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SOME EXPANSIONS ASSOCIATED WITH BESSEL AND HYPERGEOMETRIC FUNCTIONS

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1. In a recent paper we gave the expansion [4, (2.2)]

$$(1.1)\qquad \left(\frac{1}{2}z\right)^{\lambda-\mu-\nu} J_\mu(az)J_\nu(bz) = \frac{a^\mu b^\nu e^z \Gamma(\lambda)}{\Gamma(2\lambda)\Gamma(\mu+1)\Gamma(\nu+1)} \sum_{n=0}^{\infty}(-)^n \cdot$$

$$\cdot \frac{(\lambda+n)\Gamma(2\lambda+n)}{n!} I_{\lambda+n}(z)\, F\left[\begin{matrix} -n,\ 2\lambda+n : -;\ -; \\ \lambda+\frac{1}{2} : \mu+1;\ \nu+1; \end{matrix}\ -\frac{1}{8}a^2z,\ -\frac{1}{8}b^2z\right],$$

where the notation for the double hypergeometric function is due to Burchnall and Chaundy [1, pp. 112, 113] in preference to the one introduced earlier by Kampé de Fériet.

Contemplation of this result leads to a more general expansion in product of Bessel and generalised hypergeometric functions. The formula is

$$(1.2)\qquad \left(\frac{1}{2}z\right)^{\lambda-\mu} J_\mu(az)\, {}_pF_q\left[\begin{matrix}\alpha_1, \dots, \alpha_p; \\ \varrho_1, \dots, \varrho_q;\end{matrix}\ -\frac{1}{4}b^2z^2\right] = \frac{a^\mu e^z \Gamma(\lambda)}{\Gamma(2\lambda)\Gamma(\mu+1)} \sum_{n=0}^{\infty}(-)^n \cdot$$

$$\cdot \frac{(\lambda+n)\Gamma(2\lambda+n)}{n!} I_{\lambda+n}(z) F\left[\begin{matrix} -n,\ 2\lambda+n : -;\ \alpha_1, \dots, \alpha_p; \\ \lambda+\frac{1}{2} : \mu+1;\ \varrho_1, \dots, \varrho_q; \end{matrix}\ -\frac{1}{8}a^2z,\ -\frac{1}{8}b^2z\right],$$

and it is easy to see that (1.2) reduces to (1.1) when $p = q - 1 = 0$ and $\varrho_1 = \nu + 1$.

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To prove (1.2) we expand the functions on the left in ascending powers of z and use the formula [3, p. 25]

$$\left(\frac{1}{2}z\right)^{\mu+1}=\frac{\Gamma(\mu+1)}{\Gamma(2\mu+2)}e^{z}\sum_{m=0}^{\infty}(-)^{m}\frac{(\mu+m+1)\Gamma(2\mu+m+2)}{m!}I_{\mu+m+1}(z), \tag{1.3}$$

μ not a negative integer. We thus see that

$$\left(\frac{1}{2}z\right)^{\lambda-\mu}a^{-\mu}e^{-z}J_{\mu}(az)\,{}_{p}F_{q}\left[\begin{matrix}\alpha_1,\dots,\alpha_p;\\ \varrho_1,\dots,\varrho_q;\end{matrix}\ -\frac{1}{4}b^2z^2\right]=$$

$$=\sum_{r,s=0}^{\infty}\frac{(-)^{r+s}e^{-z}\left(\frac{1}{2}z\right)^{\lambda+2r+2s}(\alpha_1)_s\dots(\alpha_p)_s}{r!\,s!\,\Gamma(\mu+r+1)(\varrho_1)_s\dots\dots(\varrho_q)_s}a^{2r}b^{2s}=\sum_{r,s=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-)^{r+s+m}}{r!\,s!\,m!}\left(\frac{1}{2}a^2z\right)^{r}\cdot$$

$$\cdot\left(\frac{1}{2}b^2z\right)^{s}\frac{\Gamma(\lambda+r+s)(\lambda+r+s+m)\Gamma(2\lambda+2r+2s+m)(\alpha_1)_s\dots(\alpha_p)_s}{\Gamma(\mu+r+1)\ \Gamma(2\lambda+2r+2s)\ (\varrho_1)_s\dots\dots(\varrho_q)_s}I_{\lambda+r+s+m}(z)=$$

