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## On certain definite integrals involving Legendre's polynomials


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# ON CERTAIN DEFINITE INTEGRALS INVOLVING LEGENDRE'S POLYNOMIALS

*Nota* (*) di S. K. CHATTERJEE (*a Calcutta*)

**1.** - The object of the present note is to evaluate certain definite integrals involving Legendre's Polynomials. Using the symbol $\{u(x)\}_r$ for $\dfrac{d^r u(x)}{dx^r}$, we write

$$P_n(x) = \frac{1}{2^n n\,!}\,\frac{d^n}{dx^n}(x^2-1)^n = \frac{1}{2^n n\,!}\,\{(x^2-1)^n\}_n . \tag{1.1}$$

**2.** - Let $m$, $n$, $r$ and $s$ be four positive integers such that $1 \leqq m \leqq n$ and $1 \leqq r \leqq m$, $1 \leqq s \leqq n$.

Then

$$\int_{-1}^{1} \frac{d^r P_m(x)}{dx^r} \cdot \frac{d^s P_n(x)}{dx^s}\, dx \tag{2.1}$$

$$= \frac{1}{2^{m+n} m\,!\; n\,!} \int_{-1}^{1} \{(x^2-1)^m\}_{m+r} \{(x^2-1)^n\}_{n+s}\, dx .$$

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**3.** - CASE I. - If $2 \leqq r+s \leqq m$ we have

$$(3.1) \qquad \int_{-1}^{1} \{(x^2-1)^m\}_{m+r} \{(x^2-1)^n\}_{n+s}\, dx$$

$$= \sum_{t=0}^{s-1} (-)^t [\{(x^2-1)^m\}_{m+r+t} \{(x^2-1)^n\}_{n+s-t-1}]_{-1}^{1}$$

$$+ (-)^s \int_{-1}^{1} \{(x^2-1)^m\}_{m+r+s} \{(x^2-1)^n\}_n\, dx.$$

The last integral vanishes on account of the orthogonal property of Legendre's Polynomials. If however $r+s \geqq m+1$, the process comes to an end earlier.

CASE II. - Let $m+1 \leqq r+s \leqq 2m$. Writing $r+s=m+\delta$ we have $1 \leqq \delta \leqq m$. In this case

$$(3.2) \qquad \int_{-1}^{1} \{(x^2-1)^m\}_{m+r} \{(x^2-1)^n\}_{n+s}\, dx$$

$$= \sum_{t=0}^{s-\delta} (-)^t [\{(x^2-1)^m\}_{m+r+t} \{(x^2-1)^n\}_{n+s-t-1}]_{-1}^{1}.$$

CASE III. - Let $r+s \geqq 2m+1$, where $r \leqq m < s \leqq n$ writing $r+\Delta=m$, we have $0 \leqq \Delta \leqq m-1$.

Here, we have

$$(3.3) \qquad \int_{-1}^{1} \{(x^2-1)^m\}_{m+r} \{(x^2-1)^n\}_{n+s}\, dx$$

$$= \sum_{t=0}^{\Delta} (-)^t [\{(x^2-1)^m\}_{m+r+t} \{(x^2-1)^n\}_{n+s-t-1}]_{-1}^{1}.$$

Now, it may be easily shewn that

$$(3.4) \qquad [\{(x^2-1)^m\}_{m+r+t} \{(x^2-1)^n\}_{n+s-t-1}]_{-1}^{1}$$

$$= 2^{m+n-r-s+1}\, m!\, n!\, (r+t)!\, (s-t-1)!$$

$$\binom{m+r+t}{r+t}\binom{m}{r+t}\binom{n+s-t-1}{s-t-1}\binom{n}{s-t-1}$$

provided $r+t \leqq m$ and $s-t-1 \geqq 0$, which is certainly true in the present cases.

4. - It follows, therefore, in case I

$$\int_{-1}^{1} \frac{d^r P_m(x)}{dx^r} \frac{d^s P_n(x)}{dx^s} dx \tag{4.1}$$

$$= K \sum_{t=0}^{s-1} (-)^t \left[ (r+t)!\,(s-t-1)! \binom{m+r+t}{r+t} \binom{m}{r+t} \binom{n+s-t-1}{s-t-1} \binom{n}{s-t-1} \right]$$

where

$$K = \frac{1}{2^{r+s-1}} [1 + (-)^{m+n-r-s}].$$

In case II

$$\int_{-1}^{1} \frac{d^r P_m(x)}{dx^r} \frac{d^s P_n(x)}{dx^s} dx \tag{4.2}$$

$$= K \sum_{t=0}^{s-\delta} (-)^t \left[ (r+t)!\,(s-t-1)! \binom{m+r+t}{r+t} \binom{m}{r+t} \binom{n+s-t-1}{s-t-1} \binom{n}{s-t-1} \right]$$

And in case III

$$\int_{-1}^{1} \frac{d^r P_m(x)}{dx^r} \frac{d^s P_n(x)}{dx^s} dx \tag{4.3}$$

$$= K \sum_{t=0}^{\Delta} (-)^t \left[ (r+t)!\,(s-t-1)! \binom{m+r+t}{r+t} \binom{m}{r+t} \binom{n+s-t-1}{s-t-1} \binom{n}{s-t-1} \right]$$

If $r = s = 1$ and $m \geqq 2$, case I gives the well known result [1] (Clare, 1898)

$$\int_{-1}^{1} \frac{dP_m(x)}{dx} \frac{dP_n(x)}{dx} dx = \frac{1 + (-)^{m+n}}{2} m(m+1). \tag{4.4}$$

From the above it is also easy to verify that [2] (Math.

