Recherche Coopérative sur Programme n° 25

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Number of Phases in One Component Ferromagnets

<http://www.numdam.org/item?id=RCP25_1978__26__31_0>

NUMBER OF PHASES IN ONE COMPONENT FERROMAGNETS[*+Δ]

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Abstract

Using a new inequality, derived here, we obtain information about the number of pure phases which can coexist in one component spin system with (many body) ferromagnetic interactions. This extends previous results [1] for spin-$\frac{1}{2}$ Ising systems to continuous spin systems.

1. Introduction

As is well known it follows from the general formalism of statistical mechanics that phase transitions, e.g. the coexistence of two phases in equilibrium or the non-analytic behavior of the free energy as a function of temperature or magnetic field, can occur strictly only in infinite systems - the proper mathematical idealization of macroscopic systems which are described thermodynamically by intensive variables [2,3]. The microscopic correlations in such a system are described by Gibbs states which are probability measures on the phase space of the system satisfying the DLR equations [3,4,5]. These states are the appropriate limits of finite volume Gibbs ensembles. Equivalently one may describe the state of the infinite system by means of correlation functions. The latter are obtained as infinite volume limits of the equilibrium correlations in a finite system with specified "boundary conditions". A pure thermodynamic phase then corresponds (loosely speaking) to a translation invariant Gibbs state μ ($\mu\epsilon I$) with correlation functions which "cluster" at infinity, i.e. correlations between different local regions of the system decay (however weakly) as the distance between these regions becomes larger and larger [3]. The latter condition is equivalent to the requirement that intensive variables be well defined, i.e. that fluctuations in "all"

intensive variables, local functions averaged over the volume of the
system, vanish as the volume tends to infinity. The coexistence of
several phases then corresponds to the existence, for a given inter-
action, temperature and magnetic field, of more than one translation
invariant solution of the DLR equations. This is the same as the
possibility of obtaining different translation invariant infinite
volume limits of the Gibbs measure (or the correlation functions) from
different boundary conditions. These states have also been shown to
be (in many cases) the solution of a variational principle minimizing
the infinite volume free energy density [3]. The latter states are
sometimes called "equilibrium states" E (E $\subset$ I).

By a very general theory [3-5] it is always possible to decompose
any Gibbs state <u>uniquely</u> into "extremal" Gibbs states; the translation
invariant extremal states corresponding to the pure phases. This means
the following: given any "observable" f then its expectation value
<f> in any I. equilibrium state can be written in the form
$<f> = \sum_{k=1}^{n} \alpha_k <f>_k$ where $<f>_k$ is the expectation value of <f> in the
<u>kth</u> pure phase, $0 \leq \alpha_k \leq 1$, and $\sum_{k=1}^{n} \alpha_k = 1$, i.e. α_k measures the fraction
of volume occupied by the <u>kth</u> phase. The crucial point here is that
the α_k are independent of the observable f: n thus clearly
represents the total number of phases which can coexist (at a given
temperature and magnetic field) and the question then is to determine
n. (The Gibbs phase rule states that for an m-component fluid $n \leq m+2$,
but this is far from proven and does not apply to spin systems with
general interactions [3,6].)

This lecture is devoted mainly to the description of some new
results regarding the number of possible phases in one component spin
system with ferromagnetic interactions. We consider first the
case of spin $\frac{1}{2}$ Ising systems. These are the simplest non-trivial
systems for which such results can be derived in a mathematically
rigorous way. The main new result is that for such a system with even
spin interactions (pair, quadruple,etc.) there can coexist, at zero
magnetic field, only two phases (up and down magnetization) at all
temperatures at which the energy is continuous in the temperature
In particular, there are no intervals of temperature, below the
critical temperature T_c, at which three or more phases can coexist.
This extends results previously known only for the two dimensional
spin $\frac{1}{2}$ Ising system with nearest neighbor pair interactions [7] and for
higher dimension spin $\frac{1}{2}$ Ising systems only at low temperatures [8].
We then indicate how similar results can be obtained also
for general, bounded and unbounded, one component spin systems. For

the unbounded case there are still some gaps in the argument relating
invariant Gibbs states to solutions of the variational principle, e.g.
for what class of states are the two equivalent. It appears however
that this is a soluble technical problem and that our results may
be extended also to the field theory case.

The main results are derived in section 3. They are based on a
new inequality for ferromagnetic systems which is derived for spin $\frac{1}{2}$
Ising systems in section 2. Section 4 is devoted to proving a similar
inequality for general spin systems.

