A VARIATIONAL APPROACH TO COMPLEX MONGE-AMPÈRE EQUATIONS

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ABSTRACT

We show that degenerate complex Monge-Ampère equations in a big cohomology class of a compact Kähler manifold can be solved using a variational method, without relying on Yau's theorem. Our formulation yields in particular a natural pluricomplex analogue of the classical logarithmic energy of a measure. We also investigate Kähler-Einstein equations on Fano manifolds. Using continuous geodesics in the closure of the space of Kähler metrics and Berndtsson's positivity of direct images, we extend Ding-Tian's variational characterization and Bando-Mabuchi's uniqueness result to singular Kähler-Einstein metrics. Finally, using our variational characterization we prove the existence, uniqueness and convergence as $k \to \infty$ of k-balanced metrics in the sense of Donaldson both in the (anti)canonical case and with respect to a measure of finite pluricomplex energy.

CONTENTS

Introduction

Solving degenerate complex Monge-Ampère equations has been the subject of intensive studies in the past decade, in connection with the search for canonical models and metrics for complex algebraic varieties (see e.g. [Koł98, Tian, Che00, Don05a, Siu08, BCHM10, EGZ09, SoTi08]).

Many of these results ultimately relied on the seminal work of Yau [Yau78], which involved a continuity method and difficult *a priori* estimates to construct smooth solutions to non-degenerate Monge-Ampère equations. But the final goal and outcome of some of these results was to produce singular solutions in degenerate situations, and the main aim of the present paper is to show that one can use the *direct methods* of the calculus of variations to obtain such weak solutions.[1] Our approach is to some extent a complex analogue of the method used by Aleksandrov to provide weak solutions to the Minkowski

[1] As usual with variational methods, *smoothness* of the solution does not follow from our approach, and still requires the techniques of [Yau78].

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problem [Ale38], i.e. the existence of compact convex hypersurfaces of $\mathbf{R}^n$ with prescribed Gaussian curvature.

We obtain in particular more natural proofs of the main results of [GZ07, EGZ09, BEGZ10], together with several new results to be described below.

Weak solutions to the Calabi conjecture and balanced metrics

Previous results. — Consider for the moment a compact Kähler n-dimensional manifold (X, ω), normalized by $\int \omega^n = 1$. Denote by $\mathcal{M}_X$ the set of all probability measures on X. Given a probability measure $\mu \in \mathcal{M}_X$ with smooth positive density, it was proved in [Yau78] that there exists a unique Kähler form η in the cohomology class of ω such that $\eta^n = \mu$. More singular measures $\mu \in \mathcal{M}_X$ were later considered in [Koł98]. In that case, η is to be replaced by an element of the set $\mathcal{T}(X, \omega)$ of all closed positive $(1, 1)$-currents T cohomologous to ω, which can thus be written $T = \omega + dd^c\varphi$ where φ is an ω-psh function, the *potential* of T (defined up to a constant). The positive measure T^n had been defined by Bedford-Taylor for φ bounded [BT82], and Kołodziej showed the existence of a unique $T \in \mathcal{T}(X, \omega)$ with continuous potential such that $T^n = \mu$, when μ has $L^{1+\varepsilon}$-density with respect to Lebesgue measure [Koł98].

In order to deal with more singular measures, one first needs to extend the Monge-Ampère operator $T \mapsto T^n$ beyond currents with bounded potentials. Even though this operator cannot be extended in a reasonable way to the whole of $\mathcal{T}(X, \omega)$, it was shown in [GZ07, BEGZ10], using a construction of [BT87], that one can in fact define the *non-pluripolar* product of arbitrary closed positive $(1, 1)$-currents $T_1, \dots, T_p$ on X. It yields a closed positive (p, p)-current

$$\langle T_1 \wedge \cdots \wedge T_p \rangle$$

putting no mass on pluripolar sets and whose cohomology class is bounded in terms of the cohomology classes of the T_j's only. In particular, given $T \in \mathcal{T}(X, \omega)$ we get a positive measure $\langle T^n \rangle$ putting no mass on pluripolar sets and of total mass

$$\int \langle T^n \rangle \le \int \omega^n = 1.$$

Equality holds if T has bounded potential; more generally, currents $T \in \mathcal{T}(X, \omega)$ for which equality holds are said to have *full Monge-Ampère mass*, in which case it is licit to simply write $T^n = \langle T^n \rangle$. Now the main result of [GZ07] states that every non-pluripolar measure $\mu \in \mathcal{M}_X$ is of the form $\mu = T^n$ for some $T \in \mathcal{T}(X, \omega)$ of full Monge-Ampère mass, which is furthermore unique, as was later shown in this generality in [Din09].

The proofs of the above results from [Koł98, GZ07] eventually reduce by regularization to the smooth case treated in [Yau78]. Our first goal in the present article is to solve singular Monge-Ampère equations by the direct method of the calculus of variations, independently of [Yau78].

The variational approach. — Denote by $\mathcal{T}^1(X,\omega)$ the set of all currents $T \in \mathcal{T}(X,\omega)$ with full Monge-Ampère mass and whose potential is furthermore integrable with respect to T^n. According to [GZ07, BEGZ10], currents T in $\mathcal{T}^1(X,\omega)$ are characterized by the condition $J(T) < +\infty$, where J denotes a natural extension of Aubin's J-functional [Aub84] obtained as follows. One first considers the *Monge-Ampère energy functional* [Aub84, Mab86], defined on smooth ω-psh functions φ by

$$E(\varphi) := \frac{1}{n+1}\sum_{j=0}^{n}\int \varphi\big(\omega + dd^c\varphi\big)^j \wedge \omega^{n-j}.$$

Using integration by parts, it is easy to show that the Gâteaux derivative of E at φ is given by integration against $(\omega + dd^c\varphi)^n$. This implies in particular that E is non-decreasing on smooth ω-psh functions, and a computation of its second derivative (see (2.3) below) also shows that E is *concave*. This functional extends by monotonicity to arbitrary ω-psh functions by setting

$$E(\varphi) := \inf\big\{E(\psi) \mid \psi \text{ smooth } \omega\text{-psh}, \psi \ge \varphi\big\} \in [-\infty, +\infty[,$$

and the J-functional is in turn defined by

$$J(T) := \int \varphi\omega^n - E(\varphi)$$

for $T = \omega + dd^c\varphi$. By translation invariance, this is independent of the choice of φ, hence descends to a convex, lower semicontinuous functional

$$J : \mathcal{T}(X,\omega) \to [0, +\infty]$$

which induces an exhaustion function on $\mathcal{T}^1(X,\omega) = \{J < +\infty\}$, in the sense that $\{J \le C\}$ is compact for each $C > 0$.

Now observe that the functional $\varphi \mapsto E(\varphi) - \int \varphi\, d\mu$ also descends to a concave functional

$$F_\mu : \mathcal{T}^1(X,\omega) \to]-\infty, +\infty]$$

by translation invariance, and set

$$E^*(\mu) := \sup_{\mathcal{T}^1(X,\omega)} F_\mu.$$

This yields a convex lower semicontinuous functional

$$E^* : \mathcal{M}_X \to [0, +\infty],$$

which is essentially the *Legendre transform* of E and will be called the *pluricomplex energy*.

Indeed, when (X, ω) is the complex projective line $\mathbf{P}^1$ endowed with its Fubiny-Study metric, $E^*(\mu)$ coincides, up to a multiplicative constant, with the logarithmic energy of the signed measure $\mu - \omega$ of total mass 0 (cf. Section 5). We shall thus say by analogy that $\mu \in \mathcal{M}_X$ has *finite energy* when $E^*(\mu) < +\infty$.

We can now state our first main result.

Theorem **A.** — *A measure* $\mu \in \mathcal{M}_X$ *has finite energy iff* $\mu = T_\mu^n$ *with* $T_\mu \in \mathcal{T}^1(X, \omega)$, *which is then characterized as the unique maximizer of* F_μ *on* $\mathcal{T}^1(X, \omega)$. *Furthermore, any maximizing sequence* $T_j \in \mathcal{T}^1(X, \omega)$ *necessarily converges to* T_μ.

As a consequence, we will show in Corollary 4.9 how to recover the case of an arbitrary non-pluripolar measure μ [GZ07].

The proof of Theorem A splits in two parts. The first one consists in showing that any maximizer $T \in \mathcal{T}^1(X, \omega)$ of F_μ has to satisfy $T^n = \mu$, i.e. that a maximizer φ of $E(\varphi) - \int \varphi \, d\mu$ satisfies the Euler-Lagrange equation $(\omega + dd^c\varphi)^n = \mu$. This is actually non-trivial even when φ is smooth, the difficulty being that the set of ω-psh functions has a boundary, so that a maximum is *a priori* not a critical point. This difficulty is overcome by adapting to our case the approach of [Ale38]. The main technical tool here is the differentiability result of [BB10], which is the complex analogue of the key technical result of [Ale38].

The next step in the proof of Theorem A is then to show the *existence* of a maximizer for F_μ when μ is assumed to satisfy $E^*(\mu) < +\infty$. Since J is an exhaustion function on $\mathcal{T}^1(X, \omega)$, a maximizer will be obtained by showing that F_μ is *proper* with respect to J (i.e. $F_\mu \to -\infty$ as $J \to +\infty$), and that it is upper semi-continuous—the latter property being actually the most delicate part of the proof.

Conversely, it easily follows from the concavity property of F_μ that $E^*(\mu)$ is finite if $\mu = T_\mu^n$ with $T_\mu \in \mathcal{T}^1(X, \omega)$.

Donaldson's balanced metrics. — The fact that any maximizing sequence for F_μ necessarily converges to T_μ in Theorem A is one key feature of the variational approach. As we shall now explain, this fact can for example efficiently be used in the context of μ-*balanced metrics* in the sense of [Don09]. Here we assume that the cohomology class of ω is the first Chern class of an ample line bundle L, and a metric $e^{-\phi}$ on L is then said to be balanced with respect to μ if ϕ coincides with the Fubiny-Study type metric associated to the L^2-scalar product induced by ϕ and μ on the space $H^0(L)$ of global holomorphic sections. We will show:

Theorem **B.** — *Let* L *be an ample line bundle, and let* μ *and* $T_\mu \in c_1(L)$ *be as in Theorem A. Then there exists a* μ*-balanced metric* ϕ_k *on* kL *for each* k *large enough, and the normalized curvature forms* $\frac{1}{k} dd^c \phi_k$ *converge to* T_μ *in the weak topology of currents.*

The existence of balanced metrics was established in [Don09] under a stronger regularity condition for μ. The convergence result, suggested in [Don09] as an analogue of [Don01], was observed to hold for smooth positive measures μ in [Kel09] as a direct consequence of [Wan05].

The case of a big class. — Up to now, we have assumed that the cohomology class $\{\omega\} \in H^{1,1}(X, \mathbf{R})$ is Kähler, but our variational approach works just as well in the more general case of *big* cohomology classes, as considered in [BEGZ10]. Note that the case of a big class enables in particular to extend our results to the case where X is singular, since the pull-back of a big class to a resolution of singularities remains big.

The appropriate version of Theorem A will thus be proved in this more general setting, thereby extending [GZ07, Theorem 4.2] to the case of a big class; we will show in Corollary 4.9 that it then enables to recover the main result of [BEGZ10].

The variational approach also applies to Kähler-Einstein metrics, i.e. Kähler-Einstein metrics with constant Ricci curvature. We will discuss the Fano case separately below, and assume here instead that X is of *general type*, i.e. K_X is a big line bundle. A metric $e^{-\phi}$ on K_X induces a measure on X, denoted by e^{ϕ} for convenience, and we can thus consider the functional

$$\phi \mapsto E(\phi) - \log \int e^{\phi},$$

which descends to

$$F_+ : \mathcal{T}^1(X, K_X) \to \mathbf{R}$$

by translation invariance. We will then show:

Theorem **C.** — *Let* X *be a manifold of general type. Then* F_+ *is upper semicontinuous and* J*-proper on* $\mathcal{T}^1(X, K_X)$*. It achieves its maximum on* $\mathcal{T}^1(X, K_X)$ *at a unique point* $T_{KE} = dd^c\phi_{KE}$*, which satisfies the Kähler-Einstein equation*

$$\langle T^n_{KE} \rangle = e^{\phi_{KE}+c}$$

for some $c \in \mathbf{R}$*.*

The solution ϕ_{KE} therefore coincides with the singular Kähler-Einstein metric of [EGZ09, SoTi08, BEGZ10], which was proved to have *minimal singularities* in [BEGZ10]. The ingredients entering the proof of Theorem C are similar to that of Theorem A, the functional F_+ being concave by Hölder's inequality.

Singular Kähler-Einstein metrics on Fano manifolds. — Assume now that X is a Fano manifold, i.e. $-K_X$ is ample. A psh weight ϕ on $-K_X$ with full Monge-Ampère mass has zero Lelong numbers. By a result of Skoda [Sko72], $e^{-\phi}$ can thus be viewed as a measure on X, with L^p density (with respect to Lebesgue measure) for every $p < +\infty$. The functional

$$\phi \mapsto E(\phi) + \log \int e^{-\phi}$$

descends to

$$F_- : \mathcal{T}^1(X, -K_X) \to \mathbf{R},$$

which coincides up to sign with the Ding functional [Ding88]. The critical points of F_- in the space of Kähler forms $\omega \in c_1(X)$ are exactly the Kähler-Einstein metrics. Assuming that $H^0(T_X) = 0$, so that Kähler-Einstein metrics are unique by [BM87], Ding and Tian showed that the properness of F_- implies the existence of a Kähler-Einstein metric, and that a Kähler-Einstein metric is necessarily a maximizer of F_- (see [Tian]).

Even though these results are variational in spirit, their proof by Ding and Tian relied on the continuity method. Using our variational approach, we reprove these results independently of the continuity method, and without any assumption on $H^0(T_X)$.

Theorem **D.** — *Let* X *be a Fano manifold. Then a current* $T = dd^c\phi$ *in* $\mathcal{T}^1(X, -K_X)$ *is a maximizer of* F_- *iff it satisfies the Kähler-Einstein equation* $T^n = e^{-\phi+c}$ *for some* $c \in \mathbf{R}$.

If F_- *is furthermore* J*-proper,*[2] *the supremum of* F_- *is attained.*

As we shall see, these Kähler-Einstein currents automatically have continuous potentials by [Koł98]. It is an interesting problem to investigate higher regularity of these functions.[3]

A striking feature of the present situation is that F_- is *not* concave. However, E is *affine* for the L^2-metric on the space of strictly psh weights considered in [Mab87, Sem92, Don99], and it follows from Berndtsson's results on psh variation of Bergman kernels [Bern09a] that $\phi \mapsto -\log \int e^{-\phi}$ is *convex* with respect to the L^2-metric. We thus see that F_- is concave with respect to the L^2-metric, which morally explains Ding and Tian's result (compare Donaldson's analogous result for the Mabuchi functional [Don05a]).

But a main issue is of course that *smooth* geodesics do not exist in general [LV11]. The proof of Theorem D will instead rely on *continuous* geodesics ϕ_t, whose existence is easily obtained.

Using similar ideas we give a new proof of Bando-Mabuchi's uniqueness result [BM87] and extend it to the case of singular Kähler-Einstein currents:

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[2] This condition implies that $H^0(T_X) = 0$, see for instance [BBEGZ11, Theorem 5.4].

[3] The smoothness of such currents has subsequently been established in [SzTo11], building on the regularizing properties of parabolic Monge-Ampère equations proved in [SoTi09]. An alternative argument relying on the usual maximum principle was later given in [BBEGZ11].

Theorem **E.** — *Let* X *be a Fano manifold. Assume that* X *admits a smooth Kähler-Einstein metric* ω_{KE} *and that* $H^0(T_X) = 0$. *Then* ω_{KE} *is the unique maximizer of* F_- *over the whole of* $\mathcal{T}^1(X, -K_X)$.

An important step in the proof is to show that each ϕ_t in the geodesic connecting two Kähler-Einstein metrics satisfies the Kähler-Einstein equation for all t if ϕ_0 and ϕ_1 do. Even though the geodesic ϕ_t is actually known to be (almost) $\mathcal{C}^{1,1}$ [Che00, Bło09], a main technical point is that ϕ_t is in general not *strictly* psh, and one has to resort again to the differentiability result of [BB10] to infer that ϕ_t is Kähler-Einstein from the fact that it maximizes F_-.

Finally, we establish in Theorem 7.1 an analogue of Theorem B for Kähler-Einstein metrics. More specifically, let X be Fano with $H^0(T_X) = 0$ and assume that ω_{KE} is a Kähler-Einstein metric. We will show that there exists a unique k-anticanonically balanced metric $\omega_k \in c_1(X)$ in the sense of [Don09] for each $k \gg 1$ and that $\omega_k \to \omega_{KE}$ weakly. The proof of the existence of such anticanonically balanced metrics relies in a crucial way on the linear growth estimate for F_- established in [PSSW08], strengthening a deep result of Tian [Tia97]. A proof of the existence and convergence of anticanonically balanced metrics was announced in [Kel09, Theorem 5]. The existence and *uniform* convergence of canonically balanced metrics has also been obtained independently by B. Berndtsson (personal communication).

Organization of the article. — The structure of the paper is as follows.

- Section 1 is devoted to preliminary results in the big case that are extracted from [BEGZ10] and [BD12]. The only new result here is the outer regularity of the Monge-Ampère capacity in the big case.
- Section 2 is similarly a refresher on energy functionals, whose goal is to recall results from [GZ07, BEGZ10] and to extend to the singular case a number of basic properties that are probably well-known in the smooth case.
- Section 3 investigates the continuity and growth properties of the functionals defined by integrating quasi-psh functions against a given Borel measure.
- Section 4 is devoted to the proof of Theorem A in the general case of big classes. Theorem 4.1 and Theorem 4.7 are the main statements.
- Section 5 connects our pluricomplex energy of measures to more classical notions of capacity and to some results from [BB10].
- Section 6 is devoted to singular Kähler-Einstein metrics. It contains the proof of Theorems C, D and E.
- Finally, Section 7 contains our results on balanced metrics. The main result is Theorem 7.1 which treats in parallel the (anti)canonically balanced case and balanced metrics, with respect to a singular measure (Theorem B).

1. Preliminary results on big cohomology classes

In this whole section, θ denotes a smooth closed $(1,1)$-form on a compact Kähler manifold X.

1.1. *Quasi-psh functions.* — Recall that a function

$$\varphi : X \to [-\infty, +\infty[$$

is said to be *θ-psh* iff $\varphi + \psi$ is psh for every local potential ψ of θ. In particular, φ is usc, integrable, and satisfies $\theta + dd^c\varphi \geq 0$ in the sense of currents, where d^c is normalized so that

$$dd^c = \frac{i}{2\pi}\partial\overline{\partial}.$$

By the dd^c-Lemma, any closed positive $(1,1)$-current T on X cohomologous to θ can conversely be written as $T = \theta + dd^c\varphi$, for some θ-psh function φ which is furthermore unique up to an additive constant.

The set of all θ-psh functions φ on X will be denoted by $\mathrm{PSH}(X, \theta)$ and endowed with the weak topology of distributions, which coincides with the $L^1(X)$-topology. By Hartogs' lemma, the map $\varphi \mapsto \sup_X \varphi$ is continuous in the weak topology. Since the set of closed positive currents in a fixed cohomology class is compact (in the weak topology), it follows that the set of $\varphi \in \mathrm{PSH}(X, \theta)$ normalized by $\sup_X \varphi = 0$ is also compact.

We introduce the extremal function $V_\theta : X \to \mathbf{R}$, defined at $x \in X$ by

$$V_\theta(x) := \sup\left\{\varphi(x) \mid \varphi \in \mathrm{PSH}(X, \theta), \sup_X \varphi \leq 0\right\}. \tag{1.1}$$

It is a θ-psh function with *minimal singularities* in the sense of Demailly, i.e. we have $\varphi \leq V_\theta + O(1)$ for any θ-psh function φ. In fact, it is straightforward to see that the following 'tautological maximum principle' holds:

$$\sup_X \varphi = \sup_X (\varphi - V_\theta) \tag{1.2}$$

for any $\varphi \in \mathrm{PSH}(X, \theta)$.

1.2. *Ample locus and non-pluripolar products.* — The cohomology class $\{\theta\} \in H^{1,1}(X, \mathbf{R})$ is said to be *big* iff there exists a closed $(1,1)$-current

$$T_+ = \theta + dd^c\varphi_+$$

cohomologous to θ and such that T_+ is *strictly positive* (i.e. $T_+ \geq \omega$ for some (small) Kähler form ω). By Demailly's regularization theorem [Dem92], one can then furthermore assume that T_+ has *analytic singularities*, i.e. there exists $c > 0$ such that locally on X we have

$$\varphi_+ = c \log \sum_{j=1}^{N} |f_j|^2 \bmod C^\infty,$$

where $f_1, \dots, f_N$ are local holomorphic functions. Such a current T is then C^∞ (hence a Kähler form) on a Zariski open subset Ω of X, and the *ample locus* $\mathrm{Amp}(\theta)$ of θ (in fact, of its class $\{\theta\}$) is defined as the largest such Zariski open subset (which exists by the Noetherian property of closed analytic subsets, see [Bou04]).

Note that *any* θ-psh function φ with minimal singularities is locally bounded on the ample locus $\mathrm{Amp}(\theta)$, since it has to satisfy $\varphi_+ \le \varphi + O(1)$.

In [BEGZ10] the (multilinear) *non-pluripolar product*

$$(T_1, \dots, T_p) \mapsto \langle T_1 \wedge \dots \wedge T_p \rangle$$

of closed positive $(1,1)$-currents is shown to be well-defined as a closed positive (p,p)-current putting no mass on pluripolar sets. In particular, given $\varphi_1, \dots, \varphi_n \in \mathrm{PSH}(X, \theta)$ we define their mixed Monge-Ampère measure as

$$\mathrm{MA}(\varphi_1, \dots, \varphi_n) = \left\langle \left(\theta + dd^c \varphi_1\right) \wedge \dots \wedge \left(\theta + dd^c \varphi_n\right) \right\rangle.$$

It is a non-pluripolar positive measure whose total mass satisfies

$$\int \mathrm{MA}(\varphi_1, \dots, \varphi_n) \le \mathrm{vol}(\theta),$$

where the right-hand side denotes the *volume* of the cohomology class of θ. If $\varphi_1, \dots, \varphi_n$ have minimal singularities, then they are locally bounded on $\mathrm{Amp}(\theta)$, and the product

$$\left(\theta + dd^c \varphi_1\right) \wedge \dots \wedge \left(\theta + dd^c \varphi_n\right)$$

is thus well-defined by Bedford-Taylor [BT82]. Its trivial extension to X coincides with $\mathrm{MA}(\varphi_1, \dots, \varphi_n)$, and we have

$$\int \mathrm{MA}(\varphi_1, \dots, \varphi_n) = \mathrm{vol}(\theta).$$

In case $\varphi_1 = \dots = \varphi_n = \varphi$, we simply set

$$\mathrm{MA}(\varphi) = \mathrm{MA}(\varphi, \dots, \varphi),$$

and we say that φ has *full Monge-Ampère mass* iff $\int \mathrm{MA}(\varphi) = \mathrm{vol}(\theta)$. We thus see that θ-psh functions with minimal singularities have full Monge-Ampère mass, but the converse is not true.

