ESAIM: PS
January 2004, Vol. 8, p. 36–55
DOI: 10.1051/ps:2003018

ESAIM: Probability and Statistics

POINTWISE CONVERGENCE OF BOLTZMANN SOLUTIONS FOR GRAZING COLLISIONS IN A MAXWELL GAS VIA A PROBABILITISTIC INTERPRETATION

Hélène Guérin[1]

Abstract. Using probabilistic tools, this work states a pointwise convergence of function solutions of the 2-dimensional Boltzmann equation to the function solution of the Landau equation for Maxwellian molecules when the collisions become grazing. To this aim, we use the results of Fournier (2000) on the Malliavin calculus for the Boltzmann equation. Moreover, using the particle system introduced by Guérin and Méléard (2003), some simulations of the solution of the Landau equation will be given. This result is original and has not been obtained for the moment by analytical methods.

Mathematics Subject Classification. 60J75, 60H10, 60H07, 82C40.

Received June 4, 2002. Revised November 27, 2002.

1. Introduction

The Boltzmann equation [5, 6] describes the behaviour of particles in a rarefied gas. More precisely, it describes in dimension 2 the behaviour of the density $f(t, v, x)$ of particles having the velocity $v \in \mathbb{R}^2$ at time $t \geq 0$ and at point $x \in \mathbb{R}^2$. We consider in this work the spatially homogenous case, which means that the density does not depend on the position x of particles. In 1936, Landau [19] derived from the Boltzmann equation a new equation called the Fokker-Planck-Landau equation, usually considered as an approximation of the homogeneous Boltzmann equation in the limit of grazing collisions. These equations take the form

$$\frac{\partial f}{\partial t} = Q(f, f) \tag{1.1}$$

where Q is a quadratic operator depending on the nature of the collisions. In this paper, we consider the case of a Maxwell gas in dimension 2. Then the Boltzmann equation writes

$$\frac{\partial f}{\partial t} = Q_B(f, f) \tag{BE}$$

Keywords and phrases. Boltzmann equation without cutoff for a Maxwell gas, Landau equation for a Maxwell gas, nonlinear stochastic differential equations, Malliavin calculus.

[1] Université Paris 10, UFR SEGMI, Modal'X, 200 avenue de la République, 92000 Nanterre, France;
e-mail: `hguerin@ccr.jussieu.fr`

with a collision operator Q_B given by

$$Q_B(f,f)(t,v) = \int_{v_* \in \mathbb{R}^2} \int_{\theta=-\pi}^{\pi} (f(t,v')f(t,v'_*) - f(t,v)f(t,v_*))\beta(\theta)\,\mathrm{d}\theta \mathrm{d}v_*$$

where v, v_* are the pre-collisional velocities and v', v'_* the post-collisional velocities and where the cross-section β is an even positive function from $[-\pi,\pi]\backslash\{0\}$ to $\mathbb{R}^+$ such that $\int_{-\pi}^{\pi} \theta^2 \beta(\theta)\mathrm{d}\theta < \infty$.

The relation between the post-collisional velocities and the pre-collisional velocities in dimension 2 is the following

$$v' = v + A(\theta)(v - v_*) \ ; \ v'_* = v - A(\theta)(v - v_*)$$

with

$$A(\theta) = \frac{1}{2} \begin{pmatrix} \cos\theta - 1 & -\sin\theta \\ \sin\theta & \cos\theta - 1 \end{pmatrix}.$$

We are interested in cases for which the molecules in the gas interact according to an inverse power law in $\frac{1}{d^s}$ with $s \geq 2$, where d is the distance between particles. Consequently, the function β has a singularity in 0 of the form $\beta(\theta) \underset{\theta\to 0}{\sim} C\theta^{-\frac{s+1}{s-1}}$, with C a positive constant. We assume that

Assumption (A): β is an even positive function on $[-\pi,\pi]\backslash\{0\}$ of the form $\beta = \beta_0 + \beta_1$ such that

1) β_1 is an even and positive function on $[-\pi,\pi]$;
2) there exist $k_0 > 0$, $\theta_0 \in (0,\pi)$ and $r \in (1,3)$ such that $\beta_0(\theta) = \dfrac{k_0}{|\theta|^r}\mathbb{I}_{[-\theta_0,\theta_0]}(\theta)$.

The second equation we consider is the Landau equation:

$$\frac{\partial f}{\partial t} = Q_L(f,f) \tag{LE}$$

with the collision operator Q_L defined by

$$Q_L(f,f) = \frac{1}{2}\sum_{i,j=1}^{2} \frac{\partial}{\partial v_i}\left\{\int_{\mathbb{R}^2} \mathrm{d}v_* a_{ij}(v-v_*)\left[f(t,v_*)\frac{\partial f}{\partial v_j}(t,v) - f(t,v)\frac{\partial f}{\partial v_{*j}}(t,v_*)\right]\right\}$$

with $a = (a_{ij})_{1\leq i,j\leq 2}$ a nonnegative symmetric matrix of the form in the Maxwell case

$$a(z) = \Lambda |z|^2 \Pi(z) \tag{1.2}$$

where $\Pi(z)$ is the orthogonal projection on $(z)^\perp$ and Λ is a positive constant precised below.

Many authors have been interested in proving rigorously the convergence of Boltzmann to Landau, in different cases of scattering cross-section and initial data. Firstly Arsen'ev and Buryak [2] proved the convergence of solutions of the Boltzmann equation towards solutions of the Landau equation under very restrictive assumptions. Desvillettes [8] gave a mathematical framework for more physical situations, but excluding the case of Coulomb potential which has been studied by Degond and Lucquin [7]. Degond and Lucquin stated an asymptotic development of the Boltzmann kernel when the collisions become grazing. Then, Goudon [12] and Villani [23] proved in two independent works the existence of a solution of the Landau equation for soft potentials using the asymptotic of grazing collisions, with a bounded entropy and energy function as initial data. More recently, Guérin and Méléard [16] proved the convergence of solutions of the Boltzmann equation to a solution of the Landau equation for 'moderately soft' potentials with a probabilistic representation when the initial data is a probability measure with a finite fourth-order moment. All those works prove an L^1-weak convergence of the solutions. Alexandre and Villani [1] stated in a recent work a strong convergence in L^p for some soft potentials including the case of a Coulomb gas.

The aim of this paper is to prove a pointwise convergence of function-solutions of the Boltzmann equation to the function-solution of the Landau equation on $\mathbb{R}^2$ for a Maxwell gas, which is unknown by analytical methods. We recall that in the case of Maxwell molecules, there is uniqueness of the solution of the Landau equation (see for example [15], Cor. 7). Fournier [10] and Guérin [15] proved respectively from probability measure solutions the existence of weak function solutions of the Boltzmann equation and of the Landau equation when the initial data is not a Dirac measure. To this aim, they used an efficient probabilistic tool: the Malliavin calculus for processes with jumps in [10] and the Malliavin calculus for white noises in [15]. From the result of Guérin and Méléard in [16] on the convergence of the probability measure solutions following the asymptotic of grazing collisions, it seems to be natural to study the convergence of function solutions.

In the asymptotics of grazing collisions, we only consider collisions with an infinitesimal angle of deviation. To this aim, we renormalize the cross-section β of the Boltzmann equation to concentrate on such collisions. We use the approximation introduced by Desvillettes [8]: for any $\varepsilon > 0$, let β^{ε} be the function defined on $[-\varepsilon\pi, \varepsilon\pi]\setminus\{0\}$ by

$$\beta^{\varepsilon}(\theta) = \frac{1}{\varepsilon^3}\beta\left(\frac{\theta}{\varepsilon}\right) \tag{1.3}$$

We notice that the mass of the function β^{ε} concentrates on the values of θ near 0 when ε tends to 0, *i.e.* when the collisions become grazing, in the following sense:

$$\text{for any } \theta_0 > 0,\ \beta^{\varepsilon}(\theta) \underset{\varepsilon \to 0}{\longrightarrow} 0 \text{ uniformly on } \theta \geq \theta_0 \tag{1.4}$$

$$\text{and } \int_{-\varepsilon\pi}^{\varepsilon\pi} \sin\left(\frac{\theta}{2}\right)^2 \beta^{\varepsilon}(\theta)\,\mathrm{d}\theta \underset{\varepsilon \to 0}{\longrightarrow} \Lambda \tag{1.5}$$

where $\Lambda = \frac{1}{2}\int_0^{\pi} \theta^2 \beta(\theta)\mathrm{d}\theta > 0$ is the constant appearing in the expression (1.2) of the matrix a. This asymptotic (1.3) is a particular case of the one introduced by Villani in [23], and used by Guérin and Méléard in [16]. We prove here the following theorem:

Theorem 1.1. *Let β be an even function on $[-\pi,\pi]\setminus\{0\}$ satisfying Assumption (A). Assume that the initial data P_0 is a probability measure with finite moments of all orders and P_0 is not a Dirac mass.*

We define $\beta^{\varepsilon}(\theta) = \varepsilon^{-3}\beta(\theta/\varepsilon)$ and we denote by f^{ε} the function-solution of the Boltzmann equation (BE) associated with the cross-section β^{ε} (obtained by Fournier in [10]*). The function f^{ε} is of class $\mathcal{C}^{\infty}$ on $\mathbb{R}^2$ (*[10]*, Th. 3.2).*

Then the sequence $(f^{\varepsilon}(t,.))_{\varepsilon>0}$ is pointwise convergent on $\mathbb{R}^2$ as ε tends to 0 *for any $t > 0$ and the limiting function f is the function-solution of the Landau equation. Moreover, $f(t,.)$ is of class $\mathcal{C}^{\infty}$ and there is pointwise convergence of derivatives of any orders.*

This theorem states a strong convergence result of solutions of the Boltzmann equation to the solution of Landau equation for a Maxwell gas when the collisions become grazing. Goudon [12] and Villani [23] proved L^1 -weak convergence, but in the more general case of soft potentials and in dimension 3. It seems that their methods can not give a stronger result.

Theorem 1.1 gives a new proof of the existence of regular function-solution for the Landau equation *via* a probabilistic approach.

We have to restrict our study to the dimension 2 because of the nonregularity of the Boltzmann coefficients in $\mathbb{R}^3$ (see [11], Lem. 2.6). Fournier [10] built the functions f^{ε} using the Fourier transforms of the probability measure solutions. Consequently, since the Boltzmann measure-solutions converge, it suffices to prove that their Fourier transforms are uniformly bounded by integrable functions on $\mathbb{R}^2$, when the collisions become grazing to obtain the convergence of the function-solutions. The proof is based upon a careful study of the results of Fournier [10] (the details of the proof are given in Sect. 4).

In the last part of this paper, we use the Monte-Carlo algorithm following the asymptotic of grazing collisions developed by Guérin and Méléard in [16]. We firstly simulate the convergence of solutions of the Boltzmann equation to the solution of the Landau equation for a degenerate initial distribution, and then we observe the behaviour in time of the solution of the Landau equation and of its entropy.

Notations

- $\mathbb{D}_T$ will denote the Skorohod space $\mathbb{D}([0,T],\mathbb{R}^2)$ of cadlag functions from $[0,T]$ into $\mathbb{R}^2$.
- $C_b^2(\mathbb{R}^2)$ is the space of real bounded functions of class $\mathcal{C}^2$ with bounded derivatives.
- $\mathcal{M}_2(\mathbb{R})$ is the set of matrices of order 2×2. The matrix A^* is the adjoint of the matrix A and the matrix I denotes the identity matrix in $\mathcal{M}_2(\mathbb{R})$.
- The bracket $\langle .,. \rangle$ denotes the scalar product in $\mathbb{R}^2$.

