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Pseudo-valuation and pseudo norm.

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1. Introduction.

P. M. Cohn in [1] has given necessary and sufficient conditions for a non-discrete topological field to have its topology induced by a pseudo valuation. For this purpose, he introduced the notion of gauge set. In this note, using techniques somewhat analogous to those of Cohn, we study pseudo normed topological vector spaces over pseudo valuated fields. We extend the concept of gauge set to pair gauge set and use bounded sets in topological fields and topological vector spaces over topological fields. We also need the notion of topologically nilpotent elements to achieve our goal.

The main aim of this note is to give necessary and sufficient condition for a topological space V on a topological field K to have their topologies induced by a pseudo norm on V compatible with a pseudo valuation on K. The topology on K is induced by this pseudo valuation.

We give all the relevant definitions and properties to make the note self contained.

2. Pseudo valuation and pseudo normed vector spaces.

In this section we shall introduce the basic notions and give examples.

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DEFINITION 1. Let K be a field and R be the field of real numbers. A mapping ω from K to R satisfying the following properties is called a *pseudo valuation.*

(i) $\omega(a) \geqslant 0 \ \forall a \in K$ and $\omega(0) = 0$

(ii) $\omega(a - b) \leqslant \omega(a) + \omega(b)$

(iii) $\omega(a \cdot b) \leqslant \omega(a) \cdot \omega(b)$ for all $a, b \in K$.

From this definition it is easy to observe the following:

(i) $\omega(a) = 0 \Leftrightarrow a = 0$

(ii) $\omega(a) = \omega(-a)$.

EXAMPLE 1. Let K be any field. Set

$$\omega_0(a) = \begin{cases} 0 & \text{if } a = 0 \, , \\ 1 & \text{if } a \neq 0 \, . \end{cases}$$

Then it is easily seen that ω_0 is a pseudo valuation.

EXAMPLE 2. Let $K = R$ be the field of real numbers. If we set $\omega_1(a) = \max\{\omega_0(a), |a|\}$ where $|a|$ is the ordinary absolute value of a then $\omega_1(a)$ is a pseudo valuation. Setting $\omega_2(a) = c|a|$ where c is a real number larger than 1, we find ω_2 is also a pseudo valuation.

DEFINITION 2. Let (K, ω) be a field with a pseudo valuation and V be a vector space over K. By a *ω-pseudo norm* $\|\ldots\|$ on V we mean a real valued function on V such that

(i) $\|v\| \geqslant 0$ for all v in V and $\|\theta\| = 0$ where θ is the zero vector of V;

(ii) $\|a \cdot v\| \leqslant \omega(a) \cdot \|v\|$ for all a in K and v in V;

(iii) $\|v_1 - v_2\| \leqslant \|v_1\| + \|v_2\|$ for all v_1, v_2 in V;

(iv) there exists v in V such that $\|v\| \neq 0$.

We shall hereafter refer the ω-pseudo norm as just pseudo norm. We observe that $\|-v\| = \|v\|$ for all v in V and $\|v_1 + v_2\| \leqslant \|v_1\| + \|v_2\|$.

EXAMPLE 3. The field R with the ordinary absolute value, is a field with pseudo valuation. Let V be the n dimensional vector space over R. Define $\|x\| = +\sqrt{\sum a_i^2}$. Then $\|\ldots\|$ is a pseudo norm on V.

EXAMPLE 4. Let (K, ω_0) where ω_0 is the pseudo valuation defined in example 1, be a field with a pseudo valutation.

If we take for V the one dimensional vector space over K itself then setting $\|v\| = \omega_0(v)$, we get a pseudo normed vector space.

Clearly a pseudo norm on a vector space defines a topology on V and pseudo valuation defines a topology on the field in such a way that V is a topological vector space and K is a topological field. The neighbourhood bases of zero are defined as follows.

For (K, ω), the collection $\mathfrak{U} = \{U_a^r\}$ where

$$U_a^r = \left\{ b \in K \,\middle|\, \omega(b-a) < \frac{1}{r}\,, \quad r > 0 \text{ in } R \right\}$$

and for $(V, \|\ldots\|)$, the collection $\mathfrak{N} = \{N_a^r\}$ where

$$N_v^r = \left\{ u \in v \,\middle|\, \|(v-u)\| < \frac{1}{r}\,, \quad r > 0 \text{ in } R \right\}$$

give the neighbourhood basis. There is a unique topology τ_ω on K, and $\tau_{\|\ldots\|}$ on V with respect to which $\mathfrak{U}$ and $\mathfrak{N}$ are basis of neighbourhoods at a and v respectively. It is not hard to verifiy that under these topologies K becomes a topological field and V a topological vector space.