2. Inequality

Let Λ be a finite set of $|\Lambda|$ sites, which for later applications
we shall think of as a subset of a regular ν-dimensional lattice, say
$\mathbb{Z}^\nu$. Call $s_i \in R$, $i \in \Lambda$, the spin variable at the site i and define,
$s_A = \prod_{i \in \Lambda} s_i^{\ell_i}$, for $A \subseteq \Lambda$ (with the index i repeated ℓ_i times) $\ell_i \in \mathbb{Z}_+$.
We let $d\rho_i(s_i)$ be the free measure of the spin at the site i and
$\beta H = -\sum_{K \subseteq \Lambda} J_K s_K$ the energy (times the reciprocal temperature) of a spin
configuration in Λ. The Gibbs measure $d\mu(\underline{S}_\Lambda)$, $\underline{S}_\Lambda = \{s_i\}$, $i \in \Lambda$, has
the expectation values for $F(\underline{S}_\Lambda)$.

$$< F >_\mu = Z^{-1} \int F(\underline{S}_\Lambda) \exp [\sum J_K s_K] \prod_{i \in \Lambda} d\rho_i(s_i) \qquad (2.1)$$

We assume that the free measures, ρ_i, are even and have a sufficiently
strong decay as $|s_i| \to \infty$ for all the moments of μ to exist.

We wish to compare these expectations $< >_\mu$ with those obtained
from the Gibbs measure $d\mu'(\underline{S}_\Lambda)$ for a different spin system in Λ -
one having free measures $d\rho_i(s_i)$ and energy $\beta'H' = -\sum J_K' s_K$.

Lemma 1. Let $f_\alpha(s)$, $\alpha = 1, \ldots, n$ be odd monotone non-decreasing
functions of $s \in R$ and let $Q(s,s')$ be a symmetric, even, non-negative
function of s and s', $s' \in R$; $Q(s,s') = Q(s',s) = Q(-s,-s') \geq 0$. Then

$$M_i \equiv \int \int \prod_{\alpha=1}^{n} [f_\alpha(s) - f_\alpha(s')]^{k_\alpha} [f_\alpha(s) + f_\alpha(s')]^{\ell_\alpha} Q(s,s') d\rho_i(s) d\rho_i(s') \geq 0$$
$$(2.2)$$

Proof: Letting $s \leftrightarrow s'$ and $s \leftrightarrow -s$, $s' \leftrightarrow -s'$, shows (remembering
that ρ_i is an even measure) that $M_i = 0$ unless $h = \sum k_\alpha$ and $\ell = \sum \ell_\alpha$
are even integers in which case the integrand is non-negative. This
is very similar to Ginibre's proof of the GKS inequalities [9,10].

Lemma 2. Let $J_K \geq |J_K'|$ and let $f_i(s_i)$ be odd monotone and $g_i(s_i)$ be

either an odd or even bounded functions of s_i, $|g_i(s_i)| \leq \lambda_i$. Define
$f_A(s_A) = \prod_{i \in A} f_i(s_i)$, $g_A(s_A) = \prod_{i \in A} [g_i(s_i)/\lambda_i]$. Then

$$I \equiv \int [1 \pm g_B(s_B) g_B(s_B')][f_A(s_A) - f_A(s_A')] d\mu(\underline{S}_A) d\mu'(\underline{S}_A')$$

$$\equiv <f_A> - <f_A>' \pm [<g_B f_A> <g_B>' - <g_B f_A>' <g_B>] \geq 0.$$

(2.4)

Proof: Noting that $d\mu(\underline{S}_A) d\mu(\underline{S}_A') = \exp [\Sigma (J_K s_K + J_K' s_K')] \prod d\rho_i(s_i) d\rho_i(s_i')/ZZ'$
we put $J_K s_K + J_K' s_K' = \frac{1}{2}[(J_K + J_K')(s_K + s_K') + (J_K - J_K')(s_K - s_K')]$ and expand the
exponential. We then factorize $S_K \pm S_K'$, $f_A(s_A) - f_A(s_A')$, and
$[1 - g_B(s_B) g_B(s_B')]$ into products of terms of the form $(s_i \pm s_i')$,
$(f_i(s_i) \pm f_i(s_i'))$, $[1 \pm g_i(s_i) g_i(s_i')]$; e.g. $f_i(s_i) f_i(s_j) - f_i(s_i') f_i(s_j') =$
$\frac{1}{2}\{[f_i(s_i) + f_i(s_i')][f_j(s_j) - f_j(s_j')] + [f_i(s_i) - f_i(s_i')][f_j(s_j) + f_j(s_j')]\}$.
The final result is that I can be written as a sum of products of terms
of the form M_i in (2.3). By our assumption, $J_K \geq |J_K'|$, all these
terms have positive coefficients. Hence the lemma is proven.