2. Some definitions

Let β be defined by Assumption (A) and β^ε be defined by (1.3). We define the Boltzmann equation (BE^ε) associated with the cross-section β^ε:

$$\frac{\partial f}{\partial t} = Q_{B^\varepsilon}(f,f) \tag{BE^ε}$$

with

$$Q_{B^\varepsilon}(f,f)(t,v) = \int_{v_*\in\mathbb{R}^2}\int_{\theta=-\pi}^{\pi} (f(t,v')f(t,v'_*) - f(t,v)f(t,v_*))\beta^\varepsilon(\theta)\,\mathrm{d}\theta\mathrm{d}v_*.$$

The collision operators of the Boltzmann and the Landau equations preserve momentum and kinetic energy. Equations of the form (1.1) have to be understood in a weak sense, *i.e.* f is a solution of the equation if for any test functions ϕ,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\phi(v)f(t,v)\mathrm{d}v = \int_{\mathbb{R}^2}\phi(v)Q(f,f)(t,v)\mathrm{d}v.$$

As detailed for example in [10], a standard integration by parts and a compensation due to the bad integrability behaviour of β^ε yield to the definition of a function-solution of the Boltzmann equation:

Definition 2.1. Let $\varepsilon > 0$ be fixed. A function-solution of (BE^ε) is a function f^ε satisfying for any $\phi\in C_b^2(\mathbb{R}^2)$ the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} f^\varepsilon(t,v)\phi(v)\mathrm{d}v = \int_{\mathbb{R}^2\times\mathbb{R}^2} K^\phi_{\beta^\varepsilon}(v,v_*)\, f^\varepsilon(t,v)\mathrm{d}v f^\varepsilon(t,v_*)\mathrm{d}v_* \tag{2.1}$$

where $K^\phi_{\beta^\varepsilon}$ is defined by

$$\begin{aligned} K^\phi_{\beta^\varepsilon}(v,v_*) &= -b^\varepsilon\nabla\phi(v)\,.\,(v-v_*) \\ &\quad + \int_{-\varepsilon\pi}^{\varepsilon\pi}\Big(\phi\left(v + A(\theta)(v-v_*)\right) - \phi(v) - A(\theta)(v-v_*)\,.\nabla\phi(v)\Big)\beta^\varepsilon(\theta)\,\mathrm{d}\theta \end{aligned} \tag{2.2}$$

with $b^\varepsilon = \frac{1}{2}\int_{-\varepsilon\pi}^{\varepsilon\pi}(1-\cos\theta)\beta^\varepsilon(\theta)\mathrm{d}\theta$.

Using the conservation of the mass in (2.1), we introduce a definition of probability measure solutions of (BE^ε):

Definition 2.2. Let $\varepsilon > 0$ be fixed. Let P_0 be a probability measure with a finite 2-order moment. A measure family $(P_t^\varepsilon)_{t\geq 0}$ is a measure-solution of (BE^ε) if it satisfies for any $\phi\in\mathcal{C}_b^2\left(\mathbb{R}^2\right)$

$$\int_{\mathbb{R}^2}\phi(v)P_t^\varepsilon(\mathrm{d}v) = \int_{\mathbb{R}^2}\phi(v)P_0(\mathrm{d}v) + \int_0^t\int_{\mathbb{R}^2\times\mathbb{R}^2} K^\phi_{\beta^\varepsilon}(v,v_*)\,P_s^\varepsilon(\mathrm{d}v)\,P_s^\varepsilon(\mathrm{d}v_*)\,\mathrm{d}s. \tag{2.3}$$

In the same way, we give the following definition of a function-solution for the Landau equation:

Definition 2.3. A function f is a function-solution of (LE) if f satisfies for each $\phi \in C_b^2(\mathbb{R}^2)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} f(t,v)\phi(v)\,\mathrm{d}v = \int_{\mathbb{R}^2\times\mathbb{R}^2} L^{\phi}(v,v_*)\,f(t,v)\mathrm{d}vf(t,v_*)\,\mathrm{d}v \tag{2.4}$$

where L^{ϕ} is the Landau kernel defined on $\mathbb{R}^2\times\mathbb{R}^2$ by:

$$L^{\phi}(v,v_*) = \frac{1}{2}\sum_{i,j=1}^{2}\partial_{ij}^2\phi(v)\;a_{ij}(v-v_*) + \sum_{i=1}^{2}\partial_i\phi(v)\,b_i(v-v_*)$$

with $b_i(z) = \sum_{j=1}^{2}\partial_j a_{ij}(z) = -\Lambda z_i$.

We also state a definition of measure-solutions of (LE) as in Definition 2.2.

We notice that the Boltzmann kernel $K_{\beta^\varepsilon}^{\phi}$ is pointwise convergent on $\mathbb{R}^2\times\mathbb{R}^2$ to the Landau kernel L^{ϕ} when ε tends to 0 for any $\phi\in C_b^2(\mathbb{R}^2)$ (see for example [12] or [23]).

3. The convergence of the function-solutions

We give in this section the main idea of the proof of Theorem 1.1.

In all the following, P_0 is assumed to be a probability measure with a finite two-order moment and β a positive even function on $[-\pi,\pi]\backslash\{0\}$ satisfying Assumption (A).

In the probabilistic study of the Boltzmann equation, we consider in fact (2.3) as the evolution equation of the family of the time marginals of a jump process. The distribution of this process will be solution of the following nonlinear martingale problem:

Definition 3.1. Let $\varepsilon > 0$ be fixed. We say that a probability measure P^ε on $\mathbb{D}_T$ solves the nonlinear martingale problem (MP^ε) starting at P_0 if for X the canonical process under P^ε, the law of X_0 is P_0 and for any $\phi\in\mathcal{C}_b^2(\mathbb{R}^2)$,

$$\phi(X_t) - \phi(X_0) - \int_0^t\int_{\mathbb{R}^2} K_{\beta^\varepsilon}^{\phi}(X_s,v_*)\,P_s^\varepsilon(\mathrm{d}v_*)\,\mathrm{d}s \tag{3.1}$$

is a square-integrable martingale, where P_s^ε is the marginal of P^ε at time s.

Taking expectation in (3.1), we notice that if P^ε is a solution of (MP^ε),then $(P_t^\varepsilon)_{t\geq 0}$ is a measure-solution of (BE^ε).

Fournier proved in [10] the existence of a solution P^ε of (MP^ε) for any $\varepsilon > 0$. Moreover, Guérin and Méléard in [16] stated the tightness of the sequence $(P^\varepsilon)_{\varepsilon>0}$ when the collisions become grazing ($\varepsilon\to 0$) in the more general case of soft potentials and in dimension 3 (using the same arguments, the convergence theorem is still true in dimension 2). In the particular case of Maxwellian molecules, there is convergence of the sequence $(P^\varepsilon)_{\varepsilon>0}$ to the measure-solution of the Landau equation (LE) thanks to the uniqueness of this solution (see [15], Cor. 7). We will use those results under the following form:

Theorem 3.2. *Let $\beta^\varepsilon = \varepsilon^{-3}\beta(\theta/\varepsilon)$. For any $\varepsilon > 0$, there exists a solution P^ε of the martingale problem (MP^ε). Moreover, the sequence $(P_t^\varepsilon)_{\varepsilon>0}$ converges as ε goes to 0 to a distribution P_t which is the measure-solution of the Landau equation.*

Let us remark that to obtain a function-solution from a measure-solution $(P_t^\varepsilon)_{t\geq 0}$, it suffices to prove that $\forall t > 0$ P_t^ε admits a density $f^\varepsilon(t,.)$ with respect to the Lebesgue measure on $\mathbb{R}^2$. Then the function f^ε satisfies Definition 2.1. Fournier [10] stated the following theorem using the Malliavin calculus for processes with jumps:

Theorem 3.3. *Let $\varepsilon \in (0,1)$ be fixed. Assume that P_0 is not a Dirac measure.*

1) The Boltzmann equation (BE^ε) admits a function-solution f^ε with initial data P_0.

2) If P_0 belongs to L^p for any $p \geq 1$, then for any $t > 0$, for any couple $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, there exists a constant $C^\varepsilon_{t,\alpha}$ such that the following inequality holds for all $\varphi \in \mathcal{C}^\infty\left(\mathbb{R}^2\right)$ with compact support

$$\left| \int_{\mathbb{R}^2} \partial_\alpha \varphi(v)\, P^\varepsilon_t(\mathrm{d}v) \right| \leq C^\varepsilon_{t,\alpha} \left\| \varphi \right\|_\infty \tag{3.2}$$

where ∂_α denotes the partial derivative $\frac{\partial^{\alpha_1+\alpha_2}}{\partial^{\alpha_1}x_1 \partial^{\alpha_2}x_2}$. Consequently, the function-solution f^ε is infinitely differentiable on $\mathbb{R}^2$ and is given by:

$$f^\varepsilon(t,v) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \hat{P}^\varepsilon_t(x)\, \mathrm{e}^{-i<v,x>} \mathrm{d}x$$

where $\hat{P}^\varepsilon_t$ is the Fourier transform of P^ε_t.

We want to state the convergence of the function-solutions f^ε of the Boltzmann equation (BE^ε) when the grazing collisions prevail.

Thanks to the convergence of measure-solutions $(P^\varepsilon_t)_{t\geq 0}$ of the Boltzmann equation to the measure-solution $(P_t)_{t\geq 0}$ of the Landau equation (see Th. 3.2), the sequence $\left(\hat{P}^\varepsilon_t\right)_{\varepsilon>0}$ is pointwise convergent on $\mathbb{R}^2$ to the Fourier transform $\hat{P}_t$ of P_t, for any $t \geq 0$.

Approximating the functions $\varphi(v) = \mathrm{e}^{i<v,x>}$ with $x = (x_1, x_2) \in \mathbb{R}^2$, by compact support functions of class $\mathcal{C}^\infty$, we obtain from inequality (3.2) that $\forall x \in \mathbb{R}^2$ and $\forall \alpha_1, \alpha_2 \geq 2$

$$\left| \hat{P}^\varepsilon_t(x) \right| \leq \inf \left\{ 1, \frac{C^\varepsilon_{t,(\alpha_1,\alpha_2)}}{|x_1|^{\alpha_1} |x_2|^{\alpha_2}} \right\}.$$

Thus if we prove that the constants $C^\varepsilon_{t,\alpha}$ are uniformly bounded in ε by a constant $C_{t,\alpha}$ for any $\alpha \in \mathbb{N}^2$, using the Lebesgue theorem, we easily deduce that the function-solutions $f^\varepsilon(t,v)$ (and its derivatives of any orders) of the Boltzmann equation converge as ε goes to 0 to the function-solution $f(t,v) = \int_{\mathbb{R}^2} \hat{P}_t(x)\, \mathrm{e}^{i<v,x>} \mathrm{d}x$ (respectively, its derivatives) of the Landau equation (obtained in [15]) for any $v \in \mathbb{R}^2$ and $t > 0$. Consequently the theorem will be proved.

4. The proof of Theorem 1.1

We assume from now without restriction that $\underline{\varepsilon \in (0, 1/2]}$.

To state that the constants $C^\varepsilon_{t,\alpha}$ appearing in $\overline{(3.2)}$ are uniformly bounded in ε, we have to study the proof of Theorem 3.3. Fournier [10] proved the existence of function-solutions by the mean of a nonlinear stochastic differential equation giving a pathwise version of the probabilistic interpretation.

4.1. **The Pathwise approach**

Let $\varepsilon > 0$ be fixed, P_0 be a probability measure with a finite 2-order moment and β satisfy Assumption (A). Let us consider two probability spaces to highlight the nonlinearity of the equation: the first one is the abstract space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ and the second one is $([0,1], \mathcal{B}([0,1]), \mathrm{d}\alpha)$. The processes on $([0,1], \mathcal{B}([0,1]), \mathrm{d}\alpha)$ will be called α-processes, the expectation under $d\alpha$ will be denoted by E_α and the laws by $\mathcal{L}_\alpha$.

On $(\Omega, \mathcal{F}, P)$ we consider a Poisson measure $N^\varepsilon(\mathrm{d}\theta, \mathrm{d}\alpha, \mathrm{d}t)$ on $[-\pi, \pi] \times [0,1] \times [0,T]$ with intensity measure $\nu^\varepsilon(\mathrm{d}\theta, \mathrm{d}\alpha, \mathrm{d}t) = \beta^\varepsilon(\theta)\, \mathrm{d}\theta \mathrm{d}\alpha \mathrm{d}t$ and with compensated measure $\tilde{N}^\varepsilon(\mathrm{d}\theta, \mathrm{d}\alpha, \mathrm{d}t)$.

