PUBLICATIONS MATHÉMATIQUES DE L'I.H.É.S.

HOWARD GARLAND

The arithmetic theory of loop groups

Publications mathématiques de l'I.H.É.S., tome 52 (1980), p. 5-136
<http://www.numdam.org/item?id=PMIHES_1980__52__5_0>

THE ARITHMETIC THEORY OF LOOP GROUPS

by Howard GARLAND

To my father, Max Garland, on the occasion of his seventieth birthday

0.1. Introduction.

We let R denote a commutative ring with unit, and $\mathscr{L}_R$ the ring of all formal Laurent series

$$\sum_{i \geq i_0} a_i t^i, \qquad a_i \in R,$$

with coefficients in R. We let k denote a field, so that $\mathscr{L}_k$ is also a field. For any commutative ring R, with unit, we let R^* denote the units in R. We let $G_{\mathscr{L}_k}$ denote the group of $\mathscr{L}_k$-rational points of a simply connected Chevalley group G, which we take to be simple. We let

$$c_T : \mathscr{L}_k^* \times \mathscr{L}_k^* \to k^*$$

denote the tame-symbol (see § 12, (12.20)). Then as one knows from the work of Matsumoto, Moore, and Steinberg (see [13], [18], and [20]), there is, corresponding to c_T^{-1}, a central extension

$$(0.1) \qquad 1 \to k^* \to G_k^T \to G_{\mathscr{L}_k} \to 1,$$

of $G_{\mathscr{L}_k}$. Our first goal in this paper is to develop a representation theory for the group G_k^T (here, k may have arbitrary characteristic). These representations will be infinite-dimensional, with representation space defined over the field k. They are constructed using the representation theory of Kac-Moody Lie algebras (see [10], [11], [12], [15], [8], §§ 3, 6, and see § 3 of the present paper). Thus, let $B = (B_{ij})_{i,j=1,\dots,\ell}$ be a *symmetrizable Cartan matrix* and let $\mathfrak{g}(B)$ be the corresponding Kac-Moody Lie algebra (see § 3, for the definitions). Roughly speaking, one constructs $\mathfrak{g}(B)$ by mimicking the Serre presentation, with B in place of a classical Cartan matrix corresponding to a (finite-dimensional) semi-simple Lie algebra. In general, $\mathfrak{g}(B)$ is in fact infinite-dimensional.

Partially supported by NSF Grant #MCS76-10435.

Returning to the group G_k^T, one may then take the point of view that the Lie algebra of G_k^T is a $\mathfrak{g}(B)$, for a suitable choice of B. More precisely, let $\mathfrak{g}$ denote the Lie algebra of the Chevalley group G, and let Δ denote the set of all roots of $\mathfrak{g}$ (relative to a Cartan subalgebra $\mathfrak{h}$). Let $\alpha_1, \ldots, \alpha_\ell$ ($\ell = \text{rank } \mathfrak{g}$) denote the simple roots in $\mathfrak{g}$ (relative to some fixed order). For ease of description, we have assumed $\mathfrak{g}$ is in fact simple, and we let α_0 denote the highest root of Δ (relative to our fixed order). Let $\alpha_{\ell+1} = -\alpha_0$, and let $\widetilde{A}$ denote the $(\ell+1)\times(\ell+1)$ matrix $\widetilde{A} = (\widetilde{A}_{ij})_{i,j=1,\ldots,\ell+1}$, where $\widetilde{A}_{ij} = 2(\alpha_i, \alpha_j)(\alpha_i, \alpha_i)^{-1}$, $i, j = 1, \ldots, \ell+1$. We refer to such an $\widetilde{A}$ as an *affine Cartan matrix*, and note that an affine Cartan matrix is a symmetrizable Cartan matrix. Then the Lie algebra of G_k^T is a k-form of $\mathfrak{g}(\widetilde{A})$.

For k a field of characteristic zero, Kac and Moody initiated a representation theory for $\mathfrak{g}(B)$, B equal to a symmetrizable Cartan matrix (see § 6) (1). Moreover, in the special case when $B = \widetilde{A}$ is an affine Cartan matrix one can prove the existence of a Chevalley lattice for each of these Kac-Moody representations with dominant integral highest weight (see [7], § 11, and also § 6 of the present paper, which may be considered a continuation of [7]). This Chevalley lattice then allows one to develop a representation theory over an arbitrary field k, and in this way we obtain the desired representations for the group G_k^T.