$$=\frac{\Gamma(\lambda)}{\Gamma(2\lambda)\Gamma(\mu+1)}\sum_{n=0}^{\infty}(-)^{n}\frac{(\lambda+n)\Gamma(2\lambda+n)}{n!}I_{\lambda+n}(z)\ \cdot \tag{1.4}$$

$$\cdot\sum_{r+s\leqslant n}(-)^{r+s}\frac{(-n)_{r+s}(2\lambda+n)_{r+s}(\alpha_1)_s\dots(\alpha_p)_s}{r!\,s!\left(\lambda+\frac{1}{2}\right)_{r+s}(\mu+1)_r(\varrho_1)_s\dots(\varrho_q)_s}\left(\frac{1}{8}a^2z\right)^{r}\left(\frac{1}{8}b^2z\right)^{s},$$

on setting $m=n-r-s$, and this proves the result.

For $a=b$ the inner sum in (1.4) is equal to

$$\sum_{k=0}^{n}\frac{(-n)_k(2\lambda+n)_k}{\left(\lambda+\frac{1}{2}\right)_k}\left(-\frac{1}{8}a^2z\right)^{k}\sum_{r+s=k}\frac{(\alpha_1)_s\dots\dots(\alpha_p)_s}{r!\,s!\,(\mu+1)_r(\varrho_1)_s\dots(\varrho_q)_s}\ . \tag{1.5}$$

Since

$$\sum_{r+s=k}\frac{(\alpha_1)_s\dots\dots(\alpha_p)_s}{r!\,s!\,(\mu+1)_r(\varrho_1)_s\dots(\varrho_q)_s}=$$

$$=\frac{1}{k!\,(\mu+1)_k}\,{}_{p+2}F_{q}\left[\begin{matrix}-k,\ -\mu-k,\ \alpha_1,\dots,\alpha_p;\\ \varrho_1,\dots\dots,\varrho_q;\end{matrix}\ 1\right]=$$

$$=\frac{(\mu+\nu+1)_{2k}}{k!\,(\mu+1)_k(\nu+1)_k(\mu+\nu+1)_k},$$

when $p = q - 1 = 0$ and $\varrho_1 = \nu + 1$, (1.5) becomes

$$\sum_{k=0}^{n} \frac{(-n)_k(2\lambda + n)_k(\mu + \nu + 1)_{2k}}{k!\left(\lambda + \frac{1}{2}\right)_k (\mu + 1)_k(\nu + 1)_k(\mu + \nu + 1)_k} \left(-\frac{1}{8} a^2 z\right)^k,$$

so that (1.2) reduces to

$$\text{(1.6)} \qquad \left(\frac{1}{2} z\right)^{\lambda-\mu-\nu} J_\mu(az)J_\nu(az) =$$

$$= \frac{a^{\mu+\nu} e^z \Gamma(\lambda)}{\Gamma(2\lambda)\Gamma(\mu + 1)\Gamma(\nu + 1)} \sum_{n=0}^{\infty} (-)^n \frac{(\lambda + n)\Gamma(2\lambda + n)}{n!} I_{\lambda+n}(z) \cdot$$

$$\cdot {}_4F_4\left[\begin{matrix} -n,\ 2\lambda + n,\ \frac{1}{2}(\mu + \nu + 1),\ \frac{1}{2}(\mu + \nu + 2); \\ \lambda + \frac{1}{2},\ \mu + 1,\ \nu + 1,\ \mu + \nu + 1; \end{matrix} \ -\frac{1}{2} a^2 z\right].$$