A pseudo valuation ω (and a pseudo norm $\|\ldots\|$) on a field K (on a vector space V) is said to be trivial if the topology on K (on V) is discrete. Examples 1 and 2 give trivial pseudo valuation on K while example 4 illustrates a trivial pseudo norm on a vector space. Next, if $(V, \|.\|)$ is a pseudo normed vector space over a non-trivial pseudo valuated field (K, ω), then the topology induced by $\|.\|$ on V is non-discrete.

DEFINITION 3. A subset A in a topological field (K, τ) is called *bounded*, if given any neighbourhood U_1 of 0, there exists a neighbourhood U_2 of 0 such that $U_2 \cdot A \subset U_1$. Here

$$U_2 A = \{x \cdot a | x \in U_2,\, a \in A\}\,.$$

Analogously, we define a subset B of a topological vector space (V, T) over a topological field (K, τ), *bounded*, if given a neighbourhood N of θ in V there can be found a neighbourhood U of 0 in K such that $U \cdot B \subset N$ where $U \cdot B = \{a \cdot v | a \in U,\, v \in B\}$.

Remark 1. The set K in the topological field (K, τ) can not be bounded unless τ is discrete. Likewise, the set V of the pseudo normed vector space (V, T) on (K, τ) can not be bounded unless the topology τ on K is discrete.

Lemma 1. *If B_1 and B_2 are two bounded subsets of (V, τ_2) over (K, τ_1) then $B_1 + B_2$ and $a \cdot B_1$ for all $a \in K$ are also bounded.*

Proof. Since B_1 and B_2 are bounded, there exist suitable neighbourhoods U_1 and U_2 of 0 in K such that $U_i \cdot B_i \subset N'$ for any neighbourhood N' of θ in V. If N is any given neighbourhood of θ in V, then choosing N' to be such that $N' + N' \subset N$ and U to be $U_1 \cap U_2$, we find that $U(B_1 + B_2) \subset U_1 \cdot B_1 + U_2 \cdot B_2 \subset N$.

From the continuity of scalar multiplication, we get for any given neighbourhood N_1 of θ, another neighbourhood N_2 of θ and U_1 of a such that $U_1 \cdot N_2 \subset N_1$. As B_1 is bounded, there exists a U_2 such that $U_2 \cdot B_1 \subset N_2$. Therefore $U_1(U_2 \cdot B_1) \subset U_1 \cdot N_2 \subset N_1$. Thus $a \cdot B_1$ is bounded.

Corollary 1. *If A_1 and A_2 are bounded subsets of (K, τ) then so are $A_1 + A_2$ and $a \cdot A_1$ for all $a \in K$.*

Corollary 2. *If B is a bounded subset of (V, τ_2) over (K, τ_1) then $\{v + B\}$ where v in V, is also a bounded subset of V.*

The proof of Corollary 1 follows from specializing V to K in lemma 1 and of Corollary 2 from the fact that singleton sets are always bounded.

Corollary 3. *If A is a bounded subset of (K, τ) and a is any element of K, then $\{a + A\}$ is also bounded.*

Lemma 2. *Suppose that (K, ω) is a field with non-trivial pseudo valuation and $(V, \|.\|)$ is a pseudo normed vector space over K. Then*

(i) *$A \subseteq K$ is bounded $\Leftrightarrow$ there exists a real number r such that $\omega(a) \leqslant r$ for all a in A.*

(ii) *$B \subseteq V$ is bounded $\Leftrightarrow$ there exists a real number s such that $\|v\| \leqslant s$ for all v in B.*

Proof. It is sufficient to prove (ii) as (i) can be deduced by taking $V = K$ and $\|.\| = \omega$. If B is bounded, then for any given $\varepsilon > 0$, we can find a $\delta > 0$ such that if $\omega(a) < \delta$, for some $a \in K$, then $\|a \cdot v\| < \varepsilon$ for all v in B. Taking $\varepsilon = 1$ and a corresponding δ, from the non-triviality of ω we can find at least one non-zero a in K with $\omega(a) < \delta$

such that $\|v\| = \|a \cdot a^{-1} \cdot v\| \leqslant \omega(a^{-1}) \cdot \|a \cdot v\| < \omega(a^{-1})$ for all v in B. If we take $s = \omega(a^{-1})$, we get the desired result.

