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Partial Hölder continuity of solutions of quasilinear parabolic systems of second order with linear growth

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Partial Hölder Continuity of Solutions of Quasilinear Parabolic Systems of Second Order with Linear Growth.

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1. Introduction.

Let Ω be a bounded open set of R^n, $n > 2$ [1], with sufficiently smooth boundary $\partial\Omega$, for instance of class C^3. Let N be an integer $\geqslant 1$, $(|)$ and $\|\cdot\|$ the scalar product and the norm in R^N [2]. If $u\colon \Omega \to R^N$, we set $Du = (D_1 u|\ldots|D_n u)$ where $D_i = \partial/\partial x_i$. In general $p = (p^1|\ldots|p^n)$, $p^i \in R^N$, denotes a vector of R^{nN}, x is a point of R^n $t \in R$ and $X = (x, t)$.

$B(x_0, \sigma) = \{x \in R^n\colon \|x - x_0\| < \sigma\}$.

$Q = \Omega \times (-T, 0)$ with $T > 0$.

If $X_0 = (x_0, t)$, we set

$$Q(X_0, \sigma) = B(x_0, \sigma) \times (t_0 - \sigma^2, t_0).$$

We say that $Q(X_0, \sigma) \subset\subset Q$ if

$$B(x_0, \sigma) \subset\subset \Omega \qquad \text{and} \qquad \sigma^2 < t_0 + T \leqslant T$$

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(1) This is just to fix ideas; the case $n = 2$ can be dealt with by trivial modifications.

(2) In general $(|)_k$, $\|\cdot\|_k$ are the scalar product and the norm in R^k. We shall omit the index k if there is not ambiguity of writing.

$H^{k,p}$ and $H_0^{k,p}$ are the usual Sobolev spaces. If $p = 2$ we write simply H^k and H_0^k.

Let us consider the quasilinear parabolic system of second order

$$-\sum_{ij=1}^{n} D_i(A_{ij}(X, u)D_j u) + \frac{\partial u}{\partial t} = -\sum_{i=1}^{n} D_i f^i(X, u) + f^0(X, u, Du) \tag{1.1}$$

where A_{ij} are $N \times N$ matrices which are uniformly continuous and bounded in $\bar{Q} \times R^N$ and satisfy the strong ellipticity condition

$$\begin{gathered}\sum_{ij} (A_{ij}(X, u)p^j|p^i) \geqslant \nu \sum_i \|p^i\|^2 \qquad (\nu > 0)\,, \\ \forall (X, u, p) \in \bar{Q} \times R^N \times R^{nN}\,,\end{gathered} \tag{1.2}$$

$f^i(X, u)$, $i = 1, \ldots, n$, and $f^0(X, u, p)$ are vectors of R^N, measurable in $X \in Q$ and continuous in u and (u, p) respectively. Suppose that f^i, f^0 have linear growth

$$\|f^i(X, u)\| \leqslant g_i(X) + c\|u\|\,, \qquad i = 1, \ldots, n \tag{1.3}$$

$$\|f^0(X, u, p)\| \leqslant g_0(X) + c\{\|u\| + \sum_i \|p^i\|\} \tag{1.4}$$

with

$$g = \Big(\sum_i D_i g_i + g_0\Big) \in L^2(-T, 0, H^{-1}(\Omega))\,. \tag{1.5}$$

We set, for the sake of brevity,

$$Eu = -\sum_{ij} D_i(A_{ij}(X, u)D_j u) + \frac{\partial u}{\partial t}\,, \tag{1.6}$$

$$F = -\sum_i D_i f^i(X, u) + f^0(X, u, Du)\,, \tag{1.7}$$

$$a(u, \varphi) = \int_Q \sum_{ij} (A_{ij} D_j u|D_i\varphi) - \left(u \,\middle|\, \frac{\partial \varphi}{\partial t}\right) dX\,, \tag{1.8}$$

$$\langle F, \varphi\rangle = \int_Q \sum_i (f^i|D_i\varphi) + (f^0|\varphi)\, dX\,, \tag{1.9}$$

$$W(Q) = L^2(-T, 0, H_0^1(\Omega, R^N)) \cap H^1(-T, 0, L^2(\Omega, R^N))\,. \tag{1.10}$$

A solution of system (1.1) is a vector $u \in L^2(-T, 0, H^1(\Omega, R^N))$ such that

$$a(u, \varphi) = \langle F, \varphi \rangle, \quad \forall \varphi \in W(Q): \varphi(x, -T) = \varphi(x, 0) = 0 \quad \text{in } \Omega . \tag{1.11}$$

A solution of Cauchy-Dirichlet problem

$$\begin{aligned} Eu &= F && \text{in } Q \\ u &= 0 && \text{on } \partial\Omega \times (-T, 0) \\ u(x, -T) &= 0 && \text{in } \Omega \end{aligned} \tag{1.12}$$

is a vector $u \in L^2(-T, 0, H_0^1(\Omega, R^N))$ such that

$$a(u, \varphi) = \langle F, \varphi \rangle, \quad \forall \varphi \in W(Q): \varphi(x, 0) = 0 \quad \text{in } \Omega . \tag{1.13}$$