A crucial point is that the non-pluripolar Monge-Ampère operator is continuous along monotonic sequences of functions with full Monge-Ampère mass. In fact we have (cf. [BEGZ10, Theorem 2.17]):

Proposition **1.1.** — *The operator*

$$(\varphi_1, \ldots, \varphi_n) \mapsto \mathrm{MA}(\varphi_1, \ldots, \varphi_n)$$

is continuous along monotonic sequences of functions with full Monge-Ampère mass. If φ *has full Monge-Ampère mass and* $\int(\varphi - \mathrm{V}_\theta)\mathrm{MA}(\varphi)$ *is finite, then*

$$\lim_{j\to\infty}(\varphi_j - \mathrm{V}_\theta)\mathrm{MA}(\varphi_j) = (\varphi - \mathrm{V}_\theta)\mathrm{MA}(\varphi)$$

for any monotonic sequence $\varphi_j \to \varphi$.

1.3. *Regularity of envelopes.* — In case $\{\theta\} \in \mathrm{H}^{1,1}(\mathrm{X}, \mathbf{R})$ is a *Kähler* class, *smooth* θ-psh functions are abundant. On the other hand, for a general *big* class, the existence of even a *single* θ-psh function with minimal singularities that is also C^∞ on the ample locus $\mathrm{Amp}(\theta)$ is unknown. At any rate, it follows from [Bou04] that *no* θ-psh function with minimal singularities will have analytic singularities unless $\{\theta\}$ admits a *Zariski decomposition* (on some birational model of X). Examples of big line bundles without a Zariski decomposition have been constructed by Nakayama (see [Nak04, p. 136, Theorem 2.10]).

On the other hand, using Demailly's regularization theorem one can easily show that the extremal function V_θ introduced above satisfies

$$\mathrm{V}_\theta(x) = \sup\left\{\varphi(x) \mid \varphi \in \mathrm{PSH}(\mathrm{X}, \theta) \text{ with analytic singularities, } \sup_{\mathrm{X}} \varphi \leq 0\right\}$$

for $x \in \mathrm{Amp}(\theta)$, which implies in particular that V_θ is in fact *continuous* on $\mathrm{Amp}(\alpha)$. But we actually have the following much stronger regularity result on the ample locus. It was first obtained by the first named author in [Berm09] in case $\alpha = c_1(\mathrm{L})$ for a big line bundle L, and the general case is proved in [BD12].

Theorem **1.2.** — *The function* V_θ *has locally bounded Laplacian on* $\mathrm{Amp}(\theta)$.

Since V_θ is quasi-psh, this result is equivalent to the fact that the current $\theta + dd^c\mathrm{V}_\theta$ has $\mathrm{L}^\infty_{\mathrm{loc}}$ coefficients on $\mathrm{Amp}(\alpha)$, and implies in particular by Schauder's elliptic estimates that V_θ is in fact $\mathrm{C}^{2-\varepsilon}$ on $\mathrm{Amp}(\alpha)$ for each $\varepsilon > 0$.

As was observed in [Berm09], we also get as a consequence the following nice description of the Monge-Ampère measure of V_θ.

Corollary **1.3.** — *The Monge-Ampère measure* $\mathrm{MA}(\mathrm{V}_\theta)$ *has* L^∞*-density with respect to Lebesgue measure. More specifically, we have* $\theta \geq 0$ *pointwise on* $\{\mathrm{V}_\theta = 0\}$ *and*

$$\mathrm{MA}(\mathrm{V}_\theta) = \mathbf{1}_{\{\mathrm{V}_\theta = 0\}}\theta^n.$$

1.4. *Monge-Ampère capacity.* — As in [BEGZ10] we define the *Monge-Ampère (pre)capacity* with respect to the big class $\{\theta\}$ as the upper envelope of all measures $\mathrm{MA}(\varphi)$ with $\varphi \in \mathrm{PSH}(\mathrm{X}, \theta)$ such that $\mathrm{V}_\theta - 1 \leq \varphi \leq \mathrm{V}_\theta$, i.e.

$$\mathrm{Cap}(\mathrm{B}) := \sup\left\{\int_{\mathrm{B}} \mathrm{MA}(\varphi) \mid \varphi \in \mathrm{PSH}(\mathrm{X}, \theta),\ \mathrm{V}_\theta - 1 \leq \varphi \leq \mathrm{V}_\theta \text{ on } \mathrm{X}\right\} \tag{1.3}$$

for every Borel subset B of X. In what follows, we adapt to our setting some arguments of [GZ05, Theorem 3.2] (which dealt with the case where θ is a Kähler form).

Lemma **1.4.** — *If* K *is compact, the supremum in the definition of* Cap(K) *is achieved by the usc regularization of*

$$h_{\mathrm{K}} := \sup\left\{\varphi \in \mathrm{PSH}(\mathrm{X}, \theta) \mid \varphi \leq \mathrm{V}_\theta \textit{ on } \mathrm{X},\ \varphi \leq \mathrm{V}_\theta - 1 \textit{ on } \mathrm{K}\right\}.$$

Proof. — It is clear that h_{K}^* is a candidate in the supremum defining Cap(K). Conversely pick $\varphi \in \mathrm{PSH}(\mathrm{X}, \theta)$ such that $\mathrm{V}_\theta - 1 \leq \varphi \leq \mathrm{V}_\theta$ on X. We have to show that

$$\int_{\mathrm{K}} \mathrm{MA}(\varphi) \leq \int_{\mathrm{K}} \mathrm{MA}\big(h_{\mathrm{K}}^*\big).$$

Upon replacing φ by $(1-\varepsilon)\varphi + \varepsilon \mathrm{V}_\theta$ and then letting $\varepsilon > 0$ go to 0, we may assume that $\mathrm{V}_\theta - 1 < \varphi \leq \mathrm{V}_\theta$ everywhere on X. Noting that $\mathrm{K} \subset \{h_{\mathrm{K}}^* < \varphi\}$ we get

$$\begin{aligned}\int_{\mathrm{K}} \mathrm{MA}(\varphi) &\leq \int_{\{h_{\mathrm{K}}^* < \varphi+1\}} \mathrm{MA}(\varphi)\\ &\leq \int_{\{h_{\mathrm{K}}^* < \varphi+1\}} \mathrm{MA}\big(h_{\mathrm{K}}^*\big)\\ &\leq \int_{\{h_{\mathrm{K}}^* < \mathrm{V}_\theta\}} \mathrm{MA}\big(h_{\mathrm{K}}^*\big) = \int_{\mathrm{K}} \mathrm{MA}\big(h_{\mathrm{K}}^*\big)\end{aligned}$$

by the comparison principle (cf. [BEGZ10, Corollary 2.3] for a proof in our setting) and Lemma 1.5 below; the result follows. □

Lemma **1.5.** — *Let* K *be a compact subset. Then we have* $h_{\mathrm{K}}^* = \mathrm{V}_\theta - 1$ *a.e. on* K *and* $h_{\mathrm{K}}^* = \mathrm{V}_\theta$ *a.e. on* $\mathrm{X} \setminus \mathrm{K}$ *with respect to the measure* $\mathrm{MA}(h_{\mathrm{K}}^*)$.

Proof. — We have

$$h_{\mathrm{K}} \leq \mathrm{V}_\theta - 1 \leq h_{\mathrm{K}}^* \quad \text{on K}.$$

But the set $\{h_{\mathrm{K}} < h_{\mathrm{K}}^*\}$ is pluripolar by Bedford-Taylor's theorem [BT82], so it has zero measure with respect to the non-pluripolar measure $\mathrm{MA}(h_{\mathrm{K}}^*)$, and the first point follows.

On the other hand, by Choquet's lemma there exists a sequence of θ-psh functions φ_j increasing a.e. to h_K^* such that $\varphi_j \le V_\theta$ on X and $\varphi_j \le V_\theta - 1$ on K. If B is a small open ball centered at a point

$$x_0 \in \mathrm{Amp}(\theta) \cap \left\{h_K^* < V_\theta\right\} \cap (X \setminus K),$$

then we get

$$h_K \le V_\theta(x_0) - \delta \le V_\theta \quad \text{on B}$$

for some $\delta > 0$, by continuity of V_θ on $\mathrm{Amp}(\theta)$ (cf. Theorem 1.2); it follows that the function $\widehat{\varphi}_j$, which coincides with φ_j outside B and satisfies $\mathrm{MA}(\widehat{\varphi}_j) = 0$ on B, also satisfies

$$\widehat{\varphi}_j \le V_\theta(x_0) \le V_\theta \text{ on B.}$$

We infer that $\widehat{\varphi}_j$ increases a.e. to h_K^* and the result follows by Beford-Taylor's continuity theorem for the Monge-Ampère along non-decreasing sequences of locally bounded psh functions. □

By definition, a positive measure μ is absolutely continuous with respect the capacity Cap iff $\mathrm{Cap}(B) = 0$ implies $\mu(B) = 0$. This means exactly that μ is non-pluripolar in the sense that μ puts no mass on pluripolar sets. Since μ is subadditive, it is in turn equivalent to the existence of a non-decreasing, right-continuous function $F : \mathbf{R}_+ \to \mathbf{R}_+$ such that

$$\mu(B) \le F\big(\mathrm{Cap}(B)\big)$$

for all Borel sets B. Roughly speaking, the speed at which $F(t) \to 0$ as $t \to 0$ measures "how non-pluripolar" μ is.

Proposition **1.6.** — *Let* $F : \mathbf{R}_+ \to \mathbf{R}_+$ *be non-decreasing and right-continuous. Then the convex set of all positive measures* μ *on* X *with* $\mu(B) \le F(\mathrm{Cap}(B))$ *for all Borel subsets* B *is closed in the weak topology.*

Proof. — Since X is compact, the positive measure μ is inner regular, i.e.

$$\mu(B) = \sup_{K \subset B} \mu(K)$$

where K ranges over all compact subsets of B. It follows that $\mu(B) \le F(\mathrm{Cap}(B))$ holds for every Borel subset B iff $\mu(K) \le F(\mathrm{Cap}(K))$ holds for every compact subset K. This is however not enough to conclude since $\mu \mapsto \mu(K)$ is *upper* semi-continuous in the weak topology. We are going to show in turn that

$$\mu(K) \le F\big(\mathrm{Cap}(K)\big)$$

holds for every compact subset K iff

$$\mu(\mathrm{U}) \leq \mathrm{F}\big(\mathrm{Cap}(\mathrm{U})\big)$$

for every open subset U by showing that

$$\mathrm{Cap}(\mathrm{K}) = \inf_{\mathrm{U}\supset\mathrm{K}} \mathrm{Cap}(\mathrm{U}) \tag{1.4}$$

where U ranges over all open neighbourhoods of K. Indeed since F is right-continuous this yields $\mathrm{F}(\mathrm{Cap}(\mathrm{K})) = \inf_{\mathrm{U}\supset\mathrm{K}} \mathrm{F}(\mathrm{Cap}(\mathrm{U}))$. But $\mu \mapsto \mu(\mathrm{U})$ is now *lower* semi-continuous in the weak topology, so this will conclude the proof of Proposition 1.6.

By Lemma 1.4 and 1.5

$$\mathrm{Cap}(\mathrm{K}) = \int_{\mathrm{K}} \mathrm{MA}\big(h_{\mathrm{K}}^*\big) = \int \big(\mathrm{V}_\theta - h_{\mathrm{K}}^*\big)\mathrm{MA}\big(h_{\mathrm{K}}^*\big) \tag{1.5}$$

holds for every compact subset K. Now let K_j be a decreasing sequence of compact neighbourhoods of a given compact subset K. It is straightforward to check that $h_{\mathrm{K}_j}^*$ increases a.e. to h_{K}^*, and Proposition 1.1 thus yields

$$\inf_{\mathrm{U}\supset\mathrm{K}} \mathrm{Cap}(\mathrm{U}) \geq \mathrm{Cap}(\mathrm{K}) = \lim_{j\to\infty} \mathrm{Cap}(\mathrm{K}_j) \geq \inf_{\mathrm{U}\supset\mathrm{K}} \mathrm{Cap}(\mathrm{U})$$

as desired. □

Remark **1.7.** — Since the Monge-Ampère precapacity is defined as the upper envelope of a family of Radon measures, it is automatically *inner regular*, i.e. we have for each Borel subset $\mathrm{B} \subset \mathrm{X}$

$$\mathrm{Cap}(\mathrm{B}) = \sup_{\mathrm{K}\subset\mathrm{B}} \mathrm{Cap}(\mathrm{K})$$

where K ranges over all compact subsets of B. We claim that Cap is also *outer regular*, in the sense that

$$\mathrm{Cap}(\mathrm{B}) = \inf_{\mathrm{U}\supset\mathrm{B}} \mathrm{Cap}(\mathrm{U})$$

with U ranging over all open sets containing B. To see this, let Cap^* be the outer regularization of Cap, defined on an arbitrary subset $\mathrm{E} \subset \mathrm{X}$ by

$$\mathrm{Cap}^*(\mathrm{E}) := \inf_{\mathrm{U}\supset\mathrm{E}} \mathrm{Cap}(\mathrm{U}).$$

The above argument shows that

$$\mathrm{Cap}^*(\mathrm{K}) = \mathrm{Cap}(\mathrm{K})$$

holds for every compact subset K. Using (1.5) and following word for word the second half of the proof of [GZ05, Theorem 5.2], one can further show that Cap^* is in fact an (outer regular) *Choquet capacity*, and it then follows from Choquet's capacitability theorem that Cap^* is also *inner regular* on Borel sets. We thus get

$$\begin{aligned}\mathrm{Cap}(\mathrm{B}) \leq \mathrm{Cap}^*(\mathrm{B}) &= \sup_{\mathrm{K}\subset\mathrm{B}} \mathrm{Cap}^*(\mathrm{K}) \\ &= \sup_{\mathrm{K}\subset\mathrm{B}} \mathrm{Cap}(\mathrm{K}) \leq \mathrm{Cap}(\mathrm{B}),\end{aligned}$$

which proves the claim above.

2. Finite energy classes

We let again θ be a closed smooth $(1,1)$-form with big cohomology class. It will be convenient (and harmless by homogeneity) to assume that the volume of the class is normalized by

$$\mathrm{vol}(\theta) = 1.$$

For any $\varphi_1, \dots, \varphi_n \in \mathrm{PSH}(\mathrm{X}, \theta)$ with full Monge-Ampère mass, the mixed Monge-Ampère measure $\mathrm{MA}(\varphi_1, \dots, \varphi_n)$ is thus a *probability* measure. We will denote by $\Omega := \mathrm{Amp}(\theta)$ the ample locus of θ.

2.1. *Monge-Ampère energy functional.* — We define the *Monge-Ampère energy* of a function $\varphi \in \mathrm{PSH}(\mathrm{X}, \theta)$ with minimal singularities by

$$\mathrm{E}(\varphi) := \frac{1}{n+1} \sum_{j=0}^{n} \int (\varphi - \mathrm{V}_\theta)\mathrm{MA}\big(\varphi^{(j)}, \mathrm{V}_\theta^{(n-j)}\big). \tag{2.1}$$

Note that its restriction $t \mapsto \mathrm{E}(t\varphi + (1-t)\psi)$ to line segments is a polynomial map of degree $n+1$.

Let $\varphi, \psi \in \mathrm{PSH}(\mathrm{X}, \theta)$ with minimal singularities. It is easy to show by integration by parts (cf. [BEGZ10, BB10]) that the Gâteaux derivative of E at ψ is given by

$$\mathrm{E}'(\psi) \cdot (\varphi - \psi) = \int (\varphi - \psi)\mathrm{MA}(\psi), \tag{2.2}$$

while

$$\mathrm{E}''(\psi) \cdot (\varphi - \psi, \varphi - \psi) = -n \int_\Omega d(\varphi - \psi) \wedge d^c(\varphi - \psi) \wedge \big(\theta + dd^c\psi\big)^{n-1}, \tag{2.3}$$

which shows in particular that E is concave. Integration by parts also yields the following properties proved in [BEGZ10, BB10].

Proposition **2.1.** — E *is concave and non-decreasing. For any* $\varphi, \psi \in \mathrm{PSH}(X, \theta)$ *with minimal singularities we have*

$$\mathrm{E}(\varphi) - \mathrm{E}(\psi) = \frac{1}{n+1} \sum_{j=0}^{n} \int (\varphi - \psi)\mathrm{MA}\big(\varphi^{(j)}, \psi^{(n-j)}\big) \tag{2.4}$$

and

$$\int (\varphi - \psi)\mathrm{MA}(\varphi) \leq \cdots \leq \int (\varphi - \psi)\mathrm{MA}\big(\varphi^{(j)}, \psi^{(n-j)}\big) \leq \cdots \leq \int (\varphi - \psi)\mathrm{MA}(\psi) \tag{2.5}$$

for $j = 0, \ldots, n$.

We also remark that $\mathrm{E}(\mathrm{V}_\theta) = 0$, and note the scaling property

$$\mathrm{E}(\varphi + c) = \mathrm{E}(\varphi) + c \tag{2.6}$$

for any constant $c \in \mathbf{R}$.

We now introduce the analogue of Aubin's I and J-functionals (cf. [Aub84, p. 145], [Tian, p. 67]). We introduce the *symmetric* expression

$$\mathrm{I}(\varphi, \psi) := \int (\varphi - \psi)\big(\mathrm{MA}(\psi) - \mathrm{MA}(\varphi)\big) = -\big(\mathrm{E}'(\varphi) - \mathrm{E}'(\psi)\big) \cdot (\varphi - \psi),$$

and also set

$$\mathrm{J}_\psi(\varphi) := \mathrm{E}(\psi) - \mathrm{E}(\varphi) + \int (\varphi - \psi)\mathrm{MA}(\psi),$$

so that J_ψ is convex and non-negative by concavity of E. For $\psi = \mathrm{V}_\theta$ we simply write $\mathrm{J} := \mathrm{J}_{\mathrm{V}_\theta}$. Proposition 2.1 shows that $\mathrm{E}(\psi) - \mathrm{E}(\varphi)$ is the mean value of a non-decreasing sequence whose extreme values are $\int (\psi - \varphi)\mathrm{MA}(\psi)$ and $\int (\psi - \varphi)\mathrm{MA}(\varphi)$, and it follows for elementary reasons that

$$\frac{1}{n+1}\mathrm{I}(\varphi, \psi) \leq \mathrm{J}_\psi(\varphi) \leq \mathrm{I}(\varphi, \psi). \tag{2.7}$$

Simple algebraic identities involving integration by parts actually show as in [Tian, p. 58] that

$$\mathrm{J}_\psi(\varphi) = \sum_{j=0}^{n-1} \frac{j+1}{n+1} \int_\Omega d(\varphi - \psi) \wedge d^c(\varphi - \psi) \wedge \big(\theta + dd^c\psi\big)^j \wedge \big(\theta + dd^c\varphi\big)^{n-1-j} \tag{2.8}$$

and

$$\mathrm{I}(\varphi,\psi)=\sum_{j=0}^{n-1}\int_\Omega d(\varphi-\psi)\wedge d^c(\varphi-\psi)\wedge\big(\theta+dd^c\psi\big)^j\wedge\big(\theta+dd^c\varphi\big)^{n-1-j}. \tag{2.9}$$

As opposed to $\mathrm{I}(\varphi,\psi)$, the expression $\mathrm{J}_\psi(\varphi)$ is not symmetric in (φ,ψ). However, we have:

Lemma **2.2.** — *For any two* $\varphi,\psi\in\mathrm{PSH}(\mathrm{X},\theta)$ *with minimal singularities we have*

$$n^{-1}\mathrm{J}_\psi(\varphi)\le\mathrm{J}_\varphi(\psi)\le n\mathrm{J}_\psi(\varphi).$$

Proof. — By Proposition 2.1 we have

$$\begin{aligned} n\int(\varphi-\psi)\mathrm{MA}(\varphi)&+\int(\varphi-\psi)\mathrm{MA}(\psi)\\ &\le(n+1)\big(\mathrm{E}(\varphi)-\mathrm{E}(\psi)\big)\\ &\le\int(\varphi-\psi)\mathrm{MA}(\varphi)+n\int(\varphi-\psi)\mathrm{MA}(\psi)\end{aligned}$$

and the result follows immediately. □

Proposition **2.3.** — *For any* $\varphi,\psi\in\mathrm{PSH}(\mathrm{X},\theta)$ *with minimal singularities and any* $0\le t\le 1$ *we have*

$$\mathrm{I}\big(t\varphi+(1-t)\psi,\psi\big)\le nt^2\mathrm{I}(\varphi,\psi).$$

Proof. — We expand out

$$\begin{aligned}\int&(\varphi-\psi)\mathrm{MA}\big(t\varphi+(1-t)\psi\big)\\ &=(1-t)^n\int(\varphi-\psi)\mathrm{MA}(\psi)\\ &\quad+\sum_{j=1}^{n}\binom{n}{j}t^j(1-t)^{n-j}\int(\varphi-\psi)\mathrm{MA}\big(\varphi^{(j)},\psi^{(n-j)}\big)\\ &\ge(1-t)^n\int(\varphi-\psi)\mathrm{MA}(\psi)+\big(1-(1-t)^n\big)\int(\varphi-\psi)\mathrm{MA}(\varphi)\end{aligned}$$

by (2.5). This yields

$$\mathrm{I}\big(t\varphi+(1-t)\psi,\psi\big)\le t\big(1-(1-t)^n\big)\mathrm{I}(\varphi,\psi),$$

and the result follows by convexity of $(1-t)^n$. □

Note that by definition of I and J we have

$$\lim_{t\to 0_+} \frac{2}{t^2} \mathrm{J}_\psi\big(t\varphi + (1-t)\psi\big) = \lim_{t\to 0+} \frac{1}{t^2} \mathrm{I}\big(t\varphi + (1-t)\psi, \psi\big)$$
$$= -\mathrm{E}''(\psi)\cdot(\varphi-\psi, \varphi-\psi).$$

2.2. *Finite energy classes.* — As in [BEGZ10, Definition 2.9], it is natural to extend $\mathrm{E}(\varphi)$ by monotonicity to an *arbitrary* $\varphi \in \mathrm{PSH}(\mathrm{X}, \theta)$ by setting

$$\mathrm{E}(\varphi) := \inf\big\{\mathrm{E}(\psi) \mid \psi \in \mathrm{PSH}(\mathrm{X},\theta) \text{ with minimal singularities, } \psi \geq \varphi\big\}. \tag{2.10}$$

By [BEGZ10, Proposition 2.10] we have

Proposition **2.4.** — *The extension*

$$\mathrm{E} : \mathrm{PSH}(\mathrm{X}, \theta) \to [-\infty, +\infty[$$

so defined is concave, non-decreasing and usc.

As a consequence, E is continuous along decreasing sequences, and $\mathrm{E}(\varphi)$ can thus be more concretely obtained as the limit of $\mathrm{E}(\varphi_j)$ for any sequence of $\varphi_j \in \mathrm{PSH}(\mathrm{X}, \theta)$ with minimal singularities such that φ_j decreases to φ pointwise. One can for instance take $\varphi_j = \max\{\varphi, \mathrm{V}_\theta - j\}$.

