PERIODIC ORBITS AND CHAIN-TRANSITIVE SETS OF C^1-DIFFEOMORPHISMS

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ABSTRACT

We prove that the chain-transitive sets of C^1-generic diffeomorphisms are approximated in the Hausdorff topology by periodic orbits. This implies that the homoclinic classes are dense among the chain-recurrence classes.

This result is a consequence of a global connecting lemma, which allows to build by a C^1-perturbation an orbit connecting several prescribed points. One deduces a weak shadowing property satisfied by C^1-generic diffeomorphisms: any pseudo-orbit is approximated in the Hausdorff topology by a finite segment of a genuine orbit. As a consequence, we obtain a criterion for proving the tolerance stability conjecture in Diff^1(M).

CONTENTS

0. Introduction

0.1. *The shadowing lemma in hyperbolic dynamics*

In the study of differentiable dynamics, a remarkably successful theory, starting from the early sixties with Smale [Sm$_2$], describes a large class of systems: the *uniformly hyperbolic systems*. Their dynamics may exhibit complicated behavior but are well understood. For instance, Smale's spectral decomposition theorem asserts that the recurrent dynamics breaks down into finitely many invariant *basic sets*. Each of them is undecomposable (it contains a dense orbit), locally maximal (the only invariant subsets in a neighborhood are contained in the basic set itself) and contains a dense set of periodic points.

It also appears that these systems satisfy some stability properties, meaning that the orbit behavior does not change under small perturbations. Let Diff^r(M) be the space of C^r diffeomorphism of a compact riemannian manifold M, endowed with the C^r-topology. A diffeomorphism $f \in$ Diff^r(M) is called *structurally stable* in Diff^r(M) if any diffeomorphism $g \in$ Diff^r(M) that is close to f is conjugate to f through a homeomorphism of M. Palis and Smale conjectured in [PS] that the structurally stable diffeomorphisms in Diff^r(M) are the hyperbolic C^r-diffeomorphisms that satisfy the strong transversality condition: the stable and the unstable manifolds are transverse.

Robin, de Melo and Robinson [Rob, D, Rob$_2$, Rob$_3$] have then proven that the hyperbolic diffeomorphisms that satisfy the strong transversality condition are structurally stable and also that the strong transversality condition is necessary for the structural stability. At the end of the eighties, Mañé [M$_3$] finished to prove the conjecture for the C^1-diffeomorphisms.

An important tool of the hyperbolic theory is provided by the *shadowing lemma*. Conley and Bowen have introduced [B, C] the notion of *ε-pseudo-orbits* of a dynamical system f. These are sequences of points $\{z_n\}$ in M that generalize the orbits, since errors (of size smaller than ε) are allowed at each iterations. More precizely, for each n, the distance between $f(z_n)$ and z_{n+1} is smaller than ε. Such a pseudo-orbit is δ-shadowed by an orbit $\{f^n(x)\}$ if for any integer n, the points z_n and $f^n(x)$ are at distance less than δ. For any hyperbolic set K, the shadowing property is satisfied: for any scale $\delta > 0$, there exists $\varepsilon > 0$ such that to any ε-pseudo-orbit $\{z_n\}$ in K one can associate a genuine orbit $\{f^n(x)\}$ which δ-shadows the pseudo-orbit. Two important consequences of the shadowing lemma may be mentioned:

– If two periodic orbits in K have points that are close, their stable and unstable manifolds intersect, implying that the two orbits are included in a same transitive set. This can be used to prove Smale's spectral theorem.

– One easily gets a form of stability of the hyperbolic systems, that is weaker than the structural stability. If the diffeomorphism g is close enough to the map f, then, any orbit of g that stays in a neighborhood of the hyperbolic set K is shadowed by an orbit of the unperturbed system f. Let us now consider the *non-wandering set* $\Omega(f)$ of f, which supports, in some sense, the non-trivial dynamics. (A point is wandering if one of its neighborhoods is disjoint from all its iterates; $\Omega(f)$ is the set of points that are not wandering.) One says that f is Ω-*stable* in Diff^r(M) if for any diffeomorphism g that is close to f in Diff^r(M), the induced dynamics of f and g on $\Omega(f)$ and $\Omega(g)$ are conjugate by a homeomorphism. Using the shadowing property, one can prove that the hyperbolic systems whose basic sets don't have cycles (roughly speaking, this means that the basic sets are strictly ordered by the dynamics) are Ω-stable in Diff^r(M). This result was originally proven by Smale in [Sm$_2$]. Improving Mañé's result on the structural stability, Palis [P] proved conversely that the hyperbolic systems that don't have cycles are the only Ω-stable systems in Diff^1(M).

In parallel to these works, it appeared however that the hyperbolic diffeomorphisms fail to be dense in the set of differentiable systems. When the dimension of the manifold M is equal or larger than 3, Abraham, Smale and Simon [AS, Si] have shown that Diff^1(M) contains non-empty open sets of non-hyperbolic dynamics. These examples are related to the existence of periodic orbits that have different indices (the dimension of their stable space) but contained in the same transitive set, produc-

ing some so-called heterodimensional cycles. Many generalizations of these examples were built in [Sh$_2$,M$_1$,BD$_1$]. On surfaces, by looking at the homoclinic tangencies, Newhouse [N$_2$,N$_3$] discovered an other phenomena that produces open sets of non-hyperbolic diffeomorphisms. His result applies only in the C^r topologies with $r \geq 2$. It was later generalized to manifolds of higher dimensions, see [PV,Rom,GST].

0.2. *The generic dynamics*

In view of these results, it is also important to focus on the systems that are far from the hyperbolic dynamics. Some of them exhibit very degenerate phenomena (for instance, some coincide with the identity on an open set), but that can disappear by small perturbations of the dynamics. With the theory of generic dynamics, we forget these pathological systems, which represent a small part of the space of differentiable dynamics, and try to give a description of a large class of the remaining diffeomorphisms. In this paper we are interested by sets of diffeomorphisms that are residual for the Baire category (i.e. that contains a countable intersection of dense and open subsets of Diff^r(M)). The results on the hyperbolic systems may indicate what kind of properties can be looked for: spectral decomposition, weaker forms of stability, shadowing properties, invariant splittings of the tangent bundle, ... but new phenomena having some kind of persistence should also be introduced, in particular when tangencies or heterodimensional cycles occur: for example, the existence of infinitely many sinks or sources (Newhouse phenomenon).

Let us give an example of a generic property, proven by Kupka and Smale [K, Sm$_1$]. For any diffeomorphism in a residual subset of Diff^r(M), the periodic orbits are all hyperbolic. Moreover their stable and unstable manifolds intersect transversally. Pugh then proved [Pu$_2$] that for C^1-generic diffeomorphisms, the periodic points are dense in the non-wandering set. It is a consequence of his closing lemma [Pu$_1$]. By C^1-small perturbations, it is possible to create periodic points close to any non-wandering point. His techniques only work in the C^1 topology and explain why Mañé's theorem on the stability and most of the genericity results deal with the space Diff^1(M). Quite recently, Hayashi [H] improved this result and proved a connecting lemma that opened the door to many developments in C^1-generic dynamics. Let us consider two points p, q whose orbits accumulate – one in the future and the other one in the past – on a same non-periodic point. Then, p and q can be connected by a segment of orbit for an arbitrarily small C^1-perturbation of the dynamics. Using these techniques, we were able, with Bonatti, to get a connecting lemma for pseudo-orbits [BC].

Connecting lemma for pseudo-orbits. — *Let us consider a diffeomorphism f whose periodic orbits are hyperbolic. Let x, x' be two points that may be connected by ε-pseudo-orbits of f for any constant $\varepsilon > 0$.*

Then, there exist arbitrarily small C^1*-perturbations g of f, such that x and x' belong to the same orbit of g.*

As for the structural and the Ω-stabilities, the shadowing property is not generic. Bonatti, Díaz and Turcat have proven in [BDT] that there exists a non-empty open set of C^1-diffeomorphisms over a 3-dimensional manifolds where the shadowing property fails[1]. One of the main results of this paper however shows that a weaker property holds C^1-generically. The pseudo-orbits may be approximated by genuine orbits if one forgets the time parameterization and uses the Hausdorff topology. Let us recall that two compact sets $\mathscr{O}$ and $\mathscr{X}$ are δ-close for the Hausdorff distance if the δ-neighborhood of $\mathscr{X}$ contains $\mathscr{O}$, and the δ-neighborhood of $\mathscr{O}$ contains $\mathscr{X}$.

Theorem **1.** — *There exists a residual subset* $\mathscr{G}_{\mathbf{shadow}}$ *of* $\mathrm{Diff}^1(M)$ *such that any diffeomorphism* $f \in \mathscr{G}_{\mathbf{shadow}}$ *satisfies the following weak shadowing property:*

For any $\delta > 0$*, there is* $\varepsilon > 0$ *such that to any* ε*-pseudo-orbit* $\mathscr{X} = \{z_0, z_1, ..., z_n\}$*, one can associate a segment of orbit* $\mathscr{O} = \{x, f(x), ..., f^m(x)\}$ *which is* δ*-close to* $\mathscr{X}$ *for the Hausdorff distance.*

Moreover if the pseudo-orbit $\mathscr{X}$ *is periodic (i.e.* $z_n = z_0$*), then, the point x can be chosen m-periodic.*

The weak shadowing property we get is different and stronger than the usual weak shadowing property introduced by Corless and Pilyugin in [CP] and studied by Sakai[2]. Among other results, Sakai proves [Sa] that if M is a surface, the interior of the set of the C^1-diffeomorphisms which have the weak shadowing property are the diffeomorphisms that satisfy the axiom A and the no-cycle condition. A more detailed exposition is given in [Pi].

In the proof of Theorem 1, the connecting lemma for pseudo-orbits is needed but is not sufficient. It allows to show that for each pair of points (z, z') in the pseudo-orbit $\mathscr{X}$, there is a segment of orbit $\mathscr{O}_{z,z'}$ that intersects the balls centered at z and z', with radius δ; but the segment of orbit could be different for each pair (z, z'). In some sense, the connecting lemma for pseudo-orbits is semi-global. One connects two different points $z, z' \in M$ by a perturbation which is global in M, but we don't have any control on the support of the orbit that joints z to z'. (In comparison, the closing lemma and Hayashi's connecting lemma are local results: the perturbations are local.)

For proving Theorem 1, we need a new C^1-perturbation result which is global. In particular, we will discuss the following question:

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[1] The dynamics of these diffeomorphisms is transitive and the non-wandering set is the whole manifold. Some generalizations of this result exist [YY,AD]. Hence, even in restriction to the non-wandering set, the shadowing property fails on these examples.

[2] These authors require that for any ε-pseudo-orbit $\{z_n\}_{n\in\mathbf{Z}}$, there exists $x \in M$ such that $\{z_n\}_{n\in\mathbf{Z}}$ is contained in the δ-neighborhood of the orbit $\{f^n(x)\}_{n\in\mathbf{Z}}$. Consequently, a transitive diffeomorphism has this weak shadowing property.

Question. — Let $U_1, ..., U_k$ *be some (open) regions of the manifold* M. *Which assumptions would imply the existence of an orbit that crosses all these regions?*

We first define the sets that may be approximated by orbits by perturbation. Let $\mathscr{X}$ be a compact set which is invariant by f. For any points x, x' in $\mathscr{X}$, we say that x' is a *weak iterate* of x' in $\mathscr{X}$ (this will be denoted by $x \prec_{\mathscr{X}} x'$) if for any non-empty open sets U and U' that contain x and x' respectively and for any neighborhood W of $\mathscr{X}$, there exists a segment of orbit $\{z, f(z), ..., f^n(z)\}$ contained in W, such that z belongs to U, $f^n(z)$ belongs to U' and n is greater than or equal to 1. The set $\mathscr{X}$ is a *weak orbit* of f if any distinct points x, x' in $\mathscr{X}$ may be compared by the relation $\prec_{\mathscr{X}}$. For example, the closure and the ω-limit set of any orbit are weak orbits. Sometimes, one will consider weak orbits that are "dynamically ordered": one requires that $\prec_{\mathscr{X}}$ is transitive (so that it is a total ordering).

We will also need the following technical generic assumption:

(A) For any integer $n \geq 1$, the periodic points of f of period n are isolated in M.

This condition holds for instance when all the periodic orbits are hyperbolic. Kupka and Smale's theorem thus imply that (A) is generic in any space $\mathrm{Diff}^r(M)$ for $r \geq 1$.

The announced global perturbation result is the following:

Theorem **2.** *— Let f be a diffeomorphism that satisfies Condition (A), $\mathscr{U}$ a neighborhood of f in* $\mathrm{Diff}^1(M)$ *and $\mathscr{X}$ a weak orbit of f such that $\prec_{\mathscr{X}}$ is transitive. Then, for any $\eta_0 > 0$, there exist a diffeomorphism $g \in \mathscr{U}$ and a full orbit $\mathscr{O}(x) = \{g^n(x), n \in \mathbf{Z}\}$ of g whose closure is η_0-close to $\mathscr{X}$ for the Hausdorff distance.*

Of course, the orbit $\mathscr{O}(x)$ we obtain in this theorem is in general not periodic. (For instance if the weak orbit $\mathscr{X}$ of f contains a source p and a sink q, then the orbit $\mathscr{O}$ of g still accumulates on p in the past and on q in the future.) So, we have another version of the result which deals with weak orbits that have a recurrence property: we will say that a compact and invariant set $\mathscr{X}$ is *weakly transitive* if[3] for any non-empty open sets U and V that intersect $\mathscr{X}$ and any neighborhood W of $\mathscr{X}$, there exists a segment of orbit $\{x, f(x), ..., f^n(x)\}$ contained in W and such that x belongs to U, $f^n(x)$ belongs to V and n is greater than or equal to 1. (Note that by choosing U = V, one sees that $\mathscr{X}$ is contained in the non-wandering set $\Omega(f)$.) As an example, any transitive set is a weakly transitive set. Moreover any weakly transitive set $\mathscr{X}$ is also a weak orbit whose relation $\mathscr{X}$ is transitive.

Theorem **3.** *— Let f be a diffeomorphism that satisfies Condition (A), $\mathscr{U}$ a neighborhood of f in* $\mathrm{Diff}^1(M)$ *and $\mathscr{X}$ a weakly transitive set of f. Then, for any $\eta_0 > 0$, there exist a diffeomorphism $g \in \mathscr{U}$ and a periodic orbit $\mathscr{O}$ of g that is η_0-close to $\mathscr{X}$ for the Hausdorff distance.*

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[3] An equivalent definition is: for any x, x' in $\mathscr{X}$, the point x' is a weak iterate of x in $\mathscr{X}$.

In the next two sections we discuss the consequences of these theorems for the stability and the spectral decomposition of the C^1-generic diffeomorphisms. The reader will find other corollaries in Section 3.

0.3. *Tolerance stability*

An important goal of dynamical systems is to describe how the dynamical invariants change under perturbations. This leads to the notion of stability. Once people have discovered that the sets of structurally stable and Ω-stable diffeomorphisms are not dense, they tried to find weaker forms of stability satisfied by a larger class of systems (see [Sh_1]). At the beginning of the seventies, following an idea of Zeeman, Takens formulated [T_1] an interesting notion of stability: the tolerance stability, which asserts that the orbit structure of a system varies only a little under small perturbations.

More precisely, for $r \geq 0$, we say that a diffeomorphism $f \in \mathrm{Diff}^r(M)$ of M is *tolerance stable in* $\mathrm{Diff}^r(M)$ if for any $\varepsilon > 0$, there exists a neighborhood $\mathscr{U}$ of f in $\mathrm{Diff}^r(M)$ which satisfies the following: for any diffeomorphisms g and g' in $\mathscr{U}$, and any orbit $\mathrm{O} = \{g^n(x), n \in \mathbf{Z}\}$ of g, there exists an orbit $\mathrm{O}' = \{g'^n(x'), n \in \mathbf{Z}\}$ of g' such that O is contained in the ε-neighborhood of O′ and O′ is contained in the ε-neighborhood of O (i.e. the closures of O and O′ are ε-close for the Hausdorff topology). This may be also defined in the following elegant way: recall that the set of non-empty compact subsets of M, endowed with the Hausdorff distance, is a compact metric space, that we denote by $\mathscr{K}(M)$. Hence, the closures of all the orbits of a diffeomorphism f of M gives a subset of $\mathscr{K}(M)$. Taking its closure in $\mathscr{K}(M)$, we obtain an element $\mathbf{Orb}(f)$ of $\mathscr{K}(\mathscr{K}(M))$. A diffeomorphism f is tolerance stable in $\mathrm{Diff}^r(M)$ if it is a continuity point of the map $g \mapsto \mathbf{Orb}(g)$ from $\mathrm{Diff}^r(M)$ to $\mathscr{K}(\mathscr{K}(M))$. In [$T_1$] Takens stated the following conjecture[4].

Tolerance stability conjecture (Zeeman). — *For any compact manifold* M *and any* $r \geq 0$, *the tolerance stable diffeomorphisms are generic in* $\mathrm{Diff}^r(M)$.

Since the orbits of a perturbed map g close to a diffeomorphism f are pseudo-orbits of f, it is very natural to look for a new formulation of the tolerance stability using pseudo-orbits. Takens defined in [T_2] the notion of extended orbit, which is even more general than the weak orbits defined above: a compact set $\mathscr{X}$ which is invariant by f is an *extended orbit* if for any $\varepsilon > 0$ there exists an ε-pseudo-orbit contained in $\mathscr{X}$ and that is ε-dense in $\mathscr{X}$. The set of all the extended orbits is closed in $\mathscr{K}(M)$ and defines an element $\mathbf{EOrb}(f)$ of $\mathscr{K}(\mathscr{K}(M))$. If one replaces the orbits by the extended orbits in the definition of the tolerance stability, one gets the notion

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[4] In the AMS review of the paper [Wh], Takens explains that Zeeman never published this conjecture.

of extended-tolerance stability. (Equivalently, a diffeomorphism is extended-tolerance stable in Diff^r if it is a continuity point of the map $g \mapsto \mathbf{EOrb}(g)$.) Takens proved by semi-continuity arguments that the extended tolerance stable diffeomorphisms in Diff^r(M) are C^r-generic. Hence, he deduced the following criterion to get the genericity of the tolerance stability:

*Criterion **1** (Takens). — If the set of diffeomorphisms f such that $\mathbf{Orb}(f) = \mathbf{EOrb}(f)$ is residual in* Diff^r(M), *then, the tolerance stability conjecture holds in* Diff^r(M).

Takens proved in [T$_1$] some related results that motivated the conjecture. If T is a map from Diff^r(M) to a topological space X, a diffeomorphism f is T-*tolerance stable in* Diff^r(M) if it is a continuity point of T. For instance, T may associate to any diffeomorphism f, the set $\mathbf{Cl}(f)$ of invariant compact subsets of M, which is an element of X $= \mathscr{K}(\mathscr{K}(\mathrm{M}))$. He showed that the $\mathbf{Cl}$-tolerance stable diffeomorphisms are generic in Diff^1(M).

Another important example for T is the map which associates the support of the non-trivial dynamics, viewed as an element of $\mathscr{K}(\mathrm{M})$. In general, there is no canonical definition of the non-trivial dynamics. It should at least contain all the periodic orbits: their closure will be denoted by $\mathscr{P}\mathrm{er}(f)$. Another possibility for the non-trivial dynamics is the chain-recurrent set $\mathscr{R}(f)$: it contains the points that belong to periodic ε-pseudo-orbits for arbitrarily small $\varepsilon > 0$. This set is certainly the largest candidate (any other point belongs to the basin of a trapping region, as it was proved by Conley, see the next section). There are several other choices between $\mathscr{P}\mathrm{er}(f)$ and $\mathscr{R}(f)$ but we will only mention the non-wandering set $\Omega(f)$ since it played an important historical role (the reason is that by Pugh's closing lemma $\mathscr{P}\mathrm{er}(f) = \Omega(f)$ for C^1-generic diffeomorphisms). Recently, the equality $\mathscr{P}\mathrm{er}(f) = \mathscr{R}(f)$ was shown in [BC] for C^1-generic diffeomorphisms, implying that in some sense the non-trivial dynamics is well defined for the generic diffeomorphisms[5].

Using the genericity of Kupka–Smale diffeomorphisms, Takens showed in [T$_1$] that in Diff^r(M) the $\mathscr{P}\mathrm{er}$-tolerance stable diffeomorphisms are generic (he proved a stronger statement, see Theorem 8 at Section 3.2). From Pugh's closing lemma he got also the genericity of the Ω-tolerance stability in Diff^1(M). The case of the $\mathscr{R}$-tolerance stability is easier and follows from a standard argument[6].

Little progress have been done since Takens papers: using the classical shadowing lemma, Robinson [Rob$_4$] has proved the conjecture for Axiom A diffeomorphisms.

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[5] Due to this result, my opinion is that the chain-recurrent set should now be preferred to the non-wandering set. Note for example that, according to Newhouse [N$_1$], the diffeomorphisms that are Axiom A (i.e. the diffeomorphisms such that $\Omega(f)$ is hyperbolic and $\Omega(f) = \mathscr{P}\mathrm{er}(f)$) and satisfy the non-cycle condition may be simply defined as the diffeomorphisms whose chain-recurrent set is hyperbolic.

[6] One first proves that the map $f \mapsto \mathscr{R}(f)$ is upper-semi-continuous (for any map g close to f, the set $\mathscr{R}(g)$ is contained in a neighborhood of $\mathscr{R}(f)$); Baire's theorem implies that the continuity points of this map are residual, see also Section 3.

In the space of homeomorphisms $\mathrm{Diff}^0(\mathrm{M})$, Mazur has obtained [Maz] a strong form of the conjecture, improving a partial result of Odani [O].