It is known that, even if g is smooth, there are solutions of system (1.1) which fail to be Hölder continuous in Q (3). We shall prove, in section 3, the following partial Hölder continuity result:

THEOREM 1.I. *If $u \in L^2(-T, 0, H^1(\Omega, R^N))$ is a solution of system* (1.1) *and*

$$g_i \in L^p(Q), \quad i = 1, \dots, n, \quad g_0 \in L^p(-T, 0, L^{pn/(n+2)}(\Omega)) \quad \text{with } p > n + 2 , \tag{1.14}$$

then there is a set $Q_0 \subset Q$, closed in Q, such that

$$\mathcal{H}_n(Q_0) = 0 \tag{1.15}$$

$$u \in C^{0,\alpha}(Q \setminus Q_0, R^N), \qquad \forall \alpha < 1 - \frac{n+2}{p} \tag{1.16}$$

and for every open subset $A \subset\subset Q \setminus Q_0$

$$[u]_{\alpha, \bar{A}} \leqslant c \left\{ 1 + \int_Q \|u\|^2 + \sum_i \|D_i u\|^2 \, dX \right\} ^{(4)} . \tag{1.17}$$

(3) i.e. on every compact set $K \subset Q$.

(4) C depends on the $L^2(Q)$-norm of g_i, g_0 and on the distance of $\bar{A}$ from the parabolic boundary of Q.

Here $\mathcal{M}_n$ is the n-dimensional Hausdorff measure with respect to the metric

$$(1.18)\qquad \delta(X, Y) = \max\{\|x-y\|, |t-\tau|^{\frac{1}{2}}\}, \qquad X=(x,t),\ Y=(y,\tau)$$

and also Hölder continuity in (1.16) is related to this metric.

The previous result is proved also in [9] with a different technique, for the special case $f^i = f^0 = 0$ (5).

The method of this paper can be exstended to the case of systems of order $2m \geqslant 2$ and to more general growth conditions on the vectors f^i, f^0.

In order to prove theorem 1.I we need a local L^p regularity result of this kind: We set

$$(1.19)\qquad \xi = (n+2)\left(1-\frac{2}{p}\right),$$

$$(1.20)\qquad \Phi(X_0,\sigma) = \sigma^{\xi} + \int_{Q(X_0,\sigma)} \|u\|^2 + \sum_i \|D_i u\|^2 + \sigma^{-2}\|u-u_\sigma\|^2\, dX$$

where u_σ is the average of u on $Q(X_0, \sigma)$

$$u_\sigma = \fint_{Q(X_0,\sigma)} u(X)\, dX .$$

THEOREM 1.II. *If $u \in L^2(-T, 0, H^1(\Omega, R^N))$ is a solution of system* (1.1) *and* (1.14) *holds, then we can find $q > 2$ such that $\forall Q(X_0, 2\sigma) \subset\subset Q$ with $\sigma \leqslant 1$*

$$(1.21)\qquad \left(\fint_{Q(X_0,\sigma)} \sum_i \|D_i u\|^q\, dX\right)^{2/q} \leqslant c\sigma^{-(n+2)}\Phi(X_0, 2\sigma) .$$

Theorem 1.II easily follows, via Hölder inequality, from the following local L^q result proved in [5]:

THEOREM 1.III. *If u is a solution of system* (1.1), *we can find*

(5) In this case α can be every real number less than 1, because we are in the situation $g_i, g_0 \in L^\infty(Q)$.

p_0, $2 < p_0 \leqslant 2^*$ [6], *such that if*

$$
(1.22)\qquad \begin{aligned} &g_i \in L^q(Q)\,, \qquad i = 1, \ldots, n\,,\\ &g_0 \in L^q(-T, 0, L^2(\Omega))\,, \qquad 2 \leqslant q < p_0\,, \end{aligned}
$$

then $\forall Q(X_0, 2\sigma) \subset\subset Q$ *with* $\sigma \leqslant 1$

$$
(1.23)\qquad \Big(\int_{Q(X_0,\sigma)} \sum_i \|D_i u\|^q\, dX\Big)^{1/q} \leqslant c\Big(\int_{Q(X_0,2\sigma)} \sum_i |g_i|^q\, dX\Big)^{1/q} +
$$

$$
+\, c\sigma^{1+n(1/q-\frac12)}\Big\{\int_{t_0-4\sigma^2}^{t_0} \|g_0\|^q_{L^2(B(x_0,2\sigma))}\, dt\Big\}^{1/q} + c\sigma^{(n+2)(1/q-\frac12)}\{\Phi(X_0, 2\sigma)\}^{\frac12}.
$$

In fact, by Hölder inequality

$$
\Big(\int_{Q(X_0,2\sigma)} \sum_i |g_i|^q\, dX\Big)^{1/q} \leqslant c\sigma^{(n+2)(1/q-1/p)}\Big(\int_Q \sum_i |g_i|^p\, dX\Big)^{1/p}.
$$

