sum_{i=1}^{n} \|D_i u\|^2 dx \leqslant C(\nu) \left\{ \frac{1}{\sigma^2} \int_{B(y,2\sigma)} \|u\|^2 dx + [b(M)]^2 \sigma^{n+2} \right\}.$$

Hence, if σ is such that

$$[b(M)]^2 \sigma^4 \leqslant \sup_\Lambda \|u\|^2$$

we get

$$\int_{B(y,\sigma)} \sum_{i=1}^{n} \|D_i u\|^2 dx \leqslant C' \sigma^{n-2} \sup_\Lambda \|u\|^2 .$$

This proves (3.5) and the lemma.

REMARK 3.I. Let $u \in H^1 \cap L^\infty(\Lambda, R^N)$ be as in Lemma 3.I and consider a ball $B(x^0, r) \subset B(x^0, 2r) \subset \Lambda$, $0 < r \leqslant 1$. Let A^0_{ij} $(i, j = 1, ..., n)$

be $N \times N$ constant matrices satisfying the ellipticity condition

$$\sum_{i,j=1}^{n} \xi_i \xi_j (A_{ij}^0 \eta|\eta)_N \geqslant \nu \|\xi\|_n^2 \|\eta\|_N^2 , \qquad \forall \xi \in R^n, \ \forall \eta \in R^N .$$

Let v be the solution of the following Dirichlet problem

$$\begin{cases} v - u \in H_0^1(B(x_0, r), R^N) , \\ \displaystyle\int_{B(x_0,r)} \sum_{i,j} (A_{ij}^0 D_j v | D_i \varphi)\, dx = 0 \qquad \forall \varphi \in H_0^1(B(x_0, r), R^N) . \end{cases}$$

From Lemma 3.I and Theorem 1.III we draw the conclusion that

$$\sup_{B(x_0,r)} \|v\| \leqslant C^* \sup_{A} \|u\| .$$

Moreover, C^* does not depend on x^0 and r.
The last statement can be shown by a homothetical argument.

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