The tolerance stability conjecture in $\mathrm{Diff}^1(\mathrm{M})$ and Takens criterion were a motivation for proving the perturbation results and the generic weak shadowing properties of this paper. Theorem 1 seems very close to show that for C^1-generic diffeomorphisms f, one has $\mathbf{EOrb}(f) = \mathbf{Orb}(f)$: it claims that for a C^1-generic diffeomorphism f and for any $\delta > 0$, any extended orbit of f is δ-close to a segment of orbit of f for the Hausdorff distance. There remains however a serious gap which consists in replacing the segments of orbits of f by whole orbits: let us considers the invariant compact subsets of M that are limit of segments of orbits for the Hausdorff distance; this gives a new element $\mathbf{FOrb}(f)$ in $\mathscr{K}(\mathscr{K}(\mathrm{M}))$. We always have the inclusions $\mathbf{Orb}(f) \subset \mathbf{FOrb}(f) \subset \mathbf{EOrb}(f)$ but Theorem 1 implies that for a C^1-generic diffeomorphism f, one has $\mathbf{FOrb}(f) = \mathbf{EOrb}(f)$. Another consequence is that the continuity points of the map $g \mapsto \mathbf{FOrb}(g)$ are generic in $\mathrm{Diff}^1(\mathrm{M})$. So we improved Takens criterion:

Criterion **2.** — *If the set of diffeomorphisms f such that $\mathbf{Orb}(f) = \mathbf{FOrb}(f)$ is residual in $\mathrm{Diff}^1(\mathrm{M})$, then, the tolerance stability conjecture holds in $\mathrm{Diff}^1(\mathrm{M})$.*

The reader may observe that by Theorem 2, for any C^1-generic diffeomorphism f, and any extended orbit $\mathscr{X}$ of f, there are orbits $(\mathscr{O}_n)$ of C^1-small perturbations (g_n) of f whose closure are arbitrarily close to $\mathscr{X}$ in the Hausdorff topology. This is however not sufficient to obtain $\mathbf{Orb}(f) = \mathbf{FOrb}(f) = \mathbf{EOrb}(f)$ for C^1-generic diffeomorphisms by some Baire argument since the obtained orbits $\mathscr{O}_n$ are not robust under perturbations: for any generic diffeomorphism g close to a perturbation g_n, all the orbits that shadows $\mathscr{X}$ could escape after a long time and visit other regions of M. One would obtain however a control on the asymptotic behavior of the orbits $\mathscr{O}_n$ if the orbit $\mathscr{O}(x)$ in Theorem 2 can be obtained as a heteroclinic orbit between two hyperbolic periodic orbits. This would be implied by another perturbation result:

Problem **1** *(Asymptotic closing lemma).* — Let f be a diffeomorphism of a compact manifold M, $\mathscr{U}$ a C^1-neighborhood of f and x a point in M.

– Does there exist a perturbation $g \in \mathscr{U}$ such that x is on the stable manifold of a hyperbolic periodic orbit $\mathscr{O}$?
– Can we require moreover that the closure of the forward orbits of x by f and by g remain close for the Hausdorff distance? that the ω-limit set of the orbit of x by f (i.e. the accumulation set of the forward orbit of x) and the periodic orbit $\mathscr{O}$ by g are close for the Hausdorff distance?

A positive answer to these questions would also show that for a C^1-generic diffeomorphism, the stable and unstable manifolds of the hyperbolic periodic points are dense in the manifold.

0.4. *Spectral decomposition of* C^1*-generic diffeomorphisms*

In dynamical systems the periodic orbits play a particular role. Some dynamical invariants are associated to them; in general, they also can be followed after perturbation of the dynamics. Moreover, Pugh's closing lemma implies that any non-wandering point of a C^1-generic diffeomorphism is the limit of a sequence of periodic point. However, since the perturbation is local, it does not control the support of the periodic orbit. We will see below that it is interesting to answer the following global problem:

Question. — What is the class of compact sets that may be approximated by a sequence of periodic orbits?

A partial answer was given by Arnaud in [A$_2$]. She showed that the ω-limit sets of a C^1-generic diffeomorphism belong to this class of sets. However it is not known if for a generic diffeomorphism the ω-limit sets are the only possible compact sets in this class: in general the set of ω-limit sets is not closed for the Hausdorff topology. Let us also mention Mañé's ergodic closing lemma [M$_2$] which gives the measure theoretical viewpoint on the approximation by periodic orbits: it asserts that any ergodic invariant probability measure μ of a C^1-generic diffeomorphism is the limit of a sequence of invariant measures supported by periodic orbits $(\mathcal{O}_n)$. Moreover, the orbits $\mathcal{O}_n$ converge towards the support of μ for the Hausdorff topology. For the C^1-generic diffeomorphisms, Theorem 3 provides us with a complete answer of the question, in terms of chain-transitivity.

*Theorem **4.** — There exists a G_δ dense subset $\mathcal{G}_{\mathbf{rec}}$ of* $\mathrm{Diff}^1(\mathrm{M})$ *such that for any diffeomorphism $f \in \mathcal{G}_{\mathbf{rec}}$, a compact invariant set $\mathcal{X}$ is the limit (for the Hausdorff distance) of a sequence of periodic orbits if and only if $\mathcal{X}$ chain-transitive.*

This result has some consequences on the spectral decomposition of generic diffeomorphisms. Conley has given in [C] a very simple and general way to decompose the chain-recurrence set $\mathcal{R}(f)$ into disjoint and invariant compact sets, called the *chain-recurrence classes*: two points of M belong to a same chain-recurrence class if for any $\varepsilon > 0$, they belong to a same ε-pseudo-orbit which is periodic. He proved that there exists a "Lyapunov function" h associated to this decomposition: h is a continuous map from M to $\mathbf{R}$ such that:

  – h decreases along the orbits: $h \circ f(x) \leq h(x)$ for each $x \in \mathrm{M}$; moreover, the inequality is strict if and only if x does not belong to the chain-recurrence set.
  – h is constant on each chain-recurrence class and takes distinct values on distinct classes.
  – The image of the chain-recurrent set by h is totally discontinuous.

This provides us with some "generalized filtration" of the dynamics: for any distinct chain-recurrence classes E, E′, there exists a trapping region U that separates[7] E and E′.

Several recent results showed that this decomposition for C^1-generic diffeomorphisms shares some of the properties of Smale's spectral decomposition: for hyperbolic diffeomorphisms, each basic piece E is a *homoclinic class*. More precisely, it contains a periodic points P and E is the closure of all the transverse intersection points between the invariant manifolds of the orbit of P. The dynamics in each homoclinic class is transitive; the periodic points are dense. For C^1-generic diffeomorphisms, the pieces of Conley's decomposition also hold recurrent dynamics: by the result of [BC], the chain-recurrence classes are the maximal weakly transitive sets. Improving a theorem of Carballo, Morales and Pacifico [CMP], it shows also that the chain-recurrence classes E that contain a periodic point P are the *homoclinic classes*. Little is known for the other classes (which are called *aperiodic classes*): the examples of Bonatti and Díaz [BD$_2$] assert that they exist for generic diffeomorphisms in (non-empty) open subsets of $\text{Diff}^1(M)$ when the dimension of M is larger than or equal to 3. In order to obtain information on the aperiodic classes, it is useful to approximate them by the periodic orbits in the Hausdorff topology. This is again a consequence of Theorem 1.

Corollary **1.** — *There exists a G_δ dense subset $\mathscr{G}_{\mathbf{aper}}$ of $\text{Diff}^1(M)$ such that for any diffeomorphism $f \in \mathscr{G}_{\mathbf{aper}}$, the homoclinic classes of f are dense among the chain-recurrence classes for the Hausdorff topology.*

As an application of this theorem, we extend with Abdenur and Bonatti [AbBC] a dichotomy proven for homoclinic classes by Bonatti, Díaz and Pujals [BDP] to any chain-recurrence class E: either E is the limit of a sequence of periodic sinks or sources for the Hausdorff topology (Newhouse's phenomenon) or E has some weak form of hyperbolicity (a non-trivial dominated splitting of its tangent bundle). Corollary 1 is also used in [ABCD] for discussing the generic dynamics on surfaces.

Theorem 1 also explains how the (non-recurrent) dynamics of a C^1-generic diffeomorphism is organized between the chain-recurrence classes: if $E_1, ..., E_r$ are chain-recurrence classes, there exists orbits that successively visit arbitrarily small neighborhoods of them if and only if these chain-recurrence classes are connected together by pseudo-orbits with arbitrarily small jumps.

0.5. *Structure of the paper*

In the next section, we give a systematic presentation of the various definitions of weak and extended orbits. We also prove some general basic properties of weak orbits.

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[7] More precisely, U is an open set satisfying $f(\bar{U}) \subset U$; one class is contained in U and the other one in M\U.

Section 2 deals with the perturbation techniques in C¹-dynamics: we recall Hayashi's connecting lemma and the connecting lemma for pseudo-orbits. We then state three perturbation properties that are the technical results of the paper. The first one (*approximation by periodic orbits*) implies directly Theorem 3 (see Section 2.4). The two others (*approximation by finite segments of orbit* and *asymptotic approximation*) will give Theorem 2, as it is shown in Section 5.3.

We discuss in Section 3 the generic consequences of these perturbation results: in particular, we prove Theorems 1, 4, Corollary 1 and Criterion 2.

The heart of the paper, contained in the last two sections, is devoted to the proofs of the three perturbation results stated at Section 2: In Section 4, we explain how to approximate a weakly transitive set by a periodic orbit. In Section 5, we prove the two other properties and deduce Theorem 2.

Acknowledgement

1. Generalized notion of orbits and recurrence

1.1. *Notations*

Let M be a compact manifold endowed with a Riemannian metric. The induced distance will be denoted by d. The open ball centered at a point $x \in M$ with radius $\eta > 0$ is written $B(x, \eta)$. The η-neighborhood of a set $X \subset M$ is the union of the balls $B(x, \eta)$ over the points $x \in X$. The closure of a subset X in a topological space is $Cl(X)$.

The space of non-empty compact sets of M will be denoted by $\mathcal{K}(M)$. This is a compact metric space for the Hausdorff distance defined by:

$$d_H(K, K') = \max\left(\max_{x \in K} d(x, K'), \max_{x' \in K'} d(x', K)\right).$$

If $d_H(K, K')$ is less than η, we say that the compact subsets K and K' of M are η-close.

The set of non-empty compact subsets of $\mathcal{K}(M)$ also may be endowed with the Hausdorff topology. This defines a compact metric space $\mathcal{K}(\mathcal{K}(M))$. One adds sometimes an isolated point, the empty set $\emptyset$ of $\mathcal{K}(M)$, and one considers the space $\mathcal{K}(\mathcal{K}(M)) \cup \{\emptyset\}$.

In this work, we consider the space $\mathrm{Diff}^1(M)$ of C^1-diffeomorphisms of M endowed with the C^1-topology.

1.2. *Generalized iterates*

In this section we will discuss how to generalize the following relation between points of M: y is a forward iterate of x by f if there exists $n \geq 1$ such that $f^n(x) = y$. We will give several examples at Section 1.5.

1.2.1.   We are interested by relations that are closed for the topology of M. This leads to the following natural notion of weak iterate, first introduced by Arnaud [A₁] and Gan and Wen [GW].

*Definition **1.** — A point y is a* weak iterate *of x by f (one denotes it by $x \prec y$) if for any neighborhoods* U *of x and* V *of y, there exists $n \geq 1$ and a point $z \in$ U *such that $f^n(z)$ belongs to* V.

Sometimes one localizes the dynamics:

- Let W be an open set. If x and y belong to W, one defines the relation $x \prec_W y$ if for any neighborhoods U and V of x and y, there exists $n \geq 1$ and a point $z \in$ U such that $f^n(z)$ belongs to V and such that the segment of orbit $(z, f(z), ..., f^n(z))$ is contained in W.
- Let K be a compact set. If x and y belong to K, one defines the relation $x \prec_K y$ if for any neighborhood W of K one has $x \prec_W y$.

If $\mathscr{X}$ is a compact set, the relation $x \prec \mathscr{X}$ will mean that for any point $x' \in \mathscr{X}$, we have $x \prec x'$. One defines in the same way the relations $x \prec_W \mathscr{X}$ and $x \prec_K \mathscr{X}$.

The relations $\prec, \prec_W, \prec_K$ are in general not transitive (cf. Example 1 of Section 1.5). We prove that the relation $\prec_K$ is closed:

*Proposition **1.** — Let us consider a sequence of compact sets (K_k) and two sequences (x_k) and (y_k) in M such that $x_k \prec_{K_k} y_k$ for each k. If $(x_k), (y_k)$ converge towards x, y and if (K_k) converges towards* K *in the Hausdorff topology, then $x \prec_K y$.*

Proof. — One considers two neighborhoods U and V of x and y respectively and W a neighborhood of K. For k large, x_k belongs to U, y_k to V and W is a neighborhood of K_k. Since $x_k \prec_{K_k} y_k$, there exists a segment of orbit that crosses U and then V and is contained in W what was to be shown.   □

The relation $\prec_K$ is invariant by f, by f^{-1} and moreover $x \prec_K y$ implies $x \prec_K f(y)$. We also have the following property:

Proposition **2.** — *Let f be a diffeomorphism and* K *a compact set. Let x, y be two points in* K *such that $x \prec_K y$. If $f(x) \neq y$ then $f(x) \prec_K y$ holds also.*

Proof. — Let W be a neighborhood of K and U, V be two neighborhoods of $f(x)$ and y respectively. Since $f(x)$ is different from x, one may assume that U and V are disjoint. Using the relation $x \prec_K y$, there exists a segment of orbit $(z_0, ..., z_n)$ in W with $n \geq 1$ such that z_0 belongs to $f^{-1}(U)$ and z_n to V. Since U is disjoint from V, z_1 does not belong to V and n is larger than or equal to 2. Hence $(z_1, ..., z_n)$ is a finite segment of orbit (not reduced to a point) contained in W such that z_1 belongs to U and z_n to V. This shows $f(x) \prec_W y$. Since W is any neighborhood of K, we have $f(x) \in K$ and $f(x) \prec_K y$.   □

1.2.2.   When we consider perturbations of the dynamics, one may want to work with a relation which is semi-continuous with respect to the map f. Introducing the pseudo-orbits, one obtains[8] the following definition.

Definition **2.** — *A point y is a* chain-iterate *of x by f (one denotes it by $x \dashv y$) if for any $\varepsilon > 0$, there exists an ε-pseudo-orbit $(z_0, ..., z_n)$ (with $n \geq 1$) such that $z_0 = x$ and $z_n = y$.*

Let E *be any[9] subset of* M. *If x and y belong to* E, *one defines the relation $x \dashv_E y$ if for any $\varepsilon > 0$, there exists an ε-pseudo-orbit $(z_0, ..., z_n)$ (with $n \geq 1$) contained in* E *such that $z_0 = x$ and $z_n = y$.*

These relations are clearly transitive. Moreover the relation $x \prec_K y$ implies the relation $x \dashv_K y$ (the converse is false, see Example 1). The following proposition shows in particular that if the set E is closed, then the relation $\dashv_E$ is closed.

Proposition **3.** — *Let us consider a sequence (f_k) of diffeomorphisms that converges towards $f \in \mathrm{Diff}^1(M)$. One also considers a sequence of compact sets (K_k) and two sequences (x_k) and (y_k) in* M *such that for each k, the point y_k is a chain-iterate of x_k by f_k. If (x_k) converges towards x and (y_k) towards y and if (K_k) converges towards* K *in the Hausdorff topology, then y is a chain-iterate of y by f.*

Proof. — One fixes $\varepsilon > 0$. For k large enough, $d_H(K_k, K)$, $d(x_k, x)$, $d(y_k, y)$ and the C⁰-distance between f_k and f are small. We consider a η-pseudo-orbit of f_k in K_k

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[8] Since we are interested here by the semi-continuity property of the relation with respect to f, one could define the following relation: y is a "generalized iterate" of x by f if there exists a sequence (f_k) which converges toward f for an appropriate topology, two sequences of points (x_k) and (y_k) which converge in M toward x and y respectively, and a sequence of integers (n_k) such that for each k, we have $n_k \geq 1$ and $f^{n_k}(x_k) = y_k$. The definition of "chain-iterates" which uses pseudo-orbits appears naturally when one considers rather random perturbations of the dynamics. It is easier to work with the definition of chain-iterate since it has the advantage to involve only the initial dynamics f (and not to depend on the choice of a topology on the set of the dynamical systems).

[9] We do not distinguish here between the cases where E is open or closed. The reason is that if one considers an ε-pseudo-orbit that joints two points x, y of E and that can be chosen arbitrarily close to E, then, for any $\varepsilon' > \varepsilon$, there also exists an ε'-pseudo-orbit that joints x to y and that is contained in E'. It is obtained by "projecting" the points of the first orbit to the set E.

that connects x_k to y_k. For η small enough, it is shadowed by an ε-pseudo-orbit in K that connects x to y what was to be shown. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

1.3. *Different kinds of recurrence*

We now discuss the various definitions for a compact invariant set K to support recurrent dynamics.

1.3.1.   The sets that satisfy the strongest form of recurrence are the periodic orbits. Since one looks for a notion which is closed for the Hausdorff topology, one considers all the compact sets that are limit of periodic orbits in the Hausdorff topology. This collection defines an element of $\mathscr{K}(\mathscr{K}(M)) \cup \{\emptyset\}$ which is denoted by **Per**(f).

1.3.2.   In the introduction, we defined the notion of weakly transitive set[10]:

Definition **3.** — *A closed set* $\mathscr{X} \subset M$ *which is invariant by f is* weakly transitive *if for any neighborhood* W *of* $\mathscr{X}$, *and any open sets* U, V *which intersect* $\mathscr{X}$, *there exists* $n \geq 1$ *and a point* $z \in U$ *such that* $f^n(z)$ *belongs to* V *and such that the segment of orbit* $(z, f(z), ..., f^n(z))$ *is contained in* W.

This can be compared to the classical definition of transitivity: $\mathscr{X}$ is *transitive* if for any open sets U and V which intersect $\mathscr{X}$, there exists $n \geq 1$ and a point $z \in U \cap \mathscr{X}$ such that $f^n(z)$ belongs to V. By Proposition 1, the set of weakly transitive sets is closed in $\mathscr{K}(M)$. Hence, this is an element of $\mathscr{K}(\mathscr{K}(M))$ which is denoted by **WTrans**(f).

One can give an equivalent definition using the relation $\prec_{\mathscr{X}}$: the set $\mathscr{X}$ is weakly transitive if and only if for any points x, y in $\mathscr{X}$, we have $x \prec_{\mathscr{X}} y$.

1.3.3.   Replacing $\prec_{\mathscr{X}}$ by the relation $\dashv_{\mathscr{X}}$ in the definition of weakly transitive set, one gets another definition:

Definition **4.** — *A closed set* $\mathscr{X} \subset M$ *which is invariant by f is* chain-transitive *if for any points* x, y *in* $\mathscr{X}$ *one has* $x \dashv_{\mathscr{X}} y$.

By Proposition 3, the sets of chain-transitive sets is closed in $\mathscr{K}(M)$, defining an element of $\mathscr{K}(\mathscr{K}(M))$ which is denoted by **CTrans**(f). Moreover, it is upper-semi-continuous with respect to f: if (f_n) converges towards f in the C^0-topology, then, the upper-limit of the compact sets **CTrans**(f_n) is contained in **CTrans**(f).

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[10] Note that a slightly different notion of weak transitivity was proposed in [A_1]: it requires only that for any open sets U, V which intersect $\mathscr{X}$, there exists $n \geq 1$ and a point $z \in U$ such that $f^n(z)$ belongs to V. In particular, the union of two different periodic orbits, contained in a same transitive set, satisfies this last condition but not our definition of weak transitivity.

Note that the *chain-recurrence classes* defined at Section 0.4 are the chain-transitive sets that are maximal for the inclusion. Two chain-recurrence classes are disjoint or equal since they also are the equivalence classes for the following relation in $\mathscr{R}(f)$: for any points $x, y \in \mathscr{R}(f)$, we define $x \vdash\dashv y$ if $x \dashv y$ and $y \dashv x$.

From the construction, we always have the inclusions

$$(1) \qquad \mathbf{Per}(f) \subset \mathbf{WTrans}(f) \subset \mathbf{CTrans}(f).$$

In general they do not coincide (cf. Examples 2 and 3).

In dynamics, one also often considers the collections of transitive sets, and of limit sets (the accumulation points of orbits). The collection of periodic orbits is contained in the collection of transitive sets which is contained in the collection of limit sets. All of them are contained in $\mathbf{WTrans}(f)$. In general neither the collection of limit sets, nor the collection of transitive sets are closed; moreover, there is no relation with $\mathbf{Per}(f)$ for the inclusion (cf. Examples 0 and 2).

1.4. *The generalized orbits*

Similarly to the various definitions of generalized iterates, one now introduces different generalizations of the orbits.

1.4.1. In the strongest form, one simply considers the collection of invariant compact sets that are limit of closure of orbits by f and one obtains an element $\mathbf{Orb}(f)$ of $\mathscr{K}(\mathscr{K}(\mathrm{M}))$.

1.4.2. The invariant compact sets that are limit of finite segments of orbits in general produce a different element $\mathbf{FOrb}(f) \in \mathscr{K}(\mathscr{K}(\mathrm{M}))$ (see Example 3).

1.4.3. More generally, from any definition of generalized iterates, one derives a definition of generalized orbit: a set K is a generalized orbit if for any pair of point x, y in K, one is a generalized iterate of the other. In particular one gets the following definitions:

Definition **5.** — *An invariant compact set $\mathscr{X}$ is a* weak orbit *(resp. an* extended orbit*) if for any points $x \neq y$ in $\mathscr{X}$ one has $x \prec_{\mathscr{X}} y$ or $y \prec_{\mathscr{X}} x$ (resp. one has $x \dashv_{\mathscr{X}} y$ or $y \dashv_{\mathscr{X}} x$).*

The notion of extended orbit was introduced by Takens in [T₂]. His definition is equivalent to ours: E is an extended orbit if and only if for any $\varepsilon > 0$, the compact set E is approximated in the Hausdorff topology by the closures of ε-pseudo-orbits.

By Propositions 1 and 3, the sets of weak orbits and of extended orbits are closed for the Hausdorff topology, defining two elements of $\mathscr{K}(\mathscr{K}(\mathrm{M}))$ that we denote by

WOrb(f) and **EOrb**(f) respectively. We always have the inclusions

$$(\mathbf{2}) \qquad \mathbf{Orb}(f) \subset \mathbf{FOrb}(f) \subset \mathbf{WOrb}(f) \subset \mathbf{EOrb}(f)$$

but in general these sets do not coincide.

It is also useful to introduce the generalized segments of orbits: in the definition of the elements of **FOrb**(f), **WOrb**(f) and of **EOrb**(f), one may also consider compact sets that are not necessarily invariant by f. This defines the following elements of $\mathscr{K}(\mathscr{K}(M))$:

$$(\mathbf{3}) \qquad \mathbf{Seg}(f) \subset \mathbf{WSeg}(f) \subset \mathbf{ESeg}(f).$$

The elements of **WSeg**(f) will be called the *weak segments of orbits*; those of **ESeg**(f), the *extended segments of orbits*.