Theorem 4.1. *(see* [10], *Th. 2.8) Let* V_0 *be a random variable with distribution* P_0*. There exists a couple of processes* $(V^\varepsilon, W^\varepsilon)$ *on* $\Omega \times [0,1]$ *satisfying the nonlinear stochastic differential equation* (SDE^ε)*:*

$$V_t^\varepsilon = V_0 + \int_0^t \int_0^1 \int_{-\varepsilon\pi}^{\varepsilon\pi} A(\theta)(V_{s-}^\varepsilon - W_{s-}^\varepsilon(\alpha))\tilde{N}^\varepsilon(\mathrm{d}s, \mathrm{d}\alpha, \mathrm{d}\theta) - b^\varepsilon \int_0^t \int_0^1 \left(V_{s-}^\varepsilon - W_{s-}^\varepsilon(\alpha)\right) \mathrm{d}\alpha \mathrm{d}s$$

with $\mathcal{L}(V^\varepsilon) = \mathcal{L}_\alpha(W^\varepsilon) = P^\varepsilon$.

Moreover $E[\sup_{0\le t\le T} |V_t^\varepsilon|^2] = E_\alpha[\sup_{0\le t\le T} |W_t^\varepsilon|^2] < \infty$*. There is uniqueness in law of* P^ε*.*

Corollary 4.2. *Thanks to Itô's formula, the measure* P^ε *is also a solution of the martingale problem* (MP^ε)*. Consequently,* $(P_t^\varepsilon)_{t\ge 0}$ *is a measure-solution of the Boltzmann equation for Maxwellian molecules.*

Moreover we easily prove (see [16], Sect. 3.3):

Lemma 4.3. *Assume that* V_0 *is a random vector in* $\mathbb{R}^2$ *belonging to* L^p *for any* $p \ge 1$*. Then for any* $T > 0$, $p \ge 1$*, there exists a constant* K_p *independent of* ε *such that*

$$E[\sup_{0\le t\le T} |V_t^\varepsilon|^p] = E_\alpha[\sup_{0\le t\le T} |W_t^\varepsilon|^p] \le K_p. \tag{4.1}$$

Using the Malliavin calculus for a stochastic differential equation driven by a Poisson process, Fournier [10] proved that each time-marginal P_t^ε satisfies (3.2) for any $t > 0$ and the coefficients $C_{t,\alpha}^\varepsilon$ depend on the Malliavin derivatives of V^ε. Consequently, to control $C_{t,\alpha}^\varepsilon$ we have to estimate the Malliavin's derivatives.

4.2. Some recalls on the Malliavin calculus

The Malliavin calculus in the case of a stochastic differential equation driven by a Poisson process, also called the stochastic calculus of variations, has been adapted to the case of the Boltzmann equation by Graham and Méléard [13] and Fournier [10] from the arguments of Bichteler, Gravereaux and Jacod in [3] and [4].

Let us consider a fixed time interval $[0,T]$, $T > 0$. Let $\varepsilon \in \left(0, \frac{1}{2}\right]$ be fixed.

Let us explain the main idea of this framework. We build a perturbation replacing θ with $\theta + <\lambda, v^\varepsilon>$ in order to obtain a new family of random measures N_λ^ε (for $\lambda \in \Lambda$, Λ being a neighborhood of 0 in $\mathbb{R}^2$ and v^ε a well-chosen predictable function from $\Omega \times [0,T] \times [-\varepsilon\theta_0, \varepsilon\theta_0] \times [0,1]$ to $\mathbb{R}^2$). Then, we build a family of probability measures $P_\lambda^\varepsilon = G_{\lambda,T}^\varepsilon . P^\varepsilon$ on Ω such that $\mathcal{L}((V_0, N_\lambda^\varepsilon) | P_\lambda^\varepsilon) = \mathcal{L}((V_0, N^\varepsilon) | P^\varepsilon)$. By this way, we obtain a perturbed process V_λ^ε satisfying $\mathcal{L}\left(V_{\lambda,t}^\varepsilon | P_\lambda^\varepsilon\right) = \mathcal{L}(V_t^\varepsilon | P^\varepsilon)$, and thus $E\left[\varphi\left(V_{\lambda,t}^\varepsilon\right) G_{\lambda,t}^\varepsilon\right] = E[\varphi(V_t^\varepsilon)]$, for any Borel bounded function φ on $\mathbb{R}^2$. Differentiating this equality at $\lambda = 0$, using an L^2-differentiate of $V_{\lambda,t}^\varepsilon$ and $G_{\lambda,t}^\varepsilon$, we finally obtain an equality of the form

$$E[\varphi'(V_t^\varepsilon) . DV_t^\varepsilon] = -E[\varphi(V_t^\varepsilon) DG_t^\varepsilon]$$

which is the first step to satisfy inequality (3.2) of Theorem 3.3.

Consequently, the constant $C_{t,\alpha}^\varepsilon$ appearing in (3.2) depends on the moments of the derivatives of V_t^ε, of $\det^{-1}(DV_t^\varepsilon)$ and of the derivatives of DG_t^ε. Under some assumptions on the initial data P_0, Fournier [10] obtained estimates of those moments. Consequently, we still have to state that those moments are uniformly bounded in ε to prove Theorem 1.1. The derivatives of V_t^ε and DG_t^ε depend strongly on the random function v^ε introduced in the perturbation. The function v^ε used by Fournier in [10] does not allow to obtain uniform bounds of the moments in $\varepsilon \in (0, 1/2]$ (see Rem. 4.3). So, we consider another perturbation which we describe now.

4.3. **The perturbation and the Malliavin derivatives**

Let δ^ε be a nonnegative even function on $[-\varepsilon\theta_0,\varepsilon\theta_0]$ defined by

$$\delta^\varepsilon(\theta)=c\,\varepsilon^{1-r}\left|\theta\right|^{r+1}\left(1-\frac{|\theta|}{\varepsilon\theta_0}\right)\tag{4.2}$$

with c a constant independent of ε such that $c\leq\left[\theta_0^r\left(\theta_0+r+2+r2^{r-1}\right)\right]^{-1}$. We notice that

$$\delta^\varepsilon(\theta)+\left|\left(\delta^\varepsilon\right)'(\theta)\right|<1.$$

Let g^ε be a $\mathbb{R}^2$-valued predictable function such that for any $\omega,t,\alpha,\varepsilon$, the map $\theta\to g^\varepsilon(\omega,t,\theta,\alpha)$ is of class $\mathcal{C}^1$ with $\|g^\varepsilon\|_\infty+\|g^{\varepsilon\prime}\|_\infty\leq 1$ where $g^{\varepsilon\prime}$ is the derivative of g^ε with respect to θ.

We then define the random function v^ε on $\Omega\times[0,T]\times[-\varepsilon\theta_0,\varepsilon\theta_0]\times[0,1]$ by

$$v^\varepsilon(\omega,t,\theta,\alpha)=g^\varepsilon(\omega,t,\theta,\alpha)\,\delta^\varepsilon(\theta).\tag{4.3}$$

We denote by $v^{\varepsilon\prime}$ the derivative of v^ε with respect to θ.

Let $\Lambda\subset B(0,1)$ be a neighbourhood of 0 in $\mathbb{R}^2$. For $\lambda\in\Lambda$, we consider the following perturbation

$$\gamma^{\varepsilon,\lambda}(\omega,t,\theta,\alpha)=\theta+\langle\lambda,v^\varepsilon(\omega,t,\theta,\alpha)\rangle\cdot$$

We notice that the map $\theta\longmapsto\gamma^{\varepsilon,\lambda}(\omega,t,\theta,\alpha)$ is an increasing bijection from $[-\varepsilon\theta_0,\varepsilon\theta_0]$ into itself (for any $\varepsilon\leq\frac{1}{2}$ and $|\theta|\leq\varepsilon\theta_0$, $|v^{\varepsilon\prime}(\theta)|<1$ thanks to the choice of c).

Recalling that $\beta=\beta_1+\beta_0$, the Poisson measure N split into N_0+N_1, where N_0 and N_1 are independent Poisson measures on $[0,T]\times[0,1]\times[-\pi,\pi]$ with intensities $\nu_0(\mathrm{d}\theta,\mathrm{d}\alpha,\mathrm{d}s)=\beta_0(\theta)\mathrm{d}\theta\mathrm{d}\alpha\mathrm{d}s$ and $\nu_1(\mathrm{d}\theta,\mathrm{d}\alpha,\mathrm{d}s)=\beta_1(\theta)\mathrm{d}\theta\mathrm{d}\alpha\mathrm{d}s$ respectively. We denote by $\tilde{N}_0$ and $\tilde{N}_1$ the associated compensated measures.

For $\lambda\in\Lambda$, we define $N_0^{\varepsilon,\lambda}=\gamma^{\varepsilon,\lambda}(N_0^\varepsilon)$ the image measure of N_0^ε by the map $\gamma^{\varepsilon,\lambda}$: if $A\subset[0,T]\times[0,1]\times[-\varepsilon\theta_0,\varepsilon\theta_0]$ is a Borel set,

$$N_0^{\varepsilon,\lambda}(\omega,A)=\int_0^T\int_0^1\int_{-\varepsilon\theta_0}^{\varepsilon\theta_0}\mathbb{I}_A\left(s,\gamma^{\varepsilon,\lambda}(\omega,s,\theta,\alpha),\alpha\right)N_0^\varepsilon(\omega,\mathrm{d}\theta,\mathrm{d}\alpha,\mathrm{d}s).$$

We consider the shift $S^{\varepsilon,\lambda}$ defined by

$$V_0\circ S^{\varepsilon,\lambda}(\omega)=V_0(\omega),\;N_0^\varepsilon\circ S^{\varepsilon,\lambda}(\omega)=N_0^{\varepsilon,\lambda}(\omega),\text{ and }N_1^\varepsilon\circ S^{\varepsilon,\lambda}(\omega)=N_1^\varepsilon(\omega).$$

Proposition 4.4. *Let $G^{\varepsilon,\lambda}$ be the Doléans-Dade martingale:*

$$G_t^{\varepsilon,\lambda}=1+\int_0^t\int_0^1\int_{-\varepsilon\theta_0}^{\varepsilon\theta_0}G_{s^-}^{\varepsilon,\lambda}\left(Y^{\varepsilon,\lambda}(s,\theta,\alpha)-1\right)\tilde{N}_0^\varepsilon(\mathrm{d}\theta,\mathrm{d}\alpha,\mathrm{d}s)$$

where $Y^{\varepsilon,\lambda}$ is the following predictable real valued function on $\Omega\times[0,T]\times[-\varepsilon\theta_0,\varepsilon\theta_0]\times[0,1]$

$$Y^{\varepsilon,\lambda}(\omega,s,\theta,\alpha)=\left(1+\langle\lambda,v^{\varepsilon\prime}(\omega,t,\theta,\alpha)\rangle\right)\frac{\beta_0^\varepsilon\left(\gamma^{\varepsilon,\lambda}(\omega,t,\theta,\alpha)\right)}{\beta_0^\varepsilon(\theta)}\cdot$$

Then $G_t^{\varepsilon,\lambda}$ is positive for any $t\in[0,T]$.