Moreover, our theory being **Z**-rational, then even when $k = \mathbf{R}$ or $\mathbf{C}$ (2), we can use the existence of a Chevalley lattice to develop a theory of arithmetic subgroups $\widehat{\Gamma}$ of G_k^T, and prove the existence of a fundamental domain for $\widehat{\Gamma}$, using Siegel sets! (§§ 17-21). We will expand on this description later on in the introduction.

It should be mentioned at the outset that the present formulation of our theory substantially differs from the original version in an earlier manuscript, and owes a great deal to the insight of the referee. Thus, in the original version, given $G_{\mathscr{L}_k}$ (k an arbitrary field), the algebra $\mathfrak{g}(\widetilde{A})$ corresponding to $G_{\mathscr{L}_k}$, and a Kac-Moody representation π of $\mathfrak{g}(\widetilde{A})$, corresponding to a dominant integral highest weight λ (see § 6 for the definition), we constructed a central extension $\widehat{G}_k^\lambda$ of a quotient of $G_{\mathscr{L}_k}$. The referee explicitly computed the symbol of $\widehat{G}_k^\lambda$ and found it to be a power of the tame symbol! (see § 12, Theorem (12.24)). Incidentally, one can always use $\widehat{G}_k^\lambda$ to construct a central extension $E^\lambda(G_{\mathscr{L}_k})$ of $G_{\mathscr{L}_k}$, rather than of a quotient of $G_{\mathscr{L}_k}$, by taking a suitable fiber product.

The referee also gave an elegant cohomological interpretation of the Kac-Moody algebra $\mathfrak{g}(\widetilde{A})$ (§§ 1-3, below). Thus, let k be a field of characteristic zero, and let $k[t, t^{-1}]$ be the ring of finite Laurent polynomials

$$\sum_{i=i_0}^{i_1} a_i t^i, \qquad a_i \in k,$$

(1) It was in this context that V. Kac proved his celebrated character formula (see [11]).

(2) Throughout this paper we let **Z**, **Q**, **R**, and **C** denote the rational integers, rational numbers, real numbers, and complex numbers, respectively.

where $i_0 \leq i_1$ are integers. Let $\tilde{\mathfrak{g}} = k[t, t^{-1}] \otimes_k \mathfrak{g}$, regarded as a Lie algebra over k (so $\tilde{\mathfrak{g}}$ is an infinite-dimensional Lie algebra over k: we call it the *loop algebra* (of $\mathfrak{g}$)). Then the referee proved that $H^2(\tilde{\mathfrak{g}}, k)$, the 2nd Lie algebra cohomology of $\tilde{\mathfrak{g}}$ with respect to trivial action on k, has dimension one. The referee's proof is given in § 2. The idea is to compute the "symbol" of an arbitrary central extension; i.e., to develop a Lie algebra analogue for the theory of Matsumoto, Moore, and Steinberg ([13], [18] and [20]). The Lie algebra version is simpler.

On the other hand Kac and Moody knew that $\mathfrak{g}(\tilde{A})$ was a central extension of $\tilde{\mathfrak{g}}$ ($\mathfrak{g}$ simple) with one-dimensional center. It then follows easily that $\mathfrak{g}(\tilde{A})$ is the *universal covering* $\hat{\mathfrak{g}}$ (see § 1, Definition (1.6)) of $\tilde{\mathfrak{g}}$ (see § 3, Theorem (3.14)). Also, if one replaces $\tilde{\mathfrak{g}}$ by $\tilde{\mathfrak{g}}^c = \mathfrak{g} \otimes_k \mathscr{L}_k$, then (as observed by the referee) the argument of § 2 can be adopted to construct a universal covering $\hat{\mathfrak{g}}^c \supset \hat{\mathfrak{g}}$ of $\tilde{\mathfrak{g}}^c$, but now with $\hat{\mathfrak{g}}^c$ universal in an appropriate category (see § 5, Remarks (5.11)). The universality of $\hat{\mathfrak{g}}^c$ then allows one, by Galois descent, to construct a central extension $\hat{\mathfrak{l}}$ of a semi-simple Lie algebra $\mathfrak{l}$ over $\mathscr{L}_k$, which splits over an unramified Galois extension $\mathscr{L}_{k'}$ of $\mathscr{L}_k$ (see Theorem (C24) of Appendix III, and see the Remark following Theorem (C24)).