By Proposition 3, the sets **EOrb**(f) and **ESeg**(f) vary upper-semi-continuously with respect to the dynamics f. For **FOrb**(f) and **Seg**(f), we have:

*Proposition **4**. — The maps $f \mapsto \mathbf{Seg}(f)$ and $f \mapsto \mathbf{FOrb}(f)$ are lower-semi-continuous.*

Proof. — Let $\mathscr{X}$ be any element of **Seg**(f). It is arbitrarily close to a segment of orbit $\{x, f(x), ..., f^n(x)\}$ for the Hausdorff topology. For g close to f, the orbit $\{x, g(x), ..., g^n(x)\}$ belongs to **Seg**(g) and is close to $\{x, f(x), ..., f^n(x)\}$, hence, close to $\mathscr{X}$, showing the lower-semi-continuity of $f \mapsto \mathbf{Seg}(f)$. One obtains the result for **FOrb**(f) just by noting that **FOrb**(f) coincides with the subset of **Seg**(f) whose elements are invariant by f. $\qquad \square$

Note also that we have the inclusions

$$\mathbf{Per}(f) \subset \mathbf{Orb}(f) \subset \mathbf{FOrb}(f) \subset \mathbf{Seg}(f),$$
$$\mathbf{WTrans}(f) \subset \mathbf{WOrb}(f) \subset \mathbf{WSeg}(f),$$
$$\mathbf{CTrans}(f) \subset \mathbf{EOrb} \subset \mathbf{ESeg}(f).$$

1.5. *Examples*

0. Among all the elements of $\mathscr{K}(\mathscr{K}(M)) \cup \{\emptyset\}$ defined above, **Per**(f) is the only one that can be the empty set: this happens for instance with the irrational rotations of the circle. In this case, the collection of transitive sets (here the whole circle) is not contained in **Per**(f).

1. Let f_1 be an orientation-preserving diffeomorphism of the circle $\mathbf{T}^1$ having three semi-stable fixed points p_1, p_2 and p_3 (see the first example of Figure 1)). The set **Per**(f_1) equals $\{\{p_1\}, \{p_2\}, \{p_3\}\}$. Up to reordering the three fixed
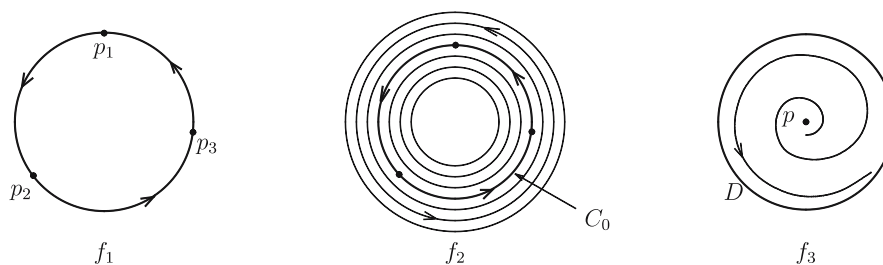
points, one has $p_1 \prec p_2$, $p_2 \prec p_3$ and $p_3 \prec p_1$ but the other relations between these points are not satisfied: in particular $p_1 \prec p_3$ is false showing that in this case $\prec$ is not transitive and that $\prec$ and $\dashv$ may differ.

2. Let f_2 be a diffeomorphism of the torus, foliated by circles $(C_t)_{t\in\mathbf{T}^1}$, such that (see the second example of Figure 1)):
   – On one circle, C_0, the induced dynamics coincides with the former example.
   – On the circles $C_t \neq C_0$ the dynamics is conjugate to a rotation.
   – The rotation number ρ_t on the circles C_t is monotonic in t.
   In this example, the elements of $\mathbf{Per}(f_2) = \mathbf{WTrans}(f_2)$ are the periodic orbits of f_2 and each circle C_t. In particular C_0 is in $\mathbf{Per}(f_2)$, but is not a limit set; the set C_0 is weakly transitive but is not transitive. Note also that despite C_0 is weakly transitive for f_2, the dynamics f_1 induced by f_2 on C_0 is not weakly transitive.
   The set $\mathbf{CTrans}(f_2)$ is larger: it contains for example the whole torus.

3. Let f_3 be a surface diffeomorphism which coincides with the identity outside an open disk D. In D, f_3 has a fixed point p. Any other orbit in D converges towards p in the past and accumulates on the whole boundary of D in the future (see the third example of Figure 1)). On this example, the boundary of D belongs to $\mathbf{FOrb}(f_3)$ but not to $\mathbf{Orb}(f_3)$. The boundary of D belongs to $\mathbf{WTrans}(f_3)$ but not to $\mathbf{Per}(f_3)$.

1.6. *Properties of weak orbits*

1.6.1. *Weak orbits and periodic points.* — The periodic orbits are the simplest weak orbits. We analyze here the other weak orbits. (Recall that a diffeomorphism satisfies Condition (A) if for any integer $n \geq 1$, the fixed points of f^n are isolated.)

Lemma **1.** — *Let $\mathscr{X} \in \mathbf{WSeg}(f)$ be a weak segment of orbit of a diffeomorphism f that satisfies Condition (A). If $\mathscr{X}$ is not a periodic orbit, the non-periodic points are dense in $\mathscr{X}$.*

Proof. — We first claim that the periodic orbits are not isolated in $\mathscr{X}$. The proof is by contradiction: let us consider a periodic isolated point x in $\mathscr{X}$; the periodic orbit $\mathscr{O}$ of x is isolated from $\mathscr{X} \setminus \mathscr{O}$. By our assumption on $\mathscr{X}$, the set $\mathscr{X} \setminus \mathscr{O}$ is non-empty and we will see that this gives a contradiction. One can choose two neighborhoods U and V of $\mathscr{O}$ and $\mathscr{X} \setminus \mathscr{O}$ respectively such that $f(\text{U}) \cap \text{V}$ and $f(\text{V}) \cap \text{U}$ are both empty. By definition of weak segment of orbit, there exists a finite segment of orbit contained in the neighborhood $\text{U} \cup \text{V}$ of $\mathscr{X}$ that crosses U and V. This is impossible by our choice of U and V. This shows the claim.

Let us now consider the set P_n of periodic points contained in $\mathscr{X}$ with period n. This set is closed and finite by the assumption (A) on the diffeomorphism. Since the points of P_n are not isolated in $\mathscr{X}$, the set P_n has empty interior in $\mathscr{X}$. By Baire's theorem the complement of the union $\cup_n \text{P}_n$ is dense in $\mathscr{X}$. Hence, the non-periodic points are dense.   □

Proposition **5.** — *Let $\mathscr{X}$ be a weak orbit of a diffeomorphism f that satisfies Condition (A). If $\mathscr{X}$ is not a periodic orbit, then for any $\eta_0 > 0$ there exists a subset X of $\mathscr{X}$ which has the following properties:*

1. X *is finite (and contains at least two points);*
2. *the Hausdorff distance between $\text{Cl}(\cup_n f^n(\text{X}))$ and $\mathscr{X}$ is less than η_0;*
3. X *has no periodic point;*
4. *the orbits of any two distinct points of X are disjoint.*

Proof. — By compactness of $\mathscr{X}$ and by Lemma 1, there exists a finite subset X of non-periodic points in $\mathscr{X}$ which is η_0-close to $\mathscr{X}$. Up to removing some points of X, the orbit of X is η_0-close to $\mathscr{X}$ but X intersects each orbit of $\mathscr{X}$ in at most one point. This gives the result.   □

1.6.2. *Weak orbits and maximal invariant subsets.* — Any weak orbit $\mathscr{X}$, such that $\prec_{\mathscr{X}}$ is an ordering, contains points y that are maximal for $\prec_{\mathscr{X}}$:

Proposition **6.** — *Let be $\mathscr{X}$ a weak orbit such that the relation $\prec_{\mathscr{X}}$ is transitive. Then, $\mathscr{X}$ contains an invariant closed set A such that for any $x \in \mathscr{X}$ and $y \in \text{A}$, one has $x \prec_{\mathscr{X}} y$.*

Remark. — Any non-empty subset of A will also satisfy the proposition. Hence, one may choose A minimal for the inclusion, so that the induced dynamics on A is minimal.

Proof. — One considers the family $\mathscr{F}$ of closed and invariant non-empty subsets K of $\mathscr{X}$ such that for any $x \in \mathscr{X} \setminus \text{K}$ and $y \in \text{K}$ one has $x \prec_{\mathscr{X}} y$. This family is non-empty (it contains $\mathscr{X}$) and partially ordered by inclusion. One easily checks that Zorn's lemma can be applied and one considers a minimal element $\text{A} \subset \mathscr{X}$. We will

now prove that for any points x, y in A, one has $x \prec_{\mathscr{X}} y$. This will concludes the proof.

Let us consider two points x, y in A and assume by contradiction that $x \prec_{\mathscr{X}} y$ does not hold. One then considers the set A′ of points $z \in$ A such that $x \prec_{\mathscr{X}} z$. This set is closed (Proposition 1), non-empty (it contains $\omega(x)$) and moreover $f(A') \subset A'$. It is smaller than A since y belongs to $A \setminus A'$. Since $\mathscr{X}$ is a weak orbit, for any point $z' \in A'$ and any point $z \in A \setminus A'$ one has $z \prec_{\mathscr{X}} z'$ or $z' \prec_{\mathscr{X}} z$. However, the second case is impossible: by transitivity of $\prec_{\mathscr{X}}$, one would get $x \prec_{\mathscr{X}} z$ so that $z \in A'$ by definition of A′. This contradicts our choice of z. Hence $z \prec_{\mathscr{X}} z'$. For any point $z' \in A'$ and any point $z \in \mathscr{X} \setminus A$ one also has $z \prec_{\mathscr{X}} z'$ by definition of A. We proved that for any $z \in \mathscr{X} \setminus A'$ and any $z' \in A'$, one has $z \prec_{\mathscr{X}} z'$.

Let us define $A_0 = \cap_{n \in \mathbf{N}} f^n(A')$. Since $f(A') \subset A'$, the set A_0 is the intersection of a decreasing sequence of non-empty compact sets. Hence, A_0 is non-empty, compact and invariant. Moreover, for any $z \in \mathscr{X} \setminus A_0$ and $z' \in A_0$, there exists $n \in \mathbf{N}$ such that $f^{-n}(z)$ belongs to $\mathscr{X} \setminus A'$ and $f^{-n}(z')$ to $A_0 \subset A'$. One thus gets $f^{-n}(z) \prec_{\mathscr{X}} f^{-n}(z')$ so that $z \prec_{\mathscr{X}} z'$. Hence A_0 is in the family $\mathscr{F}$ of compact sets defined above and A is not a minimal element of f. This is a contradiction. This ends the proof. □

1.6.3. *Weak orbits that are not weakly transitive.* — We now analyze weak orbits that are not weakly transitive.

*Proposition **7**.* — *Let $\mathscr{X}$ a weak orbit that is not weakly transitive and such that the relation $\prec_{\mathscr{X}}$ is transitive. Then, for any $\eta_0 > 0$, there exist two weak orbits $\mathscr{X}_\alpha$ and $\mathscr{X}_\omega$ whose relations $\prec_{\mathscr{X}_\alpha}$ and $\prec_{\mathscr{X}_\omega}$ are transitive, and a point $z_0 \in \mathscr{X}$ such that:*

1. *$\mathscr{X}_\alpha$, $\mathscr{X}_\omega$ and the orbit $\mathscr{O}(z_0)$ of z_0 are pairwise disjoint and contained in $\mathscr{X}$.*
2. *The set $\mathrm{Cl}(\mathscr{X}_\alpha \cup \mathscr{O}(z_0) \cup \mathscr{X}_\omega)$ is η_0-close to $\mathscr{X}$ in the Hausdorff topology.*
3. *For any $x \in \mathscr{X}_\alpha$ and $z \in \mathscr{X} \setminus \mathscr{X}_\alpha$ we have $x \prec_{\mathscr{X}} z$ but the relation $z \prec_{\mathscr{X}} x$ is not satisfied.*
4. *For any $y \in \mathscr{X}_\omega$ and $z \in \mathscr{X} \setminus \mathscr{X}_\omega$ we have $z \prec_{\mathscr{X}} y$ but the relation $y \prec_{\mathscr{X}} z$ is not satisfied.*

Remark. — By Lemma 1, if f satisfies Condition (A), the point z_0 may be modified and chosen non-periodic.

Proof. — One first builds $\mathscr{X}_\omega$: by Proposition 6, there exists an invariant closed subset A of $\mathscr{X}$ such that for any $x \in \mathscr{X}$ and $y \in$ A we have $x \prec_{\mathscr{X}} y$. One defines $\mathscr{X}_\omega$ as the set of points $y \in \mathscr{X}$ such that for any $x \in$ A, one has $x \prec_{\mathscr{X}} y$ and $y \prec_{\mathscr{X}} x$. This set is closed, invariant and contains A.

By transitivity of $\prec_{\mathscr{X}}$, for any $x \in \mathscr{X}$ and $y \in \mathscr{X}_\omega$ we have $x \prec_{\mathscr{X}} y$. Moreover $y \prec_{\mathscr{X}} x$ implies $x \in \mathscr{X}_\omega$. We then prove that $\mathscr{X}_\omega$ is a weakly transitive set, hence

a weak orbit (whose relation $\prec_{\mathscr{X}_\omega}$ is transitive). Since $\mathscr{X}$ is not weakly transitive, this also shows that $\mathscr{X}_\omega$ is not equal to $\mathscr{X}$.

Claim **1.** — The set $\mathscr{X}_\omega$ is a weakly transitive set.

Proof. — For any two points $x, y \in \mathscr{X}_\omega$, we have $x \prec_{\mathscr{X}} y$. Let us assume by contradiction that the relation $x \prec_{\mathscr{X}_\omega} y$ is not satisfied: in particular, there are segments of orbits that connect small neighborhoods U of x to small neighborhoods V of y, that are contained in arbitrarily small neighborhoods W of $\mathscr{X}$, but that are not contained in arbitrarily small neighborhoods of $\mathscr{X}_\omega$. When the size of the neighborhoods U, V, W decreases, these segments of orbits have an accumulation point z which belongs to $\mathscr{X} \setminus \mathscr{X}_\omega$. Moreover, we obtain $x \prec_{\mathscr{X}} z \prec_{\mathscr{X}} y$. This implies by transitivity of $\prec_{\mathscr{X}}$ that for any $\tilde{z}$ in $\mathscr{X}_\omega$ one has $\tilde{z} \prec_{\mathscr{X}} z$ and $z \prec_{\mathscr{X}} \tilde{z}$. Hence, by definition of $\mathscr{X}_\omega$ (and using A $\subset \mathscr{X}_\omega$), one gets $z \in \mathscr{X}_\omega$ which is a contradiction.   □

One then builds $\mathscr{X}_\alpha$: since $\mathscr{X}_\omega$ is not equal to $\mathscr{X}$, one chooses a finite set $\{z_1, ..., z_r\}$ in $\mathscr{X} \setminus \mathscr{X}_\omega$ such that $\{z_1, ..., z_r\} \cup \mathscr{X}_\omega$ is η_0-close to $\mathscr{X}$ in the Hausdorff topology. Using that $\mathscr{X}$ is a weak orbit, one can order this set as

$$z_1 \prec_{\mathscr{X}} z_2 \prec_{\mathscr{X}} ... \prec_{\mathscr{X}} z_r \prec_{\mathscr{X}} \mathscr{X}_\omega.$$

One denotes by B the set of points $x \in \mathscr{X}$ such that $x \prec_{\mathscr{X}} z_r$. This is a closed and non-empty set which satisfies B $\subset f(\text{B})$. The set $\mathscr{X}_\alpha$ is defined as the decreasing intersection $\mathscr{X}_\alpha = \cap_{n \geq 0} f^{-n}(\text{B})$. This is an invariant and non-empty compact set.

Claim **2.** — For any points $x \in \mathscr{X}_\alpha$ and $z \in \mathscr{X}$, the relation $z \prec_{\mathscr{X}} x$ implies that z belongs to $\mathscr{X}_\alpha$.

Proof. — For any $n \geq 0$, we have $f^n(x) \in \mathscr{X}_\alpha \subset \text{B}$; hence, $f^n(x) \prec_{\mathscr{X}} z_r$. Using $f^n(z) \prec_{\mathscr{X}} f^n(x)$, one also gets $f^n(z) \prec_{\mathscr{X}} z_r$ so that $f^n(z)$ belongs to B. One has $z \in \cap_{n \geq 0} f^{-n}(\text{B})$ and thus $z \in \mathscr{X}_\alpha$, proving the claim.   □

This claim and the fact that $\mathscr{X}$ is a weak orbit show that for any point $x \in \mathscr{X}_\alpha$ and $z \in \mathscr{X} \setminus \mathscr{X}_\alpha$, we have $x \prec_{\mathscr{X}} z$. One also gets that $\mathscr{X}_\alpha$ is a weak orbit whose relation $\prec_{\mathscr{X}_\alpha}$ is transitive: since $\mathscr{X}$ is a weak orbit whose relation is transitive, it is enough to show that for any points $x, y \in \mathscr{X}_\alpha$ the relation $x \prec_{\mathscr{X}} y$ implies $x \prec_{\mathscr{X}_\alpha} y$. The argument is similar to the proof of the first claim (One argues by contradiction: if $x \prec_{\mathscr{X}_\alpha} y$ does not hold, there exists a point $z \in \mathscr{X}$ such that $x \prec_{\mathscr{X}} z \prec_{\mathscr{X}} y$ and such that $z \notin \mathscr{X}_\alpha$. This is impossible since $z \prec_{\mathscr{X}} y$ and $y \in \mathscr{X}_\alpha$ imply by the second claim that z belongs to $\mathscr{X}_\alpha$.)

Let us note that $\mathscr{X}_\alpha$ and $\mathscr{X}_\omega$ are disjoint. Otherwise, there would exist $y \in \text{B}$ such that for any point $x \in \mathscr{X}_\omega$ we have $x \prec_{\mathscr{X}} y$. As $y \prec_{\mathscr{X}} z_r$ by definition of B we get

also $x \prec_{\mathscr{X}} z_r$ by transitivity of $\prec_{\mathscr{X}}$. We also have $z_r \prec_{\mathscr{X}} x$. These two relations imply that z_r belongs to $\mathscr{X}_\omega$. This is a contradiction.

Since $\mathscr{X}$ is a weak orbit and since the two invariant subsets $\mathscr{X}_\alpha$ and $\mathscr{X}_\omega$ are disjoint, the set $\mathscr{X} \setminus (\mathscr{X}_\alpha \cup \mathscr{X}_\omega)$ is non-empty. By the properties proven for $\mathscr{X}_\alpha$ and $\mathscr{X}_\omega$, we have shown the items 1), 3) and 4) of the proposition. In particular, for any $x \in \mathscr{X}_\alpha, y \in \mathscr{X}_\omega$ and $z \in \mathscr{X} \setminus (\mathscr{X}_\alpha \cup \mathscr{X}_\omega)$,

$$x \prec_{\mathscr{X}} z \prec_{\mathscr{X}} y.$$

One then introduces the point z_0. Two cases are possible:

– either z_r belongs to $\mathscr{X}_\alpha$ and the point z_0 is arbitrarily chosen in $\mathscr{X} \setminus (\mathscr{X}_\alpha \cup \mathscr{X}_\omega)$;
– or z_r does not belong to $\mathscr{X}_\alpha$ and one sets $z_0 = z_r$.

It follows directly from this definition that the orbit $\mathscr{O}(z_0)$ of z_0 is disjoint from $\mathscr{X}_\alpha \cup \mathscr{X}_\omega$. We now prove that $\mathscr{X}_\alpha \cup \mathscr{O}(z_0)$ contains the set X and, by our choice of the set X, this will imply the second item of the proposition: let us consider any point $z_i \in$ X; we have $z_i \prec_{\mathscr{X}} z_r$. If $z_r \in \mathscr{X}_\alpha$, the last claim shows that z_i is also included in $\mathscr{X}_\alpha$. If z_i does not belong to the orbit of z_r, one can apply Proposition 2: for each $n \geq 0$, we have $f^n(z_i) \prec_{\mathscr{X}} z_r$ so that z_i again belongs to $\mathscr{X}_\alpha = \cap_{n \geq 0} f^n(\mathrm{B})$. In remaining case $z_r \notin \mathscr{X}_\alpha$ but z_i belongs to the orbit of $z_r = z_0$: we have $z_i \in \mathscr{O}(z_0)$. In any of these three cases, one has $x_i \in \mathscr{X}_\alpha \cup \mathscr{O}(z_0)$ as required.

The proof of Proposition 7 is now complete.   □

2. Connecting lemmas

In this section, we recall Hayashi's connecting lemma and the connecting lemma for pseudo-orbits. We then state the technical versions of the perturbation results given in the introduction (Theorems 2 and 3).

2.1. *Support of perturbations*

Let f be a diffeomorphism and $\mathscr{U}$ a neighborhood of f in $\mathrm{Diff}^1(\mathrm{M})$. Two perturbations $g_1, g_2 \in \mathscr{U}$ of f have *disjoint supports* if $g_i = f$ outside some open set U_i for $i = 1, 2$ and $\mathrm{U}_1 \cap \mathrm{U}_2 = \emptyset$. In this case, one defines the *composed perturbation* g by $g = f$ outside $\mathrm{U}_1 \cup \mathrm{U}_2$, $g = g_1$ on U_1 and $g = g_2$ on U_2.

In general g does not belong to $\mathscr{U}$. However, there exists (see the *flexibility of the lift axiom* in Section 2 of [PR]) a basis of neighborhoods $\mathscr{U}$ of f which have this property:

(F) *For any perturbations $g_1, g_2 \in \mathscr{U}$ of f with disjoint supports the composed perturbation g remains in $\mathscr{U}$.*

2.2. *Hayashi's connecting lemma*

In the proof of the perturbation lemmas we will use Hayashi's connecting lemma. The original proof appeared in [H] but other references are given in [WX,A_1]. We need a slightly more general statement than the version of [A_1], in order to perturb at the same time in different domains. However the proof is the same.

*Theorem **5** (Hayashi's connecting lemma). — Let f_0 be a diffeomorphism of a compact manifold* M, *and* $\mathcal{U}$ *a neighborhood of* f_0 *in* $\mathrm{Diff}^1(M)$. *Then there exists* $N \geq 1$ *such that for any* $z \in M$ *which is not a periodic orbit of period less than or equal to* N, *and for any open neighborhood* U *of* z, *some smaller neighborhood* $V \subset U$ *of* z *has the following property:*

For any diffeomorphism $f \in \mathrm{Diff}^1(M)$ *that coincides with* f_0 *on* $U \cup \cdots \cup f_0^{N-1}(U)$, *for any points* $p, q \in M \setminus (U \cup \cdots \cup f_0^N(U))$ *and any integers* $n_p, n_q \geq 1$ *such that* $f^{n_p}(p)$ *belongs to* V *and* $f^{-n_q}(q)$ *to* $f_0^N(V)$ *there is a diffeomorphism* $g \in \mathrm{Diff}^1(M)$ *arbitrarily close to* f *which satisfies:*

- *g coincides with f on* $M \setminus (U \cup \cdots \cup f_0^{N-1}(U))$. *The diffeomorphism* g_0 *that coincides with* g *on* $U \cup \cdots \cup f_0^{N-1}(U)$ *and coincides with* f_0 *elsewhere belongs to* $\mathcal{U}$.
- *g sends a positive iterate $g^m(p)$ of p on q.*
- *The orbit $(p, \cdots, g^m(p))$ can be cut in three parts:*
  a. the beginning $(p, \cdots, g^{m'}(p))$ has support in

$$\{p, \cdots, f^{n_p}(p)\} \cup U \cup \cdots \cup f_0^N(U);$$

  b. the central part $(g^{m'}(p), \cdots, g^{m'+N}(p))$ has support in

$$U \cup \cdots \cup f_0^N(U);$$

  c. and the end $(g^{m'+N}(p), \cdots, g^m(p))$ has support in

$$U \cup \cdots \cup f_0^N(U) \cup \{f^{-n_q}(q), \cdots, q\}.$$

Obviously, the same statement holds if one shrinks the neighborhood V of z.