Proof. Let us notice that

$$\left|Y^{\varepsilon,\lambda}(s,\theta,\alpha)-1\right|\leq|\lambda|\,d^\varepsilon(\theta)$$

with $d^{\varepsilon}(\theta) = \delta^{\varepsilon}(\theta) + |\delta^{\varepsilon\prime}(\theta)| + r2^{r-1}\frac{\delta^{\varepsilon}(\theta)}{|\theta|}$. According to Appendix (Lem. 6.2), $d^{\varepsilon} \in \bigcap_{p\geq 2} L^{p}\left(\beta_0^{\varepsilon}(\theta)\,\mathrm{d}\theta\right)$ with moments uniformly bounded in ε. Consequently, $G^{\varepsilon,\lambda}$ is well defined and if

$$M_t^{\varepsilon,\lambda} = 1 + \int_0^t \int_0^1 \int_{-\varepsilon\pi}^{\varepsilon\pi} \left(Y^{\varepsilon,\lambda}(s,\theta,\alpha) - 1\right) \tilde{N}_0^{\varepsilon}(\mathrm{d}\theta, \mathrm{d}\alpha, \mathrm{d}s)$$

then (see Jacod and Shiryaev [18], p. 59),

$$G_t^{\varepsilon,\lambda} = \mathrm{e}^{M_t^{\varepsilon,\lambda}} \prod_{s\leq t} \left(1 + \Delta M_s^{\varepsilon,\lambda}\right) \mathrm{e}^{-\Delta M_s^{\varepsilon,\lambda}}.$$

Moreover, since $\varepsilon \leq 1/2$, for $|\theta| \leq \varepsilon\theta_0$

$$\begin{aligned} \left|Y^{\varepsilon,\lambda}(s,\theta,\alpha) - 1\right| &\leq d^{\varepsilon}(\theta) \leq \frac{1}{2} c\theta_0^r \left[\theta_0 + r + 2 + r2^{r-1}\right] \\ &\leq 1/2 \end{aligned}$$

thanks to the choice of c (see (4.2)). Thus, the jumps of $M^{\varepsilon,\lambda}$ are greater than $-1/2$ which implies that $G_t^{\varepsilon,\lambda}$ is positive.

□

Let $P^{\varepsilon,\lambda}$ be the probability measure defined by $P^{\varepsilon,\lambda} = G_T^{\varepsilon,\lambda}.P^{\varepsilon}$. Using the Girsanov theorem for random measures, we notice that $P^{\varepsilon,\lambda} \circ \left(S^{\varepsilon,\lambda}\right)^{-1} = P^{\varepsilon}$ (for more details see [10], Prop. 3.7). We consider now the perturbed process $V^{\varepsilon,\lambda} = V^{\varepsilon} \circ S^{\varepsilon,\lambda}$. Following Fournier [10], Section 3, and Appendix (Lem. 6.2), we notice that $V^{\varepsilon,\lambda}$ and $G^{\varepsilon,\lambda}$ belong to L^p for any $p \geq 1$ with bounded moments in ε, and they are differentiable at $\lambda = 0$. We give the expressions of their derivatives:

- the derivative of $G^{\varepsilon,\lambda}$ at $\lambda = 0$ is the following random vector in $\mathbb{R}^2$

$$DG_t^{\varepsilon} = \int_0^t \int_0^1 \int_{-\varepsilon\theta_0}^{\varepsilon\theta_0} \left(v^{\varepsilon\prime}(s,\theta,\alpha) - r\frac{v^{\varepsilon}(s,\theta,\alpha)}{\theta}\right) \tilde{N}_0^{\varepsilon}(\mathrm{d}\theta, \mathrm{d}\alpha, \mathrm{d}s);$$

- the derivative of V_t^{ε} is a 2×2 matrix which satisfies the equation

$$\begin{aligned} DV_t^{\varepsilon} &= -\frac{b^{\varepsilon}}{2}\int_0^t DV_s^{\varepsilon} ds + \int_0^t \int_0^1 \int_{-\varepsilon\pi}^{\varepsilon\pi} A(\theta) DV_{s^-}^{\varepsilon}\, \tilde{N}^{\varepsilon}(\mathrm{d}\theta, \mathrm{d}\alpha, \mathrm{d}s) \\ &+ \int_0^t \int_0^1 \int_{-\varepsilon\theta_0}^{\varepsilon\theta_0} A'(\theta)(V_{s^-} - W_{s^-}(\alpha))(v^{\varepsilon}(s,\theta,\alpha))^* \, N_0^{\varepsilon}(\mathrm{d}\theta, \mathrm{d}\alpha, \mathrm{d}s) \end{aligned} \tag{4.4}$$

which can be also written

$$DV_t^{\varepsilon} = M_t^{\varepsilon}.H_t^{\varepsilon} \tag{4.5}$$

where M^{ε} is the following invertible Doléans-Dade martingale

$$M_t^{\varepsilon} = I - \frac{b^{\varepsilon}}{2}\int_0^t M_s^{\varepsilon} \mathrm{d}s + \int_0^t \int_0^1 \int_{-\varepsilon\pi}^{\varepsilon\pi} A(\theta) M_{s^-}^{\varepsilon}\, \tilde{N}^{\varepsilon}(\mathrm{d}\theta, \mathrm{d}\alpha, \mathrm{d}s) \tag{4.6}$$

and

$$H_t^{\varepsilon} = \int_0^t \int_0^1 \int_{-\varepsilon\theta_0}^{\varepsilon\theta_0} \left(M_{s^-}^{\varepsilon}\right)^{-1} (I + A(\theta))^{-1} A'(\theta) \left(V_{s^-}^{\varepsilon} - W_{s^-}^{\varepsilon}(\alpha)\right) (v^{\varepsilon}(s,\theta,\alpha))^* \, N_0^{\varepsilon}(\mathrm{d}\theta, \mathrm{d}\alpha, \mathrm{d}s). \tag{4.7}$$

We want to state that the moments of the derivatives of V_t^ε, of $\det^{-1}(DV_t^\varepsilon)$ and of the derivatives of DG_t^ε are uniformly bounded in ε. We will just give here a detailed proof of the term $\det^{-1}(DV_t^\varepsilon)$. We easily obtain the bounds for the two other terms studying the construction of DG_t^ε and of DV_t^ε, using the definition (4.3) of v^ε and the bounds given in Appendix (Lem. 6.2).

The derivatives of V_t^ε and of DG_t^ε depend strongly on v^ε. The choice of v^ε is important. The moments of DG_t^ε are uniformly bounded in ε, if there exists a positive constant K_1 independent of ε such that

$$\int_0^{\varepsilon\theta_0} \left(\delta^\varepsilon(\theta) + |\delta^{\varepsilon\prime}(\theta)| + r\frac{\delta^\varepsilon(\theta)}{\theta}\right)^2 \beta_0^\varepsilon(\theta)\mathrm{d}\theta \le K_1.$$

The moments of DV_t^ε are uniformly bounded in ε, if there exists a positive constant K_2 independent of ε such that

$$\int_0^{\varepsilon\theta_0} \delta^\varepsilon(\theta)\beta_0^\varepsilon(\theta)\mathrm{d}\theta \le K_2.$$

Nevertheless, the integral $\int_0^{\varepsilon\theta_0} \delta^\varepsilon(\theta)\beta_0^\varepsilon(\theta)\mathrm{d}\theta$ must not tend to 0 as ε goes to 0. If not, the variable DV_t^ε converges to 0 in L^2 as ε tends to 0 (see Expression (4.4) of DV_t^ε) , and we have no hope to obtain uniform bounds for the term $\det^{-1}(DV_t^\varepsilon)$.

In the sequel, we will consider more precisely the perturbation v^ε defined by

$$v^\varepsilon(t,\theta,\alpha) = \bar g\left(V_{s^-}^\varepsilon - W_{s^-}^\varepsilon(\alpha), M_{s^-}^\varepsilon, \theta\right)\delta^\varepsilon(\theta)$$

with for any $x \in \mathbb{R}^2$, $y \in \mathcal{M}_2(\mathbb{R})$

$$\begin{aligned}\bar g(x,y,\theta) &= (A'(\theta)x)^* \left((I + A(\theta))^{-1}\right)^* \left(y^{-1}\right)^* \zeta(x,y,\theta)\\ \zeta(x,y,\theta) &= h(A'(\theta)x)\,k(I+A(\theta))\,k(y)\end{aligned}$$

where δ^ε is defined by (4.2) and the functions h and k satisfy the following assumptions:

- h is the function from $\mathbb{R}^2$ to $(0,1]$ defined by $h(x) = \left(1+|x|^2\right)^{-1}$;
- k is a function from $\mathcal{M}_2(\mathbb{R})$ to $[0,1]$ such that $k(y) = 0$ if and only if $\det y = 0$ and such that the map

$$y \longmapsto \begin{cases} \left(y^{-1}\right)^* k(y) & \text{if } \det y \neq 0 \\ 0 & \text{if } \det y = 0\end{cases} \tag{4.8}$$

is of class $\mathcal{C}_b^\infty$ from $\mathcal{M}_2(\mathbb{R})$ to itself.

Consequently, the process H^ε introduced in (4.5) writes

$$\begin{aligned}H_t^\varepsilon = &\int_0^t\int_0^1\int_{-\varepsilon\pi}^{\varepsilon\pi} \left(M_{s^-}^\varepsilon\right)^{-1}\Gamma\left(V_{s^-}^\varepsilon - W_{s^-}^\varepsilon(\alpha),\theta\right)\left[\left(M_{s^-}^\varepsilon\right)^{-1}\right]^*\\ &\times\zeta\left(V_{s^-}^\varepsilon - W_{s^-}^\varepsilon(\alpha), M_{s^-}^\varepsilon,\theta\right)\delta^\varepsilon(\theta)\,N_0^\varepsilon(\mathrm{d}\theta,\mathrm{d}\alpha,\mathrm{d}s)\end{aligned}$$

with for any $x \in \mathbb{R}^2$,

$$\Gamma(x,\theta) = (I + A(\theta))^{-1}(A'(\theta)x)(A'(\theta)x)^*\left((I+A(\theta))^{-1}\right)^*.$$

4.4. Study of $\det^{-1}(DV_t^\varepsilon)$

Since the derivative of V_t^ε can be written as $DV_t^\varepsilon = M_t^\varepsilon.H_t^\varepsilon$ for any $t \ge 0$, we study independently the term M_t^ε and the term H_t^ε.

Theorem 4.5. *Assume* (A) *and* $P_0 \in \cap_{p<\infty} L^p$. *For every* $t \geq 0$, $(\det M_t^\varepsilon)^{-1}$ *admits moments of all orders uniformly in* ε.

Proof. By [10] (Th. 3.20), M_t^ε is invertible and its inverse $(M_t^\varepsilon)^{-1}$ satisfies the equation

$$\begin{aligned}(M_t^\varepsilon)^{-1} &= I - \frac{b^\varepsilon}{2}\int_0^t (M_s^\varepsilon)^{-1}\,\mathrm{d}s - \int_0^t\int_0^1\int_{-\varepsilon\pi}^{\varepsilon\pi} (M_{s-}^\varepsilon)^{-1}\,(I+A(\theta))^{-1}A(\theta)\,\tilde{N}^\varepsilon(\mathrm{d}\theta,\mathrm{d}\alpha,\mathrm{d}s)\\ &\quad + \int_0^t\int_0^1\int_{-\varepsilon\pi}^{\varepsilon\pi} (M_{s-}^\varepsilon)^{-1}\,A(\theta)\,(I+A(\theta))^{-1}A(\theta)\,\beta^\varepsilon(\theta)\,\mathrm{d}\theta\mathrm{d}\alpha\mathrm{d}s \end{aligned} \tag{4.9}$$

with

$$(I+A(\theta))^{-1}A(\theta) = \frac{\sin\theta}{\cos\theta+1}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$A(\theta)\,(I+A(\theta))^{-1}A(\theta) = \frac{1}{2}\frac{\sin\theta}{\cos\theta+1}\begin{pmatrix} -\sin\theta & 1-\cos\theta \\ \cos\theta - 1 & -\sin\theta \end{pmatrix}.$$

Since $\int_0^\pi \theta^2\beta(\theta)\,\mathrm{d}\theta < \infty$, the sequence $(b^\varepsilon)_{\varepsilon>0}$ is bounded.

We notice that,

$$\int_{-\varepsilon\pi}^{\varepsilon\pi}\left(\frac{|\sin\theta|}{\cos\theta+1}\right)^p \beta^\varepsilon(\theta)\,\mathrm{d}\theta = \varepsilon^{-2}\int_{-\pi}^{\pi}\left(\frac{|\sin\varepsilon\theta|}{\cos\varepsilon\theta+1}\right)^p \beta(\theta)\,\mathrm{d}\theta.$$

For any $\varepsilon \in (0,\frac{1}{2}]$, the function $\theta \longmapsto \frac{|\sin\varepsilon\theta|}{\cos\varepsilon\theta+1}\beta(\theta)$ is continuous on $[-\pi,\pi]\setminus\{0\}$ and for ε small enough, $\frac{|\sin\varepsilon\theta|}{\cos\varepsilon\theta+1} \leq \varepsilon\theta$.