In the same way, as $pn/(n+2) > 2$ and we can suppose $q \leqslant p$,

$$
\Big(\int_{t_0-4\sigma^2}^{t_0} \|g_0\|^q_{L^2(B(x_0,2\sigma))}\, dt\Big)^{1/q} \leqslant c\sigma^{2(1/q-1/p)+(n/2)(1-2(n+2)/pn)}\Big(\int_{-T}^{0} \|g_0\|^p_{L^{pn/(n+2)}(\Omega)}\, dt\Big)^{1/p}.
$$

From (1.23), as $\sigma \leqslant 1$, we then get

$$
\Big(\int_{Q(X_0,\sigma)} \sum_i \|D_i u\|^q\, dX\Big)^{1/q} \leqslant c\sigma^{(n+2)(1/q-1/p)} + c\sigma^{(n+2)(1/q-\frac12)}[\Phi(X_0, 2\sigma)]^{\frac12}
$$

and therefore (1.21).

2. Some lemmas.

We list in this section a few lemmas that will be used in the rest of the work.

We set $Q(\sigma) = Q(0, \sigma)$ and $B(\sigma) = B(0, \sigma)$.

[6] $2^* = 2n/(n-2)$ is the Sobolev exponent.

LEMMA 2.I. *For every* $u \in L^2(-\sigma^2, 0, H^1(B(\sigma), R^N)) \cap H^{\frac{1}{2}}(-\sigma^2, 0, L^2(B(\sigma), R^N))$ *the following inequality holds*

$$\int_{Q(\sigma)} \|u - u_{Q(\sigma)}\|^2 \, dX \leqslant \tag{2.1}$$
$$\leqslant c\sigma^2 \left\{ \int_{Q(\sigma)} \sum_i \|D_i u\|^2 \, dX + \int_{-\sigma^2}^0 dt \int_{-\sigma^2}^0 d\xi \int_{B(\sigma)} \frac{\|u(x,t) - u(x,\xi)\|^2}{|t-\xi|^2} \, dx \right\}.$$

Inequality (2.1) is well known; we give the proof for the reader's convenience:

$$\int_{Q(\sigma)} \|u - u_{Q(\sigma)}\|^2 \, dX \leqslant \fint_{Q(\sigma)} dx\, dt \int_{Q(\sigma)} \|u(x,t) - u(y,\xi)\|^2 \, dy\, d\xi \leqslant$$
$$\leqslant c \left\{ \int_{-\sigma^2}^0 dt \fint_{B(\sigma)} dx \int_{B(\sigma)} \|u(x,t) - u(y,t)\|^2 \, dy + \right.$$
$$\left. + \sigma^{-2} \int_{-\sigma^2}^0 dt \int_{-\sigma^2}^0 d\xi \int_{B(\sigma)} \|u(y,t) - u(y,\xi)\|^2 \, dy \right\} \leqslant$$
$$\leqslant c\sigma^2 \left\{ \int_{Q(\sigma)} \sum_i \|D_i u\|^2 \, dX + \int_{-\sigma^2}^0 dt \int_{-\sigma^2}^0 d\xi \int_{B(\sigma)} \frac{\|u(y,t) - u(y,\xi)\|^2}{|t-\xi|^2} \, dy \right\}.$$

Let B_{ij} be $N \times N$ constant matrices which satisfy the strong ellipticity condition

$$\sum_{ij} (B_{ij} p^j | p^i) \geqslant \nu \sum_i \|p^i\|^2, \qquad \nu > 0, \tag{2.2}$$
$$\forall p \in R^{nN}.$$

Let f^i belong to $L^2(Q(\sigma), R^N)$ and let $v \in L^2(-\sigma^2, 0, H^1(B(\sigma), R^N))$ be a solution of the following parabolic system

$$\int_{Q(\sigma)} \sum_{ij} (B_{ij} D_j v | D_i \varphi) - \left(v \,\middle|\, \frac{\partial \varphi}{\partial t} \right) dX = \int_{Q(\sigma)} \sum_i (f^i | D_i \varphi) \, dX, \tag{2.3}$$
$$\forall \varphi \in C_0^\infty(Q(\sigma), R^N).$$

LEMMA 2.II. *If hypotheses* (2.2), (2.3) *hold, then for every* $\tau \in (0,1)$

$$\int_{Q(\tau\sigma)} \sum_i \|D_i v\|^2 \, dX \leqslant c \left\{ \tau^{n+2} \int_{Q(\sigma)} \sum_i \|D_i v\|^2 \, dX + \int_{Q(\sigma)} \sum_i \|f^i\|^2 \, dX \right\}, \tag{2.4}$$

$$\int_{Q(\tau\sigma)} \|v - v_{\tau\sigma}\|^2 \, dX \leqslant c \left\{ \tau^{n+4} \int_{Q(\sigma)} \|v - v_\sigma\|^2 \, dX + \tau^2 \sigma^2 \int_{Q(\sigma)} \sum_i \|f^i\|^2 \, dX \right\}. \tag{2.5}$$