The main difference with [A_1] comes from the fact that the integer N is uniform (this is explicit in [We]). The statement in [A_1] also considers only the case $f = f_0$ (so that $g = g_0$). However, the dynamics outside $U \cup \cdots \cup f_0^{N-1}(U)$ is not used in the proof. This is the reason why one can replace f_0 by any map f which coincides with f_0 on $U \cup \cdots \cup f_0^{N-1}(U)$. For a different formulation of the connecting lemma, see also [BC, Theorem 2.1].

2.3. *Connecting lemma for pseudo-orbits*

In order to handle with the pseudo-orbits, one will use a refinement of Hayashi's connecting lemma which is the connecting lemma for pseudo-orbits proven in [BC].

*Theorem **6** (Connecting lemma for pseudo-orbits [BC]). — Let f be a diffeomorphism whose periodic orbits are hyperbolic (in particular it satisfies Condition (A)). Let $\mathcal{U}$ be a neighborhood of f in* Diff¹(M).

Then, for any compact set K ⊂ M *and any neighborhood* W *of* K, *for any points x and y in* K *such that* $x \dashv_K y$, *there exists a perturbation* $g \in \mathcal{U}$ *of f whose support is in* W *and an integer* $n \geq 1$ *such that* $g^n(x) = y$. *Moreover the points* $x, g(x), ..., g^n(x)$ *belong to* W.

This is proven in [BC] in the case K = W = M but the proof in the general case is the same: the idea is to cover a large part of M by disjoint "perturbation domains", i.e. by open sets V given by Hayashi's connecting lemma. Let us consider a pseudo-orbit with small jumps and that connects two points x and y. The perturbation domains allow to remove the jumps and build an orbit between x and y for a perturbation of f.

The diameter of the perturbation domains can be chosen arbitrarily small (in particular, if a perturbation domain intersects the closed set K in Theorem 6, it should be contained in W). Thus, if one considers a pseudo-orbit contained in K that connects two points $x, y \in K$ such that $x \dashv_K y$, one can use the perturbation domains that intersect K to create a segment of orbit joining x to y. Since the perturbation domains used and the pseudo-orbits considered are contained in W, the support of the perturbation g and the segment of orbit between x and y for g will also be included in W as in Theorem 6.

2.4. *Global connecting lemmas*

We now state the technical perturbation results proven in the paper. Their proofs are based on Hayashi's connecting lemma and are independent from the connecting lemma for pseudo-orbit.

*Proposition **8** (Approximation by periodic orbits). — Let f be a diffeomorphism and $\mathcal{U}$ a neighborhood of f in* Diff¹(M). *Then, there exists an integer* $N \geq 1$ *with the following property:*

   If W ⊂ M *is an open set and* X *a finite set of points in* W *such that*

   – the points $f^j(x)$ for $x \in X$ and $j \in \{0, ..., N\}$ are pairwise distinct and contained in W,
   – for any $x, x' \in X$, we have $x \prec_W x'$,

then, for any $\eta > 0$ there exist a perturbation $g \in \mathcal{U}$ of f with support in the union of the open sets $f^j(B(x, \eta))$, for $x \in X$ and $j \in \{0, ..., N-1\}$, and a periodic orbit $\mathcal{O}$ of g, contained in W, *which crosses all the balls $B(x, \eta)$ with $x \in X$.*

This proposition gives immediately the Theorem 3 stated in the introduction:

Proof of Theorem 3. — Let f be a diffeomorphism that satisfies Condition (A), $\mathcal{X}$ be a weakly transitive set of *f* and $\eta_0 > 0$ a constant. One will assume that $\mathcal{X}$ is

not a periodic orbit (otherwise, the conclusion of the theorem is trivially satisfied). We denote by W the η_0-neighborhood of $\mathscr{X}$ in M. By Proposition 5, since Condition (A) is satisfied, there exists some finite set $X \subset \mathscr{X}$, such that every point $z \in \mathscr{X}$ belongs to one ball $B(f^k(x), \eta_0)$ with $x \in X$ and $k \in \mathbf{Z}$. Moreover the orbits of points of X are pairwise disjoint and non-periodic.

Let us consider an arbitrarily small neighborhood $\mathscr{U}$ of f in $\mathrm{Diff}^1(M)$ and a constant $\eta \in (0, \eta_0)$ small enough, so that any closed set $K \subset W$, which intersects all the balls $B(x, \eta)$ with $x \in X$ and which is invariant by a diffeomorphism $g \in \mathscr{U}$, is η_0-close to $\mathscr{X}$ in the Hausdorff topology.

One now applies Proposition 8 and one obtains the perturbation g of f and the periodic orbit $\mathscr{O}$ as announced in Theorem 3.   □

The two next propositions will imply Theorem 2 (see Section 5.3).

Proposition **9** *(Approximation by finite segments of orbits).* — *Let f be a diffeomorphism and $\mathscr{U}$ a neighborhood of f in $\mathrm{Diff}^1(M)$. Then, there exists an integer $N \geq 1$ with the following property:*

If $W \subset M$ is an open set and $X = \{p_1, ..., p_r\}$ a finite set of points in W such that

– *the points $f^j(p_k)$ for $k \in \{1, ..., r\}$ and $j \in \{0, ..., N\}$ are pairwise distinct and contained in W,*
– *for any $k, \ell \in \{1, ..., r\}$, with $k < \ell$, we have $p_k \prec_W p_\ell$,*

then, for any $\eta > 0$ there exist a perturbation $g \in \mathscr{U}$ of f with support in the union of the open sets $f^j(B(p_k, \eta))$, for $k \in \{1, ..., r\}$ and $j \in \{0, ..., N-1\}$, and a finite segment of orbit $(p_1, g(p_1), ..., g^n(p_1))$ of g, contained in W, which crosses all the balls $B(p_k, \eta)$ with $k \in \{1, ..., r\}$.

Remark. — The segment of orbit built in the proof may visit the points of X in a different order than $p_1, p_2, ..., p_r$.

Proposition **10** *(Asymptotic approximation).* — *Let f be a diffeomorphism that satisfies Condition (A) and $\mathscr{U}$ a neighborhood of f in $\mathrm{Diff}^1(M)$.*
We consider:

– *an invariant closed set A,*
– *an open neighborhood $\hat{W}_A$ of A,*
– *a point $x \in \hat{W}_A$ such that $x \prec_{\hat{W}_A} A$ (i.e. for every $\varepsilon > 0$, there exists a segment of orbit $(y, f(y), ..., f^n(y))$ contained in $\hat{W}_A$ which connects $B(x, \varepsilon)$ to the ε-neighborhood of A).*

Then, for any neighborhoods U of x and W_A of A, there exists a perturbation $g \in \mathscr{U}$ of f and a point $z \in U$ such that:

– the support of the perturbation is contained in $W_A \setminus A$ *(hence f and g coincide on* A*);*
– the forward orbit of z by g is contained in $\hat{W}_A$ *and its ω-limit set is contained in* A.

2.5. *Conservative dynamics*

The perturbation results stated above hold also in different settings, and in particular in the space $\text{Diff}^1_v(M)$ of C¹-diffeomorphisms that preserve a volume form v, or in the space $\text{Diff}^1_\omega(M)$ of C¹-diffeomorphisms that preserve a symplectic form ω. More generally, Pugh and Robinson have given in [PR] some conditions that should be satisfied by a space of diffeomorphisms so that the C¹-perturbation techniques can be used.

Kupka and Smale's theorem on periodic points of generic diffeomorphisms has been adapted by Robinson [Rob₁] to symplectic and conservative diffeomorphisms: in these settings, the periodic orbits of a generic diffeomorphism may be elliptic and not only hyperbolic. In [ArBC], we have shown that the connecting lemma for pseudo-orbits remains in $\text{Diff}^1_v(M)$ and $\text{Diff}_\omega(M)$ if one considers any diffeomorphism whose periodic orbits are hyperbolic or elliptic. As a consequence, the generic consequences proven here are also satisfied by the generic conservative or symplectic diffeomorphisms.

3. Generic properties of generalized orbits

We state and prove in this section several generic properties in $\text{Diff}^1(M)$. The proofs will use two ingredients:

- the perturbation results (the global perturbations theorems proven in the present paper and the connecting lemma for pseudo-orbits),
- the fact that the set of continuity points of any lower-semi-continuous (or upper-semi-continuous) map $\Gamma\colon \text{Diff}^1(M) \to \mathscr{K}(\mathscr{K}(M)) \cup \{\emptyset\}$ is a dense G_δ subset of $\text{Diff}^1(M)$.

The G_δ dense subset of diffeomorphisms whose periodic orbits are all hyperbolic will be denoted by $\mathscr{G}_A$. In particular, all the diffeomorphisms in $\mathscr{G}_A$ satisfy Condition (A).

3.1. *Generic comparison of weak and chain-iterates*

For any compact set K, the relation $x \prec_K y$ implies $x \dashv_K y$. The converse is true generically:

*Theorem **7** ([BC]). — The set of diffeomorphisms such that, for any compact set* K, *the relations $x \prec_K y$ and $x \dashv_K y$ are equivalent, contains a G_δ dense subset $\mathscr{G}_{\textbf{chain}}$ of* $\text{Diff}^1(M)$.

In particular for any diffeomorphism $f \in \mathscr{G}_{\mathbf{chain}}$, *we have the equalities*

$$\mathbf{WTrans}(f) = \mathbf{CTrans}(f), \quad \mathbf{WOrb}(f) = \mathbf{EOrb}(f),$$
$$\mathbf{WSeg}(f) = \mathbf{ESeg}(f).$$

This theorem is a direct consequence of the connecting lemma for pseudo-orbits (Theorem 6) and has been proven in [BC] in the case $K = M$. For completeness we give below the proof in the semi-local situation.

Since the relation $\dashv_K$ is transitive for any compact set K, one deduces the following property. (In the case $K = M$, it was proven by Arnaud [A_1] and Gan and Wen [GW] as a consequence of Hayashi's connecting lemma.)

Corollary **2.** — *For any diffeomorphism f in the G_δ dense subset $\mathscr{G}_{\mathbf{chain}}$ of* $\mathrm{Diff}^1(M)$, *and for any compact set* K, *the relation $\prec_K$ is transitive.*

Proof of Theorem 7. — For any diffeomorphism f, one denotes by $R_\prec(f)$ (resp. by $R_\dashv(f)$) the sets of triples $(x, y, K) \in M \times M \times \mathscr{K}(M)$ such that $x \dashv_K y$ (resp. $x \prec_K y$) for the dynamics of f. By Propositions 1 and 3, these sets are closed.

Claim **3.** — The set $R_\prec(f)$ varies lower-semi-continuously with f.

Let us consider a triple $(x, y, K) \in R_\prec(f)$. There exists in $R_\prec(f)$ a pair of the form $(z, f^n(z))$, $n \geq 1$, such that z and $f^n(z)$ are close to x and y respectively and the segment $\{z, f(z), ..., f^n(z)\}$ is contained in a small neighborhood of K. One deduces that the compact set $K_f = K \cup \{z, f(z), ..., f^n(z)\}$ is close to K for the Hausdorff topology. For any small perturbation g of f, the set $K_g = K \cup \{z, g(z), ..., g^n(z)\}$ is close to K in the Hausdorff topology and we have $z \prec_{K_g} g^n(z)$ for the map g. Hence, the triple $(z, g^n(z), K_g)$ is close to (x, y, K) in $M \times M \times \mathscr{K}(\mathscr{K}(M))$ and belongs to $R_\prec(g)$, showing the lower-semi-continuity of $R_\prec(f)$ and the claim.

Consequently, the set of continuity points of $f \mapsto R_\prec(f)$ is a dense G_δ subset $\mathscr{G}_\prec$ of $\mathrm{Diff}^1(M)$. We set $\mathscr{G}_{\mathbf{chain}} = \mathscr{G}_A \cap \mathscr{G}_\prec$ where $\mathscr{G}_A$ is the set of diffeomorphisms whose periodic orbits are all hyperbolic. We now prove that for any $f \in \mathscr{G}_{\mathbf{chain}}$, we have $R_\prec(f) = R_\dashv(f)$, which implies the theorem.

Note that we always have the inclusion $R_\prec(f) \subset R_\dashv(f)$. Let f be a diffeomorphism in $\mathscr{G}_{\mathbf{chain}}$, K a compact subset of M and x, y two points of M such that $x \dashv_K y$ for f. Let us also assumes by contradiction that the relation $x \prec_K y$ does not hold for f. Since $f \in \mathscr{G}_\prec$ is a continuity point of the map $g \mapsto R_\prec(g)$, there exist some neighborhoods U, V, W of x, y, K in M and a neighborhood $\mathscr{U}$ of f such that: for any diffeomorphism $g \in \mathscr{U}$, there is no segment of orbit $\{z, g(z), ..., g^n(z)\}$ with $n \geq 1$, that is contained in W and such that z belongs to U and $g^n(z)$ to V. However, since f also belongs to $\mathscr{G}_A$, one can apply the connecting lemma for pseudo-orbits (Theorem 6) and obtain a contradiction. $\square$

3.2. *Generic comparison of orbits and weak orbits (proof of Theorem 4 and Criterion 2)*

We will now prove that for C^1-generic diffeomorphisms, any chain-transitive set is the Hausdorff limit of a sequence of periodic orbits.

Proposition **11.** — *There exists a G_δ dense subset $\mathscr{G}_{\mathbf{rec}}$ of* Diff^1(M) *such that for any diffeomorphism $f \in \mathscr{G}_{\mathbf{rec}}$ we have*

$$\mathbf{Per}(f) = \mathbf{WTrans}(f) = \mathbf{CTrans}(f).$$

In particular, we obtain the conclusion of Theorem 4.

Similarly, the extended orbits are approached by finite segments of orbits in the Hausdorff topology:

Proposition **12.** — *There exists a G_δ dense subset $\mathscr{G}_{\mathbf{orbit}}$ of* Diff^1(M) *such that for any diffeomorphism $f \in \mathscr{G}_{\mathbf{orbit}}$ we have*

$$\mathbf{Seg}(f) = \mathbf{WSeg}(f) = \mathbf{ESeg}(f),$$
$$\mathbf{FOrb}(f) = \mathbf{WOrb}(f) = \mathbf{EOrb}(f).$$

Contrary to the periodic case, we were unable to answer the following problem:

Problem **2.** — *Does generically the sets* $\mathbf{Orb}(f)$ *and* $\mathbf{FOrb}(f)$ *coincide?*

A positive answer would imply that $\mathbf{Orb}(f) = \mathbf{EOrb}(f)$ generically in Diff^1(M). Hence, Takens criterion of Section 0.3 would be satisfied, proving the tolerance stability conjecture. In other terms, we have shown Criterion 2.

The first proposition will be proven by using a result of F. Takens in [T$_1$], which is a consequence of Kupka–Smale genericity theorem [K, Sm$_1$]: the **Per**-tolerance stable diffeomorphisms are generic in Diff^1(M).

Theorem **8** *(Takens [T$_1$], Theorem II).* — *The set of continuity points of the map $f \mapsto$* $\mathbf{Per}(f)$ *from* Diff^1(M) *to* $\mathscr{K}(\mathscr{K}(M)) \cup \{\emptyset\}$ *contains a G_δ dense subset $\mathscr{G}_{\mathbf{Per}}$ of* Diff^1(M).

Proof of Proposition 11. — Let us define $\mathscr{G}_{\mathbf{rec}} = \mathscr{G}_{\mathrm{A}} \cap \mathscr{G}_{\mathbf{Per}} \cap \mathscr{G}_{\mathbf{chain}}$. We consider $f \in \mathscr{G}_{\mathbf{rec}}$. By Theorem 7, we have the equality $\mathbf{WTrans}(f) = \mathbf{CTrans}(f)$. We also have the inclusion $\mathbf{Per}(f) \subset \mathbf{WTrans}(f)$ and we will assume by contradiction that the inequalities does not occur: $\mathbf{Per}(f) \neq \mathbf{WTrans}(f)$. Hence, there exists a weakly transitive invariant compact set K for f which is not accumulated by any periodic orbit. There is a neighborhood U of the point $K \in \mathscr{K}(M)$ such that $\mathbf{Per}(f)$ does not intersect U.

By Theorem 3, there is a perturbation g of f and a periodic orbit $\mathscr{O}$ of g that is close to K for the Hausdorff topology, and belongs to U. This contradicts the fact

that f is a continuity point of the map $g \mapsto \textbf{Per}(g)$. We thus showed that the set $\mathscr{G}_{\textbf{rec}}$ satisfies the conclusion of Proposition 11.   □

The second proposition is also proven by using a continuity property. By Proposition 4, the map $f \mapsto \textbf{Seg}(f)$ is lower-semi-continuous, so that the **Seg**-tolerance stable diffeomorphisms (see the definition of T-tolerance stability at Section 0.3) are generic in $\mathrm{Diff}^1(M)$:

Proposition **13.** — *The set of continuity points of the map* $f \mapsto \textbf{Seg}(f)$ *from* $\mathrm{Diff}^1(M)$ *to* $\mathscr{K}(\mathscr{K}(M))$ *contains a* G_δ *dense subset* $\mathscr{G}_{\textbf{Seg}}$ *of* $\mathrm{Diff}^1(M)$.

Proof of Proposition 12. — We define the residual set $\mathscr{G}_{\textbf{orbit}} = \mathscr{G}_A \cap \mathscr{G}_{\textbf{Seg}} \cap \mathscr{G}_{\textbf{chain}}$ and we consider $f \in \mathscr{G}_{\textbf{orbit}}$. By Theorem 7 and since f belongs to $\mathscr{G}_{\textbf{chain}}$, we have the equality $\textbf{WSeg}(f) = \textbf{ESeg}(f)$. We also have the inclusion $\textbf{Seg}(f) \subset \textbf{WSeg}(f)$. Let us assume by contradiction that $\textbf{Seg}(f) \neq \textbf{WSeg}(f)$: there exists a weak segment of orbit $\mathscr{X}$ for f which is not the Hausdorff limit of finite segments of orbits of f. Since f is a continuity point of $g \mapsto \textbf{Seg}(g)$, there is a neighborhood $\mathscr{U}_0$ of f and a constant $\eta_0 > 0$ such that the maps $g \in \mathscr{U}_0$ have no finite segment of orbit η_0-close to $\mathscr{X}$ for the Hausdorff distance. We denote by W the η_0-neighborhood of $\mathscr{X}$ in M.

Claim **4.** — There is a finite subset $X = \{x_0, ..., x_r\}$ of $\mathscr{X}$ such that:

1. $\mathscr{X}$ is $\frac{\eta_0}{2}$-close for the Hausdorff distance to the closure of

$$\{f^n(x_0)\}_{n \geq 0} \cup \{f^{-n}(x_r)\}_{n \geq 0} \cup \{f^n(x_k), n \in \textbf{Z} \text{ and } k \in \{1, ..., r-1\}\};$$

2. X has no periodic point; the orbit of any two distinct points of X are disjoint;
3. for any $k, \ell \in \{0, ..., r\}$ with $k < \ell$, we have $x_k \prec_W x_\ell$.

Proof. — Similarly to the proof of Proposition 5, one builds using Lemma 1 and Condition (A) a finite subset $\tilde{X}$ of $\mathscr{X}$ and such that

– $\tilde{X}$ is $\frac{\eta_0}{2}$-close to $\mathscr{X}$;
– $\tilde{X}$ has no periodic point.

Since f belongs to $\mathscr{G}_{\textbf{chain}}$, the relation $\prec_{\mathscr{X}}$ is transitive: one can order the set $\tilde{X}$ as a sequence $(\tilde{x}_0, ..., \tilde{x}_s)$ such that, for any $k < \ell$ in $\{0, ..., s\}$, we have $\tilde{x}_k \prec_{\mathscr{X}} \tilde{x}_\ell$.

Note that all the points of $\tilde{X}$ can not belong to a same orbit: otherwise f would have a finite segment of orbit contained in $\mathscr{X}$ which is $\frac{\eta_0}{2}$-close to $\mathscr{X}$ for the Hausdorff topology, contradicting our assumptions on $\mathscr{X}$ and η_0. By Proposition 2, we see that for any $k, \ell \in \{0, ..., s\}$, if $\tilde{x}_k$ is an iterate of $\tilde{x}_0$ and $\tilde{x}_\ell$ is not an iterate of $\tilde{x}_0$ then, we have $\tilde{x}_k \prec_{\mathscr{X}} \tilde{x}_\ell$. This shows that, up to reordering the set X, one may assume that the iterates of $\tilde{x}_0$ contained in $\tilde{X}$ are the first elements of the sequence $\{\tilde{x}_0, ..., \tilde{x}_s\}$. In particular, the points $\tilde{x}_0$ and $\tilde{x}_s$ do not belong to a same orbit.

One now removes from $\tilde{X}$ some points and defines a smaller set $X \subset \tilde{X}$ so that $\tilde{x}_0$ and $\tilde{x}_s$ belongs to X, the points of X have different orbits and any point in $\tilde{X}$ has some iterate in X. One can order X as a sequence $(x_0, ..., x_r)$ such that $\tilde{x}_0 = x_0$, $\tilde{x}_s = x_r$ and such that, for any $k < \ell$ in $\{0, ..., r\}$, we have $x_k \prec_{\mathscr{X}} x_\ell$. The set $\mathscr{X}$ is not invariant by f, but by Proposition 2, the whole orbits of the points $x_1, ..., x_{r-1}$, the positive orbit of x_0 and the negative orbit of x_r are contained in $\mathscr{X}$.