Consersely, suppose there exists an s with $\|v\| \leqslant s$ for all v in B. Then, given any $\varepsilon > 0$, if we take $\delta = \varepsilon/(1+s)$ then whenever $\omega(a) < \delta$ for some a in K then

$$\|a \cdot v\| \leqslant \omega(a) \cdot \|v\| < \frac{\varepsilon}{1+s} \cdot s < \varepsilon$$

for all v in B. Thus $U_a^\delta \cdot B \subset U_0^\varepsilon$.

3. Pair gauge sets and pair gauge functions.

P. M. Cohn introduced in [1] the concepts of gauge set and gauge function to study pseudo valuations on a field K. Here we extend these notions to study pseudo normed vector spaces over pseudo valuated fields.

DEFINITION 4. Let V be a vector space over a field K. An ordered pair (H, G) of sets where $H \subset V$ and $G \subset K$ will be called a pair *gauge set* if the following five conditions hold.

(i) $H = -H$ and $-1 \in G$

(ii) $G \cdot H \subseteq H$ and $G \cdot G \subseteq G$ where $G \cdot H = \{a \cdot v | a \in G,\ v \in H\}$ and $G \cdot G = \{a \cdot b | a, b \in G\}$.

(iii) There exists a c in K with $H + H \subseteq c \cdot H$ and $G + G \subseteq c \cdot G$. Here $H + H = \{v + w | v, w \in H\}$ and $G + G = \{a + b | a, b \in G\}$.

(iv) $H(G)$ is a proper subset of $V(K)$.

(v) There exists a non zero element d in G such that $V = \bigcup_{n=1}^{\infty} d^{-n} \cdot H$ and $K = \bigcup_{n=1}^{\infty} d^{-n} \cdot G$.

Any element d satisfying condition (v) of the above definition will be called a *gauge element* of the pair gauge set (H, G).

EXAMPLE 5. Let (K, ω) be a non-trivial pseudo valuated field with $\omega(1) = 1$, and $(V, \|.\|)$ be a pseudo normed vector space over (K, ω). If we take $H = \{v \in V | \|v\| \leqslant 1\}$ and $G = \{a \in k | \omega(a) \leqslant 1\}$, then (H, G) gives rise to a pair gauge set and any non-zero element d in G with $\omega(d) < 1$ is a gauge element.

LEMMA 3. *If* (H, G) *is a pair gauge set of a vector space* V *over* K *and* d *is a gauge element of* (H, G), *then*

(i) $\ldots d^2 \cdot H \subset d \cdot H \subset H \subset d^{-1} \cdot H d^{-1} \cdot H \subset d^{-2} \cdot H \ldots$ *and*

(ii) $\ldots d^2 \cdot G \subset d \cdot G \subset G \subset d^{-1} \cdot G \subset \ldots$

are strictly ascending sequences and $\bigcap_1^\infty d^n \cdot G = \{0\}$.

PROOF. We only prove (i). From d in G and $G \cdot H \subseteq H$, we derive that $d^n H = d^{n-1} \cdot (d \cdot H) \subseteq d^{n-1} H$. for all n. Strict inequality results from the fact that $V = \bigcup_1^\infty d^{-n} H$ and H is a proper subset of V.

REMARK 2. If d is a gauge element of a pair gauge set (H, G) then so is $d^t (t > 0)$. We prove it thus: $c \in K = \bigcup d^{-n} \cdot G$ implies that there exists an integer $t > 0$ such that $c \in d^{-t} G$. Hence $G + G \subseteq c \cdot G \subseteq (d^{-t} \cdot G) \cdot \cdot G \subseteq d^{-t} \cdot G$ and $H + H \subseteq c \cdot H \subseteq (d^{-t} \cdot G) \cdot H \subseteq d^{-t} \cdot H$.

Therefore, when the gauge element is not fixed in advance, we may choose d such that $G + G \subseteq d^{-1} \cdot G$ and $H + H \subseteq d^{-1} \cdot H$.