Consequently, the sequence $\left(\int\left(\frac{|\sin\theta|}{\cos\theta+1}\right)^p\beta^\varepsilon(\theta)\,\mathrm{d}\theta\right)_{\varepsilon\in(0,1/2]}$ is bounded for any $p \geq 2$.

Using the same arguments, we notice that the integrals

$$\int_{-\varepsilon\pi}^{\varepsilon\pi}\left(\frac{\sin^2\theta + |\sin\theta\,(1-\cos\theta)|}{\cos\theta+1}\right)^p\beta^\varepsilon(\theta)\,\mathrm{d}\theta = \varepsilon^{-2}\int_{-\pi}^{\pi}\left(\frac{\sin^2\varepsilon\theta + |\sin\varepsilon\theta\,(1-\cos\varepsilon\theta)|}{\cos\varepsilon\theta+1}\right)^p\beta(\theta)\,\mathrm{d}\theta$$

are uniformly bounded in ε, $\varepsilon \in (0,\frac{1}{2}]$, for any $p \geq 1$.

Then, using usual estimates, Gronwall's lemma in (4.9), we easily deduce that for any $p \geq 1$, there exists a constant K_p (independent of ε) such that $\forall\varepsilon \in (0,\frac{1}{2}]$,

$$E\left[(M_t^\varepsilon)^{-p}\right] \leq K_p.$$

Thus $(\det M_t^\varepsilon)^{-1}$ is uniformly bounded in ε in L^p for any $t \geq 0$. ☐

Theorem 4.6. *Assume that* (A) *is satisfied and* $V_0 \in \underset{p\geq 1}{\cap} L^p$. *For every* $t \geq 0$ $(\det H_t^\varepsilon)^{-1}$ *admits moments of all orders uniformly in* ε.

Lemma 4.7. *The map* $(\varepsilon, t, Y) \longmapsto \mathcal{L}(\langle V_t^\varepsilon, Y\rangle)$ *is weakly continuous on* $\left[0,\frac{1}{2}\right]\times[0,T]\times\left\{Y\in\mathbb{R}^2 : |Y| = 1\right\}$ *where* $P_t^0 = \mathcal{L}\left(V_t^0\right)$ *is the measure-solution of the Landau equation at time* t.

Proof. Let $(\varepsilon_n, t_n, Y_n)$ be a sequence such that $(\varepsilon_n, t_n, Y_n) \underset{n\to\infty}{\to} (\varepsilon, t, Y)$ in $[0,\frac{1}{2}]\times[0,T]\times\left\{Y\in\mathbb{R}^2 : |Y| = 1\right\}$.

Let $\psi \in \mathcal{C}_b^2(\mathbb{R})$ and we define ψ_Y on $\mathbb{R}^2$ of class $\mathcal{C}_b^2$ by $v \longmapsto \psi_Y(v) = \psi(\langle v, Y\rangle)$. We consider the sequence

$$d_n = E\left[\psi_Y(V_t^\varepsilon) - \psi_{Y_n}\left(V_{t_n}^{\varepsilon_n}\right)\right].$$

We want to state that $d_n \longrightarrow 0$ as n goes to $+\infty$.

Let $\left(Z^1, Z^2\right)$ be the canonical process on $\mathbb{D}_T \times \mathbb{D}_T$. Let us define $P^{\varepsilon_n}_{t_n} = \mathcal{L}\left(V^{\varepsilon_n}_{t_n}\right)$.

If $\varepsilon > 0$: Since the family of time marginal $\left(P^{\varepsilon_n}_{t_n}\right)_{n\geq 0}$ of the probability measure P^{ε_n} is a solution of (2.3), we notice that :

$$\begin{aligned} d_n &= E\left[\psi_Y\left(V_0\right) - \psi_{Y_n}\left(V_0\right)\right] + \int_{t_n}^{t} E_{P^\varepsilon\otimes P^\varepsilon}\left[K^{\psi_Y}_{\beta^\varepsilon}\left(Z^1_s, Z^2_s\right)\right]\mathrm{d}s \\ &\quad + \int_0^{t_n}\left(E_{P^\varepsilon\otimes P^\varepsilon}\left[K^{\psi_Y}_{\beta^\varepsilon}\left(Z^1_s, Z^2_s\right)\right] - E_{P^{\varepsilon_n}\otimes P^{\varepsilon_n}}\left[K^{\psi_{Y_n}}_{\beta^{\varepsilon_n}}\left(Z^1_s, Z^2_s\right)\right]\right)\mathrm{d}s \\ &= A_n + B_n + C_n. \end{aligned}$$

Since ψ is globally Lipschitz, obviously A_n tends to 0 as n goes to $+\infty$.

We rewrite the term C_n under the form:

$$\begin{aligned} C_n &= \int_0^{t_n}\left(E_{P^\varepsilon\otimes P^\varepsilon}\left[K^{\psi_Y}_{\beta^\varepsilon}\left(Z^1_s, Z^2_s\right)\right] - E_{P^{\varepsilon_n}\otimes P^{\varepsilon_n}}\left[K^{\psi_{Y_n}}_{\beta^{\varepsilon_n}}\left(Z^1_s, Z^2_s\right)\right]\right)\mathrm{d}s \\ &= \int_0^{t_n}\left(E_{P^\varepsilon\otimes P^\varepsilon}\left[K^{\psi_Y}_{\beta^\varepsilon}\left(Z^1_s, Z^2_s\right)\right] - E_{P^{\varepsilon_n}\otimes P^{\varepsilon_n}}\left[K^{\psi_Y}_{\beta^\varepsilon}\left(Z^1_s, Z^2_s\right)\right]\right)\mathrm{d}s \\ &\quad + \int_0^{t_n} E_{P^{\varepsilon_n}\otimes P^{\varepsilon_n}}\left[K^{\psi_Y-\psi_{Y_n}}_{\beta^\varepsilon}\left(Z^1_s, Z^2_s\right)\right]\mathrm{d}s + \int_0^{t_n} E_{P^{\varepsilon_n}\otimes P^{\varepsilon_n}}\left[K^{\psi_{Y_n}}_{\beta^\varepsilon-\beta^{\varepsilon_n}}\left(Z^1_s, Z^2_s\right)\right]\mathrm{d}s. \end{aligned}$$

We easily prove the convergence of the law $P^{\varepsilon_n}\otimes P^{\varepsilon_n}$ to $P^\varepsilon\otimes P^\varepsilon$ when n goes to $+\infty$.

For any $\phi \in \mathcal{C}^2_b(\mathbb{R})$, $\varepsilon > 0$ fixed, the function $(v, v_*) \longmapsto K^\phi_{\beta^\varepsilon}(v, v_*)$ is continuous and a simple computation shows that for any $v, v^* \in \mathbb{R}^2$

$$\left|K^\phi_{\beta^\varepsilon}(v, v_*)\right| \leq C\left\|\phi''\right\|_\infty\left(\int |\theta|^2\beta^\varepsilon(\theta)\,\mathrm{d}\theta\right)|v - v_*|^2 + |b^\varepsilon|\left\|\phi'\right\|_\infty|v - v_*|. \tag{4.10}$$

Using the bounds (4.1) of the moment of V^ε, we deduce that B_n and C_n converge to 0 as n goes to $+\infty$. So $d_n \to 0$ when n tends to $+\infty$.

Thus the function $(\varepsilon, t, Y) \to \mathcal{L}\left(\langle V^\varepsilon_t, Y\rangle\right)$ is weakly continuous on $(0, \frac{1}{2}] \times [0, T] \times \left\{Y \in \mathbb{R}^2 : |Y| = 1\right\}$.

If $\varepsilon = 0$: As $\left(P^{\varepsilon_n}_{t_n}\right)_{n\geq 0}$ and $\left(P^0_t\right)_{t\geq 0}$ are measure-solutions of the Boltzmann equation and of the Landau equation respectively, we rewrite d_n:

$$\begin{aligned} d_n &= E\left[\psi_Y\left(V_0\right) - \psi_{Y_n}\left(V_0\right)\right] + \int_{t_n}^{t} E_{P^0\otimes P^0}\left[L^{\psi_Y}\left(Z^1_s, Z^2_s\right)\right]\mathrm{d}s \\ &\quad + \int_0^{t_n}\left(E_{P^0\otimes P^0}\left[L^{\psi_Y}\left(Z^1_s, Z^2_s\right)\right] - E_{P^{\varepsilon_n}\otimes P^{\varepsilon_n}}\left[K^{\psi_{Y_n}}_{\beta^{\varepsilon_n}}\left(Z^1_s, Z^2_s\right)\right]\right)\mathrm{d}s \\ &= A'_n + B'_n + C'_n. \end{aligned}$$

As in the previous case, we divide the term C'_n into three parts

$$\begin{aligned} C'_n &= \int_0^{t_n}\left(E_{P^0\otimes P^0}\left[L^{\psi_Y}\left(Z^1_s, Z^2_s\right)\right] - E_{P^{\varepsilon_n}\otimes P^{\varepsilon_n}}\left[K^{\psi_{Y_n}}_{\beta^{\varepsilon_n}}\left(Z^1_s, Z^2_s\right)\right]\right)\mathrm{d}s \\ &= \int_0^{t_n} E_{P^0\otimes P^0}\left[L^{\psi_Y}\left(Z^1_s, Z^2_s\right) - K^{\psi_Y}_{\beta^{\varepsilon_n}}\left(Z^1_s, Z^2_s\right)\right]ds + \int_0^{t_n} E_{P^{\varepsilon_n}\otimes P^{\varepsilon_n}}\left[K^{\psi_Y-\psi_{Y_n}}_{\beta^{\varepsilon_n}}\left(Z^1_s, Z^2_s\right)\right]\mathrm{d}s \\ &\quad + \int_0^{t_n}\left(E_{P^0\otimes P^0}\left[K^{\psi_Y}_{\beta^{\varepsilon_n}}\left(Z^1_s, Z^2_s\right)\right] - E_{P^{\varepsilon_n}\otimes P^{\varepsilon_n}}\left[K^{\psi_Y}_{\beta^{\varepsilon_n}}\left(Z^1_s, Z^2_s\right)\right]\right)\mathrm{d}s. \end{aligned}$$

We notice that for any $\phi \in \mathcal{C}_b^2(\mathbb{R})$, $v, v_* \in \mathbb{R}^2$

$$L^{\phi}(v, v_*) \leq C\left(\|\phi''\|_{\infty} |v - v_*|^2 + \|\phi'\|_{\infty} |v - v_*|\right).$$

Using the same arguments as above, the convergence of the Boltzmann kernel to the Landau kernel and the convergence of measure-solutions of the Boltzmann equation to the measure solution of the Landau equation, we obtain the convergence of d_n to 0 as $n \to +\infty$. Consequently, $\mathcal{L}\left(\langle V_{t_n}^{\varepsilon_n}, Y_n\rangle\right) \underset{n\to\infty}{\to} \mathcal{L}\left(\langle V_t^0, Y\rangle\right)$.

Finally, the map $(\varepsilon, t, Y) \longmapsto \mathcal{L}\left(\langle V_t^{\varepsilon}, Y\rangle\right)$ is weakly continuous on $[0, \frac{1}{2}] \times [0, T] \times \left\{Y \in \mathbb{R}^2 : |Y| = 1\right\}$. □

We now state a technical lemma of nondegeneracy of the law of V_t^{ε}:

Lemma 4.8. *Assume that (A) is satisfied,* $V_0 \in \cap_{p<\infty} L^p$ *and* $E[V_0] = 0$. *Let* $t_0 > 0$ *be fixed. There exists* $\eta > 0$, $q > 0$ *and* $\xi > 0$ *(depending on* t_0*) such that for any* $\varepsilon \in [0, \frac{1}{2}]$, *for any* $t \in [t_0, T]$ *and for any* $X, Y \in \mathbb{R}^2$ *with* $|Y| = 1$,

$$P\left(\langle V_t^{\varepsilon} - X, Y\rangle^2 > \eta, |V_t^{\varepsilon}|^2 < \xi\right) > q$$

where $\mathcal{L}(V_t^0)$ *is the solution of the Landau equation at time* t.