We can rewrite (2.4) in the form

$$<f_A> - <f_A>' \geq |<g_B f_A> <g_B>' - <g_B f_A>' <g_B>| \geq 0.$$

(2.5)

It now follows from (2.5) that

Corollary 3: Let $J_K \geq |J_K'|$, $<f_A> = <f_A>'$ and $<g_B> = <g_B>' \neq 0$, then
$<f_A g_B> = <f_A g_B>'$.

Corollary 3 is particularly useful for the case of spin $\frac{1}{2}$ Ising systems
which correspond to having $d\rho_i(s_i) = \frac{1}{2}\delta(|s_i| - 1)$. Setting
$f_i(s_i) = g_i(s_i) = s_i$ (and writing $s_i \equiv \sigma_i = \pm 1$ to emphasize that we are
dealing with a special case) we may use the following basic group
property for the $\sigma_A = \prod_{i \in A} \sigma_i$, (this is just like s_A with $\ell_i = 1$, $\forall i \in A$,
since $\sigma_i^2 = 1$) $\sigma_A \sigma_B = \sigma_C$ with $C = A \Delta B, A \Delta B$ the symmetric difference
between A, $B \subset \Lambda$. This yields the additional results.

Corollary 4: Let $J_K \geq |J_K'|$. Then $<\sigma_A> = <\sigma_A>'$ and $<\sigma_B> = <\sigma_B>' \neq 0$ imply
$<\sigma_A \sigma_B> = <\sigma_A \sigma_B>'$ for all $A, B \subset \Lambda$.

Corollary 5: Let $J_K \geq |J_K'|$. Then: (1) $<\sigma_i> = <\sigma_i>' \neq 0$ for all the one
site sets $i \in \Lambda$ implies $<\sigma_A> = <\sigma_A>'$ for all $A \subset \Lambda$.
(2) $<\sigma_i \sigma_j> = <\sigma_i \sigma_j>' \neq 0$ for all $i, j \in \Lambda$ implies $<\sigma_E> = <\sigma_E>'$ for all
sets E containing an even number of sites, $|E|$ even.

Proof: By Corollary 3 $<\sigma_i> = <\sigma_i>' \neq 0$ and $<\sigma_j> = <\sigma_j>' \neq 0$ implies
$<\sigma_i \sigma_j> = <\sigma_i \sigma_j>'$. Furthermore since $J_K \geq 0$ it follows from the GKS
inequalities that $<\sigma_A \sigma_B> \geq <\sigma_A> <\sigma_B> \geq 0$. Hence $<\sigma_i \sigma_j> \geq <\sigma_i> <\sigma_j> \geq 0$. The

rest follows by induction. The proof of (2) is similar since $<\sigma_B>=<\sigma_B\sigma_k\sigma_\ell\sigma_k\sigma_\ell>$ for all $B\subset\Lambda$.

The proof of Corollary's 4 and 5 for spins with general measures ρ_i is a bit more complicated. It is postponed to section 4 following the discussion in the next sections of some consequences of these inequalities.

3. Equilibrium States for Spin $\frac{1}{2}$ Systems

We shall now use the inequalities derived in the last section to obtain information about the number of equilibrium states for infinite Ising systems. To do this we assume that the interactions are translation invariant $J_A=\beta\Phi_{A+x}$ where $A+x$ is the set A translated by a lattice vector x. In particular for the one point sets $A=i\epsilon\mathbb{Z}^\nu$, $\beta\Phi_i=h$, the magnetic field (times β) and for $|A|=2$, $J_{\{i,j\}}=\beta\Phi(i-j)$, etc... The energy of a spin configuration σ_Λ in $\Lambda\subset\mathbb{Z}^\nu$ will depend on the specified values of the spins outside Λ, i.e. we consider the spins outside Λ to be fixed and act as boundary conditions for the spins in Λ. A particular boundary condition "b" then corresponds to a lattice spin configuration σ^b such that $\sigma_i=\sigma_i^b$ for $i\epsilon\Lambda_c$. (Generally $\sigma_i^b=\pm1$; $\sigma_i^0=0$ correspond to zero b.c.). We then have, corresponding to Eq. (1),