With each pair gauge set (H, G) and gauge element d, we associate a pair of extended integer valued functions φ and f on V and K respectively, as follows:

(1) If $v \in V$ and $v \in \bigcap_{n=0}^\infty d^n \cdot H$ then $\varphi(v) = \infty$:
if $v \in d^n \cdot H$ and $v \notin d^{n+1} \cdot H$ then $\varphi(v) = n$.

(2) Suppose $a \in K$ and $a \neq 0$. Then we set $f(a) = m$ if $a \in d^m \cdot G$ and $a \notin d^{m+1} \cdot G$. We also set $f(0) = \infty$.

(φ, f) is called *pair gauge function* associated with the pair gauge set (H, G) and gauge element d.

Let V be a vector space over K and (H, G) be a pair gauge set with d as a gauge element. If (φ, f) is the pair gauge function associated with (H, G) and d, then we have

(i) $\varphi(v) = \varphi(-v)$ for all $v \in V$ and $f(-1) = 0$.

(ii) $\varphi(a \cdot v) \geqslant f(a) + \varphi(v)$ for all a *in* K and $v \in V$, and $f(a \cdot b) \geqslant f(a) + f(b)$ where a and b vary over K.

(iii) There exists a positive integer t such that

$$\varphi(v_1 + v_2) \geqslant \operatorname{Min}\{\varphi(v_1), \varphi(v_2)\} - t, \quad \text{for all } v_1, v_2 \in V$$

and

$$f(a+b) \geqslant \mathrm{Min}\{f(a), f(b)\} - t \qquad \text{for all } a, b \in K .$$

(iv) $f(d) = -f(d^{-1}) = 1$

(v) $f(0) = \infty$, $\varphi(\theta) = \infty$.

Given any pair of extended integer valued functions (φ, f) on V and K satisfying the above properties with some non zero d in K, we get a pair of sets H and G given by $H = \{v \in V | \varphi(v) \geqslant 0\}$ and $G = \{a \in K | f(a) \geqslant 0\}$.

One can see without difficulty that (H, G) is a pair gauge set with d as the gauge element and (φ, f) is the pair gauge function associated with (H, G) and d.

Remark 3. If t is as in (iii) above and we take d^t as gauge element instead of d, then the pair gauge functions (φ', f') associated with (H, G) and d^t satisfy the inequalities of (iii) with t replaced by 1, That is to say $\varphi'(v_1 + v_2) \geqslant \mathrm{Min}\{\varphi'(v_1), \varphi'(v_2)\} - 1$ and

$$f'(a+b) \geqslant \mathrm{Min}\{f'(a), f'(b)\} - 1 .$$

4. The pseudo valuation and pseudo norm defined by a pair gauge set.

Let V be a vector space over a field K, (H, G) be any pair gauge set with gauge element d and (φ, f) the corresponding gauge function. By remark 3 above we may choose a d so that $t = 1$. (Here t is as in the properties of the pair gauge function). Let now e be a real number such that $1 < e \leqslant 2$. We set

$$\psi(v) = \begin{cases} e^{-\varphi(v)} & \text{if } \varphi(v) \neq \infty , \\ 0 & \text{if } \varphi(v) = \infty \end{cases}$$

and

$$g(a) = \begin{cases} e^{-f(a)} & \text{if } f(a) \neq \infty , \\ 0 & \text{if } f(a) = \infty . \end{cases}$$

Obviously ψ and g are real valued functions on V and K respectively. Moreover g satisfies the condition of a pseudo valuation with $g(a+b) \leqslant e.\ \mathrm{Max}\{g(a), g(b)\}$ for all a, b in K. And ψ fulfills the condition of a pseudo norm with

$$\psi(v_1+v_2)\leqslant e\cdot \mathrm{Max}\{\psi(v_1),\ \psi(v_2)\} \text{ for all } v_1, v_2 \text{ in } V.$$

LEMMA 4. *If v_1, v_2 in V are such that $\psi(v_1)\leqslant\psi(v_2)$ and $\psi(v_1+v_2)>$ $>\psi(v_1)+\psi(v_2)$ then $e\cdot\psi(v_1)\leqslant\psi(v_2)$ where e and ψ are as above.*

PROOF. Suppose $\psi(v_1)=\psi(v_2)$. Then

$$\psi(v_1+v_2)\leqslant e\cdot \mathrm{Max}\{\psi(v_1),\ \psi(v_2)\}\leqslant 2\cdot\psi(v_1)=\psi(v_1)+\psi(v_2)\ .$$