PROOF. In $Q(\sigma)$, $v = V + W$ where W is the solution of Cauchy-Dirichlet problem

$$\begin{gathered} W \in L^2(-\sigma^2, 0, H_0^1(B(\sigma), R^N)), \\ \int_{Q(\sigma)} \sum_{ij} (B_{ij} D_j W | D_i \varphi) - \left(W \left| \frac{\partial \varphi}{\partial t} \right. \right) dX = \int_{Q(\sigma)} \sum_i (f^i | D_i \varphi) \, dX, \\ \forall \varphi \in W(Q(\sigma)) \colon \varphi(x, 0) = 0 \text{ in } B(\sigma), \end{gathered} \tag{2.6}$$

whereas

$$\begin{gathered} V \in L^2(-\sigma^2, 0, H^1(B(\sigma), R^N)), \\ \int_{Q(\sigma)} \sum_{ij} (B_{ij} D_j V | D_i \varphi) - \left(V \left| \frac{\partial \varphi}{\partial t} \right. \right) dX = 0, \qquad \forall \varphi \in C_0^\infty(Q(\sigma), R^N). \end{gathered} \tag{2.7}$$

It is known [11] that $W \in H^{\frac{1}{2}}(-\sigma^2, 0, L^2(B(\sigma), R^N))$ and

$$\int_{Q(\sigma)} \sum_i \|D_i W\|^2 \, dX + \int_{-\sigma^2}^0 dt \int_{-\sigma^2}^0 d\xi \int_{B(\sigma)} \frac{\|W(x,t) - W(x,\xi)\|^2}{|t - \xi|^2} \, dx \leqslant c \int_{Q(\sigma)} \sum_i \|f^i\|^2 \, dX \tag{2.8}$$

where c is invariant with respect to the homothetical transformation

$$x = \sigma y, \quad t = \sigma^2 \xi \tag{2.9}$$

V verifies the following inequalities, as shown in [1]: $\forall \tau \in (0,1)$

$$\int_{Q(\tau\sigma)} \sum_i \|D_i V\|^2 \, dX \leqslant c \tau^{n+2} \int_{Q(\sigma)} \sum_i \|D_i V\|^2 \, dX, \tag{2.10}$$

$$\int_{Q(\tau\sigma)} \|V - V_{\tau\sigma}\|^2 \, dX \leqslant c \tau^{n+4} \int_{Q(\sigma)} \|V - V_\sigma\|^2 \, dX, \tag{2.11}$$

here c is invariant under transformation (2.9).

(2.4) follows from (2.10) and (2.8) in a standard way. On the other hand, from (2.11) we get

$$\int_{Q(\tau\sigma)} \|v - v_{\tau\sigma}\|^2 \, dX \leqslant \tag{2.12}$$

$$\leqslant c\left\{\tau^{n+4}\int_{Q(\sigma)} \|v - v_\sigma\|^2 \, dX + \int_{Q(\tau\sigma)} \|W - W_{\tau\sigma}\|^2 \, dX + \tau^2 \int_{Q(\sigma)} \|W - W_\sigma\|^2 \, dX\right\}.$$

But lemma 2.I and (2.8) imply that

$$\int_{Q(\tau\sigma)} \|W - W_{\tau\sigma}\|^2 \, dX + \tau^2 \int_{Q(\sigma)} \|W - W_\sigma\|^2 \, dX \leqslant c\tau^2\sigma^2 \int_{Q(\sigma)} \sum_i \|f^i\|^2 \, dX. \tag{2.13}$$

Then (2.5) follows from (2.12), (2.13).

LEMMA 2.III. *If* $v \in L^2(Q(\sigma), R^N)$ *then for every* $\tau \in (0,1)$

$$\int_{Q(\tau\sigma)} \|v\|^2 \, dX \leqslant c\left\{\tau^{n+2}\int_{Q(\sigma)} \|v\|^2 \, dX + \int_{Q(\sigma)} \|v - v_\sigma\|^2 \, dX\right\} \tag{2.14}$$

(2.14) is a trivial consequence of the following estimate

$$\int_{Q(\tau\sigma)} \|v\|^2 \, dX \leqslant c\left\{ \int_{Q(\sigma)} \|v - v_\sigma\|^2 \, dX + \operatorname{meas} Q(\tau\sigma) \|v_\sigma\|^2\right\}.$$

LEMMA 2.IV. *Let* φ, ψ *be non negative functions defined in* $(0, \sigma]$, *let* α *be positive,* $A > 1$, B *and* $M \geqslant 0$ *and suppose that* $\forall \tau \in (0,1)$ *and* $\forall \varrho \leqslant \sigma$

$$\begin{aligned} \varphi(\tau\varrho) &\leqslant A\tau^\alpha \varphi(\varrho) + \psi(\varrho), \\ \psi(\tau\varrho) &\leqslant B\tau^\alpha \psi(\varrho) + M, \end{aligned} \tag{2.15}$$

then $\forall \tau \in (0,1)$ *and* $\forall \varepsilon \in (0, \alpha)$

$$\varphi(\tau\sigma) \leqslant A\tau^{\alpha-\varepsilon}\{\varphi(\sigma) + KB\psi(\sigma)\} + CM \tag{2.16}$$

where K, C *depend on* A, α, ε.

The proof is the same as in lemma 2.IV of [4].