$$H(\underline{\sigma}_\Lambda;b)=-\sum_{B\ni\{0\}}\sum_x \Phi_B\sigma_{B+x}, \quad \Phi_B\geq0 \tag{3.1}$$

where $\{0\}$ designates the origin and the sum over x goes over all x such that $\{B+x\}\cap\Lambda$ is not empty, i.e., at least some of the sites in $B+x$ are in Λ. We assume from now on that $\Phi_B\geq0$, i.e. positive ferromagnetic interactions. It is then clear that μ_+ corresponding to plus b.c., $\sigma_i^+=1$, 'dominates' all other b.c.. Hence defining $<\sigma_A>(\beta,h;b,\Lambda)$ as the expectation value of σ_A, $A\subset\Lambda$, for the Hamiltonian (3.1) at reciprocal temperature β and magnetic field h we can identify $<\sigma_A>$ of Sec. 2 with $<\sigma_A>(\beta,h;+,\Lambda)$ and $<\sigma_A>'$ with $<\sigma_A>(\beta,h;b,\Lambda)$ for any other boundary condition. (Our notation implies the "physicist" point of view where β and $h=\{\beta\Phi_i\}$ are independent "externally controlled" variables while Φ_K, $|K|\geq2$, are "given" interactions).

It follows from the GKS inequalities [10,11] that

$$\lim_{\Lambda\uparrow\mathbb{Z}} <\sigma_A>(\beta,h;+,\Lambda)=<\sigma_A>(\beta,h;+) \tag{3.2}$$

exist and are translation invariant

$$<\sigma_{A+x}>(\beta,h;+)=<\sigma_A>(\beta,h;+).\qquad(3.3)$$

To avoid unnecessary complications we assume that the interactions are of "finite range", $\Phi_B=0$ unless $B\subset N$, N bounded. The thermodynamic free energy per site, $\psi(\beta,h)=\lim\ \{|\Lambda|^{-1}\ln\ Tr(exp[-\beta H(\underline{\sigma}_\Lambda;b)])\}$ then exists and is independent of b.

We shall write $<\sigma_i>(\beta,h;+)=m(\beta,h;+)$, the magnetization per site with $+$ b.c.. For more general boundary conditions, (including a superposition, with specified weights, of different $\underline{\sigma}^b$) the limit $\Lambda\uparrow\mathbb{Z}^\nu$ might have to be taken along subsequences to obtain infinite volume correlation functions $<\sigma_A>(\beta,h;b)$ which need not, in general, be translation invariant [12]. It is however always possible to average over translations to obtain translation invariant correlation functions. The set of correlations, $<\sigma_A>(\beta,h;b)$, $A\subset\mathbb{Z}^\nu$, obtained from $<\sigma_A>(\beta,h;b,\Lambda)$ as $\Lambda\uparrow\mathbb{Z}$ define an infinite volume Gibbs measure. These measures are identical to the ones which satisfy the DLR equations and the translation invariant ones are identical to the solutions of a variational principle (minimizing the free energy per unit volume) [3-5]. We shall sometimes write $<\sigma_A>_+$ for $\int\sigma_A\mu_+(d\underline{\sigma})$, $\mu_+\in I$ being the measure obtained with $+$ b.c..

These considerations also lead to an identification of the $<\sigma_A>_\mu$, $\mu\in I$, with derivatives of the free energy density $\Psi(\underline{J})$ with respect to $J_A(=\beta\Phi_A)$ [3-5].

$$\Psi(\underline{J})=\lim_{\Lambda\uparrow\mathbb{Z}^\nu}\ |\Lambda|^{-1}\ln\ Z(\underline{J};b,\Lambda),\qquad(3.4)$$

and we have used $\underline{J}$ for the argument of Ψ to emphasize that Ψ can be thought of as a function of "all possible" potentials J_K. $\Psi(\underline{J})$, being a convex function of each J_A, will be differentiable for almost all values of J_A (keeping the other interactions fixed).

We are now ready to state our first theorem about the number of possible equilibrium states.