This contradicts the hypothesis. Hence $\psi(v_1)<\psi(v_2)$. In case $\psi(v_1)=0$, then $e\cdot\psi(v_1)=0<\psi(v_2)$. If not, then from the definition of ψ, we get $\varphi(v_2)<\varphi(v_1)$ so that $1+\varphi(v_2)\leqslant\varphi(v_1)$ as φ is an integer valued function. Consequently,

$$e^{1-\varphi(v_1)}\leqslant e^{-\varphi(v_2)} \qquad \text{which gives } e\cdot\psi(v_1)\leqslant\psi(v_2)\ .$$

LEMMA 5. *Suppose $v=v_1+v_2+\ldots+v_n$ is a decomposition of v in V. Then we can find another decomposition of v as $v=w_1+w_2+\ldots+w_m$ such that $\sum_{j=1}^{m}\psi(w_j)\leqslant\sum_{i=1}^{n}\psi(v_i)$ and $\psi(v)\leqslant e\cdot\psi(w_m)$ where ψ and e are as above.*

PROOF. In case $\psi(v)\leqslant\psi(v_1)+\ldots+\psi(v_n)$, then take $v=w_1$ and $w_i=0$ for $i=2, 3, \ldots, m$. Then the result follows. Otherwise, we claim that there is a decomposition of v as $v=w_1+w_2+\ldots+w_m$ such that

$$\sum_{j=1}^{m}\psi(w_j)\leqslant\sum_{i}\psi(v_i) \quad\text{and}\quad \psi(w_i+w_j)>\psi(w_i)+\psi(w_j) \quad \text{for all } i, j=1, 2, \ldots, m \text{ and } i\neq j. \tag{A}$$

If $v=\sum v_i$ itself has the property (A) we keep it. Otherwise, there exist i, j such that $\psi(v_i+v_j)\leqslant\psi(v_i)+\psi(v_j)$.

Then

$$\begin{aligned} v &= (v_i+v_j)+v_1+\ldots+v_{i-1}+\ldots+v_{j-1}+v_{j+1}+\ldots+v_n \\ &= v_1'+v_2'+\ldots+v_{n-1}'\,. \qquad \text{Here } v_1'=v_i+v_j \text{ etc.} \end{aligned}$$

In this decomposition $\psi(v_1')+\ldots+\psi(v_{n-1}')\leqslant\sum_{i=1}^{n}\psi(v_1)$. If this decomposition of v as a sum of v_i' satisfies (A) we are done. In the contrary case we shall continue with this process till we get the desired type of decomposition. Note that $m \geqslant 2$. Arrange the w_j such that $\psi(w_1) \leqslant \leqslant \psi(w_2) \leqslant \ldots \leqslant \psi(w_m)$. Then by lemma 3, $e \cdot \psi(w_j) \leqslant \psi(w_{j+1})$ for $j = 1, 2, \ldots, m-1$. We now claim that $\psi(w_1 + \ldots + w_t) \leqslant e \cdot \psi(w_t)$ for all $t = 1$ $2, \ldots, m$. This follows by induction and we get $\psi(v) = \psi(\sum w_j) \leqslant \leqslant e \cdot \psi(w_m)$.

THEOREM 1. *Let V be a vector space over K, (H, G) pair gauge set on V and (ψ, g) be two real valued functions defined as above.*
If we set

$$\omega(a) = \text{Inf.} \sum g(a_i) , \qquad \sum_i a_i = a , \qquad a_i \in K$$

and

$$\|v\| = \text{Inf.} \sum \psi(v_i) , \qquad \sum_j v_j = v , \qquad v_j \in V$$

where the infima are taken over all possible decompositions of $a = \sum a_i$ and $v = \sum v_j$ then ω is a pseudo valuation on K and $\|\ldots\|$ is a pseudo norm on V satisfying $\omega(a) \leqslant g(a) \leqslant e \cdot \omega(a)$ for all a in K, and

$$\|v\| \leqslant \psi(v) \leqslant e \cdot \|v\| \ \text{for all } v \text{ in } V .$$