Following [Ceg98] and [GZ07] we introduce

Definition **2.5.** — *The domain of* E *is denoted by*

$$\mathcal{E}^1(\mathrm{X}, \theta) := \big\{\varphi \in \mathrm{PSH}(\mathrm{X}, \theta), \mathrm{E}(\varphi) > -\infty\big\},$$

and its image in the set $\mathcal{T}(\mathrm{X}, \theta)$ *of all positive currents cohomologous to* θ *will be denoted by* $\mathcal{T}^1(\mathrm{X}, \theta)$. *For each* $\mathrm{C} > 0$ *we also set*

$$\mathcal{E}_\mathrm{C} := \Big\{\varphi \in \mathcal{E}^1(\mathrm{X}, \theta) \mid \sup_\mathrm{X} \varphi \leq 0, \mathrm{E}(\varphi) \geq -\mathrm{C}\Big\}.$$

Lemma **2.6.** — *For each* $\mathrm{C} > 0$ $\mathcal{E}_\mathrm{C}$ *is compact and convex.*

Proof. — Convexity follows from concavity of E. Pick $\varphi \in \mathrm{PSH}(\mathrm{X}, \theta)$ with $\sup_\mathrm{X} \varphi \leq 0$. We then have $\varphi \leq \mathrm{V}_\theta$ by (1.2) and it follows from the definition (2.1) of E that

$$\mathrm{E}(\varphi) \leq \int (\varphi - \mathrm{V}_\theta)\mathrm{MA}(\mathrm{V}_\theta) \leq \sup_\mathrm{X}(\varphi - \mathrm{V}_\theta) = \sup_\mathrm{X} \varphi,$$

using (1.2) again. Since E is usc, we thus see that $\mathcal{E}_\mathrm{C}$ is a closed subset of the compact set

$$\Big\{\varphi \in \mathrm{PSH}(\mathrm{X}, \theta) \mid -\mathrm{C} \leq \sup_\mathrm{X} \varphi \leq 0\Big\},$$

and the result follows. ∎

Lemma **2.7.** — *The integral*

$$\int (\varphi_0 - V_\theta)\mathrm{MA}(\varphi_1, \dots, \varphi_n)$$

is finite for every $\varphi_0, \dots, \varphi_n \in \mathcal{E}^1(X, \theta)$*; it is furthermore uniformly bounded in terms of* C *for* $\varphi_0, \dots, \varphi_n \in \mathcal{E}_C$.

Proof. — Upon passing to the canonical approximants, we may assume that $\varphi_0, \dots, \varphi_n$ have minimal singularities. Set $\psi := \frac{1}{n+1}(\varphi_0 + \cdots + \varphi_n)$ and observe that $V_\theta - \varphi_0 \le (n+1)(V_\theta - \psi)$. Using the convexity of $-E$ it follows that

$$\begin{aligned}\int (V_\theta - \varphi_0)\mathrm{MA}(\psi) &\le (n+1)\int (V_\theta - \psi)\mathrm{MA}(\psi)\\ &\le (n+1)^2\left|E(\psi)\right|\\ &\le (n+1)\big(\left|E(\varphi_0)\right| + \cdots + \left|E(\varphi_n)\right|\big).\end{aligned}$$

On the other hand, we easily get by expanding out

$$\mathrm{MA}(\psi) \ge c_n \mathrm{MA}(\varphi_1, \dots, \varphi_n)$$

with $c_n > 0$ only depending on n and the result follows. □

The following characterization of functions in $\mathcal{E}^1(X, \theta)$ follows from [BEGZ10, Proposition 2.11].

Proposition **2.8.** — *Let* $\varphi \in \mathrm{PSH}(X, \theta)$. *The following properties are equivalent:*

- $\varphi \in \mathcal{E}^1(X, \theta)$.
- φ *has full Monge-Ampère mass and* $\int (\varphi - V_\theta)\mathrm{MA}(\varphi)$ *is finite.*
- *We have*

$$\int_{t=0}^{+\infty} dt \int_{\{\varphi = V_\theta - t\}} \mathrm{MA}\big(\max\{\varphi, V_\theta - t\}\big) < +\infty.$$

Functions in $\mathcal{E}^1(X, \theta)$ can almost be characterized in terms of the capacity decay of sublevel sets:

Lemma **2.9.** — *Let* $\varphi \in \mathrm{PSH}(X, \theta)$. *If*

$$\int_{t=0}^{+\infty} t^n \,\mathrm{Cap}\{\varphi < V_\theta - t\} dt < +\infty$$

then $\varphi \in \mathcal{E}^1(X, \theta)$. *Conversely, for each* $\varphi \in \mathcal{E}^1(X, \theta)$

$$\int_{t=0}^{+\infty} t \,\mathrm{Cap}\{\varphi < V_\theta - t\} dt$$

is finite, and uniformly bounded in terms of C *for* $\varphi \in \mathcal{E}_C$.

Note that if φ is an arbitrary θ-psh function then $\mathrm{Cap}\{\varphi < \mathrm{V}_\theta - t\}$ usually decreases no faster that $1/t$ as $t \to +\infty$.

Proof. — The proof is adapted from [GZ07, Lemma 5.1]. Observe that for each $t \geq 1$ the function $\varphi_t := \max\{\varphi, \mathrm{V}_\theta - t\}$ satisfies $\mathrm{V}_\theta - t \leq \varphi_t \leq \mathrm{V}_\theta$. It follows that

$$t^{-1}\varphi_t + \left(1 - t^{-1}\right)\mathrm{V}_\theta$$

is a candidate in the supremum defining Cap, and hence

$$\mathrm{MA}(\varphi_t) \leq t^n \,\mathrm{Cap}\,.$$

Now the first assertion follows from Proposition 2.8.

In order to prove the converse we apply the comparison principle. Pick a candidate

$$\psi \in \mathrm{PSH}(\mathrm{X}, \theta), \qquad \mathrm{V}_\theta - 1 \leq \psi \leq \mathrm{V}_\theta$$

in the supremum defining Cap. For $t \geq 1$ we have

$$\{\varphi < \mathrm{V}_\theta - 2t\} \subset \left\{t^{-1}\varphi + \left(1 - t^{-1}\right)\mathrm{V}_\theta < \psi - 1\right\} \subset \{\varphi < \mathrm{V}_\theta - t\}$$

thus the comparison principle (cf. [BEGZ10, Corollary 2.3]) implies

$$\begin{aligned}
&\int_{\{\varphi<\mathrm{V}_\theta-2t\}} \mathrm{MA}(\psi)\\
&\quad\leq \int_{\{\varphi<\mathrm{V}_\theta-t\}} \mathrm{MA}\left(t^{-1}\varphi + \left(1-t^{-1}\right)\mathrm{V}_\theta\right)\\
&\quad\leq \int_{\{\varphi<\mathrm{V}_\theta-t\}} \mathrm{MA}(\mathrm{V}_\theta) + \sum_{j=1}^{n} \binom{n}{j} t^{-j} \int_{\{\varphi<\mathrm{V}_\theta-t\}} \mathrm{MA}\left(\varphi^{(j)}, \mathrm{V}_\theta^{(n-j)}\right)\\
&\quad\leq \int_{\{\varphi<\mathrm{V}_\theta-t\}} \mathrm{MA}(\mathrm{V}_\theta) + \mathrm{C}_1 t^{-1} \sum_{j=1}^{n} \int_{\{\varphi<\mathrm{V}_\theta-t\}} \mathrm{MA}\left(\varphi^{(j)}, \mathrm{V}_\theta^{(n-j)}\right)
\end{aligned}$$

since $t \geq 1$, and it follows that

$$\int_{t=0}^{+\infty} t\,\mathrm{Cap}\{\varphi < \mathrm{V}_\theta - t\} \leq \mathrm{C}_2 + \mathrm{C}_3 \int (\mathrm{V}_\theta - \varphi)^2 \mathrm{MA}(\mathrm{V}_\theta)$$

since $\mathrm{E}(\varphi) \geq -\mathrm{C}$ and $\mathrm{Cap} \leq 1$. But $\mathrm{MA}(\mathrm{V}_\theta)$ has L^∞-density with respect to Lebesgue measure by Corollary 1.3, and it follows from the uniform version of Skoda's theorem [Zer01] that there exists $\varepsilon > 0$ and $\mathrm{C}_1 > 0$ such that

$$\int e^{-\varepsilon\varphi} \mathrm{MA}(\mathrm{V}_\theta) \leq \mathrm{C}_1$$

for all φ in the compact subset $\mathcal{E}_\mathrm{C}$ of $\mathrm{PSH}(\mathrm{X}, \theta)$. This implies in turn that $\int (\mathrm{V}_\theta - \varphi)^2 \mathrm{MA}(\mathrm{V}_\theta)$ is uniformly bounded for $\varphi \in \mathcal{E}_\mathrm{C}$, and the result follows. □

Remark **2.10.** — It is not true in general that $\int_{t=0}^{+\infty} t^n \operatorname{Cap}\{\varphi < V_\theta - t\}dt < +\infty$ for all $\varphi \in \mathcal{E}^1(X, \theta)$. Indeed, [CGZ08, Example 3.4] exhibits a function $\varphi \in \mathcal{E}^1(X, \theta)$ (with $X = \mathbf{P}^1 \times \mathbf{P}^1$ and θ the product of the Fubini-Study metrics) such that $\varphi|_B$ is not in $\mathcal{E}^1(B)$ for some open ball $B \subset X$. By [BGZ09, Proposition B], we thus have $\int_{t=0}^{+\infty} t^n \operatorname{Cap}_B\{\varphi < -t\}dt = +\infty$, hence also $\int_{t=0}^{+\infty} t^n \operatorname{Cap}\{\varphi < -t\}dt = +\infty$ since the local and global capacities are comparable (see [GZ05, Proposition 3.10]).

Corollary **2.11.** — *If* $A \subset X$ *is a (locally) pluripolar subset, then there exists* $\varphi \in \mathcal{E}^1(X, \theta)$ *such that* $A \subset \{\varphi = -\infty\}$.

Proof. — Since $\{\theta\}$ is big, there exists a proper modification $\mu : X' \to X$ and an effective $\mathbf{R}$-divisor E on X' such that $\mu^*\theta - [E]$ is cohomologous to a Kähler form ω on X'. By the Kähler version of Josefson's theorem [GZ05, Theorem 6.2], we may thus find a positive current T in the class of ω whose polar set contains A. The push-forward $\mu_*(T + [E])$ is then a positive current in the class of θ, and its potential $\varphi \in \operatorname{PSH}(X, \theta)$ therefore satisfies $A \subset \{\varphi = -\infty\}$. Now let $\chi : \mathbf{R} \to \mathbf{R}$ be a smooth, convex and non-decreasing function such that $\chi(-\infty) = -\infty$ and $\chi(s) = s$ for all $s \geq 0$. If φ is θ-psh, then so is

$$\varphi_\chi := \chi \circ (\varphi - V_\theta) + V_\theta,$$

and A is contained in the poles of φ_χ. On the other hand, we can clearly make $\operatorname{Cap}\{\varphi_\chi < V_\theta - t\}$ tend to 0 as fast as we like when $t \to \infty$ by choosing χ with a sufficiently slow decay at $-\infty$. It thus follows from Lemma 2.9 that $\varphi_\chi \in \mathcal{E}^1(X, \theta)$ for an appropriate choice of χ, and the result follows. Actually $\chi(t) = -\log(1 - t)$ is enough (compare [GZ07, Example 5.2]). □

3. Action of a measure on psh functions

3.1. *Finiteness.* — Given a probability measure μ on X and $\varphi \in \operatorname{PSH}(X, \theta)$ we set

$$L_\mu(\varphi) := \int_\Omega (\varphi - V_\theta)d\mu \qquad (\mathbf{3.1})$$

where $\Omega := \operatorname{Amp}(\theta)$ denotes the ample locus. Note that $L_\mu(\varphi) = \int_X (\varphi - V_\theta)d\mu$ when μ is non-pluripolar, since $X \setminus \Omega$ is in particular pluripolar.

The functional $L_\mu : \operatorname{PSH}(X, \theta) \to [-\infty, +\infty[$ so defined is obviously affine, and it satisfies the scaling property

$$L_\mu(\varphi + c) = L_\mu(\varphi) + c$$

for any $c \in \mathbf{R}$.

In the special case where $\mu = \mathrm{MA}(V_\theta)$ we will simply write

$$(\mathbf{3.2}) \qquad L_0(\varphi) := L_{\mathrm{MA}(V_\theta)}(\varphi) = \int (\varphi - V_\theta)\mathrm{MA}(V_\theta),$$

so that

$$J = L_0 - E$$

holds by definition.

Lemma **3.1.** — L_μ *is usc on* $\mathrm{PSH}(X,\theta)$. *For each* $\varphi \in \mathrm{PSH}(X,\theta)$, *the map* $\mu \mapsto L_\mu(\varphi)$ *is also usc.*

Proof. — Let $\varphi_j \to \varphi$ be a convergent sequence of functions in $\mathrm{PSH}(X,\theta)$. Hartogs' lemma implies that φ_j is uniformly bounded from above, hence so is $\varphi_j - V_\theta$. Since we have

$$\varphi = \left(\limsup_{j\to\infty} \varphi_j\right)^* \geq \limsup_{j\to\infty} \varphi_j$$

everywhere on X we get as desired

$$L_\mu(\varphi) \geq \limsup_{j\to\infty} L_\mu(\varphi_j)$$

by Fatou's lemma. The second assertion follows directly from the fact that $\varphi - V_\theta$ is usc on Ω, which is true since V_θ is continuous on Ω. □

Lemma **3.2.** — *Let* $\varphi \in \mathrm{PSH}(X,\theta)$ *and set* $\mu := \mathrm{MA}(\varphi)$.

(i) *If* φ *has minimal singularities then* L_μ *is finite on* $\mathrm{PSH}(X,\theta)$.
(ii) *If* $\varphi \in \mathcal{E}^1(X,\theta)$ *then* L_μ *is finite on* $\mathcal{E}^1(X,\theta)$.

Proof. — (ii) follows directly from Lemma 2.7. We prove (i). Let $\psi \in \mathrm{PSH}(X,\theta)$. We can assume that $\psi \leq 0$, or equivalently $\psi \leq V_\theta$. Assume first that ψ also has minimal singularities. If we set $\Omega := \mathrm{Amp}(\theta)$, then we can integrate by parts using [BEGZ10, Theorem 1.14] to get

$$\int_\Omega (V_\theta - \psi)\left(\theta + dd^c\varphi\right)^n = \int_\Omega (V_\theta - \psi)\left(\theta + dd^c V_\theta\right) \wedge \left(\theta + dd^c\varphi\right)^{n-1}$$
$$+ \int_\Omega (\varphi - V_\theta) dd^c(V_\theta - \psi) \wedge \left(\theta + dd^c\varphi\right)^{n-1}.$$

The second term is equal to

$$\int_\Omega (\varphi - V_\theta)\big(\theta + dd^c V_\theta\big) \wedge \big(\theta + dd^c\varphi\big)^{n-1} - \int_\Omega (\varphi - V_\theta)\big(\theta + dd^c\psi\big) \wedge \big(\theta + dd^c\varphi\big)^{n-1},$$

and each of these integrals is controled by $\sup_X |\varphi - V_\theta|$. By iterating integration by parts as above we thus get

$$\int (V_\theta - \psi)\mathrm{MA}(\varphi) \le n \sup_X |\varphi - V_\theta| + \int (V_\theta - \psi)\mathrm{MA}(V_\theta).$$

This inequality remains valid for any $\psi \in \mathrm{PSH}(X, \theta)$, as we see by applying it to the canonical approximants $\max\{\psi, V_\theta - j\}$ and letting $j \to \infty$. Since $\mathrm{MA}(V_\theta)$ has L^∞ density with respect to Lebesgue measure by Corollary 1.3, $\int (V_\theta - \psi)\mathrm{MA}(V_\theta)$ is finite for any θ-psh function ψ, hence so is $\int (V_\theta - \psi)\mathrm{MA}(\varphi)$. □

3.2. *Properness and coercivity.* — The J-functional is translation invariant, hence descends to a non-negative, convex and lower semicontinuous function $J : \mathcal{T}(X, \theta) \to [0, +\infty]$ which is finite precisely on $\mathcal{T}^1(X, \theta)$. It actually defines an *exhaustion* function of $\mathcal{T}^1(X, \theta)$:

Lemma **3.3.** — *The function* $J : \mathcal{T}^1(X, \theta) \to [0, +\infty[$ *is an exhaustion of* $\mathcal{T}^1(X, \theta)$ *in the sense that each sublevel set* $\{J \le C\} \subset \mathcal{T}^1(X, \theta)$ *is compact.*

Proof. — By Lemma 3.2 there exists $A > 0$ such that

$$\sup_X \varphi - A \le \int \varphi \mathrm{MA}(V_\theta) \le \sup_X \varphi.$$

Now pick $T \in \{J \le C\}$ and write it as $T = \theta + dd^c\varphi$ with $\sup_X \varphi = 0$. We then have

$$J(T) = \int \varphi \mathrm{MA}(V_\theta) - E(\varphi) \le C$$

thus $E(\varphi) \ge -C - A$. This means that the closed set $\{J \le C\}$ is contained in the image of $\mathcal{E}_{C+A}$ by the quotient map

$$\mathrm{PSH}(X, \theta) \to \mathcal{T}(X, \theta).$$

The result now follows since $\mathcal{E}_{C+A}$ is compact by Lemma 2.6. □

The next result extends part of [GZ07, Lemma 2.11].

Proposition **3.4.** — *Let* $L : \mathrm{PSH}(X, \theta) \to [-\infty, +\infty[$ *be a convex and non-decreasing function satisfying the scaling property* $L(\varphi + c) = L(\varphi) + c$ *for* $c \in \mathbf{R}$.

(i) *If* L *is finite valued on a given compact convex subset* $\mathcal{K}$ *of* $\mathrm{PSH}(\mathrm{X},\theta)$*, then* L *is automatically bounded on* $\mathcal{K}$*.*

(ii) *If* L *is finite valued on* $\mathcal{E}^1(\mathrm{X},\theta)$*, then*

$$\sup_{\mathcal{E}_\mathrm{C}} |\mathrm{L}| = \mathrm{O}\big(\mathrm{C}^{1/2}\big) \tag{3.3}$$

as $\mathrm{C}\to+\infty$.

Proof. — (i) There exists $\mathrm{C}>0$ such that

$$\sup_\mathrm{X}(\varphi-\mathrm{V}_\theta) = \sup_\mathrm{X}\varphi \le \mathrm{C}$$

for all $\varphi\in\mathcal{K}$, thus L is uniformly bounded above by $\mathrm{L}(\mathrm{V}_\theta)+\mathrm{C}$. Assume by contradiction that $\mathrm{L}(\varphi_j)\le -2^j$ for some sequence $\varphi_j\in\mathcal{K}$. We then consider $\varphi := \sum_{j\ge1} 2^{-j}\varphi_j$, which belongs to $\mathcal{K}$ by Lemma 3.5 below. By (1.2) we have

$$\varphi \le \sum_{j=1}^{\mathrm{N}} 2^{-j}\varphi_j + 2^{-\mathrm{N}}(\mathrm{V}_\theta+\mathrm{C})$$

for each N, and the right-hand side is a (finite) convex combination of elements in $\mathrm{PSH}(\mathrm{X},\theta)$. The properties of L thus imply

$$-\infty < \mathrm{L}(\varphi) \le \sum_{j=1}^{\mathrm{N}} 2^{-j}\mathrm{L}(\varphi_j) + 2^{-\mathrm{N}}(\mathrm{L}(\mathrm{V}_\theta)+\mathrm{C}) = -\mathrm{N} + 2^{-\mathrm{N}}(\mathrm{L}(\mathrm{V}_\theta)+\mathrm{C})$$

and we reach a contradiction by letting $\mathrm{N}\to+\infty$.

(ii) By (i) we have $\sup_{\mathcal{E}_\mathrm{C}}|\mathrm{L}|<+\infty$ for all $\mathrm{C}>0$. Note also that $\mathrm{L}(\varphi)\le\mathrm{L}(\mathrm{V}_\theta)$ for $\varphi\in\mathcal{E}_\mathrm{C}$. If $\sup_{\mathcal{E}_\mathrm{C}}|\mathrm{L}| = \mathrm{O}(\mathrm{C}^{1/2})$ fails as $\mathrm{C}\to+\infty$, then there exists a sequence $\varphi_j\in\mathcal{E}^1(\mathrm{X},\theta)$ with $\sup_\mathrm{X}\varphi_j=0$ such that

$$t_j := \big|\mathrm{E}(\varphi_j)\big|^{-1/2}\to 0$$

and

$$t_j\mathrm{L}(\varphi_j)\to-\infty. \tag{3.4}$$

We claim that there exists $\mathrm{C}>0$ such that for any $\varphi\in\mathrm{PSH}(\mathrm{X},\theta)$ with $\sup_\mathrm{X}\varphi=0$ and $t:=|\mathrm{E}(\varphi)|^{-1/2}\le 1$ we have

$$\mathrm{E}\big(t\varphi+(1-t)\mathrm{V}_\theta\big)\ge -\mathrm{C}.$$

Indeed, $\int(\varphi-\mathrm{V}_\theta)\mathrm{MA}(\mathrm{V}_\theta)$ is uniformly bounded when $\sup_\mathrm{X}\varphi=0$ (for instance by (i)), and the claim follows from Proposition 2.3 applied to $\psi=\mathrm{V}_\theta$.

The claim implies that $t_j\varphi_j + (1-t_j)V_\theta \in \mathcal{E}_C$ for all $j \gg 1$, hence

$$t_j L(\varphi_j) + (1-t_j)L(V_\theta) \geq L\big(t_j\varphi_j + (1-t_j)V_\theta\big) \geq \inf_{\mathcal{E}_C} L > -\infty$$

by convexity of E, which contradicts (3.4). □

Lemma **3.5.** — *Let* $\varphi_j \in \mathcal{K}$ *be a sequence in a compact convex subset* $\mathcal{K}$ *of* $\mathrm{PSH}(X,\theta)$. *Then* $\varphi := \sum_{j\geq 1} 2^{-j}\varphi_j$ *belongs to* $\mathcal{K}$.

Proof. — By Hartogs' lemma $\sup_X \varphi$ is uniformly bounded for $\varphi \in \mathcal{K}$, thus we may assume upon translating by a constant that $\sup_X \varphi \leq 0$ for each $\varphi \in \mathcal{K}$. Let μ be a smooth volume form on X. Then $\int \varphi_j\, d\mu$ is uniformly bounded since $\mathcal{K}$ is a compact subset of $L^1(X)$. It thus follows that $\int \varphi\, d\mu$ is finite by Fatou's lemma. But since φ is a decreasing limit of functions in $\mathrm{PSH}(X,\theta)$ we either have $\varphi \in \mathrm{PSH}(X,\theta)$ or $\varphi \equiv -\infty$, and the latter case is excluded by $\int \varphi\, d\mu > -\infty$. □

We will now interpret Proposition 3.4 as a *coercivity* condition. Since our convention is to *maximize* certain functionals in our variational approach, we shall use the following terminology.

Definition **3.6.** — *A function* $F : \mathcal{T}^1(X,\theta) \to \mathbf{R}$ *will be said to be*

(i) J-proper *if* $F \to -\infty$ *as* $J \to +\infty$.
(ii) J-coercive *if there exists* $\varepsilon > 0$ *and* $A > 0$ *such that*

$$F \leq -\varepsilon J + A$$

on $\mathcal{T}^1(X,\theta)$.

Any function F on $\mathcal{T}^1(X,\theta)$ is induced by a function on $\mathcal{E}^1(X,\theta)$ of the form $E - L$ where L satisfies as above the scaling property. The J-coercivity of F reads

$$E - L \leq -\varepsilon(L_0 - E) + A$$

where $\varepsilon > 0$ can of course be assumed to satisfy $\varepsilon < 1$ since $J \geq 0$. Since we have

$$L_0(\varphi) = \sup_X \varphi + O(1)$$

uniformly for $\varphi \in \mathrm{PSH}(X,\theta)$ the J-coercivity of F is then easily seen to be equivalent to the growth condition

$$\sup_{\mathcal{E}_C} |L| \leq (1-\varepsilon)C + O(1) \tag{3.5}$$

as $C \to +\infty$.