LEMMA 2.V. *Let* φ, ω_1 *defined in* $(0, d]$, *and* ω_2, *defined in* $(0, +\infty)$ *be non negative and nondecreasing functions. Let* A, α *be positive constants and* $0 \leqslant \beta < \alpha$. *Suppose that* $\forall \tau \in (0,1)$ *and* $\forall \sigma \in (0, d]$

$$\varphi(\tau\sigma) \leqslant \{A\tau^{\alpha} + \omega_1(\sigma) + \omega_2(\sigma^{-\beta}\varphi(\sigma))\}\cdot\varphi(\sigma) . \tag{2.17}$$

If for a fixed $\varepsilon \in (0, \alpha - \beta)$ *we can find* $\sigma_\varepsilon \in (0, d]$ *such that*

$$\omega_1(\sigma_\varepsilon) + \omega_2(\sigma_\varepsilon^{-\beta}\varphi(\sigma_\varepsilon)) < (1 + A)^{-\alpha/\varepsilon} \tag{2.18}$$

then $\forall \tau \in (0,1)$

$$\varphi(\tau\sigma_\varepsilon) \leqslant B\tau^{\alpha-\varepsilon}\varphi(\sigma_\varepsilon) , \qquad B = (1 + A)^{(\alpha-\varepsilon)/\varepsilon} . \tag{2.19}$$

See for example [6], lemma 1.IV.

3. The partial Hölder continuity theorem.

In this section we prove theorem 1.I.

Suppose hypotheses (1.2), (1.3), (1.4), (1.14) hold and let $u \in L^2(-T, 0, H^1(\Omega, R^N))$ be a solution in Q of system (1.1).

As

$$p > n + 2 \Rightarrow g_i \in L^2(Q) , \qquad i = 0, \dots, n$$

we get from (1.3), (1.4)

$$f^0(X, u, Du) \in L^2(Q, R^N) \quad \text{and} \quad f^i(X, u) \in L^2(Q, R^N) , \qquad i = 1, \dots, n .$$

Then [11] and a standard localizing argument (see [5], n. 4) imply that for every cylinder $Q^* = \Omega^* \times (-\lambda T, 0)$ with $\Omega^* \subset\subset \Omega$ and $\lambda \in (0,1)$

$$u \in H^{\frac{1}{2}}(-\lambda T, 0, L^2(\Omega^*, R^N))$$

and

$$\int_{Q^*} \sum_i \|D_i u\|^2 \, dX + \int_{-\lambda T}^{0} dt \int_{-\lambda T}^{0} d\eta \int_{\Omega^*} \frac{\|u(x,t) - u(x,\eta)\|^2}{|t-\eta|^2} \, dx \leqslant \tag{3.1}$$
$$\leqslant c\left\{1 + \int_Q \|u\|^2 + \sum_i \|D_i u\|^2 \, dX\right\} .$$

Here c depends on the $L^2(Q)$-norms of g_i, g_0 and on the distance of Q^* from the parabolic boundary of Q (7).

Hence, by lemma 2.I, $\forall Q(X_0, \sigma) \subset\subset Q$

$$\sigma^{-2}\int\limits_{Q(X_0,\sigma)} \|u - u_\sigma\|^2 \, dX \leqslant c\left\{1 + \int\limits_{Q} \|u\|^2 + \sum_i \|D_i u\|^2 \, dX\right\} \tag{3.2}$$

where c depends on the $L^2(Q)$-norms of g_i, g_0 and on the distance of $Q(X_0, \sigma)$ from the parabolic boundary of Q.

For the hypotheses we formulated on matrices $A_{ij}(X, u)$ we can find a bounded continuous function $\omega(\eta)$, defined for $\eta \geqslant 0$, which is increasing, concave, such that $\omega(0) = 0$ and $\forall X, Y \in \bar{Q}$ and $\forall u, v \in R^N$

$$\left\{\sum_{ij} \|A_{ij}(X, u) - A_{ij}(Y, v)\|^2\right\}^{\frac{1}{2}} \leqslant \omega\big(\delta^2(X, Y) + \|u - v\|^2\big) \tag{3.3}$$

where $\delta(X, Y)$ is the parabolic distance (1.18).

We can now prove the following lemma:

LEMMA 3.I. *If u is a solution of system* (1.1) *under the hypotheses* (1.14), *then* $\forall Q(X_0, \sigma) \subset\subset Q$ *with* $\sigma \leqslant 2$, $\forall \tau \in (0, 1)$ *and* $\forall \lambda \in (n, \xi)$

$$\Phi(X_0, \tau\sigma) \leqslant K\Phi(X_0, \sigma)\{\tau^\lambda + \sigma^\varepsilon + [\omega(c\sigma^{-n}\Phi(X_0, \sigma))]^{1-2/r}\} \tag{3.4}$$

where

$$\varepsilon = 2\left(1 - \frac{2}{p}\right) \tag{3.5}$$

and ω is defined as in (3.3).