The set $\{f^n(x_0), n \geq 0\} \cup \{f^{-n}(x_r), n \geq 0\} \cup \{f^n(x_k), n \in \mathbf{Z}$ and $k \in \{1, ..., r-1\}\}$ is contained in $\mathscr{X}$ and contains $\tilde{X}$. Hence, it is $\frac{\eta_0}{2}$-close to $\mathscr{X}$ for the Hausdorff distance.   □

By the first property of the claim, there exists a neighborhood $\mathscr{U}$ of f contained in $\mathscr{U}_0$ and a constant $\eta \in (0, \eta_0)$ such that any finite segment of orbit $\{z, g(z), ..., g^n(z)\}$ with $g \in \mathscr{U}$ and $z \in B(x_0, \eta)$ which intersects all the balls $B(x_k, \eta)$ with $x_k \in X$, intersects all the balls $B(x, \eta_0)$ with $x \in \mathscr{X}$.

By the two other properties of the claim and Theorem 9, there is a perturbation g of f_0 in $\mathscr{U}$ and a finite segment of orbit $\mathscr{S} = \{x_0, g(x_0), ..., g^n(x_0)\}$ of g contained in the η_0-neighborhood W of $\mathscr{X}$ which intersects all the balls $B(x, \eta)$ with $x \in X$. In particular, $\mathscr{S}$ is η_0-close to $\mathscr{X}$ for the Hausdorff distance. This contradicts the choice of $\mathscr{U}_0$ and η_0, proving the equality $\mathbf{Seg}(f) = \mathbf{WSeg}(f)$ for f in $\mathscr{G}_{\mathbf{orbit}}$.

Since the elements of $\mathbf{FOrb}(f)$ and $\mathbf{WOrb}(f)$ are the elements of $\mathbf{Seg}(f)$ and $\mathbf{WSeg}(f)$ respectively that are invariant by f, one also gets the equality $\mathbf{FOrb}(f) = \mathbf{WOrb}(f)$ for $f \in \mathscr{G}_{\mathbf{orbit}}$.   □

3.3. *The weak shadowing property (proof of Theorem 1)*

For any $\varepsilon > 0$, we consider the set of finite segments of ε-pseudo-orbits. Taking the closure in $\mathscr{K}(M)$ we get a set $\mathbf{ESeg}_\varepsilon(f)$. The family $(\mathbf{ESeg}_\varepsilon(f))_{\varepsilon>0}$ is ordered for the inclusion and the intersection $\cap_\varepsilon \mathbf{ESeg}_\varepsilon(f)$ coincides with $\mathbf{ESeg}(f)$.

For any diffeomorphism f in the residual set $\mathscr{G}_{\mathbf{Orb}}$ given by Propositions 11 and 12 and any $\delta > 0$ we consider $\varepsilon > 0$ such that the Hausdorff distance in $\mathscr{K}(\mathscr{K}(M))$ between $\mathbf{ESeg}(f)$ and $\mathbf{ESeg}_\varepsilon(f)$ is less than $\frac{\delta}{2}$.

Let $\{z_0, ..., z_n\}$ be any ε-pseudo-orbit of f. It belongs to $\mathbf{ESeg}_\varepsilon(f)$, hence there exists an element $K \in \mathbf{ESeg}(f)$ such that the Hausdorff distance between K and $\{z_0, ..., z_n\}$ in $\mathscr{K}(M)$ is less than $\frac{\delta}{2}$. By Proposition 12, the set K also belongs to $\mathbf{Seg}(f)$ so that there exists a finite segment of orbit $\{x, f(x), ..., f^m(x)\}$ which is $\frac{\delta}{2}$-close to K. This shows that $\{z_0, ..., z_n\}$ and $\{x, ..., f^m(x)\}$ are δ-close for the Hausdorff distance.

One can consider similarly for each $\varepsilon > 0$ the set of periodic ε-pseudo-orbits $\{z_0, ..., z_n\}$ with $n \geq 1$ and $z_n = z_0$. Their closure in $\mathscr{K}(M)$ defines the set $\mathbf{CTrans}_\varepsilon(f)$. By Proposition 11, one can make a similar argument as above with the

sets $\mathscr{G}_{\mathbf{rec}}$, $(\mathbf{CTrans}_\varepsilon(f))_{\varepsilon>0}$, $\mathbf{CTrans}(f)$ and $\mathbf{Per}(f)$ instead of $\mathscr{G}_{\mathbf{Orb}}$, $(\mathbf{ESeg}_\varepsilon(f))_{\varepsilon>0}$, $\mathbf{ESeg}(f)$ and $\mathbf{Seg}(f)$ to get the second part of Theorem 1.

Hence, Theorem 1 holds with the G_δ and dense subset $\mathscr{G}_{\mathbf{shadow}} = \mathscr{G}_{\mathbf{rec}} \cap \mathscr{G}_{\mathbf{orbit}}$ of $\mathrm{Diff}^1(\mathrm{M})$.

3.4. *Tolerance stability properties*

We were unable to decide if the continuity points of the map $f \mapsto \mathbf{Orb}(f)$ are generic in $\mathrm{Diff}^1(\mathrm{M})$ (which is the original Zeeman's tolerance stability conjecture) but we get the property for all the other sets introduced in the paper:

Proposition **14.** — *There is a residual subset of* $\mathrm{Diff}^1(\mathrm{M})$ *whose elements are* **Per***-,* **FOrb***-,* **Seg***-,* **CTrans***-,* **EOrb***-,* **ESeg***-,* **WTrans***-,* **WOrb***- and* **WSeg***-tolerance stable.*

Proof. — For **Per**, this was proven by Takens (see Theorem 8 above), the case of **FOrb** and **Seg** comes from the fact they depend lower-semi-continuously in the dynamics (Proposition 4) and the case of **CTrans**, **EOrb**, **ESeg** from the fact that they vary upper-semi-continuously with the dynamics (Proposition 3).

The remaining sets: **WTrans**, **WOrb** and **WSeg** are always controlled by the other ones (using the inclusions (1), (2), (3)). Since generically the inclusions (1), (2), (3) are equalities and since we have already proven the proposition for the six first sets, one deduces the proposition for the last ones. $\qquad\qquad\square$

3.5. *Approximations of aperiodic classes (proof of Corollary 1)*

Let f be a diffeomorphism in $\mathscr{G}_{\mathrm{A}} \cap \mathscr{G}_{\mathbf{rec}}$ and let E be a aperiodic class of f. In order to prove Corollary 1 we show that E is the limit of a sequence of homoclinic classes for the Hausdorff topology. We fix some $\varepsilon > 0$.

We recall an upper-semi-continuity property of the chain-recurrence classes.

Lemma **2.** — *There exists a neighborhood* U *of* E *such that any chain-recurrence class* E' *of* f *which intersects* U *is contained in the ε-neighborhood of* E.

Proof. — One proves the lemma by contradiction. Let (E_n) be a sequence of chain-recurrence classes which converges (for the Hausdorff topology) towards a compact set K, that intersects E but is not contained in E. By Proposition 3, K is chain-transitive and contained in a chain-recurrence class. But K intersects E and the chain-recurrence classes are disjoint or equal. So, K is contained in E, which is a contradiction. $\qquad\qquad\square$

By Proposition 11, there exists a periodic orbit $\mathcal{O}$ of $f \in \mathcal{G}_{\mathbf{rec}}$, which is close to E in the Hausdorff topology: $\mathcal{O}$ is contained in U and E is contained in the ε-neighborhood of $\mathcal{O}$. Since f belongs to $\mathcal{G}_{\mathrm{A}}$, the periodic orbit $\mathcal{O}$ is hyperbolic.

The chain-recurrence class E′ which contains $\mathcal{O}$ intersects U and, by Lemma 2, is contained in the ε-neighborhood of E. The homoclinic class H which contains $\mathcal{O}$ is included in E′, hence in the ε-neighborhood of E. Since H contains $\mathcal{O}$, we also get that the ε-neighborhood of H contains E. We thus found a homoclinic class H of f which is ε-close to E for the Hausdorff distance. Consequently, E is accumulated by homoclinic classes.

We proved Corollary 1 with $\mathcal{G}_{\mathbf{aper}} = \mathcal{G}_{\mathrm{A}} \cap \mathcal{G}_{\mathbf{rec}}$.

4. Approximation of weakly transitive sets by periodic orbits: proof of Proposition 8

In this part, one considers a diffeomorphism f and a neighborhood $\mathcal{U}$ of f in Diff¹(M). By shrinking $\mathcal{U}$ if necessary, one may assume that it satisfies the Condition (F): for any perturbations $g_1, g_2 \in \mathcal{U}$ of f with disjoint support, the composed perturbation also belongs to $\mathcal{U}$.

The connecting lemma (Theorem 5) associates to f and $\mathcal{U}$ some integer N ≥ 1 that will be fixed from now on and used all along this part.

In order to prove Proposition 8, we consider in M a finite set X and an open set W that verify the assumptions of the proposition. We also choose a constant $\eta > 0$. All the segments of orbits we will consider in the following will be contained in W.

4.0. *Sketch of the proof of Proposition 8*

The main ingredient of the proof is Hayashi's connecting lemma. The natural idea is the following: one cyclically orders the points of X $= \{p_1, p_2, ..., p_r\}$. At each of these points p_i, one can apply the connecting lemma and associate two neighborhoods $V_i \subset U_i$ (contained in W): these neighborhoods may be chosen contained in arbitrarily small balls $B(p_i, \varepsilon_2)$ and $B(p_i, \varepsilon_1)$ respectively, centered at p_i and of radii $0 < \varepsilon_2 < \varepsilon_1$. Using the definition of weak iterate, one chooses for each i a segment of orbit Z_i (contained in W) that connects V_i to V_{i+1} (where the subscripts are taken modulo k) and then tries to use the connecting lemma to connect all the Z_i together by perturbing f in each of the domains U_i and their N−1 first iterates: the composed perturbation g belongs to $\mathcal{U}$ thanks to the Condition (F); moreover, if the domains U_i have been chosen small enough, one would get a periodic orbit that intersects all the balls $B(p_i, \eta)$ and that is contained in W.

Several difficulties arise in this naive approach. A first problem appears when a segment of orbit Z_i intersects a neighborhood U_j of index j different from i and

$i + 1$: the perturbation with support in U_j may break the segment Z_i and we are not sure anymore to get a periodic orbit for the composed perturbation g. To over-pass this problem, one notes that in this case the segment of orbit Z_i, that connects the neighborhoods of p_i and p_{i+1} together, also intersects the neighborhood of p_j: it may not be necessary to consider the segments Z_{j-1}, Z_j and to perturb f around p_j in order to create a periodic orbit with the required support. More precisely, one will choose a smaller set $X' \subset X$ of points where the connecting lemma will be applied. One orders it as $X' = \{x_0, ..., x_s\}$ and one chooses the neighborhoods $V_i \subset U_i$ at each point x_i and then the segments of orbits Z_i connecting U_i to U_{i+1}. As explained above, each segment Z_i may intersect a small neighborhood of several points in X (not only x_i and x_{i+1}): the set of points that are "visited" by Z_i will be denoted by $X_{i,i+1}$. Although X' is smaller than X, one can hope to create a periodic orbit that visit all the points of X if we have

$$(\mathbf{4}) \qquad X = X' \cup X_{0,1} \cup \cdots \cup X_{s-1,s} \cup X_{s,0}.$$

In order to prevent any segment of orbit Z_j to intersect the domains U_j with $j \neq i, i+1$, one will choose X' with the smallest possible cardinality s.

We now meet a second difficulty: when one applies the connecting lemma in a domain U_i to the segments of orbits Z_{i-1} and Z_i, one gets a new segment of orbit that connects U_{i-1} to U_{i+1}, and that is contained in $Z_{i-1} \cup \left(\bigcup_{0 \leq k \leq N} f^k(U_i) \right) \cup Z_i$. However, it should be noted that the connecting lemma can realize some shortcuts: some parts of the orbits Z_{i-1} and Z_i may be forgotten; hence, the new segment of orbit may not visit the neighborhood of some of the points of $X_{i-1,i} \cup X_{i,i+1}$. As a consequence, the periodic orbit that one builds may miss some of the points of X, even if the condition (4) is satisfied. To avoid this difficulty, one requires that among all the segments of orbits that connect U_i to U_{i+1}, the orbit Z_i minimizes the cardinality of the set $X_{i,i+1}$ of points it visits. By doing that, any shorter segment of orbit will visit exactly the same set $X_{i,i+1}$.

We now give the plan of the proof:

First step (Section 4.1). The construction of the minimizing sets $X_{i,i+1}$ is done before choosing the orbits Z_i, and the set X': more precisely, one introduce a minimizing visit set $X_{x,x'}$ for each pair $x \neq x'$ in X.

Second step (Section 4.2). One then chooses the set $X' = \{x_0, ..., x_s\}$. In the best case, each set $X_{i,i+1}$ does not contains any point $x_j \in X'$ with $j \neq i, i+1$, so that Z_i and U_j will be disjoint. In general, it is not possible to have this property, but we show (Proposition 16) that one can always have half of it: for each $i, j \in \{0, ..., s\}$ such that $i + 1 < j$, the point x_j does not belong to $X_{i,i+1}$.

Third step (Section 4.3). From the second step, we have to ensure that for each $j < i < s$, the neighborhood U_j does not intersect the segment of orbit Z_i. This is done

in Proposition 17 where one builds inductively the neighborhoods $V_i \subset U_i$ and the segments of orbits Z_i. One starts with the largest indices $i = s, s - 1, ..., 0$.

Fourth step (Section 4.4). It now remains to perturb in each domain U_i in order to connect all the Z_i. One obtains a segment of orbit that visits all the points of the set X.

However, the most delicate part of the proof consists in proving that this segment can be closed and that one can obtain a periodic orbit:

- As explained, the segment of orbit Z_s should be chosen at the beginning of the third step, so that, the domains U_j, with $j \in \{1, ..., s - 1\}$ can be build disjoint from Z_s. One problem is the following: Z_s should connect V_s to the open set V_0 but V_0 will be constructed only at the end of step 3 ... This paradox is solved by noting (Corollary 4) that the set $Y \subset \{x_0\} \cup X_{0,1}$ of points in X that are not contained in $X_{1,2} \cup \cdots \cup X_{s-1,s}$ is non-empty. In the simplest case x_0 belongs to Y. Hence, if one chooses *a priori* the neighborhoods V_0 and U_0 equal to $B(x_0, \varepsilon_2)$ and $B(x_0, \varepsilon_1)$, they will be disjoint from the orbits $Z_1, ..., Z_{s-1}$; so, without breaking the orbit at intermediate iterates, one can close the orbit by perturbing in U_0.
- In general, this set Y may not contain the point x_0. One has to choose a new point x_{s+1} in Y where the orbit will be closed: the choice of this point is not given *a priori*, but will depend on the construction of all the segments $Z_1, ..., Z_{s-1}$. For this reason, at the beginning of the construction, one has to consider all the possible choices for x_{s+1}: for each point $y \in Y$, one defines two neighborhoods $V_y \subset U_y$ and one introduces a segment of orbit Z_y that connects V_s to V_y. This allows in the following to take the other neighborhoods U_j disjoint from all the Z_y. At the end, one chooses x_{s+1} in Y and one defines $Z_s = Z_{x_{s+1}}$, $V_{s+1} = V_{x_{s+1}}$ and $U_{s+1} = U_{x_{s+1}}$. A perturbation in U_{s+1} will connect the orbit Z_{s+1} to the orbit Z_0 and close the orbit.
- Note that we have to ensure that the orbit Z_0 crosses the neighborhood V_{s+1} in order to perform this last perturbation. By our construction, the neighborhoods V_{s+1} and U_{s+1} can be chosen *a priori* equal to the balls $B(x_{s+1}, \varepsilon_2)$ and $B(x_{s+1}, \varepsilon_1)$ respectively. In other terms, the points of $X_{0,1}$ should be at distance less than ε_2 from Z_0. This was not the case for the other segments of orbits Z_j, with $j \in \{1, ..., s - 1\}$: the points in $X_{j,j+1}$ are at distance less than η from Z_j. This shows that several scales should be considered in the proof:
  - In order to obtain the minimizing property of the sets $X_{i,i+1}$, the (largest) scale η given in the statement of the proposition will be replaced by a smaller constant $\varepsilon_0 > 0$.
  - The (smaller) scales $\varepsilon_2 < \varepsilon_1$ give the size of the domains $V_y \subset U_y$ chosen *a priori* for the connecting lemma at points of Y.

- One has also to control the points that are visited at scale ε_2 by the segment Z_0. For this purpose, the endpoints of Z_0 will be taken close to x_0 and x_1 (see Section 4.1): the scale here is $\varepsilon_3 \in (0, \varepsilon_2)$.

  – The choice of the point x_{s+1} is the remaining difficulty. The last perturbation connects Z_s to Z_0 in U_{s+1} which is in general different from U_0. For this reason, the minimizing properties of the set $X_{0,1}$ can not be used here to guarantee that after perturbation, the orbit visit all the points of $X_{0,1}$: in particular, when one perturbs in U_1, the connecting lemma shortcuts the segment of orbit (contained in Z_0) between U_{s+1} and U_1, so that some points in $X_{0,1}$ may be missed. This is why at Section 4.4 the perturbation in U_1 will be performed before choosing the point x_{s+1}.

4.1. *Visited sets*

In this section, we give several definitions that will be used in the following proofs. In particular, for any points $x \neq x'$ in the set X given above, we introduce a subset $X_{x,x'} \subset X$: it contains the points in X that, in some sense, can not be avoided by the orbits which come close to x and then to x'.

We will study the orbits that approach X at different scales $\varepsilon \in (0, \varepsilon_0]$ where ε_0 is a small constant in $(0, \eta)$ that will be chosen at Proposition 15. As we have explained in 4.0, four different scales $\varepsilon_0 > \varepsilon_1 > \varepsilon_2 > \varepsilon_3$ will be needed in the proof. Recall also that the integer $N \geq 1$ has been fixed above by Hayashi's connecting lemma.

Definition **6.** — *Let x, x' be two distinct points in X and U, U' two neighborhoods of x and x'.*

A finite segment of orbit $(z, f(z), ..., f^n(z))$ contained in W connects U to U' if $z \in f^N(U)$ and $f^n(z) \in U'$.

Such a segment of orbit ε-visits some $y \in X$ if some iterate $f^k(z)$ with $k \in \{0, ..., n\}$ belongs to $B(y, \varepsilon)$. We denote by $X(z, \cdots, f^n(z), \varepsilon) \subset X$ the set of points which are ε-visited by the segment of orbit $(z, ..., f^n(z))$ and by $N(z, \cdots, f^n(z), \varepsilon)$ the cardinality of this set.

The segment of orbit is said to ε-visit (resp. ε-visit exactly) some set $Y \subset X$ if $Y \subset X(z, \cdots, f^n(z), \varepsilon)$ (resp. $Y = X(z, \cdots, f^n(z), \varepsilon)$).

Note that the assumptions made in Proposition 8 on the set X ensure that for any choice of U and U', there always exists a segment of orbit connecting these neighborhoods.

We now introduce the minimizing sets $X_{x,x'}$. There is some freedom in the construction and their choice is not unique, but will be fixed in the following sections.

Proposition **15.** — *There exist a constant $\varepsilon_0 \in (0, \eta)$, and for each distinct points x, x' in X, an integer $N(x, x')$ and a set $X_{x,x'} \subset X$ of cardinality $N(x, x')$ having the following property.*

For any $\varepsilon \in (0, \varepsilon_0)$, and any distinct points $x, x' \in \mathrm{X}$, there are two neighborhoods $\mathrm{U}(x, x', \varepsilon)$ and $\mathrm{U}'(x, x', \varepsilon)$ of x and x' respectively such that:

- *any segment of orbit $(z, f(z), ..., f^n(z))$ that connects $\mathrm{U}(x, x', \varepsilon)$ to $\mathrm{U}'(x, x', \varepsilon)$ ε-visits at least $\mathrm{N}(x, x')$ points in X;*
- *for any neighborhoods $\mathrm{U} \subset \mathrm{U}(x, x', \varepsilon)$ and $\mathrm{U}' \subset \mathrm{U}'(x, x', \varepsilon)$ of x and x', there exists a finite segment of orbit that connects U to U', that is disjoint from X and that ε-visits exactly the set $\mathrm{X}_{x, x'}$.*

Proof. — Let us choose two distinct points $x, x' \in \mathrm{X}$, some neighborhoods U, U' of x and x' and a constant $\varepsilon \in (0, \eta)$. We consider $\mathrm{N}(\mathrm{U}, \mathrm{U}', \varepsilon)$ the minimum of $\mathrm{N}(z, \cdots, f^n(z), \varepsilon)$ over all the orbits $(z, \cdots, f^n(z))$ that connect U to U'. Taking U or U' smaller, this number increases, but is bounded by the cardinality of X. Thus, there exists some maximal number $\mathrm{N}(x, x', \varepsilon)$ such that for any U and U' small enough, $\mathrm{N}(x, x', \varepsilon)$ is equal to $\mathrm{N}(\mathrm{U}, \mathrm{U}', \varepsilon)$.

Note that one can assume that all the segments of orbits we used to connect U to U' are disjoint from X: the first point z may be replaced by any other point $\tilde{z}$ in a small neighborhood of z; the new segment of orbit $(\tilde{z}, \cdots, f^n(\tilde{z}))$, with the same length, will connect again U to U' and ε-visit the same set. As X is finite, one can adjust $\tilde{z}$ in order that its orbit avoids X.

As $\varepsilon > 0$ decreases, the number $\mathrm{N}(x, x', \varepsilon)$ decreases also. The minimum of $\mathrm{N}(x, x', \varepsilon)$ over all ε will be denoted by $\mathrm{N}(x, x')$. We choose now the constant $\varepsilon_0 \in (0, \eta)$ small enough so that for any x and x' in X, and any $\varepsilon \in (0, \varepsilon_0]$ we have $\mathrm{N}(x, x', \varepsilon) = \mathrm{N}(x, x')$.

As X is finite, there are only finitely many possible sets $\mathrm{X}(z, \cdots, f^n(z), \varepsilon)$ with cardinality $\mathrm{N}(x, x')$. Hence, there exists $\mathrm{X}_{x, x'} \subset \mathrm{X}$ with cardinality $\mathrm{N}(x, x')$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and any neighborhoods U and U' of x and x', some orbit connects U to U' and ε-visits exactly the set $\mathrm{X}_{x, x'}$.

For any $\varepsilon \in (0, \varepsilon_0)$, the neighborhoods $\mathrm{U}(x, x', \varepsilon)$ and $\mathrm{U}'(x, x', \varepsilon)$ of x and x' are chosen such that $\mathrm{N}(\mathrm{U}(x, x', \varepsilon), \mathrm{U}'(x, x', \varepsilon)) = \mathrm{N}(x, x')$. □

In order to justify our choice of minimizing sets $\mathrm{X}_{x, x'}$, we state the following corollary.