Proof. Fournier ([10], lem. 3.22) proved this lemma for any fixed $\varepsilon \geq 0$. So we study step by step his proof to state that η, q and ξ do not depend of ε.

Let us notice that it is enough to show that there exists $\eta > 0$, $q > 0$ such that for any $t \in [t_0, T]$, for any $\varepsilon \geq 0$ and for any $X, Y \in \mathbb{R}^2$ with $|Y| = 1$,

$$P\left(\langle V_t^{\varepsilon} - X, Y\rangle^2 > \eta\right) > 2q.$$

Indeed, since $\sup_{\varepsilon\geq 0} E\left[\sup_{0\leq t\leq T} |V_t^{\varepsilon}|^2\right] \leq K$, using Bienayme-Tchebichev's inequality, there exists $\xi > 0$ such that $P\left(|V_t^{\varepsilon}|^2 < \xi\right) > 1 - q$ and ξ does not depend of ε.

Step1. Let $t \geq t_0$, $\varepsilon \geq 0$ and $|Y| = 1$ be fixed. The distribution of V_t^{ε} admits a density with respect to the Lebesgue measure, hence the distribution of $\langle V_t^{\varepsilon}, Y\rangle$ has a density on $\mathbb{R}$. Using the conservation of the momentum, we notice that $E\left(\langle V_t^{\varepsilon}, Y\rangle\right) = E\left(\langle V_0, Y\rangle\right) = 0$.

Consequently, there exists $\eta(t, \varepsilon, Y) > 0$ and $q(t, \varepsilon, Y) > 0$ such that

$$P\left(\langle V_t^{\varepsilon}, Y\rangle > \sqrt{\eta(\varepsilon, t, Y)}\right) > 2q(\varepsilon, t, Y) \text{ and } P\left(\langle V_t^{\varepsilon}, Y\rangle < -\sqrt{\eta(\varepsilon, t, Y)}\right) > 2q(\varepsilon, t, Y).$$

Step2. Using Lemma 4.7 and Portemanteau's theorem, for any $t \in [t_0, T]$, for any $\varepsilon \in \left[0, \frac{1}{2}\right]$ and $Y \in \mathbb{R}^2$ with $|Y| = 1$, there is a neighborhood $\mathcal{V}(\varepsilon, t, Y)$ of (ε, t, Y) such that for any $(\varepsilon', t', Y') \in \mathcal{V}(\varepsilon, t, Y)$

$$P\left(\langle V_{t'}^{\varepsilon'}, Y'\rangle > \sqrt{\eta(\varepsilon, t, Y)}\right) > 2q(\varepsilon, t, Y).$$

We consider a finite covering $\cup_{i=1}^N \mathcal{V}(\varepsilon_i, t_i, Y_i)$ of the compact set $[0, \frac{1}{2}] \times [t_0, T] \times \left\{Y \in \mathbb{R}^2 : |Y| = 1\right\}$. If we define $\eta = \inf_{i\leq N} \eta(\varepsilon_i, t_i, Y_i)$ and $q = \inf_{i\leq N} q(\varepsilon_i, t_i, Y_i)$, we notice that

$$P\left(\langle V_t^{\varepsilon}, Y\rangle > \sqrt{\eta}\right) > 2q$$

for any $(\varepsilon, t, Y) \in [0, \frac{1}{2}] \times [t_0, T] \times \left\{Y \in \mathbb{R}^2 : |Y| = 1\right\}$.

In the same way, $P\left(\langle V_t^{\varepsilon}, Y\rangle < -\sqrt{\eta}\right) > 2q$ for any $t \in [t_0, T]$ and $Y \in \mathbb{R}^2$ with $|Y| = 1$.

Step3. Let $X \in \mathbb{R}^2$, $t \in [t_0, T]$, $\varepsilon \geq 0$ and $|Y| = 1$ be fixed. If $\langle X, Y\rangle \leq 0$,

$$P\left(\langle V_t^{\varepsilon} - X, Y\rangle^2 > \eta\right) \geq P\left(\langle V_t^{\varepsilon}, Y\rangle > \sqrt{\eta}\right) > 2q$$

and if $\langle X,Y\rangle > 0$,

$$P\left(\langle V_t^\varepsilon - X,Y\rangle^2 > \eta\right) \geq P\left(\langle V_t^\varepsilon ,Y\rangle < -\sqrt{\eta}\right) > 2q.$$

The lemma is proved. □

Proof of Theorem 4.6. We fix $t_0>0$, and we prove the theorem for every $t \geq t_0$ which suffices.

We choose k such that $k(y)=1$ as soon as $|\det y| \geq d_0$ with $d_0 = \inf_{|\theta|\leq\theta_0} |\det(I + A(\theta))| > 0$. First of all, we prove that $\left(\det\left(F^\varepsilon H_t^\varepsilon\right)\right)^{-1}$ belongs to L^p uniformly in ε for any $p\geq 1$ where F^ε is the random variable defined by

$$F^\varepsilon = \sup_{s\in[0,T]} \left\{ \left(1+\frac{1}{4}\left(|V_s^\varepsilon|^2+\xi\right)\right) \times \left(k\left(M_{s-}^\varepsilon\right) \left\| \left(\left(M_{s-}^\varepsilon\right)^{-1}\right)^* \right\|_{op}^2 \right)^{-1} \right\}$$

with $\left\| \left(\left(M_{s-}^\varepsilon\right)^{-1}\right)^* \right\|_{op}^2$ the operator norm of $\left(\left(M_{s-}^\varepsilon\right)^{-1}\right)^*$ and ξ defined by Lemma 4.8. To this aim, using Lemma 6.1, we estimate the quantity for $p\geq 2$

$$\begin{aligned} \mathbb{E} &= E\left[\int_{X\in\mathbb{R}^2} |X|^p \exp\left(-X^* F^\varepsilon H_t^\varepsilon X\right) \mathrm{d}X\right] \\ &= \int_{\rho=0}^\infty \int_{|Y|=1} \rho^p E\left[\exp\left(-\rho^2 F^\varepsilon \times Y^* H_t^\varepsilon Y\right)\right] \mathrm{d}Y \mathrm{d}\rho. \end{aligned}$$

Thanks to Lemma 4.8, we can state (see the proof of [10], Th. 3.24) that for $\rho>0$, $t\geq t_0$ and $Y\in\mathbb{R}^2$ with $|Y|=1$,

$$E\left[\exp\left(-\rho^2 F^\varepsilon \times Y^* H_t^\varepsilon Y\right)\right] \leq \exp\left(-q\left(t-t_0\right) \int_0^{\varepsilon\theta_0} \left(1-e^{-\eta\rho^2\delta^\varepsilon(\theta)}\right) \beta_0^\varepsilon(\theta)\,\mathrm{d}\theta\right)$$

with η independent of ε issue from Lemma 4.8. Thus, there exists a constant $K>0$ (independent of ε) such that for any $p\geq 1$, $t>t_0$ and $\varepsilon>0$

$$\mathbb{E} \leq K \int_0^\infty \rho^p \exp\left(-q\left(t-t_0\right)\int_0^{\varepsilon\theta_0} \left(1-\mathrm{e}^{-\rho^2\eta\delta^\varepsilon(\theta)}\right)\beta_0^\varepsilon(\theta)\,\mathrm{d}\theta\right)\mathrm{d}\rho$$

Moreover, using Appendix Lemma 6.3, we can write

$$\mathbb{E} \leq \frac{K}{\sqrt{\eta}}\left[\int_0^{\sqrt{k^\varepsilon}} \rho^p \exp\left(-K_1\rho^2\right)\mathrm{d}\rho + \int_{\sqrt{k^\varepsilon}}^{+\infty} \rho^p \exp\left(-K_2 \varepsilon^{-\frac{4}{r+1}} \rho^{2\frac{r-1}{r+1}}\right)\mathrm{d}\rho\right]$$

where $K_1 = qC_1\left(t-t_0\right)$, $K_2 = qC_2\left(t-t_0\right)$ are positive constants independent of ε (with C_1 and C_2 constants defined in Lem. 6.3), and $k^\varepsilon = 2^{(r+2)}\theta_0^{-(r+1)}\varepsilon^{-2}/c$.

In the following computations, we observe that the choice of the random function v^ε, and consequently of δ^ε, is really important. It is the main technical difficulty of the proof of Theorem 1.1.

Let us study the first term.

We notice that $k^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} +\infty$, thus we can write for ε small enough

$$\begin{aligned} \int_0^{\sqrt{k^\varepsilon}} \rho^p \exp\left(-K_1\rho^2\right)\mathrm{d}\rho &\leq 1 + \int_1^{\sqrt{k^\varepsilon}} \rho^{2p+1}\exp\left(-K_1\rho^2\right)\mathrm{d}\rho \\ &\leq 1 + \int_1^{k^\varepsilon} \rho^p \exp\left(-K_1\rho\right)\mathrm{d}\rho \\ &\leq C_{K_1,p}\left(1+\left(k^\varepsilon\right)^p\exp\left(-K_1\,k^\varepsilon\right)\right) \end{aligned}$$

with $C_{K_1,p}$ a positive constant independent of ε. Consequently, this integral is uniformly bounded in ε, $\varepsilon \in \left(0, \frac{1}{2}\right]$.

Let us now study the second term

$$\int_{\sqrt{k^\varepsilon}}^{+\infty} \rho^p \exp\left(-K_2 \varepsilon^{-\frac{4}{r+1}} \rho^{2\frac{r-1}{r+1}}\right) \mathrm{d}\rho.$$

We notice that $K_2 \varepsilon^{-\frac{4}{r+1}} \longrightarrow +\infty$ and $k^\varepsilon \longrightarrow +\infty$ when ε tends to 0. Let us recall that $r \in (1,3)$, then for any $q \geq 1$, for any $\varepsilon > 0$, $\rho^q \exp\left(-K_2 \varepsilon^{-\frac{4}{r+1}} \rho^{2\frac{r-1}{r+1}}\right) \longrightarrow 0$ as ρ goes to $+\infty$. Consequently, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$, for any $\rho > \sqrt{k^\varepsilon}$,

$$\rho^p \exp\left(-K_2 \varepsilon^{-\frac{4}{r+1}} \rho^{2\frac{r-1}{r+1}}\right) \leq \rho^{-2}$$

and

$$\int_{\sqrt{k^\varepsilon}}^{+\infty} \rho^p \exp\left(-K_2 \varepsilon^{-\frac{4}{r+1}} \rho^{2\frac{r-1}{r+1}}\right) \mathrm{d}\rho \quad \leq \quad \int_{\sqrt{k^\varepsilon}}^{+\infty} \rho^{-2} \mathrm{d}\rho \leq \left(k^\varepsilon\right)^{-1/2}$$

which implies that

$$\int_{\sqrt{k^\varepsilon}}^{+\infty} \rho^p \exp\left(-K_2 \varepsilon^{-\frac{4}{r+1}} \rho^{2\frac{r-1}{r+1}}\right) \mathrm{d}\rho \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

We then deduce that for any $p \geq 1$ there exists K_p independent of ε such that

$$E\left[\int_{X \in \mathbb{R}^2} |X|^p \exp\left(-X^* F^\varepsilon H_t^\varepsilon X\right) \mathrm{d}X\right] \leq K_p.$$

We conclude that for any $t > t_0$, $(\det F^\varepsilon H_t^\varepsilon)^{-1} = \left((F^\varepsilon)^2 \det H_t^\varepsilon\right)^{-1}$ belongs to L^p uniformly in ε for any $p \geq 1$.