Next, consider the group of $\mathscr{L}_k$-rational points L of a semi-simple, simply connected, linear algebraic group **L** which is defined over $\mathscr{L}_k$, and which splits over the unramified extension $\mathscr{L}_{k'}$ of $\mathscr{L}_k$. We let $G_{\mathscr{L}_{k'}}$ denote the $\mathscr{L}_{k'}$ rational points of **L**. Analogous to the situation for Lie algebras, one might ask whether one may construct from the central extension $G_{k'}^T$ of $G_{\mathscr{L}_{k'}}$, a central extension $\hat{L}$ of L, by Galois descent. Of course one would expect that such a construction could be effected from a suitable universal property of the tame symbol. Indeed, when k' is a finite field, one could probably construct $\hat{L}$ by using Moore's theorem on continuous K_2 of $\mathscr{L}_{k'}$ (see, e.g., Milnor [14], Appendix). However, in Appendix III of this paper, we rather consider the case when $\operatorname{char} k = 0$, and then we take a different approach: In order to construct the group $\hat{L}$, we utilize the universality property of $\hat{\mathfrak{g}}^c$, and we utilize the Kac-Moody representations of $\mathfrak{g}(\tilde{A})$, which correspond to dominant integral highest weights (these representations extend from $\mathfrak{g}$ to $\hat{\mathfrak{g}}^c$). However, we must now allow $\mathfrak{g}$ to be semi-simple and then give a suitable definition for $\tilde{A}$.

Now we had said that we develop a representation theory for the groups G_k^T. Indeed, for each dominant integral highest weight λ, we obtain such a representation, and we obtain a corresponding image $\hat{G}_k^\lambda$ of G_k^T (see Definition (7.21) for the definition of $\hat{G}_k^\lambda$). We define an Iwahori subgroup $\mathscr{I} \subset \hat{G}_k^\lambda$ (see Definition (7.24)). *Roughly speaking*, the subgroup $\mathscr{I}$ corresponds to the pullback of an Iwahori subgroup of $G_{\mathscr{L}_k}$ (see [9], [5], and the remark at the end of § 7).

In §§ 11, 13, 14, we construct a Tits system $(\hat{G}_k^\lambda, \mathscr{I}, N, \mathbf{S})$ (see Theorem (14.10)). Again, roughly speaking, this Tits system is the pullback of the Tits system of [9] (though our N is smaller).

In §§ 9, 10, 12, we obtain explicit information about the group $\hat{G}_k^\lambda$, considered as a central extension of the $\mathscr{L}_k$-rational points of a classical Chevalley group. In particular, in § 12, we associate to $\hat{G}_k^\lambda$, a central extension $E^\lambda(G_{\mathscr{L}_k})$ of $G_{\mathscr{L}_k}$, and compute the symbol of this central extension (see Theorem (12.24)). This symbol turns out to be a power of the tame symbol. It follows easily from this, and from the definition of the group $E^\lambda(G_{\mathscr{L}_k})$, that we obtain a homomorphism from G_k^T onto $\hat{G}_k^\lambda$, i.e., a representation of G_k^T. In § 15 we compare the $\hat{G}_k^\lambda$ as λ varies.

Now let $k=\mathbf{R}$ or $\mathbf{C}$ (from § 16 on, with the exception of the appendices, we pretty much restrict to the case when $k=\mathbf{R}$ or $\mathbf{C}$). Let V_k^λ denote the representation space corresponding to the dominant integral highest weight λ for $\mathfrak{g}(\widetilde{A})$. Then V_k^λ has a Chevalley lattice $V_{\mathbf{Z}}^\lambda$ (so in particular, $V_{\mathbf{Z}}^\lambda \subset V_k^\lambda$ is a $\mathbf{Z}$-submodule such that $V_k^\lambda = k\otimes_{\mathbf{Z}} V_{\mathbf{Z}}^\lambda$) and V_k^λ has a positive-definite, Hermitian inner product $\{\ ,\ \}$ which is *coherent* with $V_{\mathbf{Z}}^\lambda$ in the sense that $\{v_1, v_2\}\in\mathbf{Z}$, for $v_1, v_2\in V_{\mathbf{Z}}^\lambda$ (see [7], §§ 11, 12, and see §§ 6, 9 of the present paper). We let $J=\mathbf{Z}$ when $k=\mathbf{R}$, and we let J be the ring of integers in a Euclidean, imaginary quadratic field, when $k=\mathbf{C}$ (see Remark (v) following the statement of Theorem (19.3)). We set $V_J^\lambda = J\otimes_{\mathbf{Z}} V_{\mathbf{Z}}^\lambda$ and let $\hat{\Gamma}$ (resp. $\hat{K}$) be the subgroup of $\hat{G}_k^\lambda$, consisting of all elements which leave V_J^λ (resp. $\{\ ,\ \}$) invariant. We take the point of view that $\hat{K}\subset\hat{G}_k^\lambda$ is the analogue of a maximal compact subgroup, and letting $\mathscr{I}$ play the role of a parabolic subgroup, we prove the existence of an Iwasawa decomposition $\hat{G}_k^\lambda = \hat{K}\mathscr{I}$, in § 16 (see Theorem (16.8) and Lemma (16.14)). We then use this Iwasawa decomposition to define the notion of a Siegel set (see Definition (19.2), and the definition of $\mathfrak{S}_\sigma^{\mathbf{R}}\,(\sigma>0)$, preceeding Lemma (20.10)).