Theorem 6. Let $\Psi(\beta,h)$ be the infinite volume free energy per site of an Ising spin system with translation invariant interactions; $\Phi_K=\Phi_{K+x}\geq0$, $x\in\mathbb{Z}^\nu$, $\beta\Phi_{\{0\}}=h$. If the derivative of Ψ with respect to h exists (is continuous) and is positive, $\dfrac{\partial\Psi(\beta,h)}{\partial h}>0$, then there is a unique translation invariant Gibbs state. In particular $<\sigma_A>(\beta,h;b)=<\sigma_A>(\beta,h:+)=\partial\Psi/\partial J_A$ for all boundary conditions b.

Proof: Given any $\mu \varepsilon I$, $<\sigma_A>_\mu = \partial \Psi / \partial J_A$, when the latter exists [3-5], and the theorem then follows from Corollary 5 with $<\sigma_i> = \partial \Psi / \partial h$.

Remark: Theorem 6 states that differentiability of Ψ with respect to h implies differentiability of Ψ with respect to all interactions. It thus generalizes to ferromagnetic many spin interactions the results of Lebowitz and Martin-Löf [11] for the case when the interactions are such that the Fortuin, Kasteleyn and Ginibre, (FKG) inequalities hold, e.g. when only pair interactions are present, $\Phi_K = 0$, $|K| > 2$ [13]. In that case however the results are stronger; there is a unique Gibbs state, (and so $I = G$) whenever $\partial \Psi(\beta,h)/\partial h$ exists. For pair interactions this is true for all $h \neq 0$, and is always true at sufficiently high temperatures [2,3].

The positivity requirement on $\partial \Psi / \partial h$ is however not as restrictive as it might appear. First, by GKS, $<\sigma_i>(\beta,h;+) > 0$ if $h > 0$ and hence $\frac{\partial \Psi(\beta,h)}{\partial h} = 0 \Longrightarrow h = 0$. Second, if the interactions are such that $<\sigma_E>(\beta,h=0;+) > 0$ for $|E|$ even, e.g. when the nearest neighbor pair interactions is positive, then it is easy to show [14] that $<\sigma_i>(\beta,0;+) = 0 \Longrightarrow <\sigma_Q>(\beta,0;+) = 0$ for all $|Q|$ odd. This implies, by GKS, that $\Phi_K = 0$ for all $|K|$ odd. These facts in turn imply that the odd correlations vanish for all b.c. since, for $|Q|$ odd,

$$0 = <\sigma_Q>(\beta,0;+) \geq <\sigma_Q>(\beta,0;b) = -<\sigma_Q>(\beta,0;-b) \qquad (3.5)$$

where $-b$ is the b.c. obtained from b by reflection; $\sigma_i^{-b} = -\sigma_i^b$. We are therefore left, when $\partial \Psi(\beta,h)/\partial h = 0$ at $h = 0$, only with the possible nonuniqueness of the even correlation functions. We shall now consider this problem which is also, as we shall see, the central problem when $\partial \Psi(\beta,h)/\partial h$ is discontinuous at $h = 0$ and there are only even interactions, e.g. in the Ising model with ferromagnetic pair interactions.

Definition: We call a (finite) collection of bounded sets $\{K_\alpha\}$, $K_\alpha \ni \{0\}$ all α, generating for the even sets, $\{K_\alpha\} = \tilde{G}$ iff; given any bounded set $E \subset \mathbb{Z}^\nu$, $|E|$ even, we can write $\sigma_E = \prod_{n=1}^m \sigma_{\{K_{\alpha_i} + x_n\}}$, m finite, with $K_{\alpha_i} \varepsilon G$, and x_n a lattice vector (we may have $K_{\alpha_i} = K_{\alpha_j}$).

By the proof of part (2) of Corollary 5, $\tilde{G}$ will be generating iff it generates all the sets consisting of pairs of sites $\{i,j\}$. Letting e_α be the unit vector in the αth direction it is now easy to see that the ν nearest neighbor sets, $K_\alpha = \{0, e_\alpha\}$, $\alpha = 1, \ldots, \nu$ are generating, e.g. the product $(\sigma_0 \sigma_{e_1})(\sigma_{e_1} \sigma_{e_1 + e_2}) = \sigma_0 \sigma_{e_1 + e_2}$ where $e_1 + e_2$

is one of the next nearest neighbor sites of the origin, etc. .