PROOF. We prove only the pseudo norm part. It is clear that for all v in V, $\|v\| \leqslant \psi(v)$. For the other inequality, take any decomposition of v as $v = v_1 + \ldots + v_n$ and apply lemma 5, to get another decomposition of v as $v = w_1 + \ldots + w_m$ such that $\sum^m \psi(w_j) \leqslant \sum^n \psi(v_i)$ and $\psi(v) \leqslant e \cdot \psi(w_m)$. Now $\psi(v) \leqslant e \cdot \psi(w_m) \leqslant e \cdot \sum_{j=1} \psi(w_j) \leqslant e \cdot \sum_{i=1} \psi(v_i)$. Taking lower bounds over all possible decompositions of v we get $\psi(v) \leqslant e \cdot \|v\|$. We now verify that $\|\ldots\|$ is a ω-pseudo norm on V. That $\|v\| \geqslant 0$ and $\|\theta\| = 0$ are obvious from the definition. Since ψ is not identically zero on V, we see from the inequality $\psi(v) \leqslant e \cdot \|v\|$ and the fact that $e > 1$, that $\|v\|$ is not identically zero. To show that $\|a \cdot v\| \leqslant \omega(a) \cdot \|v\|$ for all a in K and v in V, we take for any $\varepsilon > 0$, a decomposition of $a = a_1 + a_2 + \ldots + a_n$, $a_i \in K$ and a decomposition of $v = v_1 + \ldots + v_m$ with $v_i \in V$ such that

$$\sum_i \omega(a_i) < \omega(a) + \varepsilon/2(\|v\| + 1)$$

and

$$\sum_j \psi(v_j) < \|v\| + \varepsilon/2(\omega(a)+1)\,.$$

Then $\sum_{i,j} \psi(a_i \cdot v_j) \leqslant \sum_{i,j} g(a_i)\cdot\psi(v_j) < \omega(a)\cdot\|v\| + \varepsilon + \varepsilon^2/4$.

As $a\cdot v = \sum_{i,j} a_i \cdot v_j$, we get $\|a\cdot v\| \leqslant \omega(a)\cdot\|v\|$.

In a similar fashion we can show that $\|v - w\| \leqslant \|v\| + \|w\|$.

5. The topology associated with pair gauge sets.

Given a pair gauge set (H, G) of a vector space V over a field K we can associate a pseudo valuation ω on K and a pseudo norm $\|...\|$ on V. Note that ω and $\|...\|$ still depend upon the choice of the gauge element d. We will now show that ω and $\|...\|$ are independent of d upto equivalence (*).

THEOREM 2. *Let (H, G) be a pair gauge set of a vector space V over K. Suppose $\|...\|$ and ω are respectively the pseudo norm and pseudo valuation on V and K. If d is any gauge element of (H, G), then $\{d^n\cdot H\}$ and $\{d^n\cdot G\}$ as n varies on the natural numbers, form a basis of neighbourhoods of θ in V and of 0 in K in the topology defined by $\|...\|$ and ω.*

PROOF. The proof that $\{d^n G\}$ forms a neighbourhood basis of 0 in K in the topology defined by ω may be found in [1]. For the vector space part we proceed thus. Set $U = \{v \in V \mid \|v\| < 1/e\}$ where $1 < e \leqslant 2$. It is easy to observe that U is a neighbourhood of θ in the topology defined by $\|...\|$. Since $\psi(v) < e\cdot\|v\|$ for all v in V, if we take $u \in U$, then $\psi(u) \leqslant e\cdot\|u\| \leqslant 1$. This means $u \in H$. Thus H itself is a neighbourhood of θ in the topology defined by $\|...\|$. Continuity of scalar multiplication shows that $\{a\cdot H\}$ for all non-zero a in K give neighbourhoods of θ. Therefore, for any choice of gauge element d of (G, H), $\{d^n\cdot H\}$ give neighbourhoods of θ in the topology defined by $\|...\|$ on V.