The special case $\nu = \mu + 1$ of (1.6) is worthy of note, since we then have

$$\text{(1.7)} \qquad \left(\frac{1}{2} z\right)^{\lambda-2\mu-1} J_\mu(az)J_{\mu+1}(az) =$$

$$= \frac{\Gamma(\lambda)a^{2\mu+1}e^z}{\Gamma(2\lambda)\Gamma(\mu + 1)\Gamma(\mu + 2)} \sum_{n=0}^{\infty} (-)^n \frac{(\lambda + n)\Gamma(2\lambda + n)}{n!} I_{\lambda+n}(z) \cdot$$

$$\cdot {}_3F_3\left[\begin{matrix} -n,\ 2\lambda + n,\ \mu + \frac{3}{2}; \\ \lambda + \frac{1}{2},\ \mu + 2,\ 2\mu + 2; \end{matrix} \ -\frac{1}{2} a^2 z\right],$$

and this yields the elegant formula

$$\text{(1.8)} \qquad \frac{\sin 2az}{2\pi a} = \left(\frac{2z}{\pi}\right)^{1/2} e^z \sum_{n=0}^{\infty} (-)^n \left(n + \frac{1}{2}\right) I_{n+1/2}(z) \cdot$$

$$\cdot {}_2F_2\left(-n, n + 1; 1, \frac{3}{2}; -\frac{1}{2} a^2 z\right),$$

when $\lambda = -\mu = \frac{1}{2}$.

2. We now give two more expansions which are similar to (1.2), namely

(2.1) $$\left(\frac{1}{2}z\right)^{\lambda-\mu} J_\mu(az)\,{}_pF_q\left[\begin{matrix}\alpha_1, \dots, \alpha_p; \\ \varrho_1, \dots, \varrho_q;\end{matrix} -\frac{1}{4}b^2z^2\right] =$$

$$= \frac{a^\mu}{\Gamma(\mu+1)} \sum_{n=0}^{\infty} \frac{(\lambda+2n)\Gamma(\lambda+n)}{n!} J_{\lambda+2n}(z) \cdot$$

$$\cdot F\left[\begin{matrix}-n, \lambda+n : -; & \alpha_1, \dots, \alpha_p; \\ \mu+1; & \varrho_1, \dots, \varrho_q;\end{matrix} a^2, b^2\right]$$

and

(2.2) $$\left(\frac{1}{2}z\right)^{\lambda-\mu} J_\mu(az)\,{}_pF_q\left[\begin{matrix}\alpha_1, \dots, \alpha_p; \\ \varrho_1, \dots, \varrho_q;\end{matrix} -\frac{1}{4}b^2z^2\right] =$$

$$= \frac{a^\mu\,\Gamma(\lambda+1)}{\Gamma(\mu+1)} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}z\right)^n}{n!} J_{\lambda+n}(z)\, F\left[\begin{matrix}-n, \lambda+1 : -; & \alpha_1, \dots, \alpha_p; \\ \mu+1; & \varrho_1, \dots, \varrho_q;\end{matrix} a^2, b^2\right].$$

To prove (2.1) we expand the first member in ascending powers of z and use Neumann expansion [5, p. 138]. The formula (2.2) can similarly be proved by using the expansion [5, p. 141]

$$\left(\frac{1}{2}z\right)^\nu = \Gamma(\nu+1) \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}z\right)^m}{m!} J_{\nu+m}(z).$$

When $p = q - 1 = 0$, the double hypergeometric function in (2.1) reduces to Appell's function F_4 which can be expressed as a product of two ${}_2F_1'$ s if $\lambda = \mu + \varrho_1$, and we have

(2.3) $$\frac{1}{2} z J_\mu(z\cos\varphi\cos\Phi) J_\nu(z\sin\varphi\sin\Phi) =$$

$$= \frac{(\cos\varphi\cos\Phi)^\mu(\sin\varphi\sin\Phi)^\nu}{\Gamma(\mu+1)\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{(\mu+\nu+2n+1)\Gamma(\mu+\nu+n+1)}{n!} J_{\mu+\nu+2n+1}(z) \cdot$$

$$\cdot\ {}_2F_1(-n, \mu+\nu+n+1; \mu+1; \cos^2\varphi)\,{}_2F_1(-n, \mu+\nu+n+1; \nu+1; \sin^2\Phi).$$

Since

$${}_2F_1(-n, \mu+\nu+n+1; \mu+1; \cos^2\varphi) = (-)^n \frac{(\nu+1)_n}{(\mu+1)_n}\,{}_2F_1(-n,$$

$$\mu+\nu+n+1; \nu+1; \sin^2\varphi),$$

(2.3) leads to Bateman's well-known expansion [5, p. 370].