It follows from part (2) of Corollary 5 that if the expectation values of σ_{K_α} in a translation invariant state μ are positive and equal to $<\sigma_{K_\alpha}>(\beta,h;+)>0$, for all $K_\alpha\varepsilon\tilde{G}$, then all the even correlation functions of μ are the same as in the + state. This will be the case for all translation invariant μ whenever Ψ is differentiable with respect to J_{K_α}, for all $K_\alpha\varepsilon K$ and $\partial\Psi/\partial J_{K_\alpha}>0$. We now show that this is equivalent to having $\Psi(\beta,h)$ differentiable with respect to β.

Theorem 7: Let the conditions of theorem 6 hold and let $\phi_{K_\alpha}>0$ for all $K_\alpha\varepsilon\tilde{G}$. If $\partial\Psi(\beta,h)/\partial\beta$ exists, i.e. the energy per site (apart from the magnetic field contribution) is continuous in β, then the expectation value of σ_E, $|E|$ even, is the same in all translation invariant states: $<\sigma_E>_\mu = <\sigma_E>_+$ for $\mu\varepsilon I$.

Proof: By the general arguments [3-5] mentioned earlier $\partial\Psi(\beta,h)/\partial\beta$ continuous implies that for every $\mu\varepsilon I$, $\sum_{K\ni\{0\}}\phi_K<\sigma_K>_+ = \sum_{K\ni\{0\}}\phi_K<\sigma_K>_\mu$.
By (7) $<\sigma_K>_+ \geq <\sigma_K>_\mu$, hence the continuity of $\frac{\partial\Psi(\beta,h)}{\partial\beta}$ implies that $<\sigma_K>_+ = <\sigma_K>_\mu$ for all $\mu\varepsilon I$ and all K such that $\phi_K>0$. In particular $\phi_{K_\alpha}>0$ for all $K_\alpha\varepsilon\tilde{G}$ and by GKS $<\sigma_{K_\alpha}>_+>0$ so part (2) of Corollary 5 implies that $<\sigma_E>_+ = <\sigma_E>_\mu$ for all $|E|$ even.

The interest of theorem 4 lies primiarly in what it tells us about the number of extremal translation invariant Gibbs states for a system with even ferromagnetic interactions, when $h=0$, and $\Psi(\beta,h)$ not differentiable at $h=0$. Since $\Psi(\beta,h)$ is now symmetric (and convex) in h the non differentiability of Ψ at $h=0$ corresponds to the existence of a spontaneous magnetization with [11].

$$\mathbf{m}^*(\beta) = \lim_{h\downarrow 0}\frac{\partial\Psi(\beta,h)}{\partial h} = -\lim_{h\uparrow 0}\frac{\partial\Psi(\beta,h)}{\partial h} = <\sigma_i>(\beta,h=0;+)=$$
$$= -<\sigma_i>(\beta,h=0;-) \qquad (3.6)$$

Here $<\sigma_A>(\beta,h;-)=(-1)^{|A|}<\sigma_A>(\beta,h;+)$ is the expectation of σ_A in the infinite volume Gibbs state μ_- obtained, as $\Lambda\uparrow\mathbb{Z}^\nu$, with "minus" boundary conditions (translation invariance is assured if $h\geq0$). As already mentioned there are cases, i.e. only pair interactions (ferromagnetic), when $h=0$ is the only place where a phase transition is possible. With more general even interactions only the symmetry $h\rightarrow-h$,

is known a priori. In a recent paper [14] we were able, using the GKS inequalities, to obtain some information about the Gibbs states of such a system at $h=0$. The following theorem greatly extends those results.

Theorem 8: Let the condition of theorem 6 hold and let $\Phi_K = 0$ for all $|K|$ odd and $\Phi_{K_\alpha} > 0$ for all $K_\alpha \epsilon \tilde{G}$. If $\partial \Psi(\beta, h=0)/\partial \beta$ exists then there are at most two extremal translation invariant Gibbs states, μ_+ and μ_-. These states coincide if $\partial \Psi(\beta, h)/\partial h$ exists at $h=0$.