Moreover, it is possible to choose k such that $F^\varepsilon \leq F_1^\varepsilon \times F_2^\varepsilon$ with

$$F_1^\varepsilon = \sup_{[0,T]} \left(1 + \frac{1}{4}\left|V_s^\varepsilon\right|^2 + \frac{\xi}{4}\right) \text{ and } F_2^\varepsilon = \sup_{[0,T]} \left(k\left(M_s^\varepsilon\right) \left\|\left(\left(M_s^\varepsilon\right)^{-1}\right)^*\right\|_{op}^2\right)^{-1}.$$

The random variable F_1^ε has moments of all orders independent of ε thanks to (4.1). From the definition (4.6) of M^ε, we easily prove that the moment of $\sup_{t\in[0,T]} |M_t^\varepsilon|$ are uniformly bounded in ε. So we obtain that F_2^ε has the same property thanks to Theorem 4.5 and the following estimate (see the proof of [10], Th. 3.24),

$$F_2^\varepsilon \leq \sup_{[0,T]} \left(1 + \left|M_s^\varepsilon\right|^8\right) \times \sup_{[0,T]} \left|\left(M_s^\varepsilon\right)^{-1}\right|^2.$$

Thus, for any $p \geq 2$, there exists $C_p > 0$ such that for any $\varepsilon \in (0, \frac{1}{2}]$,

$$\begin{aligned} E\left[\left|\det H_t^\varepsilon\right|^{-p}\right] &= E\left[\left|F^\varepsilon\right|^{2p} \times \left|\det\left(F^\varepsilon H_t^\varepsilon\right)\right|^{-p}\right] \\ &\leq E\left[\left|F^\varepsilon\right|^{4p}\right]^{\frac{1}{2}} E\left[\left|\det\left(F^\varepsilon H_t^\varepsilon\right)\right|^{-2p}\right]^{\frac{1}{2}} \\ &\leq C_p < \infty. \end{aligned}$$

The Theorem 4.6 is proved. □

Consequently, according to Theorem 4.6 and Theorem 4.5, for any $p \geq 1$ there exists a constant C_p such that for any $\varepsilon > 0$

$$E[|\det(DV_t^\varepsilon)|^{-p}] \leq C_p.$$

Then, Theorem 1.1 on the convergence of the function-solutions is proved.

5. Some numerical results

Guérin and Méléard ([16], Sect. 4), built a Monte-Carlo algorithm of simulation by a conservative particle method following the asymptotic of grazing collisions. In this section, we will use this algorithm to simulate the convergence of the function-solutions of the Boltzmann equation to the function-solution of the Landau equation. Moreover, we observe the behaviour in time of the entropy of the solution of the Landau equation (even if our theorical result gives no proof on this behaviour).

Let us consider an initial measure

$$P_0 = \frac{1}{2}\left(\delta_{(-1,1)} + \delta_{(1,-1)}\right)$$

and the following approximation β^ε of the grazing collisions

$$\beta^\varepsilon(\theta) = \frac{1}{2\pi\varepsilon^3 \sin(\frac{\theta}{2\varepsilon})^2}\mathbb{I}_{\varepsilon \leq |\frac{\theta}{\varepsilon}| \leq \pi}.$$

Let $\varepsilon > 0$ be fixed. We define $\left(V^{\varepsilon,1n}, ..., V^{\varepsilon,nn}\right)$ the n-particles system in $\left(\mathbb{R}^2\right)^n$ introduced by Guérin and Méléard [16] which is a $(\mathbb{R}^2)^n$-valued pure-jump Markov process with generator defined for $\phi \in C_b((\mathbb{R}^2)^n)$ by

$$\frac{1}{n-1}\sum_{1\leq i,j\leq n}\int_{-\pi}^{\pi}\frac{1}{2}\Big(\phi\left(v^n + \mathbf{e_i}.A(\theta)(v_i - v_j) + \mathbf{e_j}.A(\theta)(v_j - v_i)\right) - \phi(v^n)\Big)\beta^\varepsilon(\theta)\mathrm{d}\theta.$$

Here $v^n = (v_1, ..., v_n)$ denotes the generic point of $(\mathbb{R}^2)^n$ and $\mathbf{e_i} : h \in \mathbb{R}^2 \mapsto \mathbf{e_i}.h = (0, ..., 0, h, 0, ..., 0) \in (\mathbb{R}^2)^n$ with h at the i-th place.

In [16] (Th. 4.1), it is proved that the empirical measure $\mu^\varepsilon = \frac{1}{n}\sum_{i=1}^n \delta_{V^{\varepsilon,in}}$ on $\mathcal{P}\left(\mathbb{D}_T\right)$ associated with the system converges to the measure-solution P of the Landau equation when n tends to $+\infty$ and ε tends to 0. Then, for any $\phi \in \mathcal{C}_b\left(\mathbb{D}_T\right)$,

$$\frac{1}{n}\sum_{i=1}^n \phi\left(V^{\varepsilon,in}\right) \underset{\substack{n\to+\infty\\ \varepsilon\to 0}}{\longrightarrow} \int_{\mathbb{R}^2}\phi\left(v\right)P\left(\mathrm{d}v\right). \tag{5.1}$$

Let us explain how we simulate the function-solution from the particle system.

Let $t > 0$ be fixed. Thanks to the convergence of the empirical measure μ^ε, the function $g_{h_1,h_2}^{\varepsilon,n}$ on $\mathbb{R}^2$ defined by

$$x = (x_1, x_2) \longmapsto g_{h_1,h_2}^{\varepsilon,n}\left(x\right) = \frac{1}{nh_1h_2}\sum_{i=1}^n \mathbb{I}_{x_1 < V_{t,1}^{\varepsilon,in} \leq x_1 + h_1}.\mathbb{I}_{x_2 < V_{t,2}^{\varepsilon,in} \leq x_2 + h_2}$$

converges to $F_{h_1,h_2}\left(x\right) = \frac{1}{h_1h_2}\int\int_{(x_1,x_1+h_1]\times(x_2,x_2+h_2]} P_t\left(\mathrm{d}v\right)$ as $n \to +\infty$ and $\varepsilon \to 0$ for any step $h_1, h_2 > 0$. Moreover, the function $F_{h_1,h_2}\left(x\right)$ is pointwise convergent to the density $f\left(t,x\right)$ of the probability measure P_t on $\mathbb{R}^2$ when $h_1, h_2 \to 0$. Thus, the function $g_{h_1,h_2}^{\varepsilon,n}$ is an estimator of the function-solution f of the Landau equation.

For the simulations, we consider 500 000 particles and we choose the step $h_1 = h_2 = 0.1$.

We first observe the behaviour of the entropy of the solution f of the Landau equation which is defined by

$$H(t) = \int_{\mathbb{R}^2} f(t,v)\ln(f(t,v))\mathrm{d}v.$$

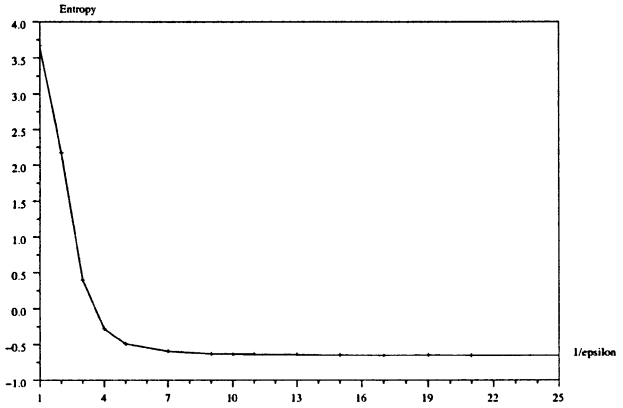

FIGURE 1. Evolution in $1/\varepsilon$ of the entropy H^{ε}.

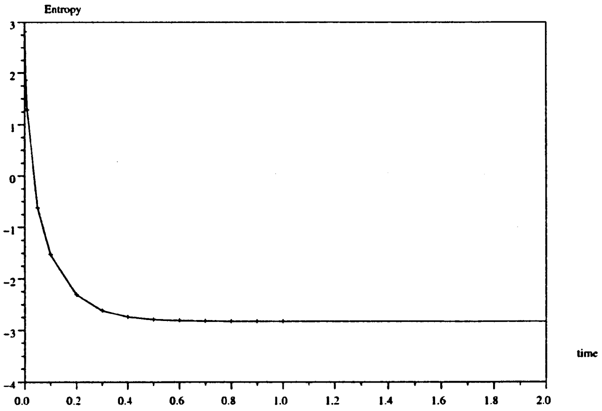

FIGURE 2. Evolution in time of the entropy H.

Replacing the density f with its estimator $g^{\varepsilon,n}_{h_1,h_2}$in the expression of H, we simulate the entropy H^{ε} associated with the Boltzmann equation and we observe in Figure 1. Its evolution in ε when $t = 0.005$.

We note that the entropy H^{ε} converges when ε tends to 0 and the choice of $\varepsilon = 0.1$ seems to be reasonable to describe the Landau behaviour.

From now, we fix $\varepsilon = 0.1$ and we observe in Figure 2. the decay in time of the entropy H of the solution of the Landau equation (see [22]). We note that the entropy converge to -2.833 when t goes to infinity.

Villani proved in [22] the convergence the function-solution f of the Landau equation to a Maxwellian function. This property is also satisfied by the solutions of the Boltzmann equation and the limited Maxwellian function is the same for the Landau and the Boltzmann equations.
As the 2-order moments of f are given by the following expression (see [22], Sect. 2.)

$$\int_{\mathbb{R}^2} v_i v_j f(t,v)\mathrm{d}v = (1 - \mathrm{e}^{-8t})\delta_{ij} \int_{\mathbb{R}^2} \frac{|v|^2}{2} P_0(\mathrm{d}v) + \mathrm{e}^{-8t} \int_{\mathbb{R}^2} v_i v_j P_0(\mathrm{d}v)$$

f converges to the following Maxwellian function when the time goes to infinity

$$M(v) = \frac{1}{2\pi}\exp\left(\frac{-|v|^2}{2}\right)\cdot$$

We notice that the entropy associated to M is equal to $-1-\ln(2\pi) \approx 2.838$ which is approximately the limit value obtained in Figure 2.

6. Appendix

We first mention a useful lemma proved in [3] (p. 92).

Lemma 6.1. *For any $p \geq 1$, there exists a constant C_p such that for any 2×2 symmetric positive matrix A,*

$$(\det A)^{-p} \leq C_p \int_{X\in\mathbb{R}^2} |X|^{4p-2}\, \mathrm{e}^{-X^*AX} \mathrm{d}X.$$

Let us now give some estimates on the function δ^ε introduced in (4.3) and defined on $[-\varepsilon\theta_0, \varepsilon\theta_0]$ by

$$\delta^\varepsilon(\theta) = c\varepsilon^{1-r}|\theta|^{r+1}\left(1-\frac{|\theta|}{\varepsilon\theta_0}\right)$$

with $c \leq \left[\theta_0^r\left(r2^{r-1}+r+2+\theta_0\right)\right]^{-1}$.

Lemma 6.2. *Assume that $\varepsilon \in \left(0, \frac{1}{2}\right]$. - $\delta^\varepsilon \in \bigcap\limits_{p\geq1} L^p\left(\beta_0^\varepsilon(\theta)\,\mathrm{d}\theta\right)$ with moments uniformly bounded in ε.*

- *Let $d^\varepsilon(\theta) = \delta^\varepsilon(\theta) + \left|(\delta^\varepsilon)'(\theta)\right| + r2^{r-1}\frac{\delta^\varepsilon(\theta)}{|\theta|}$. Then $d^\varepsilon \in \bigcap\limits_{p\geq2} L^p\left(\beta_0^\varepsilon(\theta)\,\mathrm{d}\theta\right)$ with moments uniformly bounded in ε.*

Proof. Let us recall that $\beta_0^\varepsilon(\theta) = \varepsilon^{-3}\beta_0(\theta/\varepsilon) = k_0\varepsilon^{r-3}|\theta|^{-r}\mathbb{I}_{|\theta|\leq\varepsilon\theta_0}$. Thanks to the choice of the constant c, the function δ^ε is bounded by 1. Then, it is enough to estimate its first moment:

$$\int_0^{\varepsilon\theta_0} \delta^\varepsilon(\theta)\beta_0^\varepsilon(\theta)\,\mathrm{d}\theta \quad\leq\quad ck_0\varepsilon^{-2}\int_0^{\varepsilon\theta_0}\theta\mathrm{d}\theta \leq \frac{ck_0\theta_0^2}{2}\cdot$$

Then the first point of the lemma is proved.