Now in § 3 we introduce the degree derivation $D=D_{\ell+1}$ of $\mathfrak{g}(\widetilde{A})$. From § 6 we then know that D acts on each V_k^λ, and then in § 17 we define the automorphism e^{cD}, $c\in k$, of V_k^λ. It is apparent from the definitions of e^{cD} and $\hat{G}_k^\lambda$, that e^{cD} normalizes $\hat{G}_k^\lambda$, $c\in k$. Then using our Siegel set, we construct a fundamental domain for $\hat{\Gamma}$ acting on $\hat{G}_k^\lambda e^{-rD}$, $r>0$ (see Theorem (20.14)). Our proof of Theorem (20.14) is modeled on the proof of Theorem (1.6), of [1]. However, in the infinite dimensional case it takes extra work to prove the existence of minima. Actually we pass from $\hat{\Gamma}$ to a subgroup $\hat{\Gamma}_0$, to construct the fundamental domain in Theorem (20.14). Even for $\hat{\Gamma}_0$, our fundamental domain is not exact. However, for $\hat{\Gamma}_0$, one obtains a sharp description of the self intersections when r is sufficiently large (see Theorem (21.16) and its Corollary 1—these results are analogues of the Harish-Chandra finiteness theorem and of the theorem of "transformations at ∞", in the classical theory of fundamental domains). Our proof of Theorem (21.16) is related to that of the corresponding result in [1] (see e.g., Theorem (4.4) of [1]). However, we must now contend with the existence of infinitely many Bruhat cells.

We may give an alternative point of view for the above theory of fundamental domains. Thus, for $r>0$, let $\hat{\Gamma}_r = e^{-rD}\hat{\Gamma}_0 e^{rD}$. We may then take the point of view that we are constructing a fundamental domain for $\hat{\Gamma}_r$ acting on $\hat{G}_k^\lambda$ (see Corollary 1 to Theorem (21.16), and the paragraph preceding that Corollary). Finally, if

$$\widetilde{\mathfrak{S}} = e^{rD}\mathfrak{S}_{\sigma_0}^{\mathbf{R}}(r)e^{-rD},$$

where $\mathfrak{S}^{\mathbf{R}}_{\sigma_0}(r)$ is as in Corollary 1 to Theorem (21.16), then by that corollary and the paragraph preceding it, we have that $\widetilde{\mathfrak{S}}\hat{\Gamma}_0 = \hat{G}^{\lambda}_k$, and we obtain a sharp description of the self intersections. Moreover, we may naturally regard $\widetilde{\mathfrak{S}}$ as a Siegel set constructed with $\hat{K}_r = e^{rD}\hat{K}e^{-rD}$, $r > 0$, in place of $\hat{K}$.

The paper is organized as follows: In §§ 1-3 we construct the universal covering of $\tilde{\mathfrak{g}}$ when $\mathfrak{g}$ is simple; we show in § 3 (still assuming $\mathfrak{g}$ simple) that $\mathfrak{g}(\widetilde{A})$ is the universal covering of $\tilde{\mathfrak{g}}$. In § 4 we define the Chevalley form $\mathfrak{g}_{\mathbf{Z}}(A)$ of $\mathfrak{g}(\widetilde{A})$, as in [7]. In § 5 we introduce completions of the algebras $\tilde{\mathfrak{g}}$ and $\mathfrak{g}(\widetilde{A})$. In § 6 we introduce the Kac-Moody representation theory of Kac-Moody algebras, and the Chevalley lattice $V^{\lambda}_{\mathbf{Z}}$ of [7], in the $\mathfrak{g}(\widetilde{A})$-representation space V^{λ} corresponding to a dominant integral highest weight λ. In § 7 we define the Chevalley group $\hat{G}^{\lambda}_k$ (k a field of arbitrary characteristic, and λ a dominant integral highest weight). We also introduce the Iwahori subgroup $\mathscr{I} \subset \hat{G}^{\lambda}_k$. In § 8 we study the adjoint representation of $\hat{G}^{\lambda}_k$. In §§ 9, 10, and 12, we study the groups $\hat{G}^{\lambda}_k$ as central extensions of classical Chevalley groups, and explicitly compute the symbol. In §§ 11, 13, and 14, we construct the Tits system $(\hat{G}^{\lambda}_k, \mathscr{I}, N, \mathbf{S})$ (see Theorem (14.10)). The proof given here that $(\hat{G}^{\lambda}_k, \mathscr{I}, N, \mathbf{S})$ is a Tits system, is different from the proof we gave originally, and follows a suggestion of J. Tits. Thus, in § 11, we apply the results of [4], and construct a donnée radicielle with valuation. When $\operatorname{char} k = 0$, a similar (but on the surface, slightly different) Tits system had been constructed by Marcuson in [12]. Also, with some restriction on k, such a Tits system had been constructed by Moody and Teo for the adjoint group (see [17]). Both the construction of Marcuson and of Moody-Teo were valid in the context of general Kac-Moody Lie algebras, while our Tits system is valid only for Kac-Moody algebras corresponding to affine Cartan matrices. However, our construction complements that of Marcuson, in that we make no restriction on the field k, and it complements that of Moody and Teo, in that we make no restriction on k and work with nonadjoint groups.