Let now d_1 be another gauge element of (H, G) used to define ψ on V. Then $d_1^{-1} \in d^{-t}\cdot G$ for some $t \geqslant 0$ so that $d^t \in d_1 G$. If N is any neigh-

(*) Two pseudo valuations on a field K (pseudo norms on a vector space V) are said to be equivalent if they define the same topology on $K(V)$.

bourhood of θ in the topology defined by $\|\ldots\|$ on V, then there exists a real number $\varepsilon > 0$ such that for all elements with $\|v\| < \varepsilon$, $v \in N$. Choose a positive integer n such that $e^{-n} < \varepsilon$. If $v \in d_1^n \cdot H$, then $\|v\| \leqslant \leqslant \psi(v) \leqslant e^{-n} < \varepsilon$. This shows $v \in N$. Thus $d_1^n H \subseteq N$ and hence $d^{tn} \cdot H \subseteq \subseteq N$. Thus $\{d_1^n \cdot H\}$ gives a basis of neighbourhoods at θ of V with respect to the topology defined by $\|\ldots\|$.

6. Characterization of pseudo norm topology on a vector space.

In this concluding section we give a necessary and sufficient condition for a topological vector space, (V, T_2) over a topological field (K, T_1) to have their topologics induced by a pseudo norm $\|\ldots\|$ on V and a pseudo valuation ω on K. We need the notion of a topologically nilpotent element in a topological field.

DEFINITION 5. An element a of a topological field K is said to be *topologically nilpotent* if $a^n \to 0$ in the topology of K.

THEOREM 3. *Let (K, T_1) be a non-discrete topological field and (V, T_2) be a topological vector space over (K, T_1). Then T_1 can be obtained by a non-trivial pseudo valuation ω on K and T_2 by a non-trivial ω-pseudo norm $\|\ldots\|$ on V if and only if*

(i) *K contains a non-empty open bounded set.*

(ii) *There exists a non-zero, topologically nilpotent element in K.*

(iii) *V contains a non-empty open bounded set.*

PROOF. Suppose T_1 is defined by a pseudo valuation ω on K and T_2 by a ω-pseudo norm $\|\ldots\|$ on V. Then if we set

$$A = \{a \in K | \omega(a) < 1\}$$

and

$$B = \{v \in V | \, \|v\| < 1\},$$

then it can be seen without difficulty that A and B are both non-empty open sets. Lemma 2 enables us to check that sets A and B are bounded in K and V respectively. From the non-triviality of ω, we can find a non-zero element a in K such that $\omega(a) < 1$ and this a is a topologically nilpotent element.

For establishing the converse, we adopt the following strategy. We construct from the given conditions a pair gauge set and show that the pseudo norm and pseudo valuation associated with this pair gauge set give rise to the topology T_1 on K and T_2 on V.

Let A be a non-empty bounded open set in K. Then as in [1], we construct a set G and an element $d \neq 0$ in G which is topologically nilpotent such that $\{d^n \cdot G\}$ is a neighbourhood basis of 0 in K with respect to T_1 and satisfy the conditions of a gauge set, namely, $-1 \in G$, $G \cdot G \subset G$; $G \subset K$; $G + G \subset d^{-r} \cdot G$ for some integer $r \geqslant 0$; $K = \bigcup_{1}^{\infty} d^{-n} G$.

Next we shall construct H. Let B be the given open bounded set in V. Then without loss of generality we may assume that B is a symmetric open bounded set containing θ. Thus B is a neighbourhood of θ. The collection $\{b \cdot B\}$ as b varies over the non-zero elements of K gives rise to a neighbourhood base at θ. For, if N is any neighbourhood of θ, from the boundedness of B, we can find a neighbourhood U of 0 in K such that $U \cdot B \subset N$. As T_1 is not discrete, we can find a $b \in U$, $b \neq 0$ such that $b \cdot B \subset N$.

Using continuity of scalar multiplication and the fact that B is a neighbourhood of θ we find that $(d^s \cdot G)$. $b' \cdot B \subseteq B$ for some non negative integer s and a b' in K, $b' \neq 0$. As $d^s \cdot G$ is a neighbourhood of 0 in K, so is $b' \cdot d^s \cdot G$. Hence we can find an integer $t \geqslant 0$ such that $d^t G \subseteq$ $\subseteq b' \cdot d^s \cdot G$. Therefore $d^t \cdot G \cdot B \subseteq B$.

Now set $A_1 = d^t B$. Then clearly A_1 is a symmetric, bounded open set containing θ. Consequently, $\{b \cdot A_1\}$ as b varies over the non-zero elements of K forms a neighbourhood basis of θ.