As a consequence of Proposition 3.4 we get

Corollary **3.7.** — *Let* $L : \mathcal{E}^1(X,\theta) \to \mathbf{R}$ *be a convex, non-decreasing function satisfying the scaling property. Then the function* F *on* $\mathcal{T}^1(X,\theta)$ *induced by* $E - L$ *is* J*-coercive.*

Let us finally record the following useful elementary fact.

Proposition **3.8.** — *Let* F *be a* J*-proper and usc function on* $\mathcal{T}^1(X,\theta)$. *Then* F *achieves its supremum on* $\mathcal{T}^1(X,\theta)$. *Moreover any asymptotically maximizing sequence* $T_j \in \mathcal{T}^1(X,\theta)$ *(i.e. such that* $\lim_{j\to\infty} F(T_j) = \sup F$*) stays in a compact subset of* $\mathcal{T}^1(X,\theta)$, *and any accumulation point* T *of the* T_j*'s is an* F*-maximizer.*

Proof. — Let us recall the standard argument. It is clearly enough to settle the second part. Let thus T_j be a maximizing sequence. It follows in particular that $F(T_j)$ is bounded from below, and the J-properness of F thus yields $C > 0$ such that $T_j \in \{J \le C\}$ for all j. Since $\{J \le C\}$ is compact there exists an accumulation point T of the T_j's, and $F(T_j) \to \sup F$ implies $F(T) \ge \sup F$ since F is usc. □

3.3. *Continuity.* — In the sequel, we will be interested in the upper semi-continuity of $F_\mu = E - L_\mu$ on $\mathcal{E}^1(X,\theta)$. We start with the following simple observation:

Lemma **3.9.** — *Let* $L : \mathcal{E}^1(X,\theta) \to \mathbf{R}$ *be a function satisfying the scaling property, and assume that* $F := E - L$ *is* J*-proper. If* L *is lsc on* $\mathcal{E}_C$ *for all* $C > 0$, *then* F *is usc on* $\mathcal{E}^1(X,\theta)$.

Proof. — For each $A \in \mathbf{R}$ we are to show that $\{F \ge A\}$ is closed in $\mathcal{E}^1(X,\theta)$. But properness of F yields $C > 0$ such that $\{F \ge A\} \subset \mathcal{E}_C$, and the result follows since E is usc while L is lsc on $\mathcal{E}_C$ by assumption. □

For each $\mu \in \mathcal{M}_X$, L_μ is usc on $\mathrm{PSH}(X,\theta)$ by Lemma 3.1, and we are thus reduced to understanding the continuity of L_μ on each $\mathcal{E}_C$. The next result provides a general criterion in this direction:

Theorem **3.10.** — *Let* μ *be a non-pluripolar measure and let* $\mathcal{K} \subset \mathrm{PSH}(X,\theta)$ *be a compact convex subset such that* L_μ *is finite on* $\mathcal{K}$. *The following properties are equivalent.*

(i) L_μ *is continuous on* $\mathcal{K}$.
(ii) *The map* $\tau : \mathcal{K} \to L^1(\mu)$ *defined by* $\tau(\varphi) := \varphi - V_\theta$ *is continuous.*
(iii) *The set* $\tau(\mathcal{K}) \subset L^1(\mu)$ *is uniformly integrable, i.e.*

$$\int_{t=m}^{+\infty} \mu\{\varphi \le V_\theta - t\}\,dt \to 0$$

as $m \to +\infty$, *uniformly for* $\varphi \in \mathcal{K}$.

Proof. — By the Dunford-Pettis theorem, assumption (iii) means that $\tau(\mathcal{K})$ is relatively compact in the weak topology (induced by $L^\infty(\mu) = L^1(\mu)^*$).

As a first general remark, we claim that graph of τ is closed. Indeed let $\varphi_j \to \varphi$ be a convergent sequence in $\mathcal{K}$ and assume that $\tau(\varphi_j) \to f$ in $L^1(\mu)$. We have to show that $f = \tau(\varphi)$. But $\varphi_j \to \varphi$ implies that

$$\varphi = \left(\limsup_{j\to\infty} \varphi_j\right)^*$$

everywhere on X by general properties of psh functions. On the other hand the set of points where $(\limsup_{j\to\infty} \varphi_j)^* > \limsup_{j\to\infty} \varphi_j$ is negligible hence pluripolar by a theorem of Bedford-Taylor [BT87], thus has μ-measure 0 by assumption on μ. We thus see that $\varphi = \limsup_j \varphi_j$ μ-a.e, hence $\tau(\varphi) = \limsup_j \tau(\varphi_j)$ μ-a.e. Since $\tau(\varphi_j) \to f$ in $L^1(\mu)$ there exists a subsequence such that $\tau(\varphi_j) \to f$ μ-a.e., and it follows that $f = \tau(\varphi)$ μ-a.e. as desired.

This closed graph property implies that the convex set $\tau(\mathcal{K})$ is closed in the norm topology (hence also in the weak topology by the Hahn-Banach theorem). Indeed if $\tau(\varphi_j) \to f$ holds in $L^1(\mu)$, then we may assume that $\varphi_j \to \varphi$ in $\mathcal{K}$ by compactness of the latter space, hence $f = \tau(\varphi)$ belongs to $\tau(\mathcal{K})$ by the closed graph property.

We now prove the equivalence between (i) and (ii). Observe that there exists $C > 0$ such that $\tau(\varphi) = \varphi - V_\theta \le C$ for all $\varphi \in \mathcal{K}$, since $\sup_X \varphi = \sup_X(\varphi - V_\theta)$ is bounded on the compact set $\mathcal{K}$ by Hartogs' lemma. Given a convergent sequence $\varphi_j \to \varphi$ in $\mathcal{K}$ we have $\tau(\varphi) \ge \limsup_{j\to\infty} \tau(\varphi_j)$ μ-a.e. as was explained above, thus Fatou's lemma (applied to the sequence of non-negative functions $C - \tau(\varphi_j)$) yields the asymptotic upper bound

$$\limsup_{j\to\infty} \int \tau(\varphi_j) d\mu \le \int \tau(\varphi) d\mu,$$

and the asymptotic equality case

$$\int \tau(\varphi) d\mu = \lim_{j\to\infty} \int \tau(\varphi_j) d\mu$$

holds iff $\tau(\varphi_j) \to \tau(\varphi)$ in $L^1(\mu)$. This follows from a basic lemma in integration theory, which proves the desired equivalence.

If (ii) holds, then the closed convex subset $\tau(\mathcal{K})$ is compact in the norm topology, hence also weakly compact, and (iii) holds by the Dunford-Pettis theorem recalled above.

Conversely assume that (iii) holds. We will prove (i). Let $\varphi_j \to \varphi$ be a convergent sequence in $\mathcal{K}$. We are to prove that $\int \tau(\varphi_j) d\mu \to \int \tau(\varphi) d\mu$ in $L^1(\mu)$. We may assume that $\int \tau(\varphi_j) d\mu \to L$ for some $L \in \mathbf{R}$ since $\tau(\mathcal{K})$ is bounded, and we have to show that $L = \int \tau(\varphi) d\mu$. For each k consider the closed convex envelope

$$\mathcal{C}_k := \overline{\mathrm{Conv}\left\{\tau(\varphi_j) \mid j \ge k\right\}}.$$

Each $\mathcal{C}_k$ is weakly closed by the Hahn-Banach theorem, hence weakly compact since it is contained in $\tau(\mathcal{K})$. Since $(\mathcal{C}_k)_k$ is a decreasing sequence of non-empty compact subsets, there exists $f \in \bigcap_k \mathcal{C}_k$. For each k we may thus find a finite convex combination

$$\psi_k \in \mathrm{Conv}\{\varphi_j \mid j \geq k\}$$

such that $\tau(\psi_k) \to f$ in the norm topology. Since $\varphi_j \to \varphi$ in $\mathcal{K}$ we also have $\psi_k \to \varphi$ in $\mathcal{K}$, hence $f = \tau(\varphi)$ by the closed graph property. On the other hand, $\int \tau(\psi_k) d\mu$ is a convex combination of elements of the form $\int \tau(\varphi_j) d\mu, j \geq k$, thus $\int \tau(\psi_k) d\mu \to \mathrm{L}$, and we finally get $\int \tau(\varphi) d\mu = \int f d\mu = \mathrm{L}$ as desired. $\square$

By Hölder's inequality, a bounded subset of $\mathrm{L}^2(\mu)$ is uniformly integrable in $\mathrm{L}^1(\mu)$, hence the previous result applies to yield:

Corollary **3.11.** — *Let μ be a probability measure such that*

$$\mu \leq \mathrm{A}\,\mathrm{Cap}$$

for some $\mathrm{A} > 0$. *Then* L_μ *is continuous on* $\mathcal{E}_{\mathrm{C}}$ *for each* $\mathrm{C} > 0$, *and* $\mathrm{F}_\mu = \mathrm{E} - \mathrm{L}_\mu$ *is usc on* $\mathcal{E}^1(\mathrm{X}, \theta)$.

Proof. — By (ii) of Lemma 2.9 we have

$$\int_{t=0}^{+\infty} t\mu\{\varphi < \mathrm{V}_\theta - t\} dt \leq \mathrm{A} \int_{t=0}^{+\infty} t\,\mathrm{Cap}\{\varphi < \mathrm{V}_\theta - t\} dt \leq \mathrm{C}_1$$

uniformly for $\varphi \in \mathcal{E}_{\mathrm{C}}$, and the result follows by Theorem 3.10 and Lemma 3.9. $\square$

Theorem **3.12.** — *Let* $\varphi \in \mathcal{E}^1(\mathrm{X}, \theta)$ *and set* $\mu := \mathrm{MA}(\varphi)$. *Then* L_μ *is continuous on* $\mathcal{E}_{\mathrm{C}}$ *for each* $\mathrm{C} > 0$, *and* $\mathrm{F}_\mu = \mathrm{E} - \mathrm{L}_\mu$ *is usc on* $\mathcal{E}^1(\mathrm{X}, \theta)$.

Proof. — If φ has minimal singularities, the result follows from Corollary 3.11, since we have $\mathrm{MA}(\psi) \leq \mathrm{A}\,\mathrm{Cap}$ for some $\mathrm{A} > 0$. To see this, pick $t \geq 1$ such that $\psi \geq \mathrm{V}_\theta - t$. Then $t^{-1}\varphi + (1 - t^{-1})\mathrm{V}_\theta$ is a candidate in the definition of Cap, and we get the estimate since

$$\mathrm{MA}(\varphi) \leq t^n \mathrm{MA}\big(t^{-1}\varphi + \big(1 - t^{-1}\big)\mathrm{V}_\theta\big).$$

In the general case, we write φ as the decreasing limit of its canonical approximants $\varphi_j := \max\{\varphi, \mathrm{V}_\theta - j\}$. By Proposition 1.1 we have $\mathrm{I}(\varphi_j, \varphi) \to 0$ as $k \to \infty$ and Lemma 3.13 below therefore shows that $\mathrm{L}_{\mathrm{MA}(\varphi_j)}$ converges to L_μ uniformly on $\mathcal{E}_{\mathrm{C}}$. The result follows since for each j $\mathrm{L}_{\mathrm{MA}(\varphi_j)}$ is continuous on $\mathcal{E}_{\mathrm{C}}$ by the first part of the proof. $\square$

Lemma **3.13.** — *We have*

$$\sup_{\mathcal{E}_{\mathrm{C}}} |\mathrm{L}_{\mathrm{MA}(\psi_1)} - \mathrm{L}_{\mathrm{MA}(\psi_2)}| = \mathrm{O}\big(\mathrm{I}(\psi_1, \psi_2)^{1/2}\big),$$

uniformly for $\psi_1, \psi_2 \in \mathcal{E}_{\mathrm{C}}$.

Proof. — Pick $\varphi, \psi_1, \psi_2 \in \mathcal{E}_{\mathrm{C}}$ and set for $p = 0, \dots, n$

$$a_p := \int (\varphi - \mathrm{V}_\theta)\mathrm{MA}\big(\psi_1^{(p)}, \psi_2^{(n-p)}\big).$$

Our goal is to find $\mathrm{C}_1 > 0$ only depending on C such that

$$|a_n - a_0| \leq \mathrm{C}_1 \mathrm{I}(\psi_1, \psi_2)^{1/2}.$$

It is enough to consider the case where φ, ψ_1, ψ_2 furthermore have minimal singularities. Indeed in the general case one can apply the result to the canonical approximants with minimal singularities, and we conclude by continuity of mixed Monge-Ampère operators along monotonic sequences. By integration by parts ([BEGZ10, Theorem 1.14]) we have

$$\begin{aligned} &a_{p+1} - a_p \\ &= \int_\Omega (\varphi - \mathrm{V}_\theta) dd^c(\psi_1 - \psi_2) \wedge \big(\theta + dd^c\psi_1\big)^p \wedge \big(\theta + dd^c\psi_2\big)^{n-p-1} \\ &= -\int_\Omega d(\varphi - \mathrm{V}_\theta) \wedge d^c(\psi_1 - \psi_2) \wedge \big(\theta + dd^c\psi_1\big)^p \wedge \big(\theta + dd^c\psi_2\big)^{n-p-1}, \end{aligned}$$

and the Cauchy-Schwarz inequality yields

$$|a_{p+1} - a_p|^2 \leq \mathrm{A}_p \mathrm{B}_p$$

with

$$\mathrm{A}_p := \int_\Omega d(\varphi - \mathrm{V}_\theta) \wedge d^c(\varphi - \mathrm{V}_\theta) \wedge \big(\theta + dd^c\psi_1\big)^p \wedge \big(\theta + dd^c\psi_2\big)^{n-p-1}$$

and

$$\begin{aligned} \mathrm{B}_p &:= \int_\Omega d(\psi_1 - \psi_2) \wedge d^c(\psi_1 - \psi_2) \wedge \big(\theta + dd^c\psi_1\big)^p \wedge \big(\theta + dd^c\psi_2\big)^{n-p-1} \\ &\leq \mathrm{I}(\psi_1, \psi_2) \end{aligned}$$

by (2.9). By integration by parts again we get

$$\begin{aligned} \mathrm{A}_p &= -\int_\Omega (\varphi - \mathrm{V}_\theta) dd^c(\varphi - \mathrm{V}_\theta) \wedge \big(\theta + dd^c\psi_1\big)^p \wedge \big(\theta + dd^c\psi_2\big)^{n-p-1} \\ &= \int (\varphi - \mathrm{V}_\theta)\mathrm{MA}\big(\mathrm{V}_\theta, \psi_1^{(p)}, \psi_2^{(n-p-1)}\big) \\ &\quad - \int (\varphi - \mathrm{V}_\theta)\mathrm{MA}\big(\varphi, \psi_1^{(p)}, \psi_2^{(n-p-1)}\big) \end{aligned}$$

which is uniformly bounded in terms of C only by Lemma 2.7. We thus conclude that

$$|a_n - a_0| \le |a_n - a_{n-1}| + \cdots + |a_1 - a_0| \le C_1 I(\psi_1, \psi_2)^{1/2}$$

for some $C_1 > 0$ only depending on C as desired. □

4. Variational resolution of Monge-Ampère equations

4.1. *Variational formulation.* — In this section we prove the following key step in our approach, which extends Theorem A of the introduction to the case of a big class. Recall that we have normalized the big cohomology class $\{\theta\}$ by requiring that its volume is equal to 1. We let $\mathcal{M}_X$ denote the set of all probability measures on X. For any $\mu \in \mathcal{M}_X$, $E - L_\mu$ descends to a concave functional

$$F_\mu : \mathcal{T}^1(X, \theta) \to]-\infty, +\infty].$$

Theorem **4.1.** — *Given* $T \in \mathcal{T}^1(X, \theta)$ *and* $\mu \in \mathcal{M}_X$ *we have*

$$F_\mu(T) = \sup_{\mathcal{T}^1(X,\theta)} F_\mu \iff \mu = \langle T^n \rangle.$$

Proof. — Write $T = \theta + dd^c\varphi$ and suppose that $\mu = \langle T^n \rangle$, i.e. $\mu = MA(\varphi)$. Since E is concave we have for any $\psi \in \mathcal{E}^1(X, \theta)$

$$E(\varphi) + \int (\psi - V_\theta) MA(\varphi) \ge E(\psi) + \int (\varphi - V_\theta) MA(\varphi).$$

Indeed the inequality holds when φ, ψ have minimal singularities by (2.2), and the general case follows by approximating φ by $\max\{\varphi, V_\theta - j\}$, and similarly for ψ. It follows that

$$F_\mu(T) = \sup_{\mathcal{T}^1(X,\theta)} F_\mu.$$

In order to prove the converse, we will rely on the differentiability result obtained by the first two authors in [BB10, Theorem B]. Given a usc function $u : X \to [-\infty, +\infty[$, we define its θ-psh envelope as

$$P(u) = \sup\{\varphi \in PSH(X, \theta) \mid \varphi \le u \text{ on } X\}$$

(or $P(u) :\equiv -\infty$ if u does not dominate any θ-psh function). Note that $P(u)$ is automatically usc. Indeed, its usc majorant $P(u)^* \ge P(u)$ is θ-psh and satisfies $P(u)^* \le u$ since u is usc, and it follows that $P(u) = P(u)^*$ by definition. Note also that

$$V_\theta = P(0).$$

Now let v be a continuous function on X. Since v is in particular bounded, we see that $P(\varphi + tv) \geq \varphi - O(1)$ belongs to $\mathcal{E}^1(X, \theta)$ for every $t \in \mathbf{R}$. We claim that the function $g : \mathbf{R} \to \mathbf{R}$

$$g(t) := E\big(P(\varphi + tv)\big) - L_\mu(\varphi) - t \int v d\mu$$

achieves its maximum at $t = 0$. Indeed, by monotonicity of L_μ, $P(\varphi + tv) \leq \varphi + tv$ implies

$$g(t) \leq E\big(P(\varphi + tv)\big) - L_\mu\big(P(\varphi + tv)\big),$$

which is in turn less than $E(\varphi) - L_\mu(\varphi) = g(0)$ since φ maximizes $E - L_\mu$. By Lemma 4.2 below it follows that

$$0 = g'(0) = \int v\, MA(\varphi) - \int v\, d\mu,$$

and hence $MA(\varphi) = \mu$, since this is valid for any $v \in C^0(X)$. □

Lemma **4.2.** — *Given $\varphi \in \mathcal{E}^1(X, \theta)$ and a continuous function v on X we have*

$$\frac{d}{dt}\Big|_{t=0} E\big(P(\varphi + tv)\big) = \int v\, MA(\varphi).$$

Proof. — By dominated convergence we get the following equivalent integral formulation

$$E\big(P(\varphi + v)\big) - E(\varphi) = \int_0^1 dt \int v\, MA\big(P(\varphi + tv)\big). \tag{4.1}$$

Since φ is usc, we can write it as the decreasing limit of a sequence of continuous functions u_j on X. It is then straightforward to check that, for each $t \in \mathbf{R}$, $P(\varphi + tv)$ is the decreasing limit of $P(u_j + tv)$. By [BB10, Theorem B] we have

$$E\big(P(u_j + v)\big) - E\big(P(u_j)\big) = \int_0^1 dt \int v\, MA\big(P(u_j + tv)\big)$$

for each j. By Proposition 2.4 the energy E is continuous along decreasing sequences, hence

$$E\big(P(\varphi + tv)\big) = \lim_{j \to \infty} E\big(P(u_j + tv)\big)$$

and

$$\int v\,\mathrm{MA}\big(\mathrm{P}(\varphi+tv)\big)=\lim_{j\to\infty}\int v\,\mathrm{MA}\big(\mathrm{P}(u_j+tv)\big)$$

by Proposition 1.1, since $\mathrm{P}(\varphi+tv)$ has full Monge-Ampère mass. We thus obtain (4.1) by dominated convergence, which applies since the total mass of $\mathrm{MA}(\mathrm{P}(u_j+tv))$ is equal to 1 for each j and t. □

Definition **4.3.** — *The* pluricomplex energy *of a probability measure* $\mu\in\mathcal{M}_{\mathrm{X}}$ *is defined as*

$$\mathrm{E}^*(\mu):=\sup_{\mathcal{T}^1(\mathrm{X},\theta)}\mathrm{F}_\mu\in[0,+\infty].$$

We will say that μ *has* finite energy *if* $\mathrm{E}^*(\mu)<+\infty$.

By definition, we thus have

$$\mathrm{E}^*(\mu)=\sup_{\varphi\in\mathrm{PSH}(\mathrm{X},\theta)}\left(\mathrm{E}(\varphi)-\int(\varphi-\mathrm{V}_\theta)d\mu\right),$$

which plays the role of the Legendre-Fenchel transform of E.

Since $\mathrm{E}(\mathrm{V}_\theta)=\mathrm{L}_\mu(\mathrm{V}_\theta)=0$, E^* takes non-negative values, hence defines a convex functional

$$\mathrm{E}^*:\mathcal{M}_{\mathrm{X}}\to[0,+\infty],$$

which is furthermore lower semi-continuous (in the weak topology of measures) by Lemma 3.1.

Here is a first characterization of measures μ with finite energy.

Lemma **4.4.** — *A probability measure* μ *has finite energy iff* L_μ *is finite on* $\mathcal{E}^1(\mathrm{X},\theta)$. *In that case,* μ *is necessarily non-pluripolar.*

Proof. — By Corollary 3.7, if L_μ is finite on $\mathcal{E}^1(\mathrm{X},\theta)$ then $\mathrm{F}_\mu:=\mathrm{E}-\mathrm{L}_\mu$ is J-proper on $\mathcal{T}^1(\mathrm{X},\theta)$, and bounded on each J-sublevel set; the result follows. □

The next result shows that E is in turn the Legendre transform of E^*.

Proposition **4.5.** — *For any* $\varphi\in\mathcal{E}^1(\mathrm{X},\theta)$ *we have*

$$\mathrm{E}(\varphi)=\inf_{\mu\in\mathcal{M}_{\mathrm{X}}}\big(\mathrm{E}^*(\mu)+\mathrm{L}_\mu(\varphi)\big).$$

Proof. — We have $E^*(\mu) \geq E(\varphi) - L_\mu(\varphi)$ and equality holds for $\mu = MA(\varphi)$ by Theorem 4.1. The result follows immediately. □

We can alternatively relate E^* and J as follows. If μ is a probability measure on X we define an affine functional H_μ on $\mathcal{T}(X,\theta)$ by setting

$$H_\mu(T) := \int (\varphi - V_\theta)\big(MA(V_\theta) - \mu\big)$$

with $T = \theta + dd^c\varphi$. Then we have

$$E^*(\mu) = \sup_{T\in\mathcal{T}^1(X,\omega)} \big(H_\mu(T) - J(T)\big),$$

and Theorem 4.1 combined with the uniqueness result of [BEGZ10] says that the supremum is attained (exactly) at T iff $\mu = \langle T^n\rangle$.

4.2. *The direct method of the calculus of variations.* — We will need the following technical result.