Corollary **3.** — *Let us consider any $\varepsilon \in (0, \varepsilon_0]$, any $x, x' \in \mathrm{X}$, any small neighborhoods $\mathrm{U} \subset \mathrm{U}(x, x', \varepsilon)$, $\mathrm{U}' \subset \mathrm{U}'(x, x', \varepsilon)$ of x and x', any segment of orbit $\mathrm{Z} = (z, \cdots, f^n(z))$ that connects U to U' and ε-visits exactly the set $\mathrm{X}_{x, x'}$. Then, any sub-orbit Z' of Z, that connects $\mathrm{U}(x, x', \varepsilon)$ to $\mathrm{U}'(x, x', \varepsilon)$, ε-visits exactly the set $\mathrm{X}_{x, x'}$.*

Proof. — The set Ξ of points that are ε-visited by Z' has cardinality at least $\mathrm{N}(x, x')$ by definition. On the other side Ξ is contained in the set $\mathrm{X}_{x, x'}$ of points that are ε-visited by Z. This concludes the proof since $\mathrm{X}_{x, x'}$ has cardinality $\mathrm{N}(x, x')$. □

Choice of the scales. — By having chosen the constant ε_0 small enough, one may first assume that:

a. All the balls $f^k(\mathbf{B}(x, \varepsilon_0))$ with $x \in \mathbf{X}$ and $k \in \{0, ..., \mathbf{N}\}$ are pairwise disjoint.

We will use later four different scales $0 < \varepsilon_3 < \varepsilon_2 < \varepsilon_1 < \varepsilon_0$ for the balls visited by the connecting orbits. We choose them inductively: let us assume that ε_i has been defined. For each $x \in \mathbf{X}$, the connecting lemma at x associates to $\mathbf{U}_x = \mathbf{B}(x, \varepsilon_i)$ a smaller neighborhood $\mathbf{V}_x$ of x. We choose $0 < \varepsilon_{i+1} < \varepsilon_i$ so that for each $x \in \mathbf{X}$, the ball $\mathbf{B}(x, \varepsilon_{i+1})$ is contained in $\mathbf{V}_x$ and in both sets $\mathbf{U}(x, x', \varepsilon_i)$ and $\mathbf{U}'(x', x, \varepsilon_i)$ for each $x' \in \mathbf{X} \setminus \{x\}$.

From this choice, the following additional properties are satisfied:

b. For every $x \in \mathbf{X}$, the connecting lemma may be applied at x with the time $\mathbf{N}$ and the neighborhoods $\mathbf{B}(x, \varepsilon_{i+1}) \subset \mathbf{B}(x, \varepsilon_i)$.

c. For every $x, x' \in \mathbf{X}$ and any neighborhoods $\mathbf{U} \subset \mathbf{B}(x, \varepsilon_{i+1})$, $\mathbf{U}' \subset \mathbf{B}'(x', \varepsilon_{i+1})$, there exists a finite segment of orbit that connects $\mathbf{U}$ to $\mathbf{U}'$, that is disjoint from $\mathbf{X}$, and that ε_{i+1}-visits exactly the set $\mathbf{X}_{x,x'}$.

4.2. *Combinatorics of* $\mathbf{X}$

We choose here the set $\mathbf{X}' \subset \mathbf{X}$ of points where the connecting lemma will be applied.

Proposition **16.** — *There exists a sequence* $\mathbf{X}' = \{x_0, \cdots, x_s\}$ *($s \geq 1$) of distinct points of* $\mathbf{X}$ *such that the visited sets* $\mathbf{X}_{k,k+1} = \mathbf{X}_{x_k, x_{k+1}}$ *satisfy the following properties:*

1. $\mathbf{X} = \{x_0, \cdots, x_s\} \cup \mathbf{X}_{0,1} \cup \cdots \cup \mathbf{X}_{s-1,s}$.
2. For any $2 \leq k \leq s$*, the point* x_k *does not belong to*

$$\{x_0, x_1, \cdots, x_{k-1}\} \cup \mathbf{X}_{0,1} \cup \mathbf{X}_{1,2} \cup \cdots \cup \mathbf{X}_{k-2,k-1}.$$

Proof. — One first chooses arbitrarily some point $x_0 \in \mathbf{X}$. Each time the point x_k (for $k \geq 0$) has been defined one considers inductively any point x_{k+1} in $\mathbf{X} \setminus (\{x_0, \cdots, x_k\} \cup \mathbf{X}_{0,1} \cup \cdots \cup \mathbf{X}_{k-1,k})$ unless this last set covers already all $\mathbf{X}$ (this means $k = s$). $\quad\square$

In Proposition 16, the set $\mathbf{X}'$ is not unique. The next corollary shows that some choices are better.

Corollary **4.** — *If the set* $\mathbf{X}'$ *in Proposition 16 has the smallest cardinality* s*, then, the set*

$$\mathbf{Y} = (\{x_0\} \cup \mathbf{X}_{0,1}) \setminus (\{x_1, ..., x_s\} \cup \mathbf{X}_{1,2} \cup ... \cup \mathbf{X}_{s-1,s})$$

is non-empty.

Proof. — Let us suppose that the corollary is not satisfied: the set $\{x_1, \cdots, x_s\} \cup X_{1,2} \cup \cdots \cup X_{s-1,s}$ contains all points in X. Thus, the smaller sequence $(x_1, \cdots, x_s)$ also satisfies properties (1) and (2) of Proposition 16, which is a contradiction since s is minimal.   □

Notation. — We will denote the set $X_{k,k+1}$ by $X_{x_k,x_{k+1}}$ and $X_{x_s,y}$ by $X_{s,y}$ for any $k \in \{0, ..., s-1\}$ and $y \in Y$.

4.3. *Choosing orbits and neighborhoods*

We explain in the next proposition how to choose inductively, at each point in the sets $X', Y \subset X$, a pair of neighborhoods $V \subset U$ where the connecting lemma may be applied and some segments of orbits that connects these neighborhoods (see Figure 2).

Proposition **17.** — *Let* $X' = \{x_0, ..., x_s\}$ *and* Y *be the subsets of* X *given by Proposition 16 and Corollary 4. Then, there exists*

- *a family of open sets* $U_k \subset B(x_k, \varepsilon_2)$ *for each* $k \in \{0, ..., s\}$ *and* $U_y = B(y, \varepsilon_1)$ *for each* $y \in Y$,
- *a family of smaller open sets* $V_k \subset U_k$ *for each* $k \in \{0, ..., s\}$ *and* $V_y = B(y, \varepsilon_2)$ *for each* $y \in Y$,
- *a segment of orbit* Z_k *that connects* V_k *to* V_{k+1} *for each* $k \in \{0, ..., s-1\}$,
- *a segment of orbit* Z_y *that connects* V_s *to* V_y *for each* $y \in Y$,

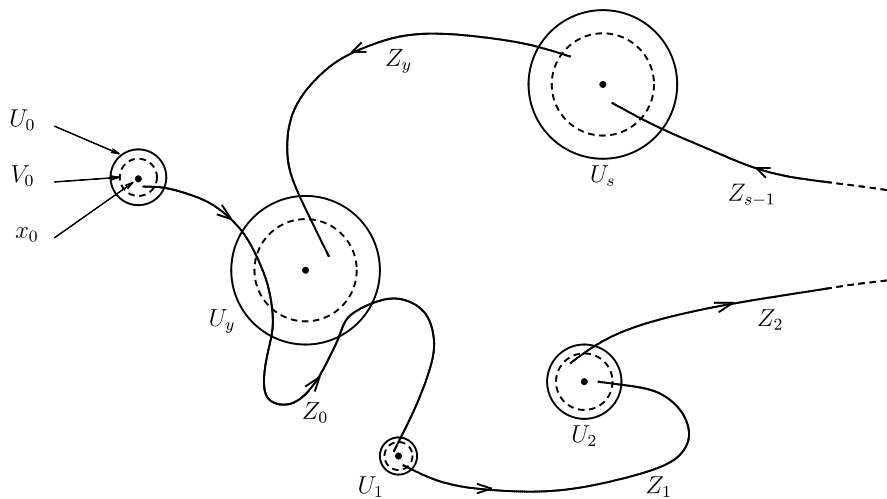*such that the following properties are satisfied:*

Fɪɢ. 2.

1. *The open set V_0 is contained in $B(x_0, \varepsilon_3)$ and U_1 in $B(x_1, \varepsilon_3)$.*
2. *The connecting lemma may be applied to the neighborhoods $V_k \subset U_k$ and to the neighborhoods $V_y \subset U_y$ with the time N.*
3. *For any $y \in Y$, the orbit Z_y does not intersect the sets U_ℓ with $\ell \in \{1, ..., s-1\}$; moreover, it ε_0-visits exactly the set $X_{s,y}$.*
4. *For any $k \in \{1, ..., s-1\}$, the orbit Z_k does not intersect the sets U_ℓ with $\ell \neq k, k+1$ nor the U_y for $y \in Y$; moreover, it ε_0-visits exactly the set $X_{k,k+1}$.*
5. *The orbit Z_0 does not intersect the sets U_k with $k \in \{2, ..., s\}$; moreover, it ε_2-visits exactly the set $X_{0,1}$.*

Proof. — We first set for any $y \in Y$,

$$U_y = B(y, \varepsilon_1), \quad V_y = B(y, \varepsilon_2),$$

and at x_s, we define

$$U_s = B(x_s, \varepsilon_2), \quad V_s = B(x_s, \varepsilon_3).$$

By Proposition 15, one chooses next for each $y \in Y$ a segment of orbit Z_y that connects V_s to V_y and ε_0-visits exactly $X_{s,y}$. This is possible by our choice of the scales ε_1 and ε_0.

We now define by induction the sets $V_k \subset U_k$ and the orbit Z_k for $k \geq 1$. Let us assume that all the sets $V_\ell \subset U_\ell$ and the orbits Z_ℓ for $k \leq \ell \leq s$ have been constructed for some $k \geq 2$. Since the orbits Z_ℓ, with $k \leq \ell \leq s$ and Z_y, with $y \in Y$ can be chosen disjoint from X (by the second item of Proposition 15), one can choose a neighborhood $U_{k-1} \subset B(x_{k-1}, \varepsilon_2)$ of x_{k-1} disjoint from

$$Z_k \cup \cdots \cup Z_{s-1} \cup \bigcup_{y \in Y} Z_y.$$

The connecting lemma at x_{k-1} associates to the neighborhood U_{k-1} an open neighborhood $V_{k-1} \subset U_{k-1}$. By Section 4.1, we choose a segment of orbit Z_{k-1} that connects V_{k-1} to V_k and ε_0-visits exactly $X_{k-1,k}$.

After s steps, the open neighborhoods $V_1 \subset U_1$ of x_1 have been defined. One may require in the last step of the previous induction that $U_1 \subset B(x_1, \varepsilon_3)$.

One now chooses a neighborhood $U_0 \subset B(x_0, \varepsilon_2)$ of x_0 disjoint from

$$Z_1 \cup ... \cup Z_{s-1}.$$

The connecting lemma associates to U_0 an open neighborhood $V_0 \subset U_0 \cap B(x_0, \varepsilon_3)$. One then introduces a segment of orbit Z_0 that connects V_0 to V_1 and ε_2-visits exactly the set $X_{0,1}$.

The first two properties of the proposition are clearly satisfied by this construction. The set of points that are visited by these segment of orbits Z_k and Z_y is also controlled.

By our choice of U_k, for any $k \in \{1, ..., s-1\}$, the open set U_k does not intersect the orbits Z_y with $y \in Y$ and for any $k \in \{0, ..., s-2\}$, the open set U_k does not intersect the orbits Z_ℓ with $\ell \in \{k+1, ..., s-1\}$.

For any $k \in \{3, ..., s\}$, the open set U_k does not intersect the orbits Z_ℓ with $\ell \in \{1, ..., k-2\}$: indeed, Z_ℓ ε_0-visits exactly the set $X_{\ell,\ell+1}$; by Proposition 16 x_k does not belong to $X_{\ell,\ell+1}$ so that Z_ℓ does not ε_0-visit x_k; moreover U_k is contained in $B(x_k, \varepsilon_0)$.

Similarly, U_k with $k \in \{2, ..., s\}$ does not intersect Z_0 since this orbit ε_2-visits exactly $X_{0,1}$ which does not contain x_k and since $U_k \subset B(x_k, \varepsilon_2)$ by construction.

We end by noting that for any $y \in Y$, the open set U_y does not intersect Z_ℓ with $\ell \in \{1, ..., s-1\}$: this is due to the fact that Z_ℓ ε_0-visits exactly the set $X_{\ell,\ell+1}$ and that by Proposition 16, the point y does not belong to $X_{\ell,\ell+1}$. This gives the property since U_y is contained in $B(y, \varepsilon_0)$.   □

4.4. *Connecting the orbits: the end of the proof of Proposition 8*

The end of the proof has now two main steps: we first choose a point x_{s+1} in Y and then apply the connecting lemma at each point x_k with $k \in \{1, ..., s+1\}$ in order to connect together the orbits $Z_1, ..., Z_s$ and $Z_{s+1} = Z_{x_{s+1}}$. Before the first step, we need however to perform a first perturbation at x_1 that connects Z_0 to Z_1.

In this section, the orbits Z_k will be written in the form $(z_k(0), ..., z_k(n_k))$ and the orbits Z_y in the form $(z_y(0), ..., z_y(n_y))$.

4.4.1. *Perturbation at* x_1. — The point $p_1 = z_0(0)$ has some positive iterate $z_0(n_0)$ in V_1. The point $q_1 = z_1(n_1)$ has some negative iterate $z_1(0)$ in $f^N(V_1)$. Hence, one can perturb f by the connecting lemma in the set $U_1 \cup \cdots \cup f^N(U_1)$. This defines a new diffeomorphism $g_1 \in \mathcal{U}$ and an integer $m_0 \geq 1$ so that the positive orbit of $z_0(0)$ now contains $z_1(n_1) = g_1^{m_0}(z_0(0))$.

By the connecting lemma, the new orbit $(z_0(0), \cdots, g_1^{m_0}(z_0(0)))$ can be cut in three parts:

a. an initial part $(z_0(0), \cdots, g_1^{n_a}(z_0(0)))$ which connects V_0 to U_1 and has support in

$$Z_0 \cup U_1 \cup \cdots \cup f^N(U_1),$$

b. some central part $(g_1^{n_a}(z_0(0)), \cdots, g_1^{n_a+N}(z_0(0)))$ with support in

$$U_1 \cup \cdots \cup f^N(U_1),$$

c. and the last part $(g_1^{n_a+N}(z_0(0)), \cdots, g_1^{m_0}(z_0(0)))$ which connects U_1 to V_2 and has support in

$$U_1 \cup \cdots \cup f^N(U_1) \cup Z_1.$$

We will also denote $g_1^{n_a}(z_0(0))$ by p and set $m_1 = m_0 - n_a$.

*Lemma **3**.* — *The initial part $(z_0(0), \cdots, p)$ ε_2-visits exactly the set $X_{0,1}$.*

Proof. — By Proposition 17, $V_0 \subset B(x_0, \varepsilon_3)$ and $U_1 \subset B(x_1, \varepsilon_3)$. By definition of the scales ε_2 and ε_3, by Corollary 3, and since Z_0 was assumed to ε_2-visits exactly the set $X_{0,1}$, this is also the case for any sub-orbit of Z_0 which connects V_0 to U_1. In particular, this is the case for the initial part $(z_0(0), \cdots, g_1^{n_a}(z_0(0)))$ which contains such a sub-orbit. □

*Lemma **4**.* — *Every sub-orbit of the last part $(g_1^N(p), \cdots, g_1^{m_1}(p))$, which connects U_1 to U_2, ε_0-visits exactly the set $X_{1,2}$.*

Proof. — The same argument as in the proof of the previous lemma shows that any sub-orbit of the last part $(g_1^{n_a+N}(z_0(0)), \cdots, g_1^{m_0}(z_0(0)))$ which connects $U_1 \subset B(x_1, \varepsilon_1)$ to $U_2 \subset B(x_2, \varepsilon_1)$ ε_0-visits exactly the set $X_{1,2}$. Indeed, such a sub-orbit contains a sub-orbit of Z_1 which ε_0-visits exactly the set $X_{1,2}$. □

4.4.2. *Choice of $x_{s+1} \in Y$.* — In this section, we define the point $x_{s+1} \in Y$. This allows to define $U_{s+1} = U_{x_{s+1}}$, $V_{s+1} = V_{x_{s+1}}$ and $Z_s = Z_{x_{s+1}}$. The other sets V_y, U_y and orbits Z_y with $y \in Y \setminus \{x_{s+1}\}$ can then be forgotten.

*Lemma **5**.* — *There exists a point $x_{s+1} \in Y$ and an integer $n_q \in \{1, ..., n_a\}$ such that the segment of orbit $(g_1^{-n_q}(p), ..., p)$ connects $V_{s+1} = V_{x_{s+1}}$ to U_1. Moreover, this segment of orbit satisfies the following properties:*

1. *It does not intersect the open sets $U_k \cup ... \cup f^N(U_k)$ for $k \in \{2, ..., s\}$.*
2. *Any sub-orbit of the form $(g_1^{-\ell}(p), ..., p)$, $\ell \le n_q$, that connects $U_{s+1} = U_{x_{s+1}}$ to U_1 ε_0-visits $Y \setminus \{x_{s+1}\}$.*

Proof. — Let $n_b \in \{0, ..., n_a\}$ be the largest integer such that $(g_1^{n_b}(z_0(0)), ..., g_1^{n_a}(z_0(0)))$ intersects every set $f^N(B(y, \varepsilon_0))$ with $y \in Y$. This exists since $(z_0(0), ..., g_1^{n_a}(z_0(0)))$ ε_2-visits $Y \subset X_{0,1}$ by Lemma 3.

Note that since n_b is maximal, $g_1^{n_b}(z_0(0))$ belongs to some unique set $f^N(B(y, \varepsilon_0))$ with $y \in Y$. One defines the point x_{s+1} by setting $x_{s+1} = y$.

By definition, n_b is also the largest integer such that $g_1^{n_b}(z_0(0))$ belongs to $f^N(B(x_{s+1}, \varepsilon_0))$ but since the orbit $(z_0(0), ..., p)$ ε_2-visits $X_{0,1}$, there exists a smaller integer $n_c \in \{0, ..., n_b\}$ such that $g_1^{n_c}(z_0(0))$ belongs to $f^N(B(x_{s+1}, \varepsilon_2))$. (There may be several choices for n_c.) One sets $n_q = n_a - n_c$.

By construction, the orbit $(g_1^{-n_q}(p), ..., p)$ connects $V_{s+1} = V_{x_{s+1}}$ to U_1 since $V_{x_{s+1}} = B(x_{s+1}, \varepsilon_2)$ by Proposition 17.

By Proposition 17, for any $k \in \{2, ..., s\}$, the open set U_k is contained in $B(x_k, \varepsilon_2)$. Since the set that $(g_1^{-n_q}(p), ..., p)$ ε_2-visits is contained in $X_{0,1}$ and since x_k does not belong to $X_{0,1}$ by Proposition 16, we deduce that $(g_1^{-n_q}(p), ..., p)$ does not intersect $U_k \cup ... \cup f^N(U_k)$.

Let us consider now a segment of orbit of the form $(g_1^{-\ell}(p), ..., p)$ that connects $U_{s+1} = U_{x_{s+1}}$ to U_1. Since $g^{n_b}(z_0(0))$ is the last point of the sequence $(g_1^{-n_a}(p), ..., p)$ that belongs to $f^N(B(x_{s+1}, \varepsilon_0))$, one gets that the orbit $(g_1^{-\ell}(p), ..., p)$, which intersects $U_{s+1} \subset B(x_{s+1}, \varepsilon_0)$, contains the orbit $(g_1^{n_b}(z_0(0)), ..., g_1^{n_a}(z_0(0)))$. By our choice of n_b, one gets that $(g_1^{-\ell}(p), ..., p)$ intersects every set $f^N(B(y, \varepsilon_0))$ with $y \in Y$. This shows that this orbit ε_0-visits every point of $Y \setminus \{x_{s+1}\}$.   □

4.4.3. *Inductive assumptions.* — One will now perturb quite independently in the open sets U_k for each $2 \le k \le s+1$. However one needs to choose carefully the order of these perturbations. Thus, we inductively build (see below) a sequence of perturbed maps g_k, $(1 \le k \le s)$ in $\mathscr{U}$ such that the following properties are satisfied:

1. g_{k+1} is a perturbation of g_k with support in

$$U_{k+1} \cup \cdots \cup f^N(U_{k+1}).$$

2. Some positive iterate $g_k^{m_k}(p)$ of p by g_k hits $z_k(n_k)$; the negative iterate $g^{-n_q}(p)$ of p belongs to $f^N(V_{s+1})$.

3. In the segment of orbit $(p, \cdots, z_k(n_k))$ by g_k, every sub-orbit that connects U_1 to U_{k+1} ε_0-visits exactly the set

$$\{x_2, ..., x_{k+1}\} \cup X_{1,2} \cup \cdots \cup X_{k,k+1}.$$

Moreover, such an orbit does not intersect the sets $U_\ell \cup ... \cup f^N(U_\ell)$ with $\ell \in \{k + 2, ..., s + 1\}$.

4. In $(g_k^{-n_q}(p), \cdots, p)$, every sub-orbit of the form $(g_k^{-m}(p), \cdots, p)$ that connects U_{s+1} to U_1 ε_0-visits all the points in $Y \setminus \{x_{s+1}\}$. Moreover, it does not intersect the sets $U_\ell \cup ... \cup f^N(U_\ell)$ with $\ell \in \{2, ..., s\}$.

4.4.4. *Definition of the perturbations g_{k+1}.* — In order to define g_{k+1} from g_k, let us consider again the point p. Recall that it does not belongs to $U_{k+1} \cup \cdots \cup f^N(U_{k+1})$ and has some iterate $g_k^{m_k}(p)$ in V_{k+1}.

Since the segment of orbit orbit Z_{k+1} under f does not intersect the support of the perturbation g_k of f, it is also an orbit of g_k that goes from $z_{k+1}(0) \in f^N(V_{k+1})$ to $q_{k+1} = z_{k+1}(n_{k+1}) \in V_{k+2}$.

One can apply the connecting lemma at x_{k+1} on the map g_k between the points p and q_{k+1}. This defines a new diffeomorphism g_{k+1}. Since the support of this new perturbation is disjoint from the previous one, g_{k+1} is contained in $\mathscr{U}$ (recall Section 2.1).

