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ON CERTAIN DEFINITE INTEGRALS INVOLVING LEGENDRE'S POLYNOMIALS

Nota () di S. K. CHATTERJEE (a Calcutta)*

1. - The object of the present note is to evaluate certain definite integrals involving Legendre's Polynomials. Using the symbol $\{u(x)\}_r$ for $\frac{d^r u(x)}{dx^r}$, we write

$$(1.1) \quad P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n = \frac{1}{2^n n!} \{ (x^2 - 1)^n \}_n.$$

2. - Let m, n, r and s be four positive integers such that $1 \leq m \leq n$ and $1 \leq r \leq m, 1 \leq s \leq n$.

Then

$$(2.1) \quad \int_{-1}^1 \frac{d^r P_m(x)}{dx^r} \cdot \frac{d^s P_n(x)}{dx^s} dx \\ = \frac{1}{2^{m+n} m! n!} \int_{-1}^1 \{ (x^2 - 1)^m \}_{m+r} \{ (x^2 - 1)^n \}_{n+s} dx.$$

(*) Pervenuta in Redazione il 24 ottobre 1956.

Indirizzo dell'A.: Department of Pure Mathematics, University, Calcutta (India).

3. - CASE I. - If $2 \leq r + s \leq m$ we have

$$\begin{aligned}
 (3.1) \quad & \int_{-1}^1 \{ (x^2 - 1)^m \}_{m+r} \{ (x^2 - 1)^n \}_{n+s} dx \\
 &= \sum_{t=0}^{s-1} (-)^t [\{ (x^2 - 1)^m \}_{m+r+t} \{ (x^2 - 1)^n \}_{n+s-t-1}]_{-1}^1 \\
 &+ (-)^s \int_{-1}^1 \{ (x^2 - 1)^m \}_{m+r+s} \{ (x^2 - 1)^n \}_n dx.
 \end{aligned}$$

The last integral vanishes on account of the orthogonal property of Legendre's Polynomials. If however $r + s \geq m + 1$, the process comes to an end earlier.

CASE II. - Let $m + 1 \leq r + s \leq 2m$. Writing $r + s = m + \delta$ we have $1 \leq \delta \leq m$. In this case

$$\begin{aligned}
 (3.2) \quad & \int_{-1}^1 \{ (x^2 - 1)^m \}_{m+r} \{ (x^2 - 1)^n \}_{n+s} dx \\
 &= \sum_{t=0}^{s-\delta} (-)^t [\{ (x^2 - 1)^m \}_{m+r+t} \{ (x^2 - 1)^n \}_{n+s-t-1}]_{-1}^1.
 \end{aligned}$$

CASE III. - Let $r + s \geq 2m + 1$, where $r \leq m < s \leq n$ writing $r + \Delta = m$, we have $0 \leq \Delta \leq m - 1$.

Here, we have

$$\begin{aligned}
 (3.3) \quad & \int_{-1}^1 \{ (x^2 - 1)^m \}_{m+r} \{ (x^2 - 1)^n \}_{n+s} dx \\
 &= \sum_{t=0}^{\Delta} (-)^t [\{ (x^2 - 1)^m \}_{m+r+t} \{ (x^2 - 1)^n \}_{n+s-t-1}]_{-1}^1.
 \end{aligned}$$

Now, it may be easily shewn that

$$\begin{aligned}
 (3.4) \quad & [\{ (x^2 - 1)^m \}_{m+r+t} \{ (x^2 - 1)^n \}_{n+s-t-1}]_{-1}^1 \\
 &= 2^{m+n-r-s+1} m! n! (r+t)! (s-t-1)! \\
 & \binom{m+r+t}{r+t} \binom{m}{r+t} \binom{n+s-t-1}{s-t-1} \binom{n}{s-t-1}
 \end{aligned}$$

provided $r + t \leq m$ and $s - t - 1 \geq 0$, which is certainly true in the present cases.

4. - It follows, therefore, in case I

$$(4.1) \quad \int_{-1}^1 \frac{d^r P_m(x)}{dx^r} \frac{d^s P_n(x)}{dx^s} dx$$

$$= K \sum_{t=0}^{s-1} (-)^t \left[(r+t)! (s-t-1)! \binom{m+r+t}{r+t} \binom{m}{r+t} \binom{n+s-t-1}{s-t-1} \binom{n}{s-t-1} \right]$$

where

$$K = \frac{1}{2^{r+s-1}} [1 + (-)^{m+n-r-s}].$$

In case II

$$(4.2) \quad \int_{-1}^1 \frac{d^r P_m(x)}{dx^r} \frac{d^s P_n(x)}{dx^s} dx$$

$$= K \sum_{t=0}^{s-s} (-)^t \left[(r+t)! (s-t-1)! \binom{m+r+t}{r+t} \binom{m}{r+t} \binom{n+s-t-1}{s-t-1} \binom{n}{s-t-1} \right]$$

And in case III

$$(4.3) \quad \int_{-1}^1 \frac{d^r P_m(x)}{dx^r} \frac{d^s P_n(x)}{dx^s} dx$$

$$= K \sum_{t=0}^{\Delta} (-)^t \left[(r+t)! (s-t-1)! \binom{m+r+t}{r+t} \binom{m}{r+t} \binom{n+s-t-1}{s-t-1} \binom{n}{s-t-1} \right]$$

If $r = s = 1$ and $m \geq 2$, case I gives the well known result [1] (Clare, 1898)

$$(4.4) \quad \int_{-1}^1 \frac{dP_m(x)}{dx} \frac{dP_n(x)}{dx} dx = \frac{1 + (-)^{m+n}}{2} m(m+1).$$

From the above it is also easy to verify that [2] (Math.