Lemma **4.6.** — *Let* ν *be a measure with finite energy and let* $A > 0$. *Then* E^* *is bounded on*

$$\{\mu \in \mathcal{M}_X \mid \mu \leq A\nu\}.$$

Proof. — By Proposition 3.4 there exists $B > 0$ such that

$$\sup_{\mathcal{E}_C} |L_\nu| \leq B\big(1 + C^{1/2}\big)$$

for all $C > 0$, hence

$$\sup_{\mathcal{E}_C} |L_\mu| \leq AB\big(1 + C^{1/2}\big)$$

for all $\mu \in \mathcal{M}_X$ such that $\mu \leq A\nu$. It follows that $E^*(\mu) = \sup_{\mathcal{E}^1(X,\theta)}(E - L_\mu)$ is bounded above by $\sup_{C>0}(AB(1 + C^{1/2}) - C) < +\infty$. □

We are now in a position to state one of our main results (see Theorem A of the introduction).

Theorem **4.7.** — *A probability measure* μ *on* X *has finite energy iff there exists* $T \in \mathcal{T}^1(X,\theta)$ *such that* $\mu = \langle T^n\rangle$. *In that case* $T = T_\mu$ *is unique and satisfies*

$$n^{-1}E^*(\mu) \leq J(T_\mu) \leq nE^*(\mu).$$

Furthermore any maximizing sequence $T_j \in \mathcal{T}^1(X,\theta)$ *for* F_μ *converges to* T_μ.

Proof. — Suppose first that $\mu = \langle T^n \rangle$ for some $T \in \mathcal{E}^1(X, \theta)$. Then μ has finite energy by Lemma 2.7 and Lemma 4.4. Uniqueness follows from [BEGZ10], where it was more generally proved that a current $T \in \mathcal{T}(X, \theta)$ with full Monge-Ampère mass is determined by $\langle T^n \rangle$ by adapting Dinew's proof [Din09] in the Kähler case.

Write $T = \theta + dd^c\varphi$. By the easy part of Theorem 4.1 we have

$$E^*(\mu) = E(\varphi) - \int (\varphi - V_\theta) MA(\varphi) = J_\varphi(V_\theta)$$

and the second assertion follows from Lemma 2.2.

Now let $T_j \in \mathcal{T}^1(X, \theta)$ be a maximizing sequence for F_μ. Since F_μ is J-proper the T_j's stay in a compact set, so we may assume that they converge towards $S \in \mathcal{T}^1(X, \theta)$, and we are to show that $S = T$. Now F_μ is usc by Theorem 3.12, thus $F_\mu(S)$ has to be equal to $\sup_{\mathcal{T}^1(X,\theta)} F_\mu$. By Theorem 4.1 we thus get

$$\langle S^n \rangle = \mu = \langle T^n \rangle$$

hence $S = T$ as desired by uniqueness.

We now come to the main point. Assume that μ has finite energy in the above sense that $E^*(\mu) < +\infty$. In order to find $T \in \mathcal{T}^1(X, \theta)$ such that $\langle T^n \rangle = \mu$, it is enough to show by Theorem 4.1 that F_μ achieves its supremum on $\mathcal{T}^1(X, \theta)$. Since F_μ is J-proper it is even enough to show that F_μ is usc, which we know holds true *a posteriori* by Theorem 3.12.

We are unfortunately unable to establish this *a priori*, thus we resort to a more indirect argument. Assume first that $\mu \le A\,\mathrm{Cap}$ for some $A > 0$. Corollary 3.11 then implies that L_μ is continuous on $\mathcal{E}_C$ for each C, hence F_μ is usc in that case, and we infer that $\mu = \langle T^n \rangle$ for some $T \in \mathcal{T}^1(X, \theta)$ as desired.

In the general case, we rely on the following result already used in [GZ07, BEGZ10] and which basically goes back to Cegrell [Ceg98].

Lemma **4.8.** — *Let μ be a probability measure that puts no mass on pluripolar subsets. Then there exists a probability measure ν with $\nu \le \mathrm{Cap}$ and such that μ is absolutely continuous with respect to ν.*

Proof. — As in [Ceg98], we apply Rainwater's generalized Radon-Nikodym theorem to the compact convex set of measures

$$\mathcal{C} := \{\nu \in \mathcal{M}_X \mid \nu \le \mathrm{Cap}\}.$$

By Proposition 1.6 this is indeed a closed subset of $\mathcal{M}_X$, hence compact. By [Rai69] there exists $\nu \in \mathcal{C}$, $\nu' \perp \mathcal{C}$ and $f \in L^1(\nu)$ such that

$$\mu = f\nu + \nu'.$$

Since μ puts no mass on pluripolar sets and $\mathcal{C}$ characterizes such sets, it follows that $\nu' = 0$. □

Since μ is non-pluripolar by Lemma 4.4, we can use Lemma 4.8 and write $\mu = f\nu$ with $\nu \le \mathrm{Cap}$ and $f \in L^1(\nu)$. Now set

$$\mu_k := (1+\varepsilon_k)\min\{f,k\}\nu$$

where $\varepsilon_k \ge 0$ is chosen so that μ_k has total mass 1. We thus have $\mu_k \le 2k\,\mathrm{Cap}$, and the first part of the proof yields $\mu_k = \langle T_k^n \rangle$ for some $T_k \in \mathcal{T}^1(X,\theta)$. On the other hand, we have $\mu_k \le 2\mu$ for all k, thus $E^*(\mu_k)$ is uniformly bounded by Lemma 4.6. By the first part of the proof, it follows that all T_k stay in a sublevel set $\{J \le C\}$. Since the latter is compact, we may assume after passing to a subsequence that $T_k \to T$ for some $T \in \mathcal{T}^1(X,\theta)$. In particular, T has full Monge-Ampère mass, and [BEGZ10, Corollary 2.21] thus yields

$$\langle T^n \rangle \ge \left(\liminf_{k\to\infty}(1+\varepsilon_k)\min(f,k)\right)\nu = \mu,$$

hence $\langle T^n \rangle = \mu$ since both measures have total mass 1. □

Using a similar argument, we can now recover the main result of [BEGZ10].

Corollary **4.9.** — *Let* μ *be a non-pluripolar probability measure on* X. *Then there exists* $T \in \mathcal{T}(X,\theta)$ *such that* $\mu = \langle T^n \rangle$.

Proof. — Using Lemma 4.8 we can write $\mu = f\nu$ with $\nu \le \mathrm{Cap}$ and $f \in L^1(\nu)$, and we set $\mu_k = (1+\varepsilon_k)\min\{f,k\}\nu$ as above. By Theorem 4.7 there exists $T_k \in \mathcal{T}^1(X,\theta)$ such that $\mu_k = \langle T_k^n \rangle$. We may assume that T_k converges to some $T \in \mathcal{T}(X,\theta)$.

We claim that T has full Monge-Ampère mass, which will imply $\langle T^n \rangle = \mu$ by [BEGZ10, Corollary 2.21], just as above. Write $T = \theta + dd^c\varphi$ and $T_k = \theta + dd^c\varphi_k$ with $\sup_X \varphi = \sup_X \varphi_k = 0$ for all k. By general Orlicz space theory [BEGZ10, Lemma 3.3], there exists a convex non-decreasing function $\chi : \mathbf{R}_- \to \mathbf{R}_-$ with a sufficiently slow decay at $-\infty$ and $C > 0$ such that

$$\int (-\chi)(\psi - V_\theta)d\mu \le \int (\psi - V_\theta)d\nu + C$$

for all $\psi \in PSH(X,\theta)$ normalized by $\sup_X \psi = 0$. Now $\int(\varphi_k - V_\theta)d\nu = L_\mu(\varphi_k)$ is uniformly bounded by Corollary 3.11, and we infer that

$$\int (-\chi)(\varphi_k - V_\theta)\mathrm{MA}(\varphi_k) \le 2\int (-\chi)(\varphi_k - V_\theta)d\mu$$

is uniformly bounded. This means that the χ-weighted energy (cf. [BEGZ10]) of φ_k is uniformly bounded (since φ_k has full Monge-Ampère mass) and we conclude that φ has finite χ-energy by semi-continuity of the χ-energy. This implies in turn that φ has full Monge-Ampère. □

5. Pluricomplex electrostatics

We assume until further notice that $\theta = \omega$ is a Kähler form (still normalized by $\int \omega^n = 1$). We then have $V_\omega = 0$.

5.1. *Pluricomplex energy of measures.* — We first record the following useful explicit formulas.

Lemma **5.1.** — *Let μ be a probability measure with finite energy, and write $\mu = (\omega + dd^c\varphi)^n$ with $\varphi \in \mathcal{E}^1(X, \omega)$. Then we have*

$$E^*(\mu) = \frac{1}{n+1} \sum_{j=0}^{n-1} \int \varphi\left((\omega + dd^c\varphi)^j \wedge \omega^{n-j} - \mu\right)$$

$$(\mathbf{5.1}) \qquad = \sum_{j=0}^{n-1} \frac{j+1}{n+1} \int d\varphi \wedge d^c\varphi \wedge \left(\omega + dd^c\varphi\right)^j \wedge \omega^{n-j}.$$

Proof. — By the easy part of Theorem 4.1 we have

$$E^*(\mu) = E(\varphi) - \int \varphi \, d\mu = J_\varphi(0)$$

and the formulas follow from the explicit formulas for E and $J_\varphi(\psi)$ given in Section 2. □

When X is a compact Riemann surface ($n = 1$), any probability measure μ may be written $\mu = \omega + dd^c\varphi$ by solving Laplace's equation. Then $E^*(\mu) < +\infty$ iff φ belongs to the Sobolev space $L^2_1(X)$, and in that case

$$2E^*(\mu) = \int \varphi(\omega - \mu) = \int d\varphi \wedge d^c\varphi$$

is nothing but the classical Dirichlet functional applied to the potential φ.

We now indicate the relation with the classical *logarithmic energy* (cf. [ST, Chapter 1]). Recall that a signed measure λ on $\mathbf{C}$ is said to have finite logarithmic energy if $(z, w) \mapsto \log|z - w|$ belongs to $L^1(|\lambda| \otimes |\lambda|)$; its logarithmic energy is then defined as

$$D(\lambda) = \int\int \log|z - w|^{-2} \lambda(dz)\lambda(dw)$$

(here D stands for Dirichlet, since the more standard notation I is already being used). When λ has finite energy, its *logarithmic potential*

$$U_\lambda(z) = \int \log|z - w|^2 \lambda(dw)$$

belongs to $L^1(|\lambda|)$, and we have

$$D(\lambda) = -\int U_\lambda(z)\lambda(dz).$$

The Fubiny-Study form ω on $X := \mathbf{P}^1$ (normalized to mass 1) has finite energy, and a simple computation in polar coordinates yields $I(\omega) = -1/2$. We also have

$$U_\omega(z) = \log\big(1 + |z|^2\big).$$

The logarithmic energy D can be polarized into a quadratic form

$$D(\lambda, \mu) := \int\int \log|z - w|^{-2}\lambda(dz)\mu(dw)$$

on the vector space of signed measures with finite energy, which then splits into the D-orthogonal sum of $\mathbf{R}\omega$ and of the space of signed measures with total mass 0. The quadratic form D is positive definite on the latter space [ST, Lemma I.1.8].

Lemma **5.2.** — *Let* $X = \mathbf{P}^1$ *and* ω *the Fubini-Study form, normalized to mass* 1. *If* μ *is a probability measure on* $\mathbf{C} \subset \mathbf{P}^1$ *then* $E^*(\mu) < +\infty$ *iff* μ *has finite logarithmic energy; in that case we have*

$$E^*(\mu) = \frac{1}{2} I(\mu, \omega) = \frac{1}{2} D(\mu - \omega).$$

Proof. — We have $\mu = \omega + dd^c(U_\mu - U_\omega)$, so the first assertion means that μ has finite logarithmic energy iff $U_\mu - U_\omega$ belongs to the Sobolev space $L^2_1(\mathbf{P}^1)$, which is a classical fact. The second assertion follows from (5.1), which yields

$$2E^*(\mu) = -\int (U_\mu - U_\omega)(\mu - \omega) = D(\mu - \omega).$$ □

5.2. *A pluricomplex electrostatic capacity.* — As in [BB10] we consider a *weighted subset* consisting of a compact subset K of X together with a continuous function $v \in C^0(K)$, and we define the *equilibrium weight* of (K, v) as the extremal function

$$P_K v := \sup{}^*\big\{\varphi \in PSH(X, \omega) \mid \varphi \le v \text{ on } K\big\}.$$

The function $P_K v$ belongs to $PSH(X, \omega)$ if K is non-pluripolar, and satisfies $P_K v \equiv +\infty$ otherwise (cf. [Sic81, GZ05]).

If K is a compact subset of $\mathbf{C}^n$ and

$$\varphi_{FS} := \log\big(1 + |z|^2\big)$$

denotes the potential on $\mathbf{C}^n$ of the Fubiny-Study metric, then $P_K(-\varphi_{FS}) + \varphi_{FS}$ coincides with Siciak's extremal function, i.e. the usc upper envelope of the family of all psh functions u on $\mathbf{C}^n$ with logarithmic growth such that $u \le 0$ on K.

The *equilibrium measure* of a non-pluripolar weighted compact set (K, v) is defined as

$$\mu_{eq}(K, v) := MA(P_K v),$$

and its *energy at equilibrium* is

$$E_{eq}(K, v) := E(P_K v).$$

The functional $v \mapsto E_{eq}(K, v)$ is concave and Gâteaux differentiable on $C^0(K)$, with directional derivative at v given by integration against $\mu_{eq}(K, v)$ by [BB10, Theorem B]. As a consequence of Theorem 4.1 we get the following related variational characterization of $\mu_{eq}(K, v)$.

Denote by $\mathcal{M}_K$ the set of probability measures on K.

Theorem **5.3.** — *If* (K, v) *is a non-pluripolar weighted compact subset, then we have*

$$E_{eq}(K, v) = \inf_{\mu \in \mathcal{M}_K} \left(E^*(\mu) + \int v \, d\mu \right)$$

and the infimum is achieved precisely for $\mu = \mu_{eq}(K, v)$.

If K *is pluripolar then* $E^*(\mu) = +\infty$ *for each* $\mu \in \mathcal{M}_K$.

Proof. — Assume first that K is non-pluripolar. The concave functional $F := E_{eq}(K, \cdot)$ is non-decreasing on $C^0(K)$ and satisfies the scaling property $F(v + c) = F(v) + c$, so its Legendre transform

$$F^*(\mu) := \sup_{v \in C^0(K)} \big(F(v) - \langle v, \mu \rangle \big)$$

is necessarily infinite outside $\mathcal{M}_K \subset C^0(K)^*$. The basic theory of convex functions thus yields

$$F(v) = \inf_{\mu \in \mathcal{M}_K} \left(F^*(\mu) + \int v \, d\mu \right),$$

and the infimum is achieved exactly at $\mu = F'(v) = \mu_{eq}(K, v)$. What we have to show is thus $F^*(\mu) = E^*(\mu)$ for any $\mu \in \mathcal{M}_K$. But on the one hand $P_K(v) \le v$ on K implies

$$\begin{aligned} \mathrm{F}^*(\mu) &\leq \sup_{v \in \mathrm{C}^0(\mathrm{K})} \left(\mathrm{E}(\mathrm{P}_\mathrm{K} v) - \int \mathrm{P}_\mathrm{K} v \, d\mu \right) \\ &\leq \sup_{\varphi \in \mathcal{E}^1(\mathrm{X},\omega)} \left(\mathrm{E}(\varphi) - \int \varphi \, d\mu \right) = \mathrm{E}^*(\mu). \end{aligned}$$

On the other hand, since ω is a Kähler form, every $\varphi \in \mathcal{E}^1(\mathrm{X}, \omega)$ can be written as a decreasing limit of *smooth* ω-psh functions φ_j by [Dem92] (see also [BK07]). For each j, the function $v_j := \varphi_j|_\mathrm{K} \in \mathrm{C}^0(\mathrm{K})$ satisfies $\varphi_j \leq \mathrm{P}_\mathrm{K}(v_j)$ hence

$$\mathrm{E}(\varphi_j) - \int \varphi_j \, d\mu \leq \mathrm{E}(\mathrm{P}_\mathrm{K} v_j) - \int v_j \, d\mu \leq \mathrm{F}^*(\mu),$$

and we infer $\mathrm{E}^*(\mu) \leq \mathrm{F}^*(\mu)$ as desired since

$$\mathrm{E}(\varphi) - \int \varphi \, d\mu = \lim_{j \to \infty} \left(\mathrm{E}(\varphi_j) - \int \varphi_j \, d\mu \right)$$

by Proposition 2.1 and monotone convergence respectively.

Now assume that K is pluripolar. If there exists $\mu \in \mathcal{M}_\mathrm{K}$ with $\mathrm{E}^*(\mu) < +\infty$, then Theorem A implies in particular that μ puts no mass on pluripolar sets, which contradicts $\mu(\mathrm{K}) = 1$. □

One can interpret Theorem 5.3 as a pluricomplex version of weighted electrostatics where K is a condenser, μ describes a charge distribution on K, $\mathrm{E}^*(\mu)$ is its internal pluricomplex energy and $\int v \, d\mu$ is the external energy induced by the field v. The equilibrium distribution $\mu_{\mathrm{eq}}(\mathrm{K}, v)$ is then the unique minimizer of the total energy $\mathrm{E}^*(\mu) + \int v \, d\mu$ of the system.

In view of Theorem 5.3, it is natural to define the *electrostatic capacity* $\mathrm{C}_\mathrm{e}(\mathrm{K}, v)$ of a weighted compact subset (K, v) by

$$-\log \mathrm{C}_\mathrm{e}(\mathrm{K}, v) = \frac{n+1}{n} \inf \left\{ \mathrm{E}^*(\mu) + \int v \, d\mu \mid \mu \in \mathcal{M}_\mathrm{K} \right\}.$$

We then have $\mathrm{C}_\mathrm{e}(\mathrm{K}, v) = 0$ iff K is pluripolar, and

$$\mathrm{C}_\mathrm{e}(\mathrm{K}, v) = \exp\left(-\frac{n+1}{n} \mathrm{E}_{\mathrm{eq}}(\mathrm{K}, v) \right)$$

when K is non-pluripolar.

Our choice of constants is guided by [BB10, Corollary A], which shows that $\mathrm{C}_\mathrm{e}(\mathrm{K}, v)$ coincides (up to a multiplicative constant) with the natural generalization of Leja-Zaharjuta's *transfinite diameter* when ω is the curvature form of a metric on an ample line bundle L over X. In particular, this result shows that the Leja-Zaharjuta transfinite diameter $d_\infty(\mathrm{K})$ of a compact subset $\mathrm{K} \subset \mathbf{C}^n$, normalized so that

$$d_\infty(t\mathrm{K}) = t d_\infty(\mathrm{K})$$

for each $t > 0$, is proportional to $\mathrm{C}_{\mathrm{e}}(\mathrm{K}, -\varphi_{\mathrm{FS}})$.

By the continuity properties of extremal functions and of the energy functional along monotone sequences, it follows that the capacity $\mathrm{C}_e(\cdot, v)$ can be extended in the usual way as an outer Choquet capacity on X which vanishes exactly on pluripolar sets. In view of Lemma 5.2, this electrostatic capacity extends the classical logarithmic capacity of a compact subset $\mathrm{K} \subset \mathbf{C}$, which is equal to

$$\exp\Big(-\inf_{\mu \in \mathcal{M}_{\mathrm{K}}} \mathrm{D}(\mu)\Big).$$

On the other hand, the *Alexander-Taylor capacity* of a weighted compact subset (K, v) may be defined by

$$\mathrm{T}(\mathrm{K}, v) := \exp\Big(-\sup_{\mathrm{X}} \mathrm{P}_{\mathrm{K}} v\Big),$$

compare [AT84, GZ05]. We thus have $\mathrm{T}(\mathrm{K}, v) = 0$ iff K is pluripolar. We have for instance

$$\mathrm{T}(\mathrm{B}_{\mathrm{R}}, 0) = \frac{\mathrm{R}}{(1+\mathrm{R}^2)^{1/2}}$$

when $\mathrm{X} = \mathbf{P}^n$ and $\mathrm{B}_{\mathrm{R}} \subset \mathbf{C}^n$ is the closed ball of radius R (cf. [GZ05, Example 4.11]). In particular, this implies $\mathrm{T}(\mathrm{B}_{\mathrm{R}}, v) \simeq \mathrm{R}$ as $\mathrm{R} \to 0$.

The two capacities compare as follows.

Proposition **5.4.** — *There exists* $\mathrm{C} > 0$ *such that*

$$\mathrm{T}(\mathrm{K}, v)^{1+\frac{1}{n}} \leq \mathrm{C}_{\mathrm{e}}(\mathrm{K}, v) \leq \mathrm{C} e^{-\inf_{\mathrm{K}} v} \mathrm{T}(\mathrm{K}, v)^{\frac{1}{n}}$$

for each weighted compact subset (K, v).

Proof. — The definition of E immediately implies that

$$\mathrm{E}_{\mathrm{eq}}(\mathrm{K}, v) = \mathrm{E}(\mathrm{P}_{\mathrm{K}} v) \leq \sup_{\mathrm{X}} \mathrm{P}_{\mathrm{K}} v,$$

hence the left-hand inequality. Conversely, set $\mathrm{M} := -\inf_{\mathrm{K}} v$. Then $v \geq -\mathrm{M}$ implies $\mathrm{P}_{\mathrm{K}} v \geq -\mathrm{M}$, and Proposition 2.1 yields

$$\int (\mathrm{P}_{\mathrm{K}} v) \omega^n - n\mathrm{M} \leq (n+1) \mathrm{E}_{\mathrm{eq}}(\mathrm{K}, v).$$

But there exists a constant $C > 0$ such that

$$\sup_X \varphi \leq \int \varphi\, \omega^n + C$$

for all $\varphi \in PSH(X, \omega)$ by compactness of $\mathcal{T}(X, \omega)$, and we get

$$\frac{1}{n} \sup_X P_K v \leq \frac{n+1}{n} E_{eq}(K, v) + M + C'.$$

□

When K lies in the unit ball of $\mathbf{C}^n \subset \mathbf{P}^n$ and

$$v(z) = -\frac{1}{2}\log\left(1 + |z|^2\right),$$

then $-\inf_K v \leq \log\sqrt{2}$, so that the above results improve on [LT83].

5.3. *Convergence in energy.* — In what follows, θ denotes again any smooth $(1, 1)$-form with big cohomology class. The symmetric functional $I(\varphi, \psi)$ introduced in Section 2 is invariant by translation in each variable, hence descends to $\mathcal{T}^1(X, \theta)$. In dimension $n = 1$ the formulas of Section 5.1 show that $I(T_1, T_2)$ is equal to the squared norm of $T_1 - T_2$ with respect to the Dirichlet quadratic form D (which is positive definite on measures of zero total mass such as $T_1 - T_2$). In higher dimensions $I^{1/2}$ no longer satisfies the triangle inequality,[4] but as we shall see it is nevertheless convenient to introduce the following convergence notion.

Definition **5.5.** — *A sequence* $T_j \in \mathcal{T}^1(X, \theta)$ *is said to* converge in energy *to* $T \in \mathcal{T}^1(X, \theta)$ *if* $I(T_j, T) \to 0$ *as* $j \to \infty$.

Using (2.7) and Lemma 2.2 it is immediate to see that T_j converges to T in energy iff T_j is a maximizing sequence for F_μ with $\mu := \langle T^n \rangle$. By Theorem 4.7 and Lemma 3.13 we thus get:

Proposition **5.6.** — *Let* T_j *converge to* T *in energy. Then* $J(T_j)$ *is uniformly bounded,* $T_j \to T$ *weakly and* $L_{\langle T_j^n \rangle} \to L_{\langle T^n \rangle}$ *uniformly on* $\mathcal{E}_C$ *for each* C. *In particular* $\langle T_j^n \rangle \to \langle T^n \rangle$ *weakly.*

We are now going to show that convergence in energy implies convergence in capacity.