Some positive iterate of p under g_{k+1} hits now q_{k+1}. Since the former orbit $p, \cdots, z_k(n_k)$ (resp. $z_{k+1}(0), \cdots, z_{k+1}(n_{k+1})$) did not intersect the set U_{k+2} (resp. the set U_1), any sub-orbit Z' of $(p, \cdots, q_{k+1})$ By g_{k+1} that connects U_1 to U_{k+2} intersects U_{k+1} by the connecting lemma. Thus, Z' contains some sub-orbits of the orbits $p, \cdots, z_k(n_k)$ and $(z_{k+1}(0), \cdots, q_{k+1})$ by g_k which connect U_1 to U_{k+1} and U_{k+1} to U_{k+2} respectively. This implies by item (3) of Section 4.4.3 for g_k and by Corollary 3 that Z' ε_0-visits exactly the set

$$\{x_2, \cdots, x_{k+2}\} \cup X_{1,2} \cup \cdots \cup X_{k+1,k+2}.$$

The orbit under g_k from $g_k^{-n_q}(p)$ to p remains unchanged since it does not intersect the support of the new perturbation. Hence, property (4) of Section 4.4.3 still holds. This ends the construction of the perturbed maps g_k, $0 \le k \le s$.

4.4.5. *The last perturbation g_{s+1}.* — One builds finally a perturbation g_{s+1} of g_s with support in $U_{s+1} \cup \cdots \cup f^N(U_{s+1})$ in order to close the orbit $(g_s^{-n_q}(p), \cdots, p, \cdots, z_s(n_s) = g_s^{m_s}(p))$.

Let us define $n_p = m_s$ in order to apply Theorem 5. Recall that p does not belongs to $U_{s+1} \cup \cdots \cup f^N(U_{s+1})$ and has some forward iterate $g_s^{n_p}(p)$ in V_{s+1}. The point $q = p$ has also some backward iterate $g_s^{-n_q}(q)$ in $f^N(V_{s+1})$.

One more time, one applies the connecting lemma (Theorem 5), at x_{s+1} on the map g_s between the points p and q. This defines a new diffeomorphism g_{k+1}. Since the support of this new perturbation is disjoint from the previous one, g_{s+1} is contained in $\mathscr{U}$ (recall Section 2.1).

4.4.6. *Conclusion of the proof.* — The map g_{s+1} sends by positive iterations p on $q = p$ so that p now is periodic. This orbit (by g_{s+1}) contains some sub-orbit of $(p, \cdots, g_s^{m_s}(p))$ by g_s connecting U_1 to U_{s+1}. Hence, by property (3) of Section 4.4.3 the orbit of p ε_0-visits the set

$$\{x_2, \cdots, x_{s+1}\} \cup X_{1,2} \cup \cdots \cup X_{s,s+1}.$$

Moreover it contains also a sub-orbit $(g_s^{-n}(p), \cdots, p)$ of $(g_s^{-n_q}(p), \cdots, p)$ that connects U_{s+1} to U_1. By property (4) of Section 4.4.3, it must ε_0-visits all the points in $Y \setminus \{x_{s+1}\}$. Note also that p belongs to $B(x_1, \varepsilon_0)$. This shows that the orbit of p ε_0-visits also x_1.

By Proposition 16 (1) and the definition of Y, the periodic orbit we have built ε_0-visit every point in X: it intersects every ball $B(x, \eta)$ with $x \in X$. On the other hand,

the periodic orbit is made of segment of orbits Z_k and Z_y and of points contained in the support of the perturbations. By construction, the periodic orbit is thus contained in the open set W introduced at the beginning of Section 4. This ends the proof of the Proposition 8.

5. Proof of the other perturbation results

5.1. *Approximation by finite segments of orbits (proof of Proposition 9)*

As in Section 4, one considers a diffeomorphism f and a neighborhood $\mathcal{U}$ of f in Diff^1(M) which satisfies Condition (F). An integer N $\geq$ 1 is given by the connecting lemma. Let W $\subset$ M be an open set containing a finite set X = $\{p_1, ..., p_r\}$ that satisfies the assumptions of Proposition 9. We also choose a constant $\eta > 0$.

The proof of the proposition is very similar but simpler than the proof of Proposition 8. One difficulty in the proof of Proposition 8 was to close the orbit. This justified the introduction of the set Y and the delicate choice of the point x_{s+1} (see Sections 4.2 and 4.4.2). In this section, only two scales $\varepsilon_1 < \varepsilon_0$ in $(0, \eta)$ will be used (see Section 4.0).

As in Section 4.1, one considers finite segments of orbits contained in W. One defines a set $X_{x,x'} \subset X$ for each pair (x, x') of points in X such that $x \prec_W x'$. Then, one adapts Proposition 16:

Proposition **18.** — *There exists a sequence* $(x_0, \cdots, x_s)$ *($s \geq 1$) of distinct points of* X *with the following properties:*

1. $x_0 = p_1$.
2. $x_k \prec_W x_{k+1}$ for any $1 \leq k \leq s - 1$;
3. X = $\{x_0, \cdots, x_s\} \cup X_{0,1} \cup \cdots \cup X_{s-1,s}$;
4. for any $2 \leq k \leq s$, the point x_k does not belong to

$$\{x_0, x_1, ..., x_{k-1}\} \cup X_{0,1} \cup X_{1,2} \cup ... \cup X_{k-2,k-1}.$$

Proof. — By assumption, the set X is ordered by $\prec_W$. One sets $x_0 = p_1$. Our hypothesis implies that for any $x \in X \setminus \{x_0\}$ we have $x_0 \prec_W x$.

One builds the sequence (x_k) inductively and assumes at each step that

(**5**) $x_k \prec_W x$ for any $x \notin \{x_0, \cdots, x_k\} \cup X_{0,1} \cup \cdots \cup X_{k-1,k}$.

Each time the point x_k has been defined, one can order

$$X \setminus \{x_1, \cdots, x_k\} \cup X_{1,0} \cup \cdots \cup X_{k-1,k}$$

as a sequence $(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_m)$ such that $\tilde{p}_i \prec_W \tilde{p}_j$ for each $0 \leq i < j \leq m$ (there may be several possibilities) and one chooses $x_{k+1} = \tilde{p}_1$. Then the inductive Assumption (5) is clearly satisfied. $\square$

In order to conclude the proof of Proposition 9, one chooses inductively (as k decreases):

1. some neighborhoods $V_k \subset U_k \subset B(x_k, \varepsilon_1)$ of x_k that satisfy the connecting lemma at x_k,
2. a segment of orbit $Z_k = (z_k(0), \cdots, z_k(n_k))$ connecting V_k to V_{k+1} that ε_0-visits exactly the set $X_{k,k+1}$ and that is contained in W.

One may assume that Z_k is disjoint from any U_ℓ with $\ell \neq k, k+1$. Then, one sets $p = p_1 = x_0$ so that p belongs to V_0.

Since $z_0(0)$ belongs to $f^N(V_0)$, the connecting lemma between the points $f^{-1}(p)$ and $f(z_0(0))$ builds a perturbation $g_0 \in \mathscr{U}$ of f with support in $U_0, ..., f^{N-1}(U_0)$ and such that $f^N(p) = z_0(0)$. Hence, the segments of orbits Z_k with $k \in \{1, ..., s-1\}$ do not intersect the support of this perturbation. One now builds a sequence of perturbations $(g_k)_{1 \leq k \leq s-1}$ in $\mathscr{U}$ such that

1. g_{k+1} belongs to $\mathscr{U}$ and is a perturbation of g_k with support in

$$U_{k+1} \cup \cdots \cup f^{N-1}(U_{k+1});$$

2. g_k sends p on $z_k(n_k) = g_k^{m_k}(p)$;
3. in the orbit $(p, \cdots, g_k^{m_k}(p))$ of g_k, every sub-orbit of the form $(p, \cdots, g_k^m(p))$ that intersects U_{k+1} ε_0-visits exactly the set

$$\{x_0, \cdots, x_{k+1}\} \cup X_{0,1} \cup \cdots \cup X_{k,k+1}.$$

The orbit $(p, \cdots, g_{s-1}^{m_{k-1}}(p))$, for the last perturbation $g = g_{s-1}$, ε_0-visits all the points of X and is contained in W as required.

The support of the perturbation g is contained in the balls $f^j(B(x_k, \eta))$ with $j \in \{0, ..., N-1\}$ and $k \in \{0, ..., s-1\}$.

Remark. — From the conclusion of the proof, one sees that one gets a stronger statement for Proposition 9: let $x \in \{p_1, ..., p_r\}$ be the last point visited by the orbit $(p_1, ..., g^n(p_1))$ (i.e. choosing n minimal, one may have $g^n(p_1) \in B(x, \eta)$). Then, the support of the perturbation g of f is only contained in the balls $f^j(B(p_k, \eta))$ with $j \in \{0, ..., N-1\}$ and $k \in \{1, ..., r\}$ such that $p_k \neq x$.

5.2. *Asymptotic approximation (proof of Proposition 10)*

Let f be a diffeomorphism that satisfies Condition (A) and $\mathscr{U}$ a neighborhood of f in $\mathrm{Diff}^1(M)$ that satisfies Condition (F). In order to control an infinite half orbit, one will perform an infinite sequence of perturbation:

– from the relation $x \prec_{\hat{W}_A} A$, one introduces a sequence of intermediary points (y_n) such that

$$x \prec_{\hat{W}_A} y_1 \prec_{W_A} y_2 \prec_{W_A} \cdots \prec_{W_A} A;$$

– one then perturbs at each point y_n in order to connect all these points together.

For the second step of the proof, one should check the convergence of the sequence of perturbation: one considers a decreasing sequence of neighborhoods $(\mathscr{V}_k)$ of Id in $\mathrm{Diff}^1(M)$ which satisfy the property (F).

(F) For any perturbations φ and φ' of Id *in $\mathscr{V}_k$ with disjoint support, the composed perturbation $\varphi \circ \varphi'$ belongs to $\mathscr{V}_k$.*

Hence the set $\mathscr{U}_k$ of diffeomorphisms g of the form $\varphi \circ f$ with $\varphi \in \mathscr{V}_k$ is a neighborhood of f which satisfies the property (F). One will assume that $\cap_k \mathscr{V}_k = \{\mathrm{Id}\}$ and $\mathrm{Cl}(\mathscr{U}_0) \subset \mathscr{U}$.

The connecting lemma associates to each pair $(f, \mathscr{U}_k)$ an integer $N_k \geq 1$.

5.2.1. *Intermediary points.* — We consider a closed invariant set A.
The following lemma allows to introduce intermediary points for the stable set:

Lemma **6.** — *For any neighborhoods $W_i \subset W_e$ of A and any point $y \in W_e$ such that $y \prec_{W_e} A$, there exists $y' \in W_i \setminus A$ such that $y \prec_{W_e} y'$ and $y' \prec_{W_i} A$. Moreover, the forward orbit of y' is contained in W_i.*

Proof. — Let us take a smaller neighborhood V of A such that $\mathrm{Cl}(V) \subset W_i$. One considers for any $k \geq 1$, a finite segment of orbit $(z_0^k, ..., z_{n(k)}^k)$ in W_e such that z_0^k belongs to the ball $B(y, 1/k)$ and $z_{n(k)}^k$ to the $1/k$-neighborhood of A. For k large enough, $z_{n(k)}^k$ belongs also to the set V. One then considers the first point $z_{m(k)}^k$ of the orbit $(z_0^k, ..., z_{n(k)}^k)$ such that $(z_{m(k)}^k, ..., z_{n(k)}^k)$ is contained in V.

One can extract a subsequence of $(z_{m(k)})_k$ and assume that it converges to a point $y' \in \mathrm{Cl}(V) \setminus A$. The sequences $(z_0^k, ..., z_{m(k)}^k)$ are contained in W_e and show that $y \prec_{W_e} y'$. Similarly, the sequences $(z_{m(k)}^k, ..., z_{n(k)}^k)$ are contained in W_i and show that $y' \prec_{W_i} A$, as required. Since the sequences $(z_{m(k)}^k, ..., z_{n(k)}^k)$ are contained in $\mathrm{Cl}(V)$, since $(z_{m(k)}^k)$ converges to y' and since $z_{n(k)}^k$ becomes arbitrarily close to the invariant set A, the difference $n(k) - m(k)$ goes to infinity and the forward orbit of y' is contained in $\mathrm{Cl}(V) \subset W_i$. □

We now introduce as in the statement of Proposition 10 two open neighborhoods $W_A \subset \hat{W}_A$ of A, with $\mathrm{Cl}(W_A) \subset \hat{W}_A$, and a point $x \in \hat{W}_A$ such that $x \prec_{\hat{W}_A} A$. We choose a neighborhood U of x. One may assume that $U \subset \hat{W}_A \setminus A$.

We build the open sets where the perturbations will be performed:

Proposition **19**. — *There exist three sequences* $(U_k)_{k\geq 1}$, $(V_k)_{k\geq 1}$ *and* $(W_k)_{k\geq 0}$ *of open sets and a sequence of finite segment of orbits* $(z_0^k, z_1^k, ..., z_{n(k)}^k)_{k\geq 0}$ *such that:*

1. (W_k) *is a decreasing sequence of neighborhoods of* A *such that* $W_0 \subset \hat{W}_A$, $W_1 \subset W_A$ *and* $\cap_k W_k = A$.
2. *The orbit* $(z_0^k, z_1^k, ..., z_{n(k)}^k)$ *is contained in* $W_k \setminus W_{k+2}$.
3. *The connecting lemma may be applied to the open sets* $V_k \subset U_k$ *with the time* N_k *for* f *and the neighborhood* $\mathcal{U}_k$; *moreover, the iterates* $U_k, f(U_k), ..., f^{N_k}(U_k)$ *are all contained in* W_k *and have their closure disjoint from* W_{k+1}.
4. *The points* $z_{n(k)}^k$ *and* z_0^{k+1} *are contained in* V_{k+1}; *the point* z_0^0 *belongs to* U.

Proof. — In the proof, it will be useful to introduce also the open sets $V_0 = U_0 = U$ (however the connecting lemma will not be used for (V_0, U_0)) and to builds a sequence $(y_k)_{k\geq 0}$ such that

– $y_k \in V_k$, moreover $y_k \prec_{W_k} y_{k+1}$ and $y_k \prec_{W_k} A$;
– when $k \geq 1$, the point y_k is not a periodic point of period less than or equal to N_k;
– when $k \geq 1$, the forward orbit of y_k is contained in W_k.

We now start the construction. The first open set W_0 is chosen equal to $\hat{W}_A$. One sets $y_0 = x$.

The constructions are then done by induction on k: one assumes that y_k, U_k, V_k, W_k and (if $k \geq 1$) $(z_0^{k-1}, z_1^{k-1}, ..., z_{n(k-1)}^{k-1})$ have been defined and one builds U_{k+1}, V_{k+1}, W_{k+1} and the sequence $(z_0^k, z_1^k, ..., z_{n(k)}^k)$.

One first chooses a neighborhood W_{k+1} of A whose closure is contained in W_k and disjoint from the closures of the iterates $U_k, ..., f^{N_k}(U_k)$ and (if $k \geq 1$) from $(z_0^{k-1}, z_1^{k-1}, ..., z_{n(k-1)}^{k-1})$. Recall that by assumption (A) on f, the periodic points of period less than or equal to N_{k+1} are finite and hence isolated. One thus can choose W_{k+1} such that $W_{k+1} \setminus A$ does not contain any periodic point of period less than or equal to N_{k+1}. One also chooses W_{k+1} in a $1/(k+1)$-neighborhood of A. For $k = 0$, one can assume that W_1 is contained in W_A.

One can then apply Lemma 6 to the point $y = y_k$ and to the open sets $W_e = W_k$ and $W_i = W_{k+1}$. This defines a new point y_{k+1} whose forward orbit is contained in W_i. Moreover $y_k \prec_{W_k} y_{k+1}$ and $y_{k+1} \prec_{W_{k+1}} A$. As y_{k+1} belongs to $W_{k+1} \setminus A$, it is not a periodic point of period less than or equal to N_{k+1}.

Since the forward orbit of y_{k+1} is contained in W_{k+1} and since y_{k+1} is not a periodic point of period less than or equal to N_{k+1}, one can build two neighborhoods $V_{k+1} \subset U_{k+1}$ of y_{k+1} where the connecting lemma can be applied with the time N_{k+1} to f and the neighborhood $\mathcal{U}_{k+1}$. Choosing U_{k+1} small enough, one can assume that the closure of the iterates $U_{k+1}, ..., f^{N_{k+1}}(U_{k+1})$ are all contained in W_{k+1} and are disjoint from A.

By the relation $y_k \prec_{W_k} y_{k+1}$, there exists a finite segment of orbit $(z_0^k, z_1^k, ..., z_{n(k)}^k)$ contained in W_k such that z_0^k belongs to V_k and $z_{n(k)}^k$ belongs to V_{k+1}. By this induction, all the required properties are satisfied. We get $A = \cap_k W_k$ since W_k is contained in a $1/k$-neighborhood of A.   □

5.2.2. *Infinite sequence of perturbations.* — One sets $z = z_0^0$. We then define a sequence of intermediate perturbations.

Proposition **20.** — *There is a sequence of perturbations* (g_k) *of* f *and an increasing sequence of integers* (n_k) *larger than* 1 *such that:*

1. *$g_0 = f$ and $n_0 = 0$.*
2. *For $k \geq 1$, $g_k = \varphi_k \circ g_{k-1}$ and φ_k is a perturbation of* Id *with support in* $U_k \cup ... \cup f^{N_k - 1}(U_k)$. *Moreover φ_k belongs to $\mathcal{V}_k$.*
3. *For $\ell \in \{0, ..., k - 1\}$, the orbit $(g_k^{n_\ell}(z), g_k^{n_\ell + 1}(z), ..., g_k^{n_{\ell+1}}(z))$ by g_k is contained in $W_\ell \setminus W_{\ell+2}$.*

Proof. — The sequence is built inductively and will satisfy the following additional property:

 – There exists an integer $m_k > n_k$ such that $g_k^{m_k}(z)$ belongs to V_{k+1}.
 – The orbit $(g_k^{n_k}(z), ..., g_k^{m_k}(z))$ is contained in $W_k \setminus W_{k+2}$.

We define $g_0 = f$ and $n_0 = 0$. By g_0, the point z has a forward iterate $f^{n(0)}(z) = z_{n(0)}^0$ in V_1. One sets $m_0 = n(0)$. The inductive assumptions are satisfied for $k = 0$.

We now assume that the perturbation g_k, the integers $(n_\ell)_{\ell \leq k}$ and the integer m_k have been defined. We explain how to define the perturbation g_{k+1} and the integers n_{k+1} and m_{k+1}.

The forward orbit of $g_k^{n_k}(z)$ has an iterate $g_k^{m_k}(z)$ in V_{k+1} and the backward orbit of $z_{n(k+1)}^{k+1}$ has an iterate z_0^{k+1} in V_{k+1}. Moreover these segments of orbits are contained in $W_k \setminus W_{k+3}$. One can apply the connecting lemma at the sets $V_{k+1} \subset U_{k+1}$ to $(g_k, \mathcal{U}_{k+1})$ since g_k and f coincide on $U_{k+1}...,f^{N_{k+1}}(U_{k+1})$. The new perturbation is of the form $g_{k+1} = \varphi_{k+1} \circ g_k$ and $\varphi_{k+1} \circ f$ belongs to $\mathcal{U}_{k+1}$. Hence, $\varphi_{k+1} \in \mathcal{V}_{k+1}$.

By the new diffeomorphism g_{k+1}, the point $g_k^{n_k}(z)$ has a forward iterate $g_{k+1}^{m_{k+1}}(z)$ equal to $z_{n(k+1)}^{k+1}$, in V_{k+2}. There also exists an iterate $g_{k+1}^{n_{k+1}}(z)$ with $n_k < n_{k+1} < m_{k+1}$ such that:

 – The segment of orbit $(g_{k+1}^{n_k}(z), ..., g_{k+1}^{n_{k+1}})$ is contained in the union of $\{g_k^{n_k}(z), ..., g_k^{m_k}\}$ with the open sets $V_{k+1}, f(V_{k+1}), ..., f^{N_{k+1}}(V_{k+1})$. Hence, it is contained in $W_k \setminus W_{k+2}$.
 – The segment of orbit $(g_{k+1}^{n_{k+1}}(z), ..., g_{k+1}^{m_{k+1}})$ is contained in the union of $\{z_0^{k+1}, ..., z_{n(k+1)}^{k+1}\}$ with the open sets $V_{k+1}, f(V_{k+1}), ..., f^{N_{k+1}}(V_{k+1})$. Hence, it is contained in $W_{k+1} \setminus W_{k+3}$.

By construction, the orbits $(z, g_k(z), ..., g_k^{n_k}(z))$ and $(z, g_{k+1}(z), ..., g_{k+1}^{n_k}(z))$ coincide. Hence, the inductive assumption are verified for $k + 1$. $\qquad \square$

We can now conclude the proof of the proposition:

End of the proof of Proposition 10. — We are now ready to introduce the perturbation g. Since for $i < j$, the supports $U_i \cup ... \cup f^{N_i-1}(U_i)$ and $U_j \cup ... \cup f^{N_j-1}(U_j)$ of the perturbations φ_i and φ_j are pairwise disjoint, since $\mathscr{V}_j \subset \mathscr{V}_i$ and since $\mathscr{U}_i$ satisfies property (F), the perturbations $\varphi_\ell \circ \varphi_{\ell+1} \circ \cdots \circ \varphi_k$ with $k \leq \ell$ belong to $\mathscr{V}_k$. This shows that the sequence $(\varphi_0 \circ ... \circ \varphi_k)_k$ converges in the C^1-topology towards a diffeomorphism $\varphi \in \mathrm{Cl}(\mathscr{V}_0)$. In other words, the sequence (g_k) converges towards a diffeomorphism $g \in \mathrm{Cl}(\mathscr{U}_0) \subset \mathscr{U}$.

The diffeomorphism g coincides with $\varphi_k \circ f$ in $U_k \cup ... \cup f^{N-1}(U_k)$ and with f elsewhere. Consequently, g is a perturbation of f with support contained in $W_1 \subset W_A$ and which coincides with f on A. From property (3) of Proposition 20, the orbit $(g^n(z))_{n \geq n_k}$ of z by g is contained in W_k. One deduces that the forward orbit of z by g is contained in $\hat{W}_A$ and that its ω-limit set is contained in A. This ends the proof of Proposition 10. $\qquad \square$

5.3. *Approximation by full orbit (proof of Theorem 2)*

Let f be a diffeomorphism that satisfies Condition (A) (the periodic points are "isolated"), $\mathscr{U}$ a neighborhood of f in $\mathrm{Diff}^1(M)$ that satisfies Condition (F) (allowing the composition of perturbations with disjoint supports), $\mathscr{X}$ a weak orbit and $\eta_0 > 0$ a small constant.

If $\mathscr{X}$ is weakly transitive, there exists a perturbation $g \in \mathscr{U}$ and a periodic orbit $\mathscr{O}$ of g which is at distance less than η_0 from $\mathscr{X}$ in the Hausdorff topology, by Theorem 3. This proves Theorem 2 in this case. Hence, we will assume that $\mathscr{X}$ is not weakly transitive.