PROOF. By theorem 1.II, we can find $r > 2$ such that $\forall Q(X_0, 2\sigma) \subset\subset Q$ with $\sigma \leqslant 1$

$$\left[\int\limits_{Q(X_0,\sigma)} \left(\sum_i \|D_i u\|^2\right)^{r/2} dX\right]^{2/r} \leqslant c\sigma^{(n+2)(2/r-1)}\Phi(X_0, 2\sigma) . \tag{3.6}$$

(7) $\Omega \times \{-T\} \cup \partial\Omega \times (-T, 0)$.

Consider $Q(X_0, 2\sigma) \subset\subset Q$ with $\sigma \leqslant 1$. For the sake of brevity we set

$$A^0_{ij} = A_{ij}(X_0, u_{Q(X_0,\sigma)}) ,$$
$$a_0(u, \varphi) = \int\limits_{Q(X_0,\sigma)} \sum_{ij} (A^0_{ij} D_j u | D_i \varphi) - \left(u \,\middle|\, \frac{\partial \varphi}{\partial t}\right) dX .$$

In $Q(X_0, \sigma)$ we write $u = v + w$ where

$$w \in L^2(t_0 - \sigma^2, t_0, H^1_0(B(x_0, \sigma), R^N)) ,$$
$$(3.7) \qquad a_0(w, \varphi) = \int\limits_{Q(X_0,\sigma)} \sum_{ij} ([A^0_{ij} - A_{ij}(X, u)] D_j u | D_i \varphi)\, dX + \int\limits_{Q(X_0,\sigma)} (f^0 | \varphi)\, dX ,$$
$$\forall \varphi \in W(Q(X_0, \sigma)) \colon \varphi(x, t_0) = 0 \qquad \text{in } B(x_0, \sigma) ,$$

whereas

$$v \in L^2(t_0 - \sigma^2, t_0, H^1(B(x_0, \sigma), R^N)) ,$$
$$(3.8) \qquad a_0(v, \varphi) = \int\limits_{Q(X_0,\sigma)} \sum_i (f^i | D_i \varphi)\, dX , \qquad \forall \varphi \in C^\infty_0(Q(X_0, \sigma), R^N) .$$

As

$$f^0(X, u, Du) \in L^2(Q)$$

there is one and only one solution w and

$$(3.9) \qquad \int\limits_{Q(X_0,\sigma)} \sum_i \|D_i w\|^2\, dX + \int\limits_{t_0-\sigma^2}^{t_0} dt \int\limits_{t_0-\sigma^2}^{t_0} d\eta \int\limits_{B(x_0,\sigma)} \frac{\|w(x, t) - w(x, \eta)\|^2}{|t - \eta|^2}\, dx \leqslant$$
$$\leqslant c \int\limits_{Q(X_0,\sigma)} \sum_{ij} \|A^0_{ij} - A_{ij}(X, u)\|^2 \cdot \sum_i \|D_i u\|^2\, dX + c\sigma^2 \int\limits_{Q(X_0,\sigma)} \|f^0(X, u, Du)\|^2\, dX .$$

By Hölder inequality

$$\int\limits_{Q(X_0,\sigma)} |g_0|^2\, dX \leqslant c \left[\int\limits_{-T}^{0} dt \left(\int\limits_{\Omega} |g_0|^{pn/(n+2)}\, dx \right)^{(n+2)/n} \right]^{2/p} \sigma^{2(1-2/p)+n-(2/p)(n+2)} .$$

Then, if we take into account (1.14) and if we set

$$\varepsilon = 2\left(1-\frac{2}{p}\right) \tag{3.10}$$

we have

$$\sigma^2\int_{Q(X_0,\sigma)}\|f^0(X,u,Du)\|^2\,dX \leqslant c\sigma^\varepsilon\Phi(X_0,\sigma)\,. \tag{3.11}$$

On the other hand, from (3.3), (3.6) and the fact that ω is concave [8], we get

$$\begin{aligned}(3.12)\qquad &\int_{Q(X_0,\sigma)}\sum_{ij}\|A^0_{ij}-A_{ij}(X,u)\|^2\cdot\sum_i\|D_iu\|^2\,dX\leqslant\\ &\leqslant\left[\int_{Q(X_0,\sigma)}\left(\sum_i\|D_iu\|^2\right)^{r/2}dX\right]^{2/r}\cdot\left[\int_{Q(X_0,\sigma)}\omega(\sigma^2+\|u-u_\sigma\|^2)\,dX\right]^{1-2/r}\leqslant\\ &\leqslant c\Phi(X_0,2\sigma)\left[\omega\left(\sigma^2+\fint_{Q(X_0,\sigma)}\|u-u_\sigma\|^2\,dX\right)\right]^{1-2/r}\leqslant{}^{(9)}\\ &\leqslant c\Phi(X_0,2\sigma)[\omega(c\sigma^{-n}\Phi(X_0,\sigma))]^{1-2/r}\,.\end{aligned}$$

From (3.9), (3.11), (3.12) and lemma 2.I we draw the conclusion that $\forall\tau\in(0,1]$

$$\begin{aligned}(3.13)\qquad &\int_{Q(X_0,\sigma)}\sum_i\|D_iw\|^2\,dX+(\tau\sigma)^{-2}\int_{Q(X_0,\tau\sigma)}\|w-w_{\tau\sigma}\|^2\,dX\leqslant\\ &\qquad\leqslant c\Phi(X_0,2\sigma)\{\sigma^\varepsilon+[\omega(c\sigma^{-n}\Phi(X_0,\sigma))]^{1-2/r}\}\,.\end{aligned}$$