[4] See however [BBEGZ11, Theorem 1.8].

Theorem **5.7.** — *Let* $T_j \to T$ *in energy. Then* $T_j \to T$ *in capacity in the following sense: if we write* $T_j = \theta + dd^c\varphi_j$ *(resp.* $T = \theta + dd^c\varphi$*) normalized by* $\int(\varphi_j - V_\theta)MA(V_\theta) = 0$ *(resp.* $\int(\varphi_j - V_\theta)MA(V_\theta)$*), then for each* $\varepsilon > 0$ *we have*

$$\lim_{j\to\infty} \mathrm{Cap}\{|\varphi_j - \varphi| \geq \varepsilon\} = 0.$$

Proof. — Let $\psi \in PSH(X, \theta)$ such that

$$-1 \leq \psi - V_\theta \leq 0.$$

Since Cap is the upper envelope of all measures $MA(\psi)$ with ψ as above, the Chebyshev inequality shows that it is enough to prove

$$\int |\varphi_j - \varphi| MA(\psi) \to 0 \tag{5.2}$$

uniformly with respect to ψ as above. We set

$$\widetilde{\varphi}_j := \max(\varphi_j, \varphi)$$

and $\mu := MA(\varphi)$. We then have

$$\int |\varphi_j - \varphi| MA(\psi) = \int (\widetilde{\varphi}_j - \varphi) MA(\psi) - 2\int (\varphi_j - \varphi) MA(\psi). \tag{5.3}$$

Now the convergence $T_j \to T$ means that φ_j is a maximizing sequence for F_μ, and it implies that $E(\varphi_j)$ is uniformly bounded by Proposition 5.6. We claim that $\widetilde{\varphi}_j$ is then also a maximizing sequence. Indeed we have

$$F_\mu(\widetilde{\varphi}_j) - F_\mu(\varphi_j) = E(\widetilde{\varphi}_j) - E(\varphi_j) + L_\mu(\varphi_j) - L_\mu(\widetilde{\varphi}_j).$$

Since E is non-decreasing we have $E(\widetilde{\varphi}_j) \geq E(\varphi_j)$, which shows that there exists $C > 0$ such that $\widetilde{\varphi}_j \in \mathcal{E}_C$ for all j. Since L_μ is continuous on $\mathcal{E}_C$ by Theorem 3.12, it follows that

$$\liminf_{j\to\infty}\big(F_\mu(\widetilde{\varphi}_j) - F_\mu(\varphi_j)\big) \geq 0$$

and $\widetilde{\varphi}_j$ is maximizing as desired. By Lemma 5.8 below we thus see that each term in the right-hand side of (5.3) tends to 0 uniformly with respect to ψ (note that all ψ as above lie in $\mathcal{E}_1$), and we are done. □

Lemma **5.8.** — *Let* $C > 0$. *Then we have*

$$\int (\varphi_1 - \varphi_2)\big(MA(\psi_1) - MA(\psi_2)\big) \to 0$$

as $I(\varphi_1, \varphi_2) \to 0$ *with* $\varphi_1, \varphi_2, \psi_1, \psi_2 \in \mathcal{E}_C$, *uniformly with respect to* ψ_1, ψ_2.

Proof. — We will use several times that $\mathrm{I}(\cdot,\cdot)$ is bounded on $\mathcal{E}_{\mathrm{C}}\times\mathcal{E}_{\mathrm{C}}$ by Lemma 2.7. Let first $\varphi_1,\varphi_2,\psi\in\mathcal{E}_{\mathrm{C}}$ and set $u:=\varphi_1-\varphi_2$ and $v:=(\varphi_1+\varphi_2)/2$. For each $p=0,\dots,n$ let

$$a_p:=\int u\,\theta_{\varphi_1}^p\wedge\theta_\psi^{n-p}$$

and

$$b_p:=\int du\wedge d^c u\wedge\theta_v^p\wedge\theta_\psi^{n-p-1}.$$

For $p=0,\dots,n-1$ we have

$$\begin{aligned}a_p&=a_{p+1}+\int udd^c(\psi-\varphi_1)\wedge\theta_{\varphi_1}^p\wedge\theta_\psi^{n-p-1}\\&=a_{p+1}-\int du\wedge d^c(\psi-\varphi_1)\wedge\theta_{\varphi_1}^p\wedge\theta_\psi^{n-p-1}\end{aligned}$$

by integration by parts. The Cauchy-Schwarz inequality yields

$$\begin{aligned}&\left(\int du\wedge d^c(\psi-\varphi_1)\wedge\theta_{\varphi_1}^p\wedge\theta_\psi^{n-p+1}\right)^2\\&\qquad\le\left(\int du\wedge d^c u\wedge\theta_{\varphi_1}^p\wedge\theta_\psi^{n-p+1}\right)\mathrm{I}(\psi,\varphi_1)\end{aligned}$$

by (2.9). Since I is bounded we thus get $\mathrm{B}>0$ such that

$$|a_p-a_{p+1}|\le \mathrm{B}b_p^{1/2},$$

for $p=0,\dots,n-1$, which yields

$$\left|\int(\varphi_1-\varphi_2)\big(\mathrm{MA}(\varphi_1)-\mathrm{MA}(\psi)\big)\right|\le\mathrm{B}\sum_{p=0}^{n-1}b_p^{1/2}.\tag{5.4}$$

On the other hand integration by parts yields

$$\begin{aligned}b_p&=\int du\wedge d^c u\wedge\theta_v^{p+1}\wedge\theta_\psi^{n-p-1}+\int du\wedge d^c u\wedge dd^c(\psi-v)\wedge\theta_v^p\wedge\theta_\psi^{n-p-1}\\&=b_{p+1}-\int du\wedge d^c(\psi-v)\wedge dd^c u\wedge\theta_v^p\wedge\theta_\psi^{n-p-1}\\&=b_{p+1}-\int du\wedge d^c(\psi-v)\wedge\theta_{\varphi_1}\wedge\theta_v^p\wedge\theta_\psi^{n-p-1}\\&\qquad+\int du\wedge d^c(\psi-v)\wedge\theta_{\varphi_2}\wedge\theta_v^p\wedge\theta_\psi^{n-p-1}.\end{aligned}$$

For $i=1,2$ we have $\theta_{\varphi_i}\le 2\theta_v$ thus

$$\begin{aligned}&\left|\int du\wedge d^c(\psi-v)\wedge\theta_{\varphi_i}\wedge\theta_v^p\wedge\theta_\psi^{n-p-1}\right|\\&\quad\le 2\left|\int du\wedge d^c(\psi-v)\wedge\theta_v^{p+1}\wedge\theta_\psi^{n-p-1}\right|\\&\quad\le 2b_{p+1}^{1/2}\mathrm{I}(\psi,v)^{1/2}\end{aligned}$$

by Cauchy-Schwarz and (2.9). Using again that I is bounded on $\mathcal{E}_C\times\mathcal{E}_C$ it follows upon possibly enlarging B that

$$b_p\le b_{p+1}+\mathrm{B}b_{p+1}^{1/2}. \tag{5.5}$$

Now there exists a numerical constant C_n such that $b_n\le C_n\mathrm{I}(\varphi_1,\varphi_2)$ by (2.9) and we thus see that there exists a continuous function $f:\mathbf{R}_+\to\mathbf{R}_+$ with $f(0)=0$ and only depending on C such that

$$\sum_{p=0}^{n-1}b_p^{1/2}\le f\big(\mathrm{I}(\varphi_1,\varphi_2)\big).$$

In view of (5.4) we have thus shown that

$$\left|\int(\varphi_1-\varphi_2)\big(\mathrm{MA}(\varphi_1)-\mathrm{MA}(\psi)\big)\right|\le f\big(\mathrm{I}(\varphi_1,\varphi_2)\big)$$

for all $\varphi_1,\varphi_2,\psi\in\mathcal{E}_C$. But we have

$$\int(\varphi_1-\varphi_2)\big(\mathrm{MA}(\varphi_2)-\mathrm{MA}(\varphi_1)\big)=\mathrm{I}(\varphi_1,\varphi_2)$$

by definition of I, so we get

$$\left|\int(\varphi_1-\varphi_2)\big(\mathrm{MA}(\psi_1)-\mathrm{MA}(\psi_2)\big)\right|\le\mathrm{I}(\varphi_1,\varphi_2)+2f\big(\mathrm{I}(\varphi_1,\varphi_2)\big),$$

which concludes the proof. □

6. Variational principles for Kähler-Einstein metrics

In this section, we use the variational approach to study the existence of Kähler-Einstein metrics on manifolds with definite first Chern class. The Ricci-flat case is an easy consequence of Theorem A. In Section 6.1 we treat the case of manifolds of general type and prove Theorem C. The more delicate case of Fano manifolds occupies the remaining sections: in Section 6.2 we construct continuous geodesics in the space of positive closed currents with prescribed cohomology class, we then prove Theorem D in

Section 6.3, while uniqueness of (singular) Kähler-Einstein metrics with positive curvature (Theorem E) is established in Section 6.4. We will use throughout the convenient language of weights, i.e. view metrics additively. We refer for instance to [BB10] for explanations.

6.1. *Manifolds of general type.* — Let X be a smooth projective variety of *general type*, i.e. such that K_X is big. A weight ϕ on K_X induces a volume form e^ϕ. By a *singular Kähler-Einstein weight* we mean a psh weight on K_X such that $\mathrm{MA}(\phi) = e^\phi$ *and* such that $\int e^\phi = \mathrm{vol}(K_X) =: V$, or equivalently such that $\mathrm{MA}(\phi)$ has *full Monge-Ampère mass*.

In [EGZ09] a singular Kähler-Einstein weight was constructed using the existence of the *canonical model*

$$X_{\mathrm{can}} := \mathrm{Proj}\Big(\bigoplus_{m\ge 0} H^0(X, mK_X)\Big)$$

provided by the fundamental result of [BCHM10]. In [Tsu10] a direct proof of the existence of a singular Kähler-Einstein weight was sketched, and the argument was expanded in [SoTi08]. In [BEGZ10], existence and uniqueness of singular Kähler-Einstein weights was established using a generalized comparison principle, and the unique singular Kähler-Einstein weight was furthermore shown to have *minimal singularities* in the sense of Demailly.

We propose here to give a direct variational proof of the existence of a singular Kähler-Einstein weight in $\mathcal{E}^1(X, K_X)$ (we therefore don't recover the full force of the result in [BEGZ10]). We proceed as before, replacing the functional $F_\mu = E - L_\mu$ with $F_+ := E - L_+$, where we have set

$$L_+(\phi) := \log \int e^\phi.$$

Proof of Theorem C. — Note that e^ϕ has L^∞-density with respect to Lebesgue measure. Indeed, if ϕ_0 is a given smooth weight on K_X, then $e^\phi = e^{\phi-\phi_0} e^{\phi_0}$, where e^{ϕ_0} is a smooth positive volume form and the *function* $\phi - \phi_0$ is bounded from above on X. Given $\phi, \psi \in \mathrm{PSH}(X, K_X)$, we can in particular consider the integral

$$\int (\phi - \psi) e^\psi,$$

since $\phi - \psi$ is integrable on X. □

Lemma **6.1.** — *The directional derivatives of* L_+ *on* $\mathrm{PSH}(X, K_X)$ *are given by*

$$\frac{d}{dt}\Big|_{t=0_+} L_+\big(t\phi + (1-t)\psi\big) = \frac{\int (\phi-\psi) e^\psi}{\int e^\psi}.$$

Proof. — By the chain rule, it is enough to show that

$$\frac{d}{dt}\Big|_{t=0_+} \int e^{t\phi+(1-t)\psi} = \int (\phi-\psi)e^{\psi}.$$

One has to be a little bit careful since $\phi-\psi$ is not bounded on X. But we have

$$\int \left(e^{t\phi+(1-t)\psi} - e^{\psi}\right) = \int \left(e^{t(\phi-\psi)} - 1\right)e^{\psi}.$$

Now $(e^{t(\phi-\psi)}-1)/t$ decreases pointwise to $\phi-\psi$ as t decreases to 0 by convexity of the exponential, and the result indeed follows by monotone convergence. □

Using this fact and arguing exactly as in Theorem 4.1 proves that

$$\text{(6.1)} \qquad F_+(\phi) = \sup_{\mathcal{E}^1(X,K_X)} F_+$$

implies

$$\text{(6.2)} \qquad MA(\phi) = e^{\phi+c}$$

for some $c \in \mathbf{R}$. Indeed apart from [BB10] the main point of the proof of Theorem 4.1 is that $E(P(\phi+v)) - L_\mu(\phi+v)$ is maximum for $v=0$ if $E-L_\mu$ is maximal at ϕ, and this only relied on the fact that L_μ is non-decreasing, which is also the case for L_+.

Conversely, E is concave while L_+ is convex by Hölder's inequality, thus F_+ is concave and (6.2) implies (6.1) as in Theorem 4.1.

In order to conclude the proof of Theorem C, we need to prove that F_+ achieves its supremum on $\mathcal{E}^1(X,K_X)$, or equivalently on $\mathcal{T}^1(X,K_X)$. Now Corollary 3.7 applies to $F_+ = E - L_+$ since L_+ is non-decreasing, convex and satisfies the scaling property, and we conclude that F_+ is J-proper as before. It thus remains to check that F_+ is upper semicontinuous.

In the present case, it is even true that L_+ is continuous on the whole of $PSH(X,K_X)$. To see this, let $\phi_j \to \phi$ be a convergent sequence in $PSH(X,K_X)$. Upon extracting a subsequence, we may assume that $\phi_j \to \phi$ a.e. Given a reference weight ϕ_0, $\sup_X(\phi_j-\phi_0)$ is uniformly bounded by Hartogs' lemma, thus $e^{\phi_j-\phi_0}$ is uniformly bounded, and we get $\int e^{\phi_j} \to \int e^{\phi}$ as desired by dominated convergence applied to the fixed measure e^{ϕ_0}.

6.2. *Continuous geodesics.* — Let ω be a semi-positive $(1,1)$-form on X. If Y is a complex manifold, then a map $\Phi : Y \to PSH(X,\omega)$ will be said to be psh (resp. locally bounded, continuous, smooth) iff the induced function $\Phi(x,y) := \Phi(y)(x)$ on $X \times Y$ is $\pi_X^*\omega$-psh (resp. locally bounded, continuous, smooth). We shall also say that Φ is *maximal* if it is psh, locally bounded and

$$\left(\pi_X^*\omega + dd^c_{(x,y)}\Phi\right)^{n+m} = 0$$

where $m := \dim Y$ and $dd^c_{(x,y)}$ acts on both variables (x,y). If Y is a radially symmetric domain in $\mathbf{C}$ and Φ is smooth on $X \times \overline{Y}$ such that $\omega + dd^c_x \Phi(\cdot, y) > 0$ for each $y \in Y$ then Φ is maximal iff $\Phi(e^t)$ is a *geodesic* for the Riemannian metric on Kähler potentials

$$\left\{\varphi \in C^\infty(X) \mid \omega + dd^c\varphi > 0\right\}$$

defined in [Mab87, Sem92, Don99].

Proposition **6.2.** — *If* $\Phi : Y \to PSH(X, \omega)$ *is a psh map, then* $E \circ \Phi$ *is a psh function on* Y *(or is indentically* $-\infty$ *on some component of* Y*). When* Φ *is furthermore locally bounded we have*

$$dd^c_y(E \circ \Phi) = (\pi_Y)_*\left(\left(\pi_X^*\omega + dd^c_{(x,y)}\Phi\right)^{n+1}\right). \tag{6.3}$$

In particular, if $\dim Y = 1$ *and* Φ *is maximal then* $E \circ \Phi$ *is harmonic on* Y.

Proof. — Assume first that Φ is smooth. Then we can consider

$$E \circ \Phi := \frac{1}{n+1}(\pi_Y)_*\left(\Phi \sum_{j=0}^{n} \left(\pi_X^*\omega + dd^c_x\Phi\right)^j \wedge \pi_X^*\omega^{n-j}\right). \tag{6.4}$$

The formula

$$dd^c_y(E \circ \Phi) = (\pi_Y)_*\left(\left(\pi_X^*\omega + dd^c_{(x,y)}\Phi\right)^{n+1}\right)$$

follows from an easy but tedious computation relying on integration by parts and will be left to the reader.

When $\Phi(x,y)$ is bounded and $\pi_X^*\omega$-psh the same argument works. Indeed integration by parts is a consequence of Stokes formula applied to a *local* relation of the form $u = dv$, and the corresponding relation in the smooth case can be extended to the bounded case by a local regularization argument.

Finally let $\Phi(x,y)$ be an arbitrary $\pi_X^*\omega$-psh function. We may then write Φ as the decreasing limit of $\max\{\Phi, -k\}$ as $k \to \infty$, and by Proposition 2.4 $E \circ \Phi$ is then the pointwise decreasing limit of $E \circ \Phi_k$, whereas

$$\left(\pi_X^*\omega + dd^c_{(x,y)}\Phi_k\right)^{n+1} \to \left(\pi_X^*\omega + dd^c_{(x,y)}\Phi\right)^{n+1}$$

by Bedford-Taylor's monotonic continuity theorem. □

Proposition **6.3.** — *Let* $\Omega \Subset \mathbf{C}^m$ *be a smooth strictly pseudoconvex domain and let* $\varphi : \partial\Omega \to PSH(X, \omega)$ *be a continuous map. Then there exists a unique continuous extension* $\Phi : \overline{\Omega} \to PSH(X, \omega)$ *of* φ *which is maximal on* Ω.

The proof is a simple adaptation of Bedford-Taylor's techniques to the present situation. Although it has recently appeared in [BD12] we include a proof as a courtesy to the reader.

Proof. — Uniqueness follows from the maximum principle. Let $\mathcal{F}$ be the set of all continuous psh maps $\Psi : \overline{\Omega} \to \mathrm{PSH}(X, \omega)$ such that $\Psi \le \varphi$ on $\partial\Omega$. Note that $\mathcal{F}$ is non-empty since it contains all sufficiently negative constant functions of (x, y). Let Φ be the upper envelope of $\mathcal{F}$. We are going to show that $\Phi = \varphi$ on $\partial\Omega$ and that Φ is continuous. The latter property will imply that Φ is $\pi_X^*\omega$-psh, and it is then standard to show that Φ is maximal on Ω by using local solutions to the homogeneous Monge-Ampère equation (compare. [Dem91, p. 17], [BB10, Proposition 1.10]).

Assume first that φ is a smooth. We claim that φ admits a smooth psh extension $\widetilde{\varphi} : \overline{\Omega} \to \mathrm{PSH}(X, \omega)$. Indeed we first cover $\overline{\Omega}$ by two open subsets U_1, U_2 such that U_1 retracts smoothly to $\partial\Omega$. We can then extend φ to a smooth map $\varphi_1 : U_1 \to \mathrm{PSH}(X, \omega)$ using the retraction and pick any constant map $\varphi_2 : U_2 \to \mathrm{PSH}(X, \omega)$. Since $\mathrm{PSH}(X, \omega)$ is convex $\theta_1\varphi_1 + \theta_2\varphi_2$ defines a smooth extension $\overline{\Omega} \to \mathrm{PSH}(X, \omega)$ (where θ_1, θ_2 is a partition of unity adapted to U_1, U_2). Now let χ be a smooth strictly psh function on $\overline{\Omega}$ vanishing on the boundary of Ω. Then $\widetilde{\varphi} := \theta_1\varphi_1 + \theta_2\varphi_2 + C\chi$ yields the desired smooth psh extension of φ for $C \gg 1$.

Since $\widetilde{\varphi}$ belongs to $\mathcal{F}$ we get in particular $\widetilde{\varphi} \le \Phi$ hence $\Phi = \varphi$ on $\partial\Omega$. Still assuming that φ is smooth, we now take care of the continuity of Φ, basically following [Dem91, p. 13]. By [Dem92] the exists a sequence Φ_k of smooth functions on $X \times \overline{\Omega}$ decreasing pointwise to the usc regularization Φ^* and such that

$$dd^c\Phi_k \ge -\varepsilon_k\big(\pi_X^*\omega + dd^c\chi\big),$$

with $\varepsilon \to 0$. Note that $\Psi_k := (1 - \varepsilon_k)(\Phi_k + \varepsilon_k\chi)$ is thus $\pi_X^*\omega$-psh. Given $\varepsilon > 0$ we have $\Phi^* < \widetilde{\varphi} + \varepsilon$ on a compact neighbourhood U of $X \times \partial\Omega$ thus $\Psi_k < \widetilde{\varphi} + \varepsilon$ on U for $k \gg 1$. It follows that $\max(\Psi_k - \varepsilon, \widetilde{\varphi})$ belongs to $\mathcal{F}$, so that $\Psi_k - \varepsilon \le \Phi$, and we get

$$\Phi \le \Phi^* \le \Phi_k \le (1 - \varepsilon_k)^{-1}(\Phi + \varepsilon) - \varepsilon_k\chi,$$

which in turn implies that Φ_k converges to Φ uniformly on $X \times \overline{\Omega}$. We conclude that Φ is continuous in that case as desired.

Let now $\varphi : \partial\Omega \to \mathrm{PSH}(X, \omega)$ be an arbitrary continuous map. By Richberg's approximation theorem (cf. e.g. [Dem92]) we may find a sequence of smooth functions $\varphi_k : \partial\Omega \to \mathrm{PSH}(X, \omega)$ such that $\sup_{X \times \partial\overline{\Omega}} |\varphi - \varphi_k| =: \varepsilon_k$ tends to 0. The corresponding envelopes Φ_k are continuous by the first part of the proof, and satisfy $\Phi_k - \varepsilon_k \le \Phi \le \Phi_k + \varepsilon_k$; this shows that $\Phi_k \to \Phi$ uniformly on $X \times \overline{\Omega}$, and continuity of Φ follows. □

6.3. *Fano manifolds.* — Let X be a Fano manifold. Our goal in this section is to prove that singular Kähler-Einstein weights, i.e. weights $\phi \in \mathcal{E}^1(X, -K_X)$ such that $\mathrm{MA}(\phi) = e^{-\phi}$, can be characterized by a variational principle.

Lemma **6.4.** — *For any compact Kähler manifold* (X,ω)*, the map* $\mathcal{E}^1(X,\omega)\to L^1(X)$ $\varphi\mapsto e^{-\varphi}$ *is continuous.*

Proof. — As already mentioned, every $\varphi\in PSH(X,\omega)$ with full Monge-Ampère mass has identically zero Lelong numbers [GZ07, Corollary 1.8], which amounts to saying that $e^{-\varphi}$ belongs to $L^p(X)$ for all $p<+\infty$ by Skoda's integrability criterion [Sko72]. Now let $\varphi_j\to\varphi$ be a convergent sequence in $\mathcal{E}^1(X,\omega)$. After passing to a subsequence we may assume that $e^{-\varphi_j}\to e^{-\varphi}$ a.e. Since $\sup_X\varphi_j$ is uniformly bounded, it follows from the uniform version of Skoda's theorem [Zer01] that $e^{-\varphi_j}$ stays in a bounded subset of $L^2(X)$. In particular, the sequence $e^{-\varphi_j}$ is uniformly integrable, and hence $e^{-\varphi_j}\to e^{-\varphi}$ in $L^1(X)$. □

Set

$$L_-(\phi):=-\log\int e^{-\phi},\qquad F_-:=E-L_-.$$

Note that L_- is now *concave* on $\mathcal{E}^1(X,-K_X)$ by Hölder's inequality, so that $E-L_-$ is merely the difference of two concave functions. However, we have the following psh analogue of Prekopa's theorem, which follows from Berndtsson's results on the psh variation of Bergman kernels and implies in particular that L_- is *geodesically convex*:

Lemma **6.5.** — *Let* $\Phi:Y\to PSH(X,-K_X)$ *be a psh map. Then* $L_-\circ\Phi$ *is psh on* Y.