Proof: By theorem 7 the differentiability of $\Psi(\beta, 0)$ implies that the $<\sigma_E>_\mu$, $|E|$ even, are the same in all $\mu \epsilon T$. If furthermore $\partial \Psi(\beta, h)/\partial h = 0$, at $h=0$, then by the remarks following theorem 6 the odd correlations vanish for all $\mu \epsilon G$ and the state $\mu \epsilon I$ is then unique. (When the FKG inequalities hold differentiability with respect to h implies differentiability with respect to β). When Ψ is not differentiable at $h=0$, $m^*(\beta)>0$, there are at least two extremal translation invariant Gibbs states, μ_+ and μ_-, [11]. Let $\mu(\beta, 0; b)$ be an invariant state then $\bar{\mu}(\beta, 0; b) \equiv \frac{1}{2}[\mu(\beta, 0; b) + \mu(\beta, 0; -b)]$ is an invariant state in which all the odd correlations vanish by symmetry. Hence $\bar{\mu}(\beta, 0; b) = \frac{1}{2}[\mu_+ + \mu_-]$ which, since invariant Gibbs states form a simplex, i.e. each state has a unique decomposition into extremal states, implies that $\mu(\beta, 0; b) = \gamma \mu_+ + (1-\gamma)\mu_-$, $0 \leq \gamma \leq 1$. This completes the proof. (The last part of the argument, which is also used in refs. [7] and [8], I heard originally from Ruelle).

Remarks: i) It follows [2] from GKS that there exists a unique β_c such that

$$m^*(\beta) = \begin{cases} 0, & \beta < \beta_c \\ >0, & \beta > \beta_c \end{cases}$$

We always have [2,15] $\beta_c \geq \beta_0 > 0$ and for $\nu \geq 2$ (with non-vanishing Φ_K), $\beta_c \leq \beta_p < \infty$ by the Peierls argument (or the more recent method of Fröhlich, Simon and Spencer [16] for $\nu \geq 3$). Using the convexity of $\Psi(\beta, 0)$ it follows from theorem 8 that with the possible exception at a countable number of values of β, there is a unique $\mu \epsilon I$ for $\beta < \beta_c$ and two extremal states $\mu \epsilon I$ for $\beta > \beta_c$. In particular there are no triple or higher order points at $h=0$ when the energy is continuous in β.

ii) The state, at $h=0$, obtained with "zero" (or periodic) b.c. $\mu_0(\mu_p)$ is translation invariant and has vanishing odd correlations

[2,15]. Hence $\mu_0 = \mu_p = \frac{1}{2}[\mu_+ + \mu_-]$. This implies in particular the existence of "long range order" in these states for $\beta > \beta_c$, i.e.

$<\sigma_i \sigma_j>_{\mu_0} \xrightarrow[|i-j| \to \infty]{} [m^*(\beta)]^2 > 0$, for $\beta > \beta_c$. (The converse of this statement, long range order $\Longrightarrow m^*(\beta) > 0$, is also true [2]).

iii) For the two dimensional Ising system with nearest neighbor pair interactions the continuity of $\partial \Psi(\beta, h=0)/\partial \beta$ follows from Onsager's [1,17] exact computation of $\Psi(\beta, 0)$. Hence theorem 6 establishes the existence of exactly two extremal states for all $\beta > \beta_c$, (β_c being here the place where the second derivative of $\Psi(\beta, 0)$ diverges logarithmically [18]). This result for the square lattice was proven earlier, using duality, by Messager and Miracle-Sole [7]. For more general Ising systems with even ferromagnetic interactions this result is known at low temperatures (not all the way to T_c) from the work of Gallavotti and Miracle-Sole and of Slawny [8]. Gallavotti and Miracle-Sole used (for nearest neighbor interactions) a beautiful version of the Peierls argument while Slawny uses the Asano-Ruelle method of locating zeros of the partition function to prove analyticity of $\psi(\underline{J})$ in the even interactions at sufficiently large β. Using the above theorem it is sufficient to establish that $\psi(\beta, 0)$ is C^1. This can be done readily if the correlation function in the + state cluster sufficiently well for $\sum_x [<\sigma_A \sigma_{B+x}>(\beta, 0; +) - <\sigma_A>(\beta, 0; +) <\sigma_{B+x}>(\beta, 0; +)] < \infty$

[18]. The latter can be easily proven for large β by a Peierls type argument [19] which actually establishes exponential clustering.

iv) Theorem 8 can be generalized, in a fairly direct way, using the ideas of Slawny, Gruber and their coworkers to non-even interactions. One then gets a larger number of extremal states: these are related to the group, acting on the spins, which leaves the Hamiltonian invariant.