We notice that the function d^ε is also bounded by 1. So we just have to study the integral $\int_0^{\varepsilon\theta_0}\left(d^\varepsilon(\theta)\right)^2\beta_0^\varepsilon(\theta)\,\mathrm{d}\theta$. The function d^ε is the sum of three terms. We already know that $\int_0^{\varepsilon\theta_0}\left(\delta^\varepsilon(\theta)\right)^2\beta_0^\varepsilon(\theta)\,\mathrm{d}\theta$ is uniformly bounded in ε. We estimate now the two other terms:

- Study of the second term:

$$(\delta^\varepsilon)'(\theta) = \frac{c(r+1)}{\varepsilon^{r-1}}\theta^r\left(1-\frac{\theta}{\varepsilon\theta_0}\right) - \frac{c}{\varepsilon^r\theta_0}\theta^{r+1} \quad \text{if } \theta \in [0, \varepsilon\theta_0].$$

Thus

$$\begin{aligned}\int_0^{\varepsilon\theta_0}\left((\delta^\varepsilon)'(\theta)\right)^2\beta_0^\varepsilon(\theta)\,\mathrm{d}\theta &\leq k_0c^2\varepsilon^{-(r+1)}\int_0^{\varepsilon\theta_0}\left((r+1)+\frac{\theta}{\varepsilon\theta_0}\right)^2\theta^r\mathrm{d}\theta \\ &\leq 2k_0c^2\theta_0^{r+1}\frac{r^2+4r+4}{r+3}\cdot\end{aligned}$$

- Study of the third term:

$$\begin{aligned}\int_0^{\varepsilon\theta_0}\left(\frac{\delta^\varepsilon(\theta)}{\theta}\right)^2\beta_0^\varepsilon(\theta)\,\mathrm{d}\theta &\leq c^2k_0\varepsilon^{-(r+1)}\int_0^{\varepsilon\theta_0}\theta^r\mathrm{d}\theta \\ &\leq \frac{c^2\theta_0^{r+1}k_0}{r+1}\cdot\end{aligned}$$

The lemma is proved. □

Lemma 6.3. *Let* $r \in (1,3)$ *and* $x > 0$*. Let* $k^{\varepsilon} = 2^{r+2}\theta_0^{-(r+1)}\varepsilon^{-2}/c$*.*

a) For any $x \geq k^{\varepsilon}$ *there exists a constant* $C_1 > 0$ *independent of* ε *such that*

$$\int_0^{\varepsilon\theta_0} \left(1 - \mathrm{e}^{-x\delta^{\varepsilon}(\theta)}\right) \beta_0^{\varepsilon}(\theta)\, \mathrm{d}\theta \geq C_1 \varepsilon^{-\frac{4}{r+1}} x^{\frac{r-1}{r+1}}.$$

b) For any $x \leq k^{\varepsilon}$ *there exists a constant* $C_2 > 0$ *independent of* ε *such that*

$$\int_0^{\varepsilon\theta_0} \left(1 - \mathrm{e}^{-x\delta^{\varepsilon}(\theta)}\right) \beta_0^{\varepsilon}(\theta)\, \mathrm{d}\theta \geq C_2 x.$$

Proof. Since $\beta_0^{\varepsilon}(\theta) = k_0 \varepsilon^{r-3} |\theta|^{-r} \mathbb{I}_{|\theta| \leq \varepsilon\theta_0}$, we write

$$\begin{aligned} I(\varepsilon, x) &= \int_0^{\varepsilon\theta_0} \left(1 - \mathrm{e}^{-x\delta^{\varepsilon}(\theta)}\right) \beta_0^{\varepsilon}(\theta)\, \mathrm{d}\theta = k_0 \varepsilon^{r-3} \int_0^{\varepsilon\theta_0} \left(1 - \mathrm{e}^{-x\delta^{\varepsilon}(\theta)}\right) \theta^{-r} \mathrm{d}\theta \\ &\geq k_0 \varepsilon^{r-3} \int_0^{\frac{\varepsilon\theta_0}{2}} \left(1 - \mathrm{e}^{-x\tilde{\delta}^{\varepsilon}(\theta)}\right) \theta^{-r} \mathrm{d}\theta \end{aligned}$$

with $\tilde{\delta}^{\varepsilon}(\theta) = \frac{c}{2}\varepsilon^{1-r}\theta^{r+1}$. We notice that $k^{\varepsilon} = 1/\tilde{\delta}^{\varepsilon}(\frac{\varepsilon\theta_0}{2})$.

We use in the proof the following inequality:

$$\text{if } x \in [0,1],\ 1 - \mathrm{e}^{-x} \geq \frac{x}{2}.$$

a) The function $\tilde{\delta}^{\varepsilon}$ is increasing and its inverse function is

$$\left(\tilde{\delta}^{\varepsilon}\right)^{-1}(y) = \left(\frac{2\varepsilon^{r-1}}{c} y\right)^{1/(r+1)} \quad \text{for } y > 0.$$

If $x \geq 1/\tilde{\delta}^{\varepsilon}(\frac{\varepsilon\theta_0}{2})$, we notice that $\left(\tilde{\delta}^{\varepsilon}\right)^{-1}\left(x^{-1}\right) \leq \frac{\varepsilon\theta_0}{2}$, thus

$$I(\varepsilon, x) \geq k_0 \varepsilon^{r-3} \int_0^{\left(\tilde{\delta}^{\varepsilon}\right)^{-1}\left(x^{-1}\right)} \left(1 - \mathrm{e}^{-x\tilde{\delta}^{\varepsilon}(\theta)}\right) \theta^{-r} \mathrm{d}\theta.$$

As $\tilde{\delta}^{\varepsilon}$ is an increasing function, $x\tilde{\delta}^{\varepsilon}(\theta) \leq 1$ for any $\theta \in \left[0, \left(\tilde{\delta}^{\varepsilon}\right)^{-1}\left(x^{-1}\right)\right]$. Thus, we conclude

$$\begin{aligned} I(\varepsilon, x) &\geq \frac{k_0}{2}\varepsilon^{r-3} x \int_0^{\left(\tilde{\delta}^{\varepsilon}\right)^{-1}\left(x^{-1}\right)} \tilde{\delta}^{\varepsilon}(\theta)\, \theta^{-r} \mathrm{d}\theta \\ &\geq \frac{k_0 c}{4}\varepsilon^{-2} x \int_0^{\left(\tilde{\delta}^{\varepsilon}\right)^{-1}\left(x^{-1}\right)} \theta \mathrm{d}\theta \\ &\geq \frac{k_0 c}{8}\left(\frac{2}{c}\right)^{\frac{2}{r+1}} \varepsilon^{-\frac{4}{r+1}} x^{\frac{r-1}{r+1}}. \end{aligned}$$

b) If $x \le 1/\tilde{\delta}^{\varepsilon}(\frac{\varepsilon\theta_0}{2})$, then we have clearly $x\tilde{\delta}^{\varepsilon}(\theta) \le 1$ and

$$\begin{aligned} I(\varepsilon, x) &\geq \frac{k_0}{2}\varepsilon^{r-3}x\int_0^{\frac{\varepsilon\theta_0}{2}}\tilde{\delta}^{\varepsilon}(\theta)\,\theta^{-r}\mathrm{d}\theta \\ &\geq \frac{k_0 c}{4}\varepsilon^{-2}x\int_0^{\frac{\varepsilon\theta_0}{2}}\theta\mathrm{d}\theta \\ &\geq \frac{k_0 c\theta_0^2}{32}x. \end{aligned}$$

□

Acknowledgements. The author thanks Sylvie Méléard for many helpful discussions during the preparation of this work.

References

[1] R. Alexandre et C. Villani, On the Landau approximation in plasma physics (in preparation).

[2] A.A. Arsenev and O.E. Buryak, On the connection between a solution of the Boltzmann equation and a solution of the Landau–Fokker–Planck equation. *Math. USSR Sbornik* **69** (1991) 465-478.

[3] K. Bichteler, J.B. Gravelreaux and J. Jacod, Malliavin calculus for processes with jumps, *Theory and Application of stochastic Processes*. Gordon and Breach, New York (1987).

[4] K. Bichteler and J. Jacod, Calcul de Malliavin pour les diffusions avec sauts, existence d'une densité pour le cas unidimensionel, in *Séminaire de probabilités XVII*. Springer, Berlin, *Lecture Notes in Math.* **986** (1983) 132-157.

[5] L. Boltzmann, Weitere studien über das wärme gleichgenicht unfer gasmoläkuler. *Sitzungsber. Akad. Wiss.* **66** (1872) 275-370. Translation: *Further Studies on the thermal equilibrium of gas molecules*, S.G. Brush Ed., Pergamon, Oxford, *Kinetic Theory* **2** (1966) 88-174.

[6] L. Boltzmann, *Lectures on gas theory.* Reprinted by Dover Publications (1995).

[7] P. Degon and B. Lucquin–Desreux, The Fokker-Planck asymptotics of the Boltzmann collision operator in the Coulomb case. *Math. Mod. Meth. Appl. Sci.* **2** (1992) 167-182.

[8] L. Desvillettes, On asymptotics of the Boltzmann equation when the collisions become grazing. *Transp. Theory Statist. Phys.* **21** (1992) 259-276.

[9] L. Desvillettes, C. Graham and S. Méléard, Probabilistic interpretation and numerical approximation of a Kac equation without cutoff. *Stochastic Process. Appl.* **84** (1999) 115-135.

[10] N. Fournier, Existence and regularity study for two-dimensional Kac equation without cutoff by a probabilistic approach. *Ann. Appl. Probab.* **10** (2000) 434-462.

[11] N. Fournier and S. Méléard, A stochastic particle numerical method for 3D Boltzmann equations without cutoff. *Math. Comput.* **70** (2002) 583-604.

[12] T. Goudon, Sur l'équation de Boltzmann homogène et sa relation avec l'équation de Landau–Fokker–Planck : influence des collisions rasantes. *C. R. Acad. Sci. Paris* **324** (1997) 265-270.

[13] C. Graham and S. Méléard, Existence and regularity of a solution of a Kac equation without cutoff using the stochastic calculus of variations. *Comm. Math. Phys.* **205** (1999) 551-569.

[14] H. Guérin, Solving Landau equation for some soft potentials through a probabilistic approach. *Ann. Appl. Probab.* **13** (2003) 515-539.

[15] H. Guérin, Existence and regularity of a weak function-solution for some Landau equations with a stochastic approach. *Stochastic Process. Appl.* **101** (2002) 303-325.

[16] H. Guérin and S. Méléard, Convergence from Boltzmann to Landau processes with soft potential and particle approximation. *J. Statist. Phys.* **111** (2003) 931-966.

[17] J. Horowitz and R.L. Karandikar, *Martingale problem associated with the Boltzmann equation, Seminar on Stochastic Processes*, 1989, E. Cinlar, K.L. Chung and R.K. Getoor Eds., Birkhäuser, Boston (1990).

[18] J. Jacod and A.N. Shiryaev, *Limit theorems for stochastic processes.* Springer (1987).

[19] E.M. Lifchitz and L.P. Pitaevskii, *Physical kinetics – Course in theorical physics.* Pergamon, Oxford 10 (1981).

[20] D. Nualart, *The Malliavin calculus and related topics.* Springer-Verlag (1995).

[21] H. Tanaka, Probabilistic treatment of the Boltzmann equation of Maxwellian molecules. *Z. Wahrsch. Verw. Geb.* **46** (1978) 67-105.

[22] C. Villani, On the spatially homogeneous Landau equation for Maxwellian molecules. *Math. Meth. Mod. Appl. Sci.* **8** (1998) 957-983.

[23] C. Villani, On a new class of weak solutions to the spatially homogeneous Boltzmann and Landau equations. *Arch. Rational Mech. Anal.* **143** (1998) 273-307.