Trip, 1897)

(4.5)
$$\int_{-1}^{1} \frac{d^2P_m(x)}{dx^2}\frac{d^2P_n(x)}{dx^2}\,dx$$
$$= \frac{(m-1)m(m+1)(m+2)}{48}\{3n(n+1)-m(m+1)+6\} \times \{1+(-)^{m+n}\}.$$

**5.** - We next evaluate the integral $\int_{-1}^{1}\prod_{r=1}^{k}\frac{dP_{n_r}(r)\,dx}{dx}$ where $n_1, n_2, \dots, n_k$ are $k$ *positive integers such that*

$1 \leqq n_1 \leqq n_2 \dots \leqq n_{k-1} \leqq n_k$ and $n_1 + n_2 + \dots n_{k-1} < n_k + k$.

Now,
$$\int_{-1}^{1}\prod_{r=1}^{k}\frac{dP_{n_r}(x)}{dx}\,dx = L\int_{-1}^{1}\prod_{r=1}^{k}\{(x^2-1)^{n_r}\}_{n_r+1}\,dx$$

where
$$L = \frac{1}{2^{\sum_{r=1}^{k} n_r}\prod_{r=1}^{k} n_r!}.$$

Again

(5.1)
$$\int_{-1}^{1}\prod_{r=1}^{k}\{(x^2-1)^{n_r}\}_{n_r+1}\,dx$$
$$=[\{(x^2-1)^{n_1}\}_{n_1+1}\{(x^2-1)^{n_2}\}_{n_2+1}\dots\{(x^2-1)^{n_{k-1}}\}_{n_{k-1}+1}\{(x^2-1)^{n_k}\}_{n_k}]_{-1}^{1}$$
$$-\int_{-1}^{1}\{(x^2-1)^{n_1}\}_{n_1+2}\{(x^2-1)^{n_2}\}_{n_2+1}\dots\{(x^2-1)^{n_{k-1}}\}_{n_{k-1}+1}\{x^2-1)^{n_k}\}_{n_k}\,dx$$
$$-\int_{-1}^{1}\{(x^2-1)^{n_1}\}_{n_1+1}(x^2-1)^{n_2}\}_{n_2+2}\dots\{(x^2-1)^{n_{k-1}}\}_{n_{k-1}+1}\{(x^2-1)^{n_k}\}_{n_k}\,dx$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$-\int_{-1}^{1}\{(x^2-1)^{n_1}\}_{n_1+1}\{(x^2-1)^{n_2}\}_{n_2+1}\dots\{(x^2-1)^{n_{k-1}}\}_{n_{k-1}+2}\{(x^2-1)^{n_k}\}_{n_k}\,dx$$

Each of the $k-1$ integrals vanishes if $n_1 + n_2 + \dots +$ $+ n_{k-1} < n_k + k$.

Also

$$(5.2) \qquad [\{(x^2-1)^{n_1}\}_{n_1+1} \{(x^2-1)^{n_2}\}_{n_2+1} \dots$$
$$\dots \{(x^2-1)^{n_{k-1}}\}_{n_{k-1}+1} \{(x^2-1)^{n_k}\}_{n_k}]_{-}^{1}$$
$$= 2^{\sum_{r=1}^{k} n_r - (k-1)} [1-(-)^{\sum_{r=1}^{k} n_r - (k-1)}] \cdot F$$

where

$$F = (n_1+1)!\,(n_2+1)! \dots (n_{k-1}+1)!\, n_k!\, n_1 n_2 \dots n_{k-1}.$$

So, under the conditions stated above

$$(5.3) \qquad \int_{-1}^{1} \prod_{r=1}^{k} \frac{dP_{n_r}(x)}{dx}\, dx = \frac{[1-(-)^{\sum_{r=1}^{k} n_r - (k-1)}]}{2^{k-1}} \prod_{r=1}^{k-1} n_r(n_r+1).$$

If $k = 2$, we again get the result [1].

$$(5.4) \qquad \int_{-1}^{1} \frac{dP_m(x)}{dx} \frac{dP_n(x)}{dx}\, dx = \frac{1+(-)^{m+n}}{2}\, m(m+1)$$

where $m = n_1$ and $n = n_2$.

The results of this paper are embodied in formulae (4,1), (4.2), (4.3) and 5.3).

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## REFERENCES

[1] WHITTAKER E. T. and WATSON G. N. - *Modern Analysis*. 1952, p. 309.

[2] WHITTAKER E. T. and WATSON G. N. - *Modern Analysis*. 1952, p. 309.