Finally define $H = \{v \in V | d^t \cdot Gv \subseteq A_1\}$. This H will serve our purpose. It is easily observed that H is an open, bounded, symmetric neighbourhood of θ. We assert that $\{d^n H\}$ forms a basis of neighbourhoods at θ. For, if N is any neighbourhood of θ, then we can find a non-zero b in K such that $b \cdot A_1 \subseteq N$. As $d^n \to 0$, $d^n \cdot b^{-1}$ also tends to zero.

Therefore, $d^n \cdot b^{-1}$ belongs to $d^t \cdot G$ for some intger t. Thus $d^n \in b \cdot$ $\cdot d^t \cdot G$. Hence $d^n \cdot H \subseteq (b \cdot d^t \cdot G) H \subseteq b(d^t \cdot GH) = bA_1 \subseteq N$.

We now show that (H, G) is a pair gauge set with d as the gauge element. Symmetry of H gives $-H = H$, and from the definition of H, $G \cdot H \subseteq H$. We have already noted that $-1 \in G$ and $G \cdot G \subseteq G$.

From the continuity of addition in V and the fact that $\{d^n \cdot H\}$ is a basis of neighbourhood of θ, we get $H + H \subseteq d^{-p} H$ for some integer $p \geqslant 0$.

Already we have seen that $G + G \subseteq d^{-r} G$. Taking $c = d^{-\max(r,p)}$, we get $H + H \subseteq c \cdot H$ and $G + G \subseteq c \cdot G$.

Now $H \neq V$ since H is bounded while V is not. Also $G \neq K$ as noted above. For any $v \in V$, $d^n v \to \theta$ as $n \to \infty$ since d is topologically nilpotent. Therefore, $d^n \cdot v \in H$ for some integer $n \geqslant 1$. Thus $v \in d^{-n} H$. Hence $V \subseteq \bigcup_{n=1}^{\infty} d^{-n} H$ and K is contained in $\bigcup d^{-n} \cdot G$. Thus (G, H) is a pair gauge set. The rest of the proof follows from theorem 2.

We conclude this note with an example of a topological vector space (V, τ) over a pseudo valuated field (K, ω) such that τ is not given by any ω-pseudo norm.

EXAMPLE 6. Let $K = R$ be the field of real numbers and $\omega(a) = c \cdot |a|$ where $|a|$ is the ordinary absolute value of a and c is a real number such that $1 \leqslant c \leqslant 2$. Clearly ω induces a pseudo valuation on K. For the vector space V we take the ring of polynomials in one variable x over K and give a topology τ for V as follows.

Set $\Omega = \{\alpha | \alpha = (\alpha_1, \alpha_2, \ldots)$ sequence of positive real numbers$\}$. For each α in Ω, define

$$N_\alpha = \left\{ f(x) = \sum_{i=1}^{n} a_i x^i \,|\, \text{ such that } \omega(a_i) < \alpha_i \text{ for all } i \right\}$$

It can be verified without difficulty that the set $\mathfrak{N} = \{N_\alpha\}$, gives rise to a neighbourhood basis for 0 in V. If τ is the topology given by $\mathfrak{N}$ then (V, τ) is a topological vector space over (K, ω). We now show that given any N_α in $\mathfrak{N}$, we can find a N_β such that for all non-zero a in K, $a \cdot N_\alpha \not\subset N_\beta$. For this, if $\alpha = (\alpha_1, \alpha_2, \ldots)$ then take $\beta = (\beta_1, \beta_2, \ldots)$ where $\beta_n = 2^n \cdot \alpha_n$ for each n. Suppose b is a non-zero element of K. Then there exists an integer m such that $\omega(b) > (\frac{1}{2})^{m-1}$. The polynomial $(\beta_m/2) x^m$ belongs to N_β, while $b \cdot \beta_m/2 \cdot x$ does not belong to N_α.

For,

$$\omega\left(\frac{b\beta_m}{2}\right) = c \cdot \left|\frac{b\beta_m}{2}\right| > \frac{1}{2}\left(\frac{1}{2}\right)^{m-1} \cdot 2^m \cdot \alpha_m = \alpha_m .$$

Thus $b \cdot N_\alpha \not\subset N_\beta$. From this one can easily infer that there does not exist a non-empty open bounded subset of V. Therefore, in view of Theorem 3, the topology τ for V can not be induced by a ω-pseudo norm.

REFERENCES

[1] P. M. Cohn, *An invariant characterization of pseudo valuation on a field*, Cambridge Phil. Soc., **50** (1954), pp. 159-177.

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