It is simpler to create and to control a forward (or a backward) orbit rather than a full orbit. Hence, the idea of the proof will be to first decompose a subset of $\mathscr{X}$ as a union $\mathscr{X}_\alpha \cup \mathscr{O}(z_0) \cup \mathscr{X}_\omega$ by using Proposition 7 and then to create indenpendently a forward orbit that is close to $\mathscr{X}_\omega$ and a backward orbit that is close to $\mathscr{X}_\alpha$ for the Hausdorff topology.

5.3.1. *Reduction to half orbits.* — The following proposition is an analogue of Theorem 2 for half orbits and "generalizes" Proposition 10.

Proposition **21.** — *Let f be a diffeomorphism that satisfies Condition (A) and $\mathscr{U}$ a neighborhood of f in $\mathrm{Diff}^1(M)$. We consider:*

  – a weak orbit $\mathscr{X}_\omega$ whose relation $\prec_{\mathscr{X}_\omega}$ is transitive,
  – a neighborhood $\hat{W}_\omega$ of $\mathscr{X}_\omega$,
  – a point $z_0 \in \hat{W}_\omega \setminus \mathscr{X}$ such that $z_0 \prec_{\hat{W}_\omega} \mathscr{X}_\omega$.

We assume moreover that:

  (*.1) The points z_0 and $\mathscr{X}_\omega$ are contained in a weak orbit $\mathscr{X}$.
  (*.2) We have $z_0 \prec_{\hat{K}_\omega} \mathscr{X}_\omega$ where $\hat{K}_\omega$ is a compact set contained in $\mathscr{X} \cap \hat{W}_\omega$.
  (*.3) For any $y \in \mathscr{X}_\omega$ and $z \in \mathscr{X} \setminus \mathscr{X}_\omega$, the relation $y \prec_{\mathscr{X}} z$ is not satisfied.

  Then, for any $\eta_0 > 0$, for any neighborhoods U_{z_0} of z_0 and W_ω of $\mathscr{X}_\omega$, there exist a perturbation $g_\omega \in \mathscr{U}$ of f with support in W_ω, a point $z_\omega \in U_{z_0}$ and an iterate $g_\omega^{n_\omega}(z_\omega)$ such that

  – the forward orbit of z_ω by g_ω is contained in $\hat{W}_\omega$,
  – the closure of the forward orbit of $g_\omega^{n_\omega}(z_\omega)$ is η_0-close to $\mathscr{X}_\omega$ for the Hausdorff topology.

Of course, one gets a similar statement for backwards half orbits if one applies the previous proposition to f^{-1}.

The proof of the proposition is postponed to the next section. The idea is to approximate the set $\mathscr{X}_\omega$ by a finite segment of orbit Z_b (by using Proposition 9). By the connecting lemma, one can connect a point z_ω, close to z_0, to the first point of Z_b, by a finite segment of orbit Z_a. By a last perturbation (given by Proposition 10), one can control the forward orbit of the last point in Z_b: it stays in a small neighborhood of $\mathscr{X}_\omega$ and accumulates on a subset.

The assumption (*) is technical: it simplifies the proof but it could maybe be removed.

End of the proof of Theorem 2. — The connecting lemma applied to $(f, \mathscr{U})$ gives an integer N. We denote by W the η_0-neighborhood of $\mathscr{X}$. We only have to consider the case where $\mathscr{X}$ is not weakly transitive: using Proposition 7, there exist some weak orbits $\mathscr{X}_\alpha$, $\mathscr{X}_\omega$ (whose relations $\prec_{\mathscr{X}_\alpha}$ and $\prec_{\mathscr{X}_\omega}$ are transitive) and a point z_0 such that:

1. The sets $\mathscr{X}_\alpha$, $\mathscr{X}_\omega$ and the orbit $\mathscr{O}(z_0)$ of z_0 are contained in $\mathscr{X}$ and are pairwise disjoint.
2. The set $\mathrm{Cl}(\mathscr{X}_\alpha \cup \mathscr{O}(z_0) \cup \mathscr{X}_\omega)$ is $\eta_0/2$-close to $\mathscr{X}$ in the Hausdorff topology.
3. For any $x \in \mathscr{X}_\alpha$ and $z \in \mathscr{X} \setminus \mathscr{X}_\alpha$ we have $x \prec_{\mathscr{X}} z$ but the relation $z \prec_{\mathscr{X}} x$ is not satisfied.
4. For any $y \in \mathscr{X}_\omega$ and $z \in \mathscr{X} \setminus \mathscr{X}_\omega$ we have $z \prec_{\mathscr{X}} y$ but the relation $y \prec_{\mathscr{X}} z$ is not satisfied.
5. The point z_0 is non-periodic (by the remark after Proposition 7).

Let $\hat{K}_\omega$ be the compact set obtained by removing from K a small open neighborhood of $\mathscr{X}_\alpha$. By item 3), for any points $x \in \mathscr{X} \setminus \hat{K}_\omega$, the relation $z_0 \prec_{\mathscr{X}} x$ does not hold. One deduces that if we have $z_0 \prec_{\mathscr{X}} y$ for some point y, then $z_0 \prec_{\hat{K}_\omega} y$. By

item 4), this is satisfied for any point $y \in \mathscr{X}_\omega$. For any neighborhood $\hat{W}_\omega$ of $\hat{K}_\omega$ (to be defined later), the assumptions of Proposition 21 are satisfied for f by the sets $\mathscr{X}_\omega$, $\hat{W}_\omega$, $\mathscr{X}$ and $\hat{K}_\omega$.

One defines similarly a compact set $\hat{K}_\alpha$ by removing from K a small neighborhood of $\mathscr{X}_\omega$, so that the assumptions of Proposition 21 are satisfied for f^{-1} by the sets $\mathscr{X}_\alpha$, $\hat{W}_\alpha$, $\mathscr{X}$ and $\hat{K}_\alpha$ where $\hat{K}_\alpha$ is any neighborhood of $\hat{K}_\alpha$. By choosing some small neighborhoods W_α, U_{z_0}, W_ω of $\mathscr{X}_\alpha$, z_0 and $\mathscr{X}_\omega$, one gets the following properties:

– For any perturbation g of f with support in $W_\alpha \cup W_\omega \cup \bigcup_{-N \le k \le N} f^k(U_{z_0})$, the $\eta_0/2$-neighborhood of the orbit of any point $z \in U_{z_0}$ by g contains the orbit of z_0 by f (by item 1)).
– The N first backward and forward iterates of U_{z_0} are pairwise disjoint and disjoint from W_α and W_ω (by items 1) and 5)).

The neighborhood $\hat{W}_\omega$ of $\hat{K}_\omega$ is taken equal to $W \setminus Cl(W_\alpha)$ and the neighborhood $\hat{W}_\alpha$ of $\hat{K}_\alpha$ is taken equal to $W \setminus Cl(W_\omega)$

Applying Proposition 21 to f and f^{-1}, one obtains two perturbation g_ω and g_α of g with support in W_ω and W_α respectively, two points z_ω and z_α in U_{z_0}, and two iterates $g_\omega^{n_\omega}(z_\omega)$, $g_\alpha^{-n_\alpha}(z_\alpha)$. The two perturbations g_α, $g_\omega \in \mathscr{U}$ have disjoint supports W_α and W_ω, hence one can introduce the composed perturbation $\hat{g} \in \mathscr{U}$. Since the forward orbit of z_ω by g_ω is contained in $\hat{W}_\omega$, it does not intersect W_α and for the same reason, the backward orbit of z_α by g_α does not intersect U_ω. Hence, these two half orbits remain unchanged by $\hat{g}$. Moreover, there are two iterates $\hat{g}^{n_\omega}(z_\omega)$ and $\hat{g}^{-n_\alpha}(z_\alpha)$ with $n_\alpha, n_\omega < N$ in W_ω and W_α respectively such that the forward orbit of $\hat{g}^{n_\omega}(z_\omega)$ and the backward orbit of $\hat{g}^{-n_\alpha}(z_\alpha)$ by $\hat{g}$ are $\eta_0/2$-close to $\mathscr{X}_\omega$ and $\mathscr{X}_\alpha$ respectively.

The connecting lemma allows to perturb $\hat{g}$ in U_{z_0} and its N first forward iterates in order to connect $\hat{g}^{-n_\alpha}(z_\alpha)$ to $\hat{g}^{n_\omega}(z_\omega)$ in an orbit which intersects U_{z_0} at some point z. The new diffeomorphism g belongs to $\mathscr{U}$ since all the perturbations have disjoint supports.

The $\eta_0/2$-neighborhood of the orbit of z by g contains the orbit of z_0 by f. Moreover, the forward orbit of $\hat{g}^{n_\omega}(z_\omega)$ by g_ω is the same as by g and the backward orbit of $\hat{g}^{-n_\alpha}(z_\alpha)$ by g_α the same as by g. Hence, the $\eta_0/2$-neighborhood of the orbit of z by g contains the sets $\mathscr{X}_\alpha$ and $\mathscr{X}_\omega$. Consequently (using item 2)), the η_0-neighborhood of $\mathscr{O}(z)$ contains $\mathscr{X}$. By construction, the orbit of z by g is contained in W, the η_0-neighborhood of $\mathscr{X}$. We thus proved that the closure of the orbit of z by g and $\mathscr{X}$ are η_0-close in the Hausdorff topology.   □

5.3.2. *Proof of Proposition 21.* — Note that if $\mathscr{X}_\omega$ supports a minimal dynamics, then, one can apply Proposition 10. One immediately gets a point $z_\omega \in U_{z_0}$ and a perturbation $g_\omega \in \mathscr{U}$ with support in W_ω such that the forward orbit of z_ω is contained in $\hat{W}_\omega$ and accumulates on a part of $\mathscr{X}_\omega$. Since $\mathscr{X}_\omega$ is minimal, it accumulates

on the whole set $\mathscr{X}_\omega$. Hence, one gets the conclusion of Proposition 21 in this case. We will now assume that $\mathscr{X}_\omega$ is not minimal.

By shrinking $\mathscr{U}$, one will assume that it satisfies Condition (F). The connecting lemma associates to $(\mathscr{U}, f)$ an integer $N \geq 1$. One can shrink the open set W_ω so that it is contained in the η_0-neighborhood of $\mathscr{X}_\omega$. We need some preliminary constructions, that are described in the next two lemmas.

*Lemma **7**. — There exist*

- *a finite set $\{p_1, ..., p_{r-1}\}$ in $\mathscr{X}_\omega$, with $r \geq 2$, and a point $p_r \in (\mathscr{X} \setminus \mathscr{X}_\omega) \cap W_\omega$,*
- *two compact sets $K_\omega \subset \mathscr{X} \cap W_\omega$ and $\hat{K}_\omega \subset \mathscr{X} \cap \hat{W}_\omega$,*
- *an invariant compact subset $A \subset \mathscr{X}_\omega$ that supports a minimal dynamics,*
- *a small neighborhood $\hat{W}_A$ of A, contained in W_ω,*

such that:

1. *The points $f^j(p_k)$ for $k \in \{1, ..., r\}$ and $j \in \{-N, ..., N\}$ are pairwise distinct.*
2. *For any $k, \ell \in \{1, ..., r\}$, with $k < \ell$, we have $p_\ell \prec_{K_\omega} p_k$.*
3. *We have $z_0 \prec_{\hat{K}_\omega} p_r$.*
4. *We have $p_1 \prec_{\hat{W}_A} A$.*
5. *The points $f^j(p_k)$ for $k \in \{2, ..., r\}$ and $j \in \{-N, ..., N\}$ are not in $\hat{W}_A$.*
6. *The union of A with the closure of the orbits of the points p_k is $\eta_0/2$-close to $\mathscr{X}_\omega$ for the Hausdorff distance.*

Proof. — By Proposition 6 and the remark which follows the proposition, there exists an invariant compact set $A \subset \mathscr{X}_\omega$ which supports a minimal dynamics and such that $x \prec_{\mathscr{X}_\omega} y$ for any $x \in \mathscr{X}_\omega$ and $y \in A$. Since $\mathscr{X}_\omega$ is not minimal, we have $A \neq \mathscr{X}_\omega$. As in Proposition 5, there exists a finite subset $\{p_2, ..., p_{r-1}\} \subset \mathscr{X}_\omega \setminus A$ of points that are non-periodic, have distinct orbits and such that the union of A with the orbits of the points $p_2, ..., p_{r-1}$ has a closure which is $\eta_0/2$-close to $\mathscr{X}_\omega$ for the Hausdorff distance. Since $\mathscr{X}_\omega$ is a weak orbit whose relation $\prec_{\mathscr{X}_\omega}$ is transitive, one may order the points such that $p_\ell \prec_{\mathscr{X}_\omega} p_k$ when $k < \ell$.

We also have $z_0 \prec_{\hat{K}_\omega} p_{r-1}$. As in Lemma 6, one deduces that there exists a point $p_r \in (\mathscr{X} \setminus \mathscr{X}_\omega) \cap W_\omega$ such that $z_0 \prec_{\hat{K}_\omega} p_r$, and such that $p_r \prec_{K_\omega} p_{r-1}$ where K_ω is a compact set contained in $W_\omega \cap \mathscr{X}$ that contains $\mathscr{X}_\omega$ (for instance, K_ω can be the intersection of $\mathscr{X}$ with any compact neighborhood of $\mathscr{X}_\omega$). If p_r has been chosen close enough to $\mathscr{X}_\omega$, it is not a periodic point with period less than or equal to $2N$, by Condition (A).

Let $\hat{W}_A$ be a neighborhood of A, contained in W_ω and that is disjoint from the points the points $f^j(p_k)$ for $k \in \{2, ..., r\}$ and $j \in \{-N, ..., N\}$. We have $p_2 \prec_{cX_\omega} A$ by definition of A. By Lemma 6, there exists a point p_1 in $\hat{W}_A$ such that $p_2 \prec_{\mathscr{X}_\omega} p_1$ and $p_1 \prec_{\hat{W}_A} A$. Moreover, if the point p_1 is chosen close enough to A, all the points $f^j(p_k)$ with $k \in \{1, ..., r\}$ and $j \in \{-N, ..., N\}$ are pairwise distinct. □

Lemma **8.** — *There exist a constant $\varepsilon > 0$ and two neighborhoods $V_r \subset U_r$ of p_r in $B(p_r, \varepsilon)$, such that:*

1. *The balls $f^j(B(p_k, \varepsilon))$ for $k \in \{1, ..., r\}$ and $j \in \{-N, ..., N\}$ are pairwise disjoint, disjoint from A, and, when $k > 1$, disjoint from $\hat{W}_A$.*
2. *The closing lemma may be applied to $(\mathscr{U}, f)$ in the neighborhoods (V_r, U_r).*
3. *Let g be any perturbation of f with support in the balls $f^j(B(p_k, \varepsilon))$ (with $k \in \{1, ..., r\}$ and $j \in \{-N, ..., N\}$) and in the ε-neighborhood of A. Then, the η_0-neighborhood of any segment of orbit by g that intersects all the balls $B(p_k, \varepsilon)$ and the ε-neighborhood of A contains $\mathscr{X}_\omega$.*
4. *In the ε-neighborhood of $\mathscr{X}$, there is no segment of orbit $(z, f(z), ..., f^n(z))$ such that z belongs to some ball $B(p_k, \varepsilon)$, with $k \in \{1, ..., r-1\}$ and $f^n(z)$ to the ball $B(p_r, \varepsilon)$.*

Proof. — Let $\varepsilon > 0$ be a small constant. By items 1) and 5) of Lemma 7, for ε small enough, the item 1) of the lemma is satisfied. Moreover, let g be a perturbation of f with support in the balls $f^j(B(p_k, \varepsilon))$ and in the ε-neighborhood of A. Since ε is small, any segment of orbit of g that intersects all the balls $B(p_k, \varepsilon)$ and the ε-neighborhood of A contains in its $\eta_0/2$-neighborhood the orbits of the points p_k, with $k \in \{1, ..., r-1\}$ by f. Since the dynamics of f on A is minimal, it contains also in its $\eta_0/2$-neighborhood the set A. By item 6) of Lemma 7, one deduces that the η_0-neighborhood of this segment of orbit contains $\mathscr{X}_\omega$. One gets item 3).

The item 4) is verified for ε small enough, otherwise one would have $p_k \prec_{\mathscr{X}} p_r$ for some $k \in \{1, ..., r-1\}$ but since the points $p_1, ..., p_{r-1}$ belong to $\mathscr{X}_\omega$ and p_r to $\mathscr{X} \setminus \mathscr{X}_\omega$, this would contradict the assumption (*) of Proposition 21. It remains to apply Hayashi's connecting lemma (Theorem 5) at p_r in order to build the neighborhoods V_r, U_r, contained in $B(p_r, \varepsilon)$. $\square$

We now finish to prove Proposition 21 (and Theorem 2).

End of the proof of Proposition 21. — Let W be the ε-neighborhood of $\mathscr{X}$. By item 3) of Lemma 7, there exists a finite segment of orbit $Z_a = (z_\omega, f(z_\omega), ..., f^{n_a}(z_\omega))$ which is contained in $W \cap \hat{W}_\omega$ and such that z_ω belongs to U_{z_0} and $f^{n_a}(z_\omega)$ to V_r. By item 4) of Lemma 8, Z_a does not intersect the balls $B(p_k, \varepsilon)$ for $k \in \{1, ..., r-1\}$.

Let $\eta > 0$ be a constant smaller than ε and such that $B(p_r, \eta)$ is contained in V_r. By items 1) and 2) of Lemma 7, one can apply Proposition 9 to f^{-1} for the set $\{p_1, ..., p_r\}$ and the open set $W \cap W_\omega$: there is a perturbation $g_b \in \mathscr{U}$ of f with support in

$$\bigcup_{1 \le k \le r} \bigcup_{1 \le j \le N} f^{-j}(B(p_k, \eta))$$

and some integer $n_b \ge 1$ such that $Z_b = (g_b^{-n_b}(p_1), g_b^{-n_b+1}(p_1), ..., p_1)$ is a finite segment of orbit which is contained in $W \cap W_\omega$ and which crosses all the balls $B(p_k, \eta)$.

Let us denote by $m_b \in \{0, ..., n_b\}$ the smallest integer such that $g_b^{-m_b}(p_1)$ belongs to $\mathrm{B}(p_r, \varepsilon)$. The iterates $(g_b^{-n_b}(p_1), ..., g_b^{-m_b}(p_1))$ can not intersect the other balls $\mathrm{B}(p_k, \varepsilon)$ with $k \in \{1, ..., r-1\}$ by item 4) of Lemma 8. Hence, the orbit $(g_b^{-m_b}(p_1), ..., p_1)$ crosses all the balls $\mathrm{B}(p_k, \varepsilon)$ with $k \in \{1, ..., r-1\}$. By the remark at the end of Section 5.1 we deduce that it was not necessary to perturb f close to the point p_r and its N first backward iterates in order to build the orbit Z_b of g_b. Thus, one can assume that the support of the perturbation g_b is contained in the union of the balls $\mathrm{B}(p_k, \varepsilon)$ with $k \in \{0, ..., r-1\}$ and their N first backward iterates. By item 1) of Lemma 8, this support is disjoint from U_r and its N backward iterates.

One chooses now at $f(p_1)$ a small neighborhood $\mathrm{U} \subset \mathrm{B}(f(p_1), \varepsilon)$ whose N first forward iterates by f are disjoint from A and from the segments of orbit Z_a by f and Z_b by g_b. One considers a smaller neighborhood $\mathrm{V} \subset \mathrm{U}$ of $f(p_1)$ such that the connecting lemma can be applied for $(f, \mathcal{U})$ in $\mathrm{V} \subset \mathrm{U}$ with the time N.

Let us introduce a neighborhood $\mathrm{W}_\mathrm{A} \subset \hat{\mathrm{W}}_\mathrm{A}$ of A which is contained in the ε-neighborhood of A and which is disjoint from the sequences Z_a and Z_b, from the set U and its N first forward iterates and (by item 1) of Lemma 8) from the balls $f^{-j}(\mathrm{B}(p_k, \varepsilon))$ with $j \in \{0, ..., \mathrm{N}\}$ and $k \in \{1, ..., r\}$. By Proposition 10 and item 4) of Lemma 7, there exists a perturbation $g_c \in \mathcal{U}$ of f in $\mathcal{U}$ whose support is contained in $\mathrm{W}_\mathrm{A} \setminus \mathrm{A}$ and a point p_0 in $f^\mathrm{N}(\mathrm{V})$ whose forward orbit $\mathrm{Z}_c = \{g_c^n(p_0), n \geq 0\}$ by g_c is contained in $\hat{\mathrm{W}}_\mathrm{A}$ and accumulates on a subset of A. Moreover, Z_c is disjoint from the support of g_b by item 1) of Lemma 8.

The perturbation g_b and g_c have disjoint supports, so that the composed perturbation $\hat{g}$ belongs to $\mathcal{U}$. By construction, the sequences Z_a, Z_b and Z_c remain segments of orbits of $\hat{g}$. One can now apply the connecting lemma a first time to $\hat{g}$ in the sets U, V between the points $\hat{g}^{-n_b}(p_1)$ and $\hat{g}^{\mathrm{N}+1}(p_0)$ and then a second time to $\hat{g}^{-1}$ in the sets $\mathrm{V}_r, \mathrm{U}_r$ between the points z_ω and p_1 (by item 2) of Lemma 8). The supports of these two perturbation are disjoint and disjoint from the support of the perturbation $\hat{g}$ of f. Hence, by composition one obtains a diffeomorphism g_ω in $\mathcal{U}$.

Note that the segments of orbit $(g_b^{-m_b}(p_1), ..., p_1)$ by $\hat{g}$ was not modified by these last perturbations. Moreover, it now belongs to the forward orbit of z_ω. In particular, the point $g_b^{-m_b}(p_1)$ is now an iterate $g_\omega^{n_\omega}(z_\omega)$ of z_ω. This also shows that the forward orbit of z_ω by g_ω intersects all the balls $\mathrm{B}(p_k, \varepsilon)$. By construction, the forward orbit of z_ω by g_ω coincides after a large iterate with the forward orbit of an iterate of Z_c; hence, the forward orbit of z_ω by g_ω intersects the ε-neighborhood of A. By Lemma 8, the ε-neighborhood of this half orbit contains the set $\mathscr{X}_\omega$. On the other hand, by our constructions, the forward orbit of z_ω by g_ω is contained in $\hat{\mathrm{W}}_\omega$ and the forward orbit of $g_\omega^{n_\omega}(z_\omega)$ is contained in W_ω (which is contained in the η_0-neighborhood of $\mathscr{X}_\omega$). Hence, the closure of the forward orbit of $g_\omega^{n_\omega}(z_\omega)$ by g_ω is η_0-close to $\mathscr{X}_\omega$ for the Hausdorff distance. This concludes the proof of Proposition 21. $\square$

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