Proof. — Consider the product family $\pi_Y:Z:=X\times Y\to Y$ and the line bundle $M:=\pi_X^*(-K_X)$, which coincides with relative anticanonical bundle of Z/Y. Then $y\mapsto\log(\int e^{-\Phi(\cdot,y)})^{-1}$ is the weight of the L^2 metric induced on the direct image bundle $(\pi_Y)_*\mathcal{O}_Z(K_{Z/Y}+M)$. The result thus follows from [Bern09a]. □

We are now ready to prove the main part of Theorem D.

Theorem **6.6.** — *Let* X *be a Fano manifold and let* $\phi\in\mathcal{E}^1(X,-K_X)$*. The following properties are equivalent.*

(i) $F_-(\phi)=\sup_{\mathcal{E}^1(X,-K_X)}F_-$.
(ii) $MA(\phi)=e^{-\phi+c}$ *for some* $c\in\mathbf{R}$.

Furthermore, these properties imply that ϕ *is continuous.*

As mentioned in the introduction, this result extends a theorem of Ding-Tian (cf. [Tian, Corollary 6.26]) to singular weights while relaxing the assumption that $H^0(T_X)=0$ in their theorem.

Proof. — The proof of (i)⇒(ii) is similar to that of Theorem 4.1: given $u \in C^0(X)$ we have

$$E\big(P(\phi+u)\big) + \log\int e^{-(\phi+u)} \le E\big(P(\phi+u)\big) + \log\int e^{-P(\phi+u)}$$
$$\le E(\phi) + \log\int e^{-\phi}$$

thus $u \mapsto E(P(\phi+u)) + \log\int e^{-(\phi+u)}$ achieves its maximum at 0. By Lemma 4.2 $MA(\phi)$ is thus equal to the differential of $u \mapsto -\log\int e^{-(\phi+u)}$ at 0 and we get $MA(\phi) = e^{-\phi+c}$ for some $c \in \mathbf{R}$ as desired.

The equation $MA(\phi) = e^{-\phi+c}$ shows in particular that $MA(\phi)$ has $L^{1+\varepsilon}$ density and we infer from [Koł98] that ϕ is continuous.

Conversely, let $\phi \in \mathcal{E}^1(X, -K_X)$ be such that $MA(\phi) = e^{-\phi+c}$ and let $\psi \in \mathcal{E}^1(X, -K_X)$. We are to show that $F_-(\phi) \ge F_-(\psi)$. By scaling invariance of F_- we may assume that $c = 0$, and by continuity of F_- along decreasing sequences we may assume that ψ is continuous. Since ϕ is continuous, Proposition 6.3 yields a radially symmetric continuous map $\Phi : \overline{A} \to PSH(X, \omega)$ where A denotes the annulus $\{z \in \mathbf{C}, 0 < \log|z| < 1\}$, such that Φ is maximal on A and coincides with ϕ (resp. with ψ) for $\log|z| = 0$ (resp. 1). The path $\phi_t := \Phi(e^t)$ is thus a "continuous geodesic" in $PSH(X, \omega)$, and $E(\phi_t)$ is an affine function of t on the segment $[0, 1]$ by Proposition 6.2. On the other hand, Lemma 6.5 implies that $L_-(\phi_t)$ is a convex function of t, thus $F_-(\phi_t)$ is concave, with $F_-(\phi_0) = F_-(\phi)$ and $F_-(\phi_1) = F_-(\psi)$. In order to show that $F_-(\phi) \ge F_-(\psi)$, it will thus be enough to show

$$(\mathbf{6.5}) \qquad \frac{d}{dt}\Big|_{t=0_+} F_-(\phi_t) \le 0.$$

Note that $\phi_t(x)$ is a convex function of t for each x fixed, thus

$$u_t := \frac{\phi_t - \phi_0}{t}$$

decreases pointwise as $t \to 0_+$ to a function v on X that is bounded from above (by $u_1 = \phi_0 - \phi_1$). The concavity of E implies

$$\frac{E(\phi_t) - E(\phi_0)}{t} \le \int u_t MA(\phi_0),$$

hence

$$(\mathbf{6.6}) \qquad \frac{d}{dt}\Big|_{t=0_+} E(\phi_t) \le \int v MA(\phi_0) = \int v e^{-\phi_0}$$

by the monotone convergence theorem (applied to $-u_t$, which is uniformly bounded below and increases to $-v$). Note that this implies in particular that $v \in L^1(X)$. On the other hand we have

$$\frac{\int e^{-\phi_t} - \int e^{-\phi_0}}{t} = -\int u_t f(\phi_t - \phi_0) e^{-\phi_0}$$

with $f(x) := (1 - e^{-x})/x$, and $f(\phi_t - \phi_0)$ is uniformly bounded on X since $\phi_t - \phi_0$ is uniformly bounded. It follows that $|u_t f(\phi_t - \phi_0)|$ is dominated by an integrable function, hence

$$\textbf{(6.7)} \qquad \frac{d}{dt}\Big|_{t=0_+} \int e^{-\phi_t} = -\int v e^{-\phi_0}$$

since $f(\phi_t - \phi_0) \to 1$. The combination of (6.6) and (6.7) now yields (6.5) as desired. □

Remark **6.7.** — Suppose that $\phi, \psi \in \mathrm{PSH}(X, -K_X)$ are smooth, with ϕ Kähler-Einstein. We would like to briefly sketch Ding-Tian's argument for comparison. Since F_- is translation invariant we may assume that they are normalized so that $\int e^{-\phi} = \int e^{-\psi} = 1$, and our goal is to show that $E(\phi) \geq E(\psi)$. By the normalization we get $\mathrm{MA}(\phi) = V e^{-\phi}$ with $V := \mathrm{vol}(-K_X) = c_1(X)^n$, and there exists a smooth weight $\tau \in \mathrm{PSH}(X, -K_X)$ such that $\mathrm{MA}(\tau) = V e^{-\psi}$ by [Yau78]. If we further assume that $H^0(T_X) = 0$ then [BM87] yields the existence of a *smooth* path $\phi_t \in \mathrm{PSH}(X, -K_X) \cap C^\infty$ with $\phi_0 = \tau$, $\phi_1 = \phi$ and

$$\textbf{(6.8)} \qquad \mathrm{MA}(\phi_t) = V e^{-(t\phi_t + (1-t)\psi)}$$

for each $t \in [0, 1]$. The argument of Ding-Tian can then be formulated as follows. The claim is that $t(E(\phi_t) - E(\psi))$ is a non-decreasing function of t, which implies $E(\phi) - E(\psi) \geq 0$ as desired. Indeed we have

$$\textbf{(6.9)} \qquad \frac{d}{dt}\big(t\big(E(\phi_t) - E(\psi)\big)\big) = E(\phi_t) - E(\psi) + t\int \dot{\phi}_t \mathrm{MA}(\phi_t).$$

On the other hand differentiating $\int e^{-(t\phi_t + (1-t)\psi)} = 1$ yields

$$0 = \frac{d}{dt}\int e^{-(t\phi_t + (1-t)\psi)} = -\int (\phi_t + t\dot{\phi}_t - \psi) e^{-(t\phi_t + (1-t)\psi)}$$

thus

$$\int (\phi_t + t\dot{\phi}_t - \psi)\mathrm{MA}(\phi_t)$$

by (6.8), and (6.9) becomes

$$\frac{d}{dt}\big(t\big(E(\phi_t) - E(\psi)\big)\big) = E(\phi_t) - E(\psi)\int (\psi - \phi_t)\mathrm{MA}(\phi_t) = J_{\phi_t}(\psi),$$

which is non-negative as desired by concavity of E.

Proof of Theorem D. — The first part of the proof of Theorem D follows from Theorem 6.6. By Lemma 6.4 F_- is usc. If it is J-proper, then its supremum is attained on a compact convex set of weights with energy uniformly bounded from below by some large constant $-C$. The conclusion thus follows from Theorem 6.6. □

As opposed to F_+, let us recall for emphasis that F_- is not necessarily J-proper (see [Tian]).

6.4. *Uniqueness of Kähler-Einstein metrics.* — This section is devoted to the proof of Theorem E, which extends in particular [BM87] in case $H^0(T_X) = 0$.

Theorem **6.8.** — *Let* X *be a Kähler-Einstein Fano manifold without non-trivial holomorphic vector field. Then* F *achieves its maximum on* $\mathcal{T}^1(X, -K_X)$ *at a unique point.*

Proof. — Let ϕ be a smooth Kähler-Einstein weight on $-K_X$, which exists by assumption. We may assume that ϕ is normalized so that $MA(\phi) = e^{-\phi}$. Now let $\psi \in \mathcal{E}^1(X, -K_X)$ be such that $MA(\psi) = e^{-\psi}$. We are going to show that $\phi = \psi$. By Kolodziej's theorem ψ is continuous, and we consider as before the continuous geodesic ϕ_t connecting $\phi_0 = \phi$ to $\phi_1 = \psi$. Theorem 6.6 implies that the concave function $F_-(\phi_t)$ achieves its maximum at $t = 0$ and $t = 1$, thus $F_-(\phi_t)$ is *constant* on $[0, 1]$. Since $E(\phi_t)$ is affine, it follows that $L_-(\phi_t)$ is also affine on $[0, 1]$, hence $L_-(\phi_t) \equiv 0$ since $L_-(\phi_0) = L_-(\phi_1) = 0$ by assumption. This implies in turn that $E(\phi_t)$ is constant. Theorem 6.6 therefore yields $MA(\phi_t) = e^{-\phi_t}$ for all $t \in [0, 1]$.

Set $v_t := \frac{\partial}{\partial t}\phi_t$, which is non-decreasing in t by convexity. One sees as in the proof of Theorem 6.6 that $v_t \in L^1(X)$ and

$$\int v_t e^{-\phi_t} = 0 \tag{6.10}$$

for all t. We claim that $v_0 = 0$, which will imply $v_t \geq v_0 = 0$ for all t, hence $v_t = 0$ a.e. for all t by (6.10), and the proof will be complete.

We are going to show by differentiating the equation $(dd^c\phi_t)^n = e^{-\phi_t}$ that

$$n dd^c v_0 \wedge \left(dd^c\phi_0\right)^{n-1} = -v_0 e^{-\phi_0} \tag{6.11}$$

in the sense of distributions, i.e.

$$n\int v_0\left(dd^c\phi_0\right)^{n-1} \wedge dd^c w = -\int w v_0\left(dd^c\phi_0\right)^n$$

for every smooth function w on X. Using $(dd^c\phi_0)^n = e^{-\phi_0}$ (6.11) means that v_0 is an eigendistribution with eigenvalue -1 of the Laplacian Δ of the (smooth) Kähler-Einstein metric $dd^c\phi_0$, and thus $v_0 = 0$ since $H^0(T_X) = 0$ (cf. [Tian, Lemma 6.12]).

We claim that

$$\left.\frac{d}{dt}\right|_{t=0_+}\int we^{-\phi_t}=-\int wv_0e^{-\phi_0} \tag{6.12}$$

and

$$\left.\frac{d}{dt}\right|_{t=0_+}\int w\left(dd^c\phi_t\right)^n=n\int v_0\left(dd^c\phi_0\right)^{n-1}\wedge dd^cw, \tag{6.13}$$

which will imply (6.11). The proof of (6.12) is handled as before: we write

$$\int w\frac{e^{-\phi_t}-e^{-\phi_0}}{t}=-\int wu_tf(\phi_t-\phi_0)e^{-\phi_0}$$

with $f(x):=(1-e^{-x})/x$ and use the monotone convergence theorem.

On the other hand, writing dd^cw as the difference of two positive $(1,1)$-forms shows by monotone convergence that (6.13) is equivalent to

$$\int w\left(\left(dd^c\phi_t\right)^n-\left(dd^c\phi_0\right)^n\right)=n\int(\phi_t-\phi_0)\left(dd^c\phi_0\right)^n\wedge dd^cw+o(t),$$

where the left-hand side can be rewritten as

$$\int(\phi_t-\phi_0)\left(\sum_{j=0}^{n-1}\left(dd^c\phi_t\right)^j\wedge\left(dd^c\phi_0\right)^{n-j-1}\right)\wedge dd^cw$$

after integration by parts. (6.13) will thus follow if we can show that

$$\int(\phi_t-\phi_0)\left(\left(dd^c\phi_t\right)^j\wedge\left(dd^c\phi_0\right)^{n-j-1}-\left(dd^c\phi_0\right)^{n-1}\right)\wedge dd^cw=o(t)$$

for $j=0,\dots,n-1$, which will in turn follow from

$$\int(\phi_t-\phi_0)dd^c(\phi_t-\phi_0)\wedge\left(dd^c\phi_t\right)^j\wedge\left(dd^c\phi_0\right)^{n-j-2}\wedge dd^cw=o(t) \tag{6.14}$$

for $j=0,\dots,n-2$. Now we have

$$\begin{aligned}&\int(\phi_t-\phi_0)dd^c(\phi_t-\phi_0)\wedge\left(dd^c\phi_t\right)^j\wedge\left(dd^c\phi_0\right)^{n-j-2}\wedge dd^cw\\&\quad=\int d(\phi_t-\phi_0)\wedge d^c(\phi_t-\phi_0)\wedge\left(dd^c\phi_t\right)^j\wedge\left(dd^c\phi_0\right)^{n-j-2}\wedge dd^cw.\end{aligned}$$

Since w is smooth and $dd^c\phi_0$ is a Kähler form we have

$$-\mathrm{C}dd^c\phi_0\le dd^cw\le\mathrm{C}dd^c\phi_0$$

for $C \gg 1$, and we see that (6.14) will follow from

$$\int d(\phi_t - \phi_0) \wedge d^c(\phi_t - \phi_0) \wedge \left(dd^c\phi_t\right)^j \wedge \left(dd^c\phi_0\right)^{n-j-1} = o(t)$$

for $j = 0, \ldots, n-1$ since $d(\phi_t - \phi_0) \wedge d^c(\phi_t - \phi_0) \wedge (dd^c\phi_t)^j \wedge (dd^c\phi_0)^{n-j-2}$ is a positive current. Now consider

$$J_{\phi_0}(\phi_t) := E(\phi_0) - E(\phi_t) + \int (\phi_t - \phi_0) MA(\phi_0).$$

Since $E(\phi_t)$ is constant, the monotone convergence theorem yields

$$\left.\frac{d}{dt}\right|_{t=0_+} J_{\phi_0}(\phi_t) = \int v_0 MA(\phi_0) = \int v_0 e^{-\phi_0} = 0.$$

By (2.8) this implies that

$$\int d(\phi_t - \phi_0) \wedge d^c(\phi_t - \phi_0) \wedge \left(dd^c\phi_t\right)^j \wedge \left(dd^c\phi_0\right)^{n-j-1} = o(t)$$

for $j = 0, \ldots, n-1$ as desired. □

7. Balanced metrics

Let A be an ample line bundle on a projective manifold X, and denote by $\mathcal{H}_k$ the space of all positive Hermitian products on the space $H^0(kA)$ of global sections of $kA = A^{\otimes k}$, which is isomorphic to the Riemannian symmetric space

$$\mathcal{H}_k \simeq GL(N_k, \mathbf{C})/U(N_k)$$

with $N_k := h^0(kA)$. We will always assume that k is taken large enough to ensure that kA is very ample. There is a natural injection

$$f_k : \mathcal{H}_k \hookrightarrow PSH(X, A) \cap C^\infty$$

sending $H \in \mathcal{H}_k$ to the Fubiny-Study type weight

$$f_k(H) := \frac{1}{k} \log \left(\frac{1}{N_k} \sum_{j=1}^{N_k} |s_j|^2 \right)$$

where (s_j) is an H-orthonormal basis of $H^0(kA)$.

On the other hand, every measure μ on X yields a map

$$h_k(\mu,\cdot) : \mathrm{PSH}(X,A) \to \mathcal{H}_k$$

by letting $h_k(\mu,\phi)$ be the L^2-scalar product on $H^0(kA)$ induced by μ and $k\phi$.

Consider the following three settings (compare [Don09]).

($\mathbf{S}_\mu$) Let μ be a probability measure with finite energy on X, and let ϕ_0 be a reference smooth strictly psh weight on A. We set

$$h_k(\phi) := h_k(\mu,\phi)$$

and

$$L(\phi) := L_\mu(\phi) = \int (\phi - \phi_0) d\mu.$$

We also let $T \in c_1(A)$ be the unique closed positive current with finite energy such that $V^{-1}T^n = \mu$ where $V := (A^n)$.

($\mathbf{S}_+$) $A = K_X$ is ample. A weight $\phi \in \mathrm{PSH}(X, K_X)$ induces a measure e^ϕ with L^∞ density on X, and we set

$$h_k(\phi) := h_k\big(e^\phi, \phi\big)$$

and

$$L(\phi) := L_+(\phi) = \log \int e^\phi.$$

We let $T := \omega_{KE}$ be the unique Kähler-Einstein metric.

($\mathbf{S}_-$) $A = -K_X$ is ample. A weight $\phi \in \mathcal{E}^1(X, -K_X)$ induces a measure $e^{-\phi}$ on X with L^p density for all $p < +\infty$, and we set

$$h_k(\phi) := h_k\big(e^{-\phi}, \phi\big)$$

and

$$L(\phi) := L_-(\phi) = -\log \int e^{-\phi}.$$

In that case we also assume that $H^0(T_X) = 0$ and that $T := \omega_{KE}$ is a Kähler-Einstein metric, which is therefore unique by [BM87] (or Theorem 6.8 above).

As in [Don09], we shall say in each case that $H \in \mathcal{H}_k$ is *k-balanced* if it is a fixed point of $h_k \circ f_k$. The maps h_k and f_k induce a bijective correspondence between the k-balanced points in $\mathcal{H}_k$ and the *k-balanced weights* $\phi \in \mathrm{PSH}(X, A)$, i.e. the fixed points of

$f_k \circ h_k$. The k-balanced points $H \in \mathcal{H}_k$ admit the following variational characterization (cf. [Don09] and Corollary 7.5 below). Consider the function D_k on $\mathcal{H}_k$ defined by

$$D_k := -\frac{1}{2kN_k} \log \det, \tag{7.1}$$

where the determinant is computed with respect to a fixed base point in $\mathcal{H}_k$. Then $H \in \mathcal{H}_k$ is k-balanced iff it maximizes the function

$$F_k := D_k - L \circ f_k \tag{7.2}$$

on $\mathcal{H}_k$. Further, there exists at most one such maximizer, up to scaling (Corollary 7.3).

Our main result in this section is the following.

Theorem **7.1.** — *In each of the three settings* $(\mathbf{S}_\mu)$, $(\mathbf{S}_+)$ *and* $(\mathbf{S}_-)$ *above, there exists for each* $k \gg 1$ *a k-balanced metric* $\phi_k \in \mathrm{PSH}(X, A)$, *unique up to a constant. Moreover in each case* $dd^c\phi_k$ *converges weakly to* T *as* $k \to \infty$.

This type of result has its roots in the seminal work of Donaldson [Don01], and the present statement was inspired by [Don09]. In fact, the existence of k-balanced metrics in case $(\mathbf{S}_\mu)$ was established in [Don09, Proposition 3] assuming that μ integrates $\log|s|$ for every section $s \in H^0(mA)$. In [Don09, p. 12], Donaldson conjectured the convergence statement in the case where μ is a smooth positive volume form, by analogy with [Don01]. The result was indeed observed to hold for such measures in [Kel09], as a special case of [Wan05] (which in turn relied on the techniques introduced in [Don01]). The settings $(\mathbf{S}_\pm)$ were introduced and briefly discussed in [Don09, §2.2.2].

The main idea of our argument goes as follows. In each case, the functional $F := E - L$ is usc and J-coercive on $\mathcal{E}^1(X, A)$ (by Corollary 3.7 in case $(\mathbf{S}_\mu)$ and $(\mathbf{S}_+)$, and by [PSSW08] in case $(\mathbf{S}_-)$), and T is characterized as the unique maximizer of F on $\mathcal{T}^1(X, A) = \mathcal{E}^1(X, A)/\mathbf{R}$, by our variational results.

The crux of the proof is Lemma 7.7 below, which compares the restriction $J \circ f_k$ of the exhaustion function of $\mathcal{E}^1(X, A)$ to $\mathcal{H}_k$ to a natural exhaustion function J_k on $\mathcal{H}_k$. This result enables us to carry over the J-coercivity of F to a J_k-coercivity property of F_k that is furthermore uniform with respect to k (Lemma 7.9). This shows on the one hand that F_k achieves its maximum on $\mathcal{H}_k$, which yields the existence of a k-balanced weight ϕ_k. On the other hand it provides a lower bound

$$F(\phi_k) \geq \sup_{\mathcal{H}_k} F_k + o(1)$$

which allows us to show that ϕ_k is a maximizing sequence for F. We can then use Proposition 3.8 to conclude that $dd^c\phi_k$ converges to T.

7.1. *Convexity properties.* — Any geodesic $t \mapsto H_t$ in $\mathcal{H}_k$ is the image of a 1-parameter subgroup of $GL(H^0(kA))$, which means that there exists a basis $S = (s_j)$ of $H^0(kA)$ and

$$(\lambda_1, \dots, \lambda_{N_k}) \in \mathbf{R}^{N_k}$$

such that $e^{\lambda_j t} s_j$ is H_t-orthonormal for each t. We will say that H_t is *isotropic* if

$$\lambda_1 = \dots = \lambda_{N_k}.$$

The isotropic geodesics are thus the orbits of the action of $\mathbf{R}_+$ on $\mathcal{H}_k$ by scaling. With this notation, there exists $c \in \mathbf{R}$ such that

$$D_k(H_t) = \frac{t}{kN_k} \sum_j \lambda_j + c \tag{7.3}$$

for all t, and we have

$$f_k(H_t) = \frac{1}{k} \log\left(\frac{1}{N_k} \sum_j e^{t\lambda_j} |s_j|^2 \right). \tag{7.4}$$

Observe that $z \mapsto f_k(H_{\Re z})$ defines a *psh map* $\mathbf{C} \to PSH(X, A)$, i.e. $f_k(H_{\Re z})$ is psh in all variables over $\mathbf{C} \times X$. We also record the formula

$$\frac{\partial}{\partial t} f_k(H_t) = \frac{1}{k} \frac{\sum_j \lambda_j e^{t\lambda_j} |s_j|^2}{\sum_j e^{t\lambda_j} |s_j|^2}. \tag{7.5}$$

The next convexity properties will be crucial to the proof of Theorem 7.1. Recall that k is assumed to be large enough to guarantee that kA is very ample.

Lemma **7.2.** — *The function* D_k *is affine on* $\mathcal{H}_k$, *and* $E \circ f_k$ *is convex. Moreover, in each of the three settings* $(\mathbf{S}_\mu)$, $(\mathbf{S}_+)$ *and* $(\mathbf{S}_-)$ *above* $L \circ f_k$ *is convex on* $\mathcal{H}_k$, *and* strictly *convex along non-isotropic geodesics.*

Proof. — The first property follows from (7.3). Let H_t be a geodesic in $\mathcal{H}_k$ and set

$$\phi_t := f_k(H_t).$$

The convexity of $t \mapsto E(\phi_t)$ follows from Proposition 6.2, since $z \mapsto \phi_{\Re z}$ is a psh map as was observed above.