4. General One Component Spin Systems

In order to generalize theorems 5-8 to arbitrary (rapidly decaying) free measures $\rho_i(ds_i)$ we first need the analog of corollarys 4 and 5 for such ρ_i's. This can be achieved by defining

$$\sigma(s;\lambda) = \begin{cases} s, & |s| \leq \lambda \\ \lambda, & |s| \geq \lambda, \quad 0 < \lambda < \infty. \end{cases} \tag{4.1}$$

and choosing in Lemma 2, $f_i(s_i) = g_i(s_i) = \sigma(s_i, \lambda_i)$. Using (2.5) the analog of Corollary 4 would now be:

Corollary 4': Let $J_K \geq |J'_K|$. Then $<\sigma_A(\underline{s}_A;\lambda_A)> = <\sigma_A(\underline{s}_A;\lambda_A)>'$ and

$<\sigma_B(\underline{s}_B;\lambda_B)> = <\sigma_B(\underline{s}_B;\lambda_B)>'$ implies $<\sigma_A(\underline{s}_A;\lambda_A)\sigma_B(\underline{s}_B;\lambda_B)> = <\sigma_A(\underline{s}_A;\lambda_A)\sigma_B(\underline{s}_B;\lambda_B)>'$

where $\sigma_C(\underline{s}_C;\lambda_C) = \prod_{i\in C}[\sigma(s_i;\lambda_i)]^{\ell_i}$, $\ell_i = 1,2,\ldots$.

To obtain the analog of Corollary 5 we make two remarks: (a) since both $\sigma(s;\lambda)$ and $s - \sigma(s;\lambda)$ are odd monotone increasing functions of ·s it follows from Lemma 2 that $<\sigma(s;\lambda)> \geq <\sigma(s;\lambda)>'$ and $<s-\sigma(s;\lambda)> \geq <s-\sigma(s;\lambda)>'$. Hence $<s_i> = <s_i>'$ implies $<\sigma(s_i;\lambda_i)> = <\sigma(s_i;\lambda_i)>'$ for all $\lambda_i \geq 0$. Similarly $<s_i s_j> = <s_i s_j>'$ implies $<\sigma(s_i;\lambda_i)\sigma(s_j;\lambda_j)> = <\sigma(s_i;\lambda_i)\sigma(s_j;\lambda_j)>'$, etc. Since $[\sigma(s_i;\lambda_i)/\lambda_i]^2 \to 1$ as $\lambda_i \to 0$, and $s_i \neq 0$,

$<\sigma(s_1;\lambda_1)[\sigma(s_2;\lambda_2)]^2\sigma(s_3;\lambda_3)> = <\sigma(s_1;\lambda_1)[\sigma(s_2;\lambda_2)]^2\sigma(s_3;\lambda_3)>'$ for all $\lambda_2 \Longrightarrow <\sigma(s_1;\lambda_1)\sigma(s_3;\lambda_3)> = <\sigma(s_1;\lambda_1)\sigma(s_3;\lambda_3)>'$ (if $\rho_2(ds_2)$ is not concentrated on $s_2 = 0$, which is an irrelevant case).

Permitting now the $\{\lambda_i\}$ in Corollary 4' to vary arbitrarily we obtain

Corollary 5': Let $J_K \geq |J'_K|$. Then: (1) $<s_i> = <s_i>'$ $\forall i \in \Lambda$ implies $\mu(d\underline{S}_\Lambda) = \mu'(d\underline{S}_\Lambda)$. (2) $<s_i s_j> = <s_i s_j>'$ for all $i,j \in \Lambda$ implies that all even moments of μ and μ' are equal.

Let us consider first the case when $\rho(dS)$ has compact support in some interval $[-R,R]$. Identifying $\pm$b.c. with $s_i^b = \pm R$, for $i \in \Lambda_c$, Theorems 6-8, remain unchanged. In the case of unbounded spins however the situation is more complicated. It was shown by Lebowitz and Presulti [20] for the case of pair interactions (but this restriction is unessential) that if one restricts oneself to "regular" Gibbs states - those for which $\mu(d\underline{S}_\Omega)$, the projection of μ on the region $\Omega \subset \mathbb{Z}^\nu$, does not grow "too fast" - then the role of $\mu_\pm$ will be played by states obtained as the infinite volume limit of Gibbs states with b.c. $s_j^b = \pm a \ln|j|$. It seems likely that the translation invariant regular Gibbs states satisfy the variational principle (indeed all the solutions of variational principle may be regular) and theorems 6-8 would then apply to these states. (Indeed, using methods similar to these, Theorem 6 has been extended recently to classical rotators [21]).

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