If we use lemma 2.II, then we get the following estimate on v: $\forall\tau\in\in(0,1)$ and $\varrho\leqslant\sigma$

$$\begin{aligned}(3.14)\qquad &\int_{Q(X_0,\tau\varrho)}\sum_i\|D_iv\|^2\,dX+(\tau\varrho)^{-2}\int_{Q(X_0,\tau\varrho)}\|v-v_{\tau\varrho}\|^2\,dX\leqslant\\ &\leqslant c\tau^{n+2}\int_{Q(X_0,\varrho)}\sum_i\|D_iv\|^2+\varrho^{-2}\|v-v_\varrho\|^2\,dX+c\int_{Q(X_0,\varrho)}\sum_i\|f^i(X,u)\|^2\,dX\,.\end{aligned}$$

[8] $\displaystyle\fint_{Q(X_0,\sigma)}\omega(\varphi)\,dX\leqslant\omega\left(\fint_{Q(X_0,\sigma)}\varphi\,dX\right)$.

[9] By (1.19) and the fact that $\sigma\leqslant 1$.

On the other and, by (1.3) and Hölder inequality

$$\int_{Q(X_0,\varrho)} \sum_i \|f^i(X, u)\|^2 \, dX \leqslant c\left\{\varrho^\xi + \int_{Q(X_0,\varrho)} \|u\|^2 \, dX\right\} = c\psi(X_0, \varrho) .$$

Lemma 2.III implies that $\forall \tau \in (0, 1)$ and $\varrho \leqslant \sigma$

$$\psi(X_0, \tau\varrho) \leqslant c\tau^\xi \psi(X_0, \varrho) + c\sigma^\varepsilon \Phi(X_0, \sigma) . \tag{3.15}$$

From (3.14), (3.15) and lemma 2.IV we conclude that $\forall \lambda \in (n, \xi)$ and $\forall \tau \in (0, 1)$

$$\begin{aligned} &\int_{Q(X_0,\tau\sigma)} \sum_i \|D_i v\|^2 + (\tau\sigma)^{-2}\|v - v_{\tau\sigma}\|^2 \, dX \leqslant \\ &\leqslant c\tau^\lambda \int_{Q(X_0,\sigma)} \sum_i \|D_i v\|^2 + \sigma^{-2}\|v - v_\sigma\|^2 \, dX + c\Phi(X_0, \sigma)\{\tau^\lambda + \sigma^\varepsilon\} . \end{aligned} \tag{3.16}$$

As $u = v + w$, from (3.13), (3.16) we get by a standard argument that $\forall \tau \in (0, 1)$

$$\begin{aligned} &\Phi(X_0, \tau\sigma) - \psi(X_0, \tau\sigma) \leqslant \\ &\qquad \leqslant c\Phi(X_0, 2\sigma)\{\tau^\lambda + \sigma^\varepsilon + [\omega(c\sigma^{-n}\Phi(X_0, 2\sigma))]^{1-2/r}\} . \end{aligned} \tag{3.17}$$

The previous inequality is trivial for $1 \leqslant \tau < 2$ and we can add $\psi(X_0, \tau\sigma)$ to the left hand side because by (3.15)

$$\psi(X_0, \tau\sigma) \leqslant c\Phi(X_0, \sigma)\{\tau^\lambda + \sigma^\varepsilon\} .$$

Therefore we have proved (3.4).

We define

$$Q_0 = \left\{X \in Q : \min\lim_{\sigma \to 0} \sigma^{-n}[\Phi(X, \sigma) - \sigma^\xi] > 0\right\} . \tag{3.18}$$

For a well known theorem (Lebesgue)

$$\lim_{\sigma \to 0} \fint_{Q(X,\sigma)} \|u(Y) - u_\sigma\|^2 \, dY = 0 \qquad \text{a.e. in } Q ,$$

$$\lim_{\sigma \to 0} \sigma^{-n} \int_{Q(X,\sigma)} \|u(Y)\|^2 + \sum_i \|D_i u(Y)\|^2 \, dY = 0 \qquad \text{a.e. in } Q$$

and then

$$\text{meas. } Q_0 = 0 \, .$$

We can even say something more. We define the Hausdorff measure $\mathcal{M}_\alpha$ with respect to the metric δ, as usual,

$$\mathcal{M}_\alpha(E) = \liminf_{\sigma \to 0} \Big\{ \sum_i \delta^\alpha(E_i) : \bigcup_i E_i \supset E \text{ and } \delta(E_i) < \sigma \Big\} \tag{3.19}$$

where $\delta(E_i)$ is the diameter of E_i with respect to δ. Then, if we argue as in [9] ([10]), we can show that

$$\mathcal{M}_n(Q_0) = 0 \, . \tag{3.20}$$

LEMMA 3.II. – *If u is a solution of system* (1.1) *and hypotheses* (1.2), (1.3), (1.4), (1.14) *are satisfied, then* $\forall X_0 \in Q \setminus Q_0$ *and* $\forall \eta \in (0, \xi - n)$ *we can find* $\sigma_\eta < 1$ *and* $r > 0$, *with* $Q(X_0, r + \sigma_\eta) \subset Q$, *such that* $\forall Y \in Q(X_0, r)$ *and* $\forall \tau \in (0, 1)$

$$\Phi(Y, \tau\sigma_\eta) \leqslant c\tau^{\xi - \eta} \Phi(Y, \sigma_\eta) \, . \tag{3.21}$$

In particular, Q_0 is closed in Q.