Let us now first consider the cases $(\mathbf{S}_\mu)$ and $(\mathbf{S}_+)$. Since $t \mapsto \phi_t(x)$ is convex for each $x \in X$, the convexity of $L(\phi_t)$ directly follows since $\phi \mapsto L(\phi)$ is convex and non-decreasing in these cases. In order to get the strict convexity along non-isotropic geodesics one however has to be slightly more precise. By (7.5) we have

$$k \frac{\partial}{\partial t} \phi_t = \sum_j \lambda_j \sigma_j(t)$$

with

$$\sigma_j(t) := \frac{e^{t\lambda_j}|s_j|^2}{\sum_i e^{t\lambda_i}|s_i|^2},$$

and a computation yields

$$\frac{k}{2}\frac{\partial^2}{\partial t^2}\phi_t = \left(\sum_j \lambda_j^2\sigma_j(t)\right) - \left(\sum_j \lambda_j\sigma_j(t)\right)^2.$$

Now the Cauchy-Schwarz inequality implies that

$$\left(\sum_j \lambda_j\sigma_j(t)\right)^2 \le \left(\sum_j \lambda_j^2\sigma_j(t)\right)\left(\sum_j \sigma_j(t)\right),$$

which shows that $\frac{\partial^2}{\partial t^2}\phi_t \ge 0$ (which we already knew) since

$$\sum_j \sigma_j(t) = 1.$$

Furthermore the equality case $\frac{\partial^2}{\partial t^2}\phi_t(x) = 0$ holds for a given $t \in \mathbf{R}$ and a given $x \in \mathrm{X}$ iff there exists $c \in \mathbf{R}$ such that for all j we have

$$\lambda_j\sigma_j(t)^{1/2} = c\sigma_j(t)^{1/2}$$

at the point x. If x belongs to the complement of the zero divisors $\mathrm{Z}_1, \dots, \mathrm{Z}_{\mathrm{N}_k}$ of the s_j's we therefore conclude that $\frac{\partial^2}{\partial t^2}\phi_t(x) > 0$ for all t unless H_t is isotropic.

Now in both cases $(\mathbf{S}_\mu)$ and $(\mathbf{S}_+)$ the map $\phi \mapsto \mathrm{L}(\phi)$ is convex and non-decreasing on $\mathrm{PSH}(\mathrm{X}, \mathrm{A})$ as we already noticed. We thus have

$$\frac{d^2}{dt^2}\mathrm{L}(\phi_t) \ge \int \left(\frac{\partial^2}{\partial t^2}\phi_t\right)\mathrm{L}'(\phi_t)$$

where $\mathrm{L}'(\phi_t)$ is viewed as a positive measure on X. This measure is in both cases non-pluripolar, thus the union of the zero divisors Z_j has zero measure with respect to $\mathrm{L}'(\phi_t)$, and it follows as desired from the above considerations that $t \mapsto \mathrm{L}(\phi_t)$ is strictly convex when H_t is non-isotropic.

We finally consider case $(\mathbf{S}_-)$. Since $z \mapsto \phi_{\Re z}$ is a psh map, the convexity of $t \mapsto \mathrm{L}(\phi_t)$ follows from Lemma 6.5, which was itself a direct consequence of [Bern09a]. Now if we assume that H_t is non-isotropic then the strict convexity follows from [Bern09b]. Indeed if $t \mapsto \mathrm{L}_-(\phi_t)$ is affine on a non-empty open interval I then [Bern09b, Theorem 2.4] implies that $c(\phi_t) = 0$ on I and that the vector field V_t that is dual to the $(0, 1)$-form

$$\bar{\partial}\left(\frac{\partial}{\partial t}\phi_t\right)$$

with respect to the metric $dd^c\phi_t$ is holomorphic for each $t \in \mathrm{I}$. Since we assume that $\mathrm{H}^0(\mathrm{T}_\mathrm{X}) = 0$ we thus get $\mathrm{V}_t = 0$. But we have by definition

$$c(\phi_t) = \frac{\partial^2}{\partial t^2}\phi_t - |\mathrm{V}_t|^2$$

where the norm of V_t is computed with respect to $dd^c\phi_t$, and we conclude that $\frac{\partial^2}{\partial t^2}\phi_t = 0$ on I. This however implies that H_t is isotropic by the first part of the proof, and we have reached a contradiction. □

Corollary **7.3.** — *The function* $\mathrm{F}_k := \mathrm{D}_k - \mathrm{L} \circ \mathrm{f}_k$ *is concave on* $\mathcal{H}_k$*, and all its critical points are proportional.*

Proof. — The first assertion follows directly from Lemma 7.2. As a consequence $\mathrm{H} \in \mathcal{H}_k$ is a criticical point of F_k iff it is a maximizer. Now let $\mathrm{H}_0, \mathrm{H}_1$ be two critical points and let H_t be the geodesic through $\mathrm{H}_0, \mathrm{H}_1$. If H_t is non-isotropic then $t \mapsto \mathrm{F}_k(\mathrm{H}_t)$ is *strictly* concave, which contradicts the fact that it is maximized at $t = 0$ and $t = 1$. So we conclude that H_t must be isotropic, which means that H_0 and H_1 are proportional. □

7.2. *Variational characterization of balanced metrics.* — Recall that a k-balanced weight ϕ is by definition a fixed point of $\mathrm{f}_k \circ \mathrm{h}_k$. The maps f_k and h_k induce a bijective correspondence between the fixed points of $\mathrm{f}_k \circ \mathrm{h}_k$ and those of

$$t_k := \mathrm{h}_k \circ \mathrm{f}_k$$

in $\mathcal{H}_k$. The following result is implicit in [Don09].

Lemma **7.4.** — *Let* $\mathrm{H} \in \mathcal{H}_k$. *Then* H *is a fixed point of* t_k *iff it is a critical point of*

$$\mathrm{F}_k = \mathrm{D}_k - \mathrm{L} \circ \mathrm{f}_k.$$

Proof. — Recall that for each geodesic H_t with $\mathrm{H}_0 = \mathrm{H}$ there exists $\lambda \in \mathbf{R}^{\mathrm{N}_k}$ and an H-orthonormal basis (s_j) such that $e^{t\lambda_j}s_j$ is H_t-orthonormal. We claim that

$$k\frac{d}{dt}\bigg|_{t=0} \mathrm{L} \circ \mathrm{f}_k(\mathrm{H}_t) = \left(\sum_j \lambda_j \|s_j\|^2_{t_k(\mathrm{H})}\right)\left(\sum_j \|s_j\|^2_{t_k(\mathrm{H})}\right)^{-1}. \tag{7.6}$$

In case $(\mathbf{S}_\mu)$ we have by (7.5)

$$\begin{aligned} k\frac{d}{dt}\bigg|_{t=0} \mathrm{L} \circ \mathrm{f}_k(\mathrm{H}_t) &= \int \frac{\sum_j \lambda_j |s_j|^2}{\sum_j |s_j|^2} d\mu \\ &= \sum_j \lambda_j \int |s_j|^2 e^{-k\mathrm{f}_k(\mathrm{H})} d\mu = \sum_j \lambda_j \|s_j\|^2_{\mathrm{h}_k \circ \mathrm{f}_k(\mathrm{H})}, \end{aligned}$$

and (7.6) follows since

$$\sum_j \|s_j\|^2_{\mathrm{h}_k\circ\mathrm{f}_k(\mathrm{H})} = 1$$

in that case. In case ($\mathbf{S}_\pm$) we find instead

$$k\frac{d}{dt}\Big|_{t=0} \mathrm{L}\circ\mathrm{f}_k(\mathrm{H}_t) = \left(\int \frac{\sum_j \lambda_j |s_j|^2}{\sum_j |s_j|^2} e^{\pm\mathrm{f}_k(\mathrm{H})}\right)\left(\int e^{\pm\mathrm{f}_k(\mathrm{H})}\right)^{-1}$$

and (7.6) again follows by writing

$$\int e^{\pm\mathrm{f}_k(\mathrm{H})} = \sum_j \int \frac{|s_j|^2}{\sum_i |s_i|^2} e^{\pm\mathrm{f}_k(\mathrm{H})} = \sum_j \|s_j\|^2_{t_k(\mathrm{H})}.$$

As a consequence of (7.6) we see that H is a critical point of $\mathrm{F}_k = \mathrm{D}_k - \mathrm{L}\circ\mathrm{f}_k$ iff

$$\textbf{(7.7)} \qquad \frac{1}{\mathrm{N}_k}\sum_j \lambda_j = \left(\sum_j \lambda_j \|s_j\|^2_{t_k(\mathrm{H})}\right)\left(\sum_j \|s_j\|^2_{t_k(\mathrm{H})}\right)^{-1}$$

holds for all H-orthonormal basis (s_j) and all $\lambda \in \mathbf{R}^{\mathrm{N}_k}$. If we choose in particular (s_j) to be also $t_k(\mathrm{H})$-orthogonal then (7.7) holds for all $\lambda \in \mathbf{R}^{\mathrm{N}_k}$ iff $\|s_j\|^2_{t_k(\mathrm{H})} = 1$ for all j, which means that $t_k(\mathrm{H}) = \mathrm{H}$. Conversely $t_k(\mathrm{H}) = \mathrm{H}$ certainly implies (7.7) since (s_j) is then $t_k(\mathrm{H})$-orthonormal, and the proof is complete. □

As a consequence of Corollary 7.3 and Lemma 7.4 we get

Corollary **7.5.** — *Up to an additive constant, there exists at most one k-balanced weight* $\phi \in \mathrm{PSH}(\mathrm{X}, \mathrm{A})$, *and* ϕ *exists iff* $\mathrm{F}_k = \mathrm{D}_k - \mathrm{L}\circ\mathrm{f}_k$ *admits a maximizer* $\mathrm{H} \in \mathcal{H}_k$, *in which case we have* $\phi = \mathrm{f}_k(\mathrm{H})$.

7.3. *Asymptotic comparison of exhaustion functions.* — Recall that we have fixed a reference smooth strictly psh weight ϕ_0 on A. We set $\mu_0 := \mathrm{MA}(\phi_0)$ and normalize the determinant (and thus the function D_k) by taking

$$\mathrm{B}_k := \mathrm{h}_k(\mu_0, \phi_0)$$

as a base point in $\mathcal{H}_k$ and setting $\det \mathrm{B}_k = 1$.

We now introduce a natural exhaustion function on $\mathcal{H}_k/\mathbf{R}_+$.

Lemma **7.6.** — *The scale-invariant function* $\mathrm{J}_k := \mathrm{L}_0\circ\mathrm{f}_k - \mathrm{D}_k$ *induces a convex exhaustion function of* $\mathcal{H}_k/\mathbf{R}_+$.

Proof. — Convexity follows from Lemma 7.2. The fact that $J_k \to +\infty$ at infinity on $\mathcal{H}_k/\mathbf{R}_+$ is easily seen and is a special case of [Don09, Proposition 3]. □

The next key estimate shows that the restriction $J \circ f_k$ of the exhaustion function J of $\mathcal{E}^1(X, A)$ to $\mathcal{H}_k$ is asymptotically bounded from above by the exhaustion function J_k. In other words the injection

$$f_k : \mathcal{H}_k \hookrightarrow \mathcal{E}^1(X, A)$$

sends each J_k-sublevel set $\{J_k \leq C\}$ into a J-sublevel set $\{J \leq C_k\}$ where C_k is only slightly larger than C.

Lemma **7.7.** — *There exists* $\varepsilon_k \to 0$ *such that*

$$J \circ f_k \leq (1 + \varepsilon_k) J_k + \varepsilon_k \text{ on } \mathcal{H}_k \tag{7.8}$$

for all k.

Before proving this result we need some preliminaries. Given any weight ϕ on A recall that the *distortion function* of $(\mu_0, k\phi)$ is defined by

$$\rho_k(\mu_0, \phi) := \sum_j |s_j|^2_{k\phi}$$

where (s_j) is an arbitrary $h_k(\mu_0, \phi)$-orthonormal basis of $H^0(kA)$, and the *Bergman measure* of $(\mu_0, k\phi)$ is then the probability measure

$$\beta_k(\mu_0, \phi) := \frac{1}{N_k} \rho_k(\mu_0, \phi)\mu_0.$$

When ϕ is smooth and strictly psh, the Bouche-Catlin-Tian-Zelditch theorem [Bou90, Cat99, Tia90, Zel98] gives

$$\lim_{k\to\infty} \beta_k(\mu_0, \phi) = \mathrm{MA}(\phi) \tag{7.9}$$

in C^∞-topology. The operator

$$P_k := f_k \circ h_k(\mu_0, \cdot)$$

satisfies by definition

$$P_k(\phi) - \phi = \frac{1}{k} \log\big(N_k^{-1} \rho_k(\mu_0, \phi)\big).$$

As a consequence, any smooth strictly psh weight ϕ is the C^∞ limit of $P_k(\phi)$.

Now pick $H \in \mathcal{H}_k$, and let $t \mapsto H_t$ be the (unique) geodesic in $\mathcal{H}_k$ such that $H_0 = B_k$ and $H_1 = H$. We denote by

$$v(H) := \frac{\partial}{\partial t}\bigg|_{t=0} f_k(H_t)$$

the tangent vector at $t = 0$ to the corresponding path $t \mapsto f_k(H_t)$. As before there exists $(\lambda_1, \dots, \lambda_{N_k}) \in \mathbf{R}^{N_k}$ and a basis (s_j) that is both B_k-orthonormal and H-orthogonal such that

$$v(H) = \frac{1}{k} \frac{\sum_j \lambda_j |s_j|^2}{\sum_j |s_j|^2}. \tag{7.10}$$

By convexity in the t-variable we note that

$$v(H) \le f_k(H_1) - f_k(H_0) = f_k(H) - P_k(\phi_0) \tag{7.11}$$

holds pointwise on X.

Lemma **7.8.** — *We have*

$$D_k(H) = \int v(H) \beta_k(\mu_0, \phi_0).$$

Proof. — Let H_t be the geodesic through B_k and H as above. On the one hand we have

$$D_k(H_t) = \frac{t}{kN_k} \sum_j \lambda_j.$$

On the other hand (7.10) yields

$$\int v(H) \beta_k(\mu_0, \phi_0) = \frac{1}{kN_k} \sum_j \lambda_j \int |s_j|^2_{k\phi_0} d\mu_0$$

and the result follows since (s_j) is B_k-orthonormal. □

We are now in a position to prove Lemma 7.7.

Proof of Lemma 7.7. — Let $H \in \mathcal{H}_k$. In what follows all O and o are meant to hold as $k \to \infty$ uniformly with respect to $H \in \mathcal{H}_k$. By scaling invariance of both sides of (7.8) we may assume that H is normalized by

$$L_0\big(f_k(H)\big) = 0,$$

so that

$$\sup_X \big(f_k(H) - \phi_0\big) \leq O(1)$$

and (7.11) yields

$$\sup_X v(H) \leq O(1) \tag{7.12}$$

since $P_k(\phi_0) = \phi_0 + O(1)$.

On the other hand Lemma 7.8 gives

$$D_k(H) = \int v(H)\mu_0 + o\big(\|v(H)\|_{L^1}\big) \tag{7.13}$$

since $\beta_k(\mu_0, \phi_0) \to MA(\phi_0) = \mu_0$ in L^∞ by Bouche-Catlin-Tian-Zelditch. Now we have

$$\begin{aligned}\|v(H)\|_{L^1} &\leq 2\sup_X v(H) - \int v(H)d\mu_0 \\ &= -D_k(H) + o\big(\|v(H)\|_{L^1}\big) + O(1)\end{aligned}$$

(by (7.12) and (7.13)) and it follows that

$$\big(1 + o(1)\big)\|v(H)\|_{L^1} \leq -D_k(H) + O(1). \tag{7.14}$$

On the other hand, the convexity of $E \circ f_k$ (Lemma 7.2) shows that

$$E \circ f_k(H) - E\big(P_k(\phi_0)\big) \geq \big\langle E'\big(P_k(\phi_0)\big), v(H)\big\rangle = \int v(H) MA\big(P_k(\phi_0)\big).$$

Now we have $E(P_k(\phi_0)) = o(1)$ since $P_k(\phi_0) = \phi_0 + o(1)$ uniformly on X and

$$\int v(H) MA\big(P_k(\phi_0)\big) = \int v(H)\mu_0 + o\big(\|v(H)\|_{L^1}\big)$$

by $\mathcal{L}^\infty$ convergence of $MA(P_k(\phi_0))$ to $MA(\phi_0) = \mu_0$. By (7.13) we thus get

$$\begin{aligned}E \circ f_k(H) &\geq D_k(H) + o\big(\|v(H)\|_{L^1}\big) + o(1) \\ &\geq \big(1 + o(1)\big)D_k(H) + o(1)\end{aligned}$$

by (7.14) and the result follows. □

7.4. *Coercivity.* — Recall that $F = E - L$ is J-coercive, i.e. there exists $0 < \delta < 1$ and $C > 0$ such that

$$F \leq -\delta J + C \tag{7.15}$$

on $\mathcal{E}^1(X, A)$. The next result uses the key estimate (7.8) to show that the J-coercivity of F carries over to a uniform J_k-coercivity estimate for $F_k = D_k - L \circ f_k$ for all $k \gg 1$.

Lemma **7.9.** — *There exists* $\varepsilon > 0$ *and* $B > 0$ *such that*

$$F_k \leq -\varepsilon J_k + B$$

holds on $\mathcal{H}_k$ *for all* $k \gg 1$.

Proof. — As discussed after Definition 3.6 (7.15) is equivalent to the linear upper bound

$$L_0 - L \leq (1 - \delta)J + C \tag{7.16}$$

which implies

$$L_0 \circ f_k - L \circ f_k \leq (1 - \delta)J \circ f_k + C.$$

On the other hand we have

$$J \circ f_k \leq (1 + \varepsilon_k)J_k + \varepsilon_k$$

by (7.8) hence

$$L_0 \circ f_k - L \circ f_k \leq (1 - \delta)(1 + \varepsilon_k)J_k + C + \varepsilon_k.$$

Since $J \geq 0$ (7.8) shows in particular that J_k bounded below on $\mathcal{H}_k$ uniformly with respect to k. For $k \gg 1$ we have $(1 - \delta)(1 + \varepsilon_k) < (1 - \varepsilon)$ and $C + \varepsilon_k < B$ for some $\varepsilon > 0$ and $B > 0$ and we thus infer

$$L_0 \circ f_k - L \circ f_k \leq (1 - \varepsilon)J_k + B.$$

It is then immediate to see that this is equivalent to the desired inequality by using $J_k = L_0 \circ f_k - D_k$. □

Note that the coercivity constants ε and B of F_k can even be taken arbitrarily close to those δ and C of F, as the proof shows.

Combining Lemma 7.9 with Lemma 7.6 yields

Corollary **7.10.** — *For each* $k \gg 1$ *the scale-invariant functional* F_k *tends to* $-\infty$ *at infinity on* $\mathcal{H}_k/\mathbf{R}_+$, *hence it achieves its maximum on* $\mathcal{H}_k$.

7.5. *Proof of Theorem 7.1.* — The existence and uniqueness of a k-balanced metric ϕ_k for $k \gg 1$ follows by combining Corollary 7.5 and Corollary 7.10. Recall that $\phi_k = \mathrm{f}_k(\mathrm{H}_k)$ where $\mathrm{H}_k \in \mathcal{H}_k$ is the unique maximizer of $\mathrm{F}_k = \mathrm{D}_k - \mathrm{L} \circ \mathrm{f}_k$ on $\mathcal{H}_k$.

In order to prove the convergence of $dd^c\phi_k$ to T we will rely on Proposition 3.8. Since T is characterized as the unique maximizer of $\mathrm{F} = \mathrm{E} - \mathrm{L}$, we will be done if we can show that

$$\liminf_{k\to\infty} \mathrm{F}(\phi_k) \geq \mathrm{F}(\psi) \tag{7.17}$$

for each $\psi \in \mathcal{E}^1(\mathrm{X}, \mathrm{A})$. As a first observation, we note that it is enough to prove (7.17) when ψ is smooth and strictly psh. Indeed, by [Dem92, BK07] we can write an arbitrary element of $\mathcal{E}^1(\mathrm{X}, \mathrm{A})$ as a decreasing sequence of smooth strictly psh weights, and the monotone continuity properties of E and L therefore show that $\sup_{\mathcal{E}^1}(\mathrm{E} - \mathrm{L})$ is equal to the sup of $\mathrm{E} - \mathrm{L}$ over all smooth strictly psh weights.

Let us now establish (7.17) for a smooth strictly psh ψ. Since $\mathrm{F}_k = \mathrm{D}_k - \mathrm{L} \circ \mathrm{f}_k$ is maximized at H_k we have in particular

$$\mathrm{F}_k(\mathrm{H}_k) \geq \mathrm{D}_k\big(\mathrm{h}_k(\mu_0, \psi)\big) - \mathrm{L}\big(\mathrm{P}_k(\psi)\big). \tag{7.18}$$

Since $\mathrm{D}_k(\mathrm{h}_k(\mu_0, \phi_0)) = 0$ the first term on the right-hand side of (7.18) writes

$$\mathrm{D}_k\big(\mathrm{h}_k(\mu_0, \psi)\big) = \int_{t=0}^{1} \left(\frac{d}{dt}\mathrm{D}_k\big(\mathrm{h}_k\big(\mu_0, t\psi + (1-t)\phi_0\big)\big)\right) dt.$$

By [BB10, Lemma 4.1] we have

$$\frac{d}{dt}\mathrm{D}_k\big(\mathrm{h}_k\big(\mu_0, t\psi + (1-t)\phi_0\big)\big) = \int (\psi - \phi_0)\beta_k\big(\phi_0, t\psi + (1-t)\phi_0\big)$$

and the Bouche-Catlin-Tian-Zelditch theorem yields

$$\mathrm{D}_k\big(\mathrm{h}_k(\mu_0, \psi)\big) \to \int_{t=0}^{1} \int (\psi - \phi_0)\mathrm{MA}\big(t\psi + (1-t)\phi_0\big) dt = \mathrm{E}(\psi)$$

(this argument being actually an easy special case of [BB10, Theorem A]). The second term on the right-hand side of (7.18) satisfies $\mathrm{L}(\mathrm{P}_k(\psi)) \to \mathrm{L}(\psi)$ since $\mathrm{P}_k(\psi) \to \psi$ uniformly. It follows that

$$\mathrm{F}_k(\mathrm{H}_k) \geq \mathrm{F}(\psi) + o(1) \tag{7.19}$$

(where $o(1)$ depends on ψ) and we will thus be done if we can show that

$$\mathrm{F}(\phi_k) - \mathrm{F}_k(\mathrm{H}_k) \geq o(1).$$

Now we have

$$\mathrm{F}(\phi_k) - \mathrm{F}_k(\mathrm{H}_k) = (\mathrm{J}_k - \mathrm{J} \circ \mathrm{f}_k)(\mathrm{H}_k) \geq -\varepsilon_k \mathrm{J}_k(\mathrm{H}_k) + o(1)$$

by (7.8) so it is enough to show that $J_k(H_k)$ is bounded from above. But we can apply the uniform coercivity estimate of Lemma 7.9 to get

$$F_k(H_k) \leq -\varepsilon J_k(H_k) + O(1)$$

for some $\varepsilon > 0$. Since the left-hand side is bounded from below in view of (7.19) we are finally done.

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