The F_4 obtained in (2.2) when $p = q - 1 = 0$ can be expressed in terms of F_2 if $\lambda = \mu + \varrho_1 - 1$, and we find that

$$(2.4) \qquad J_\mu\,(z \cos\varphi \cos\Phi) J_\nu(z \sin\varphi \sin\Phi) =$$

$$= \binom{\mu+\nu}{\nu} (\cos\varphi \cos\Phi)^\mu (\sin\varphi \sin\Phi)^\nu \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2} z\right)^n}{n\,!} J_{\mu+\nu+n}(z) \cdot$$

$$\cdot F_2(\mu+\nu+1,\ -n,\ -n;\ \mu+1, \nu+1;\ \cos^2\varphi, \sin^2\Phi)\,.$$

Rice [1]) [6, p. 62] proved this formula in a different way.

For $a = b$ and $p = q - 1 = 0$, the formulae (2.1) and (2.2) give

$$(2.5) \quad \left(\frac{1}{2} z\right)^{\lambda-\mu-\nu} J_\mu(az) J_\nu(az) = \frac{a^{\mu+\nu}}{\Gamma(\mu+1)\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{(\lambda+2n)\Gamma(\lambda+n)}{n\,!} J_{\lambda+2n}(z) \cdot$$

$$\cdot\ {}_4F_3 \left[\begin{matrix} -n,\ \lambda+n,\ \frac{1}{2}(\mu+\nu+1),\ \frac{1}{2}(\mu+\nu+2); \\ \mu+1, \qquad \nu+1, \qquad \mu+\nu+1; \end{matrix}\ 4a^2 \right]$$

and

$$(2.6) \qquad \left(\frac{1}{2} z\right)^{\lambda-\mu-\nu} J_\mu(az) J_\nu(az) = \frac{a^{\mu+\nu}\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2} z\right)^n}{n\,!} J_{\lambda+n}(z) \cdot$$

$$\cdot\ {}_4F_3 \left[\begin{matrix} -n,\ \lambda+1,\ \frac{1}{2}(\mu+\nu+1),\ \frac{1}{2}(\mu+\nu+2); \\ \mu+1, \qquad \nu+1, \qquad \mu+\nu+1; \end{matrix}\ 4a^2 \right].$$

When $\frac{1}{2}\lambda = \mu + 1 = \nu$, (2.5) yields

$$(2.7) \quad \frac{1}{2} z J_\mu(az) J_{\mu+1}(az) = \frac{a^{2\mu+1}}{\Gamma(\mu+1)\Gamma(\mu+2)} \sum_{n=0}^{\infty} \frac{(2\mu+2n+2)\Gamma(2\mu+n+2)}{n\,!} \cdot$$

$$\cdot\ J_{2\mu+2n+2}(z)\ {}_3F_2 \left[\begin{matrix} -n,\ 2\mu+n+2,\ \mu+\frac{3}{2}; \\ \mu+2,\ 2\mu+2; \end{matrix}\ 4a^2 \right],$$

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1) Rice omits $n\,!$ in the denominator on the right of (2.4).

and this further reduces to

$$\text{(2.8)}\qquad \frac{\sin az}{\pi a} = \sum_{n=0}^{\infty} \frac{n!}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}+n\right)} (2n+1)J_{2n+1}(z)P_n^{(\frac{1}{2},-\frac{1}{2})}(1-2a^2),$$

if $\mu = -\dfrac{1}{2}$.

The particular case $a = 1$ of the last formula is worthy of note. Thus we have

$$\text{(2.9)}\qquad \sin z = 2\pi \sum_{n=0}^{\infty} \frac{J_{2n+1}(z)}{\Gamma\left(\frac{1}{2}-n\right)\Gamma\left(\frac{1}{2}+n\right)},$$

which is a special case of Jacobi's expansion [5, p. 22 (4)] for $\eta = 2k\pi$, k being zero or any integer. See also [7, p. 430].