PROOF. Having chosen $X_0 \in Q \setminus Q_0$, we define ([11])

$$\begin{aligned} \omega_1(t) &= K t^\varepsilon \, , \\ \omega_2(t) &= K[\omega(ct)]^{1-2/r} \, , \\ G(X_0, \sigma) &= \omega_1(\sigma) + \omega_2\big(\sigma^{-n} \Phi(X_0, \sigma)\big) \, . \end{aligned} \tag{3.22}$$

As $X_0 \in Q \setminus Q_0$

$$\operatorname*{minlim}_{\sigma \to 0} G(X_0, \sigma) = 0 \, . \tag{3.23}$$

Having fixed $\eta \in (0, \xi - n)$, we choose $\lambda = \xi - \eta/2$ and $\eta_0 = \eta/2$.

([10]) Proof of Theorem 2.

([11]) K is the constant which appears (3.4).

Therefore

$$\lambda \in (n, \xi)\,, \qquad \eta_0 \in (0, \lambda - n)\,, \qquad \lambda - \lambda_0 = \xi - \eta\,.$$

By (3.23) we can find $\sigma_\eta < 1$ such that $Q(X_0, \sigma_\eta) \subset\subset Q$ and

$$G(X_0, \sigma_\eta) < (1 + K)^{-\lambda/\eta_0}\,. \tag{3.24}$$

As $Y \to G(Y, \sigma_\eta)$ is continuous in Q, we can find r such that $Q(X_0, r + \sigma_\eta) \subset Q$ and

$$G(Y, \sigma_\eta) < (1 + K)^{-\lambda/\eta_0}\,, \qquad \forall Y \in Q(X_0, r)\,. \tag{3.25}$$

Then, $\forall Y \in Q(X_0, r)$ and $\forall \sigma \leqslant 1$ such that $Q(Y, \sigma) \subset\subset Q$, inequality (3.1) and condition (3.25) hold; therefore hypotheses of lemma 2.IV are satisfied with

$$\varphi(\sigma) = \Phi(Y, \sigma)\,,$$

$$\alpha = \lambda\,, \qquad \beta = n\,, \qquad \varepsilon = \eta_0\,,$$

ω_1, ω_2 are defined as in (3.22).

Hence $\forall \tau \in (0, 1)$ and $\forall Y \in Q(X_0, r)$

$$\Phi(Y, \tau\sigma^\eta) \leqslant c\tau^{\lambda - \eta_0}\Phi(Y, \sigma_\eta) = c\tau^{\xi - \eta}\Phi(Y, \sigma_\eta)\,. \tag{3.26}$$

In particular

$$\lim_{\sigma \to 0} \sigma^{-n}\Phi(Y, \sigma) = 0\,.$$

Therefore

$$X_0 \in Q \setminus Q_0 \Rightarrow Q(X_0, r) \subset Q \setminus Q_0$$

which means that $Q \setminus Q_0$ is open and so Q_0 is closed in Q.

Now the partial Hölder continuity theorem easily follows from lemma 3.II. In fact, recalling the definition of Φ, from (3.21) we deduce that, if $X_0 \in Q \setminus Q_0$, $Y \in Q(X_0, r)$ and $\tau \in (0, 1)$, then

$$\int_{Q(Y, \tau\sigma_\lambda)} \|u - u_{\tau\sigma_\lambda}\|^2\, dX \leqslant c\tau^{\lambda+2}\Phi(Y, \sigma_\lambda)\,, \qquad \forall n < \lambda < \xi\,.$$

By (3.2)

$$\Phi(Y, \sigma_\lambda) \leqslant c\left\{1 + \int_Q \|u\|^2 + \sum_i \|D_i u\|^2 \, dX\right\}.$$

Therefore $\forall Y \in Q(X_0, r)$ and $\tau \in (0, 1)$

$$\int_{Q(Y,\tau\sigma_\lambda)} \|u - u_{\tau\sigma_\lambda}\|^2 \, dX \leqslant c\tau^{\lambda+2}\left\{1 + \int_Q \|u\|^2 + \sum_i \|D_i u\|^2 \, dX\right\}.$$

By [8], the previous inequality implies that $u \in C^{0,\alpha}(\overline{Q(X_0, r)})$ (12) $\forall \alpha < 1 - (n + 2)/p$ and

$$[u]^2_{\alpha, \overline{Q(X_0,r)}} \leqslant c\left\{1 + \int_Q \|u\|^2 + \sum_i \|D_i u\|^2 \, dX\right\}.$$

This proves the theorem.

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