Trip, 1897)

$$(4.5) \quad \int_{-1}^1 \frac{d^2 P_m(x)}{dx^2} \frac{d^2 P_n(x)}{dx^2} dx$$

$$= \frac{(m-1)m(m+1)(m+2)}{48} \{ 3n(n+1) - m(m+1) + 6 \}$$

$$\times \{ 1 + (-1)^{m+n} \}.$$

5. - We next evaluate the integral $\int_{-1}^1 \prod_{r=1}^k \frac{dP_{n_r}(r)}{dx}$ where

n_1, n_2, \dots, n_k are k positive integers such that

$$1 \leq n_1 \leq n_2 \dots \leq n_{k-1} \leq n_k \quad \text{and} \quad n_1 + n_2 + \dots + n_{k-1} < n_k + k.$$

Now,

$$\int_{-1}^1 \prod_{r=1}^k \frac{dP_{n_r}(x)}{dx} dx = L \int_{-1}^1 \prod_{r=1}^k \{ (x^2 - 1)^{n_r} \}_{n_r+1} dx$$

where

$$L = \frac{1}{2^{\sum_{r=1}^k n_r} \prod_{r=1}^k n_r!}.$$

Again

$$(5.1) \quad \int_{-1}^1 \prod_{r=1}^k \{ (x^2 - 1)^{n_r} \}_{n_r+1} dx$$

$$= [\{ (x^2 - 1)^{n_1} \}_{n_1+1} \{ (x^2 - 1)^{n_2} \}_{n_2+1} \dots \{ (x^2 - 1)^{n_{k-1}} \}_{n_{k-1}+1} \{ (x^2 - 1)^{n_k} \}_{n_k}]_{-1}^1$$

$$- \int_{-1}^1 \{ (x^2 - 1)^{n_1} \}_{n_1+2} \{ (x^2 - 1)^{n_2} \}_{n_2+1} \dots \{ (x^2 - 1)^{n_{k-1}} \}_{n_{k-1}+1} \{ (x^2 - 1)^{n_k} \}_{n_k} dx$$

$$- \int_{-1}^1 \{ (x^2 - 1)^{n_1} \}_{n_1+1} \{ (x^2 - 1)^{n_2} \}_{n_2+2} \dots \{ (x^2 - 1)^{n_{k-1}} \}_{n_{k-1}+1} \{ (x^2 - 1)^{n_k} \}_{n_k} dx$$

.

$$- \int_{-1}^1 \{ (x^2 - 1)^{n_1} \}_{n_1+1} \{ (x^2 - 1)^{n_2} \}_{n_2+1} \dots \{ (x^2 - 1)^{n_{k-1}} \}_{n_{k-1}+2} \{ (x^2 - 1)^{n_k} \}_{n_k} dx$$

Each of the $k - 1$ integrals vanishes if $n_1 + n_2 + \dots + n_{k-1} < n_k + k$.

Also

$$\begin{aligned}
 (5.2) \quad & [\{ (x^2 - 1)^{n_1} \}_{n_1+1} \{ (x^2 - 1)^{n_2} \}_{n_2+1} \dots \\
 & \dots \{ (x^2 - 1)^{n_{k-1}} \}_{n_{k-1}+1} \{ (x^2 - 1)^{n_k} \}_{n_k}]^{-1} \\
 & = 2^{\sum_{r=1}^k n_r - (k-1)} [1 - (-1)^{\sum_{r=1}^k n_r - (k-1)}] \cdot F
 \end{aligned}$$

where

$$F = (n_1 + 1)! (n_2 + 1)! \dots (n_{k-1} + 1)! n_k! n_1 n_2 \dots n_{k-1}.$$

So, under the conditions stated above

$$(5.3) \quad \int_{-1}^1 \prod_{r=1}^k \frac{dP_{n_r}(x)}{dx} dx = \frac{[1 - (-1)^{\sum_{r=1}^k n_r - (k-1)}]^{k-1}}{2^{k-1}} \prod_{r=1}^k n_r (n_r + 1).$$

If $k = 2$, we again get the result [1].

$$(5.4) \quad \int_{-1}^1 \frac{dP_m(x)}{dx} \frac{dP_n(x)}{dx} dx = \frac{1 + (-1)^{m+n}}{2} m(m + 1)$$

where $m = n_1$ and $n = n_2$.

The results of this paper are embodied in formulae (4.1), (4.2), (4.3) and 5.3).

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REFERENCES

[1] WHITTAKER E. T. and WATSON G. N. - *Modern Analysis*. 1952, p. 309.
 [2] WHITTAKER E. T. and WATSON G. N. - *Modern Analysis*. 1952, p. 309.