In a similar way from (2.6) we obtain

$$\text{(2.10)}\qquad \sin z = -\frac{3}{8}\pi^{1/2} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}z\right)^n}{n!} \frac{\Gamma\left(n-\frac{1}{2}\right)}{\Gamma\left(n+\frac{3}{2}\right)} J_{n+1}(z).$$

3. Finally we make use of the formula [5, p. 151]

$$\left(\frac{1}{2}z\right)^{\mu+\nu} = \frac{\Gamma(\mu+1)\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)} \sum_{m=0}^{\infty} \frac{(\mu+\nu+2m)\Gamma(\mu+\nu+m)}{m!} J_{\mu+m}(z)\, J_{\nu+m}(z),$$

so that

$$\text{(3.1)}\qquad \left(\frac{1}{2}z\right)^{\lambda_1+\lambda_2-\mu} J_\mu(az)\,{}_pF_q\left[\begin{matrix}\alpha_1, \ldots, \alpha_p; \\ \varrho_1, \ldots, \varrho_q;\end{matrix}\; -\frac{1}{4}b^2z^2\right] =$$

$$= a^\mu \frac{\Gamma(\lambda_1+1)\Gamma(\lambda_2+1)}{(\lambda_1+\lambda_2)\Gamma(\mu+1)} \sum_{n=0}^{\infty} \frac{(\lambda_1+\lambda_2+2n)(\lambda_1+\lambda_2)_n}{n!} J_{\lambda_1+n}(z)J_{\lambda_2+n}(z)\,\cdot$$

$$\cdot\, F\left[\begin{matrix}-n,\ \lambda_1+1,\ \lambda_2+1,\ \lambda_1+\lambda_2+n : -; & \alpha_1, \ldots, \alpha_p; \\ \frac{1}{2}(\lambda_1+\lambda_2+1),\ \frac{1}{2}(\lambda_1+\lambda_2+2) : \mu+1; & \varrho_1, \ldots, \varrho_q;\end{matrix}\; \frac{1}{4}a^2,\ \frac{1}{4}b^2\right].$$

If we take $b = a$ the hypergeometric function on the right is equal to

$$\sum_{k=0}^{n} \frac{(-n)_k(\lambda_1+1)_k(\lambda_2+1)_k(\lambda_1+\lambda_2+n)_k}{k!\,(\lambda_1+\lambda_2+1)_{2k}(\mu+1)_k} a^{2k} \cdot$$

$$\cdot\ {}_{p+2}F_q\left[\begin{matrix} -k,\ -\mu-k,\ \alpha_1,\ldots,\alpha_p; \\ \varrho_1,\ldots\ldots,\varrho_q; \end{matrix}\ 1\right],$$

which reduces to

$$\sum_{k=0}^{n} \frac{(-n)_k(\lambda_1+1)_k(\lambda_2+1)_k(\lambda_1+\lambda_2+n)_k(\mu+\nu+1)_{2k}}{k!\,(\lambda_1+\lambda_2+1)_{2k}(\mu+1)_k(\nu+1)_k(\mu+\nu+1)_k} a^{2k},$$

when $p = q - 1 = 0$ and $\varrho_1 = \nu + 1$.

Thus we have [2, p. 135 (4)]

$$(3.2)\qquad \left(\frac{1}{2}z\right)^{\lambda_1+\lambda_2-\mu-\nu} J_\mu(az)J_\nu(az) =$$

$$= \frac{\Gamma(\lambda_1+1)\Gamma(\lambda_2+1)a^{\mu+\nu}}{(\lambda_1+\lambda_2)\Gamma(\mu+1)\,\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{(\lambda_1+\lambda_2+2n)(\lambda_1+\lambda_2)_n}{n!} J_{\lambda_1+n}(z)J_{\lambda_2+n}(z)\cdot$$

$$\cdot\ {}_6F_5\left[\begin{matrix} -n,\ \lambda_1+1,\ \lambda_2+1,\ \lambda_1+\lambda_2+n,\ \frac{1}{2}(\mu+\nu+1),\ \frac{1}{2}(\mu+\nu+2); \\ \frac{1}{2}(\lambda_1+\lambda_2+1),\ \frac{1}{2}(\lambda_1+\lambda_2+2),\ \mu+1,\ \nu+1,\ \mu+\nu+1; \end{matrix}\ a^2\right].$$

By the methods of the preceding sections we can obtain a number of interesting special cases of the formula (3.2). Cf., e. g., [2] and [7].

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