In § 15 we study the relation among the $\hat{G}^{\lambda}_k$, as λ varies. Finally, in §§ 17-21, we construct the fundamental domain for the arithmetic group $\hat{\Gamma}_0$. In § 17 we prove the existence of minima of certain matrix coefficients of V^{λ}_k ($k = \mathbf{R}$ or $\mathbf{C}$) on $\hat{\Gamma}_0$ orbits. In § 21, we prove our theorem on self intersections (Theorem (21.16)). We mention that as a consequence of Theorem (21.16), $\hat{\Gamma}_0$ and $\hat{\Gamma}_r$ are not conjugate in $\hat{G}^{\lambda}_k$ ($k = \mathbf{R}$ or $\mathbf{C}$) for $r > 0$ sufficiently large (see Corollary (3) to Theorem (21.16)).

In Appendix I, we prove Lemma (11.2). Also, Lemma (A.1) of Appendix I is used to prove (17.1). In Appendix II, we consider the case when k is a finite field, and formulate a conjecture about the special representation of $G_{\mathscr{L}_k}$, relating this representation to a suitable V^{λ}_k (this conjecture is an analogue to the known theorem for finite Chevalley groups, relating the Steinberg representation to a highest weight module). (*Added in proof.* — J. Arnon recently proved a slightly modified version of this conjecture.)

As it stands, our theory is a theory for Chevalley groups $G_{\mathscr{L}_k}$, and their Lie algebras. In Appendix III we begin investigating how to extend our results to non-split groups. Thus, assume $\operatorname{char} k = 0$, and as before, let L be the group of $\mathscr{L}_k$-rational points of a linear algebraic group which is semi-simple and defined over $\mathscr{L}_k$. Assume further that L splits over an unramified Galois extension $\mathscr{L}_{k'}$ of $\mathscr{L}_k$. Let $G_{\mathscr{L}_{k'}}$ denote the group of $\mathscr{L}_{k'}$-rational points of L. Following a suggestion of the referee, we show (by universality!) that at the Lie algebra level, we can lift Galois automorphisms from the Lie algebra of $G_{\mathscr{L}_k}$ to that of the central extension G_k^T. We then use this Lie algebra result and representation theory, to obtain a similar result at the group level, and then, by Galois descent, construct a central extension $\hat{L}$ of L. Our method of construction of $\hat{L}$, here, is somewhat different from the method of construction proposed by the referee. As we mentioned earlier, one could probably also construct $\hat{L}$ when k is a finite field by using Moore's theorem (see Milnor [14], Appendix).

If $k = \mathbf{R}$ or $\mathbf{C}$, and if L is actually defined over k, then the referee has extended the Iwasawa decomposition of § 16 to L and to $\hat{L}$. Thus the stage is set for beginning the extension of our reduction theory for arithmetic groups to the non-split case.

It is now apparent that the restrictions on the field k vary in the course of the paper. In §§ 1, 2, 3, the meaning of k is always made explicit, and for the most part, we assume in these sections that $\operatorname{char} k = 0$. In §§ 4, 5, 6, the meaning of k is made explicit. In §§ 7, 8, and 10-15, *no restriction* is made *on the field* k (so k may have arbitrary characteristic). From § 16 through the final section, § 21, we assume k is $\mathbf{R}$ or $\mathbf{C}$. In Appendix I, no restriction is made on k. In Appendix II, we assume k is finite. In Appendix III, we assume $\operatorname{char} k = 0$.

As we have mentioned, we are indebted to the referee for communicating the universality results of §§ 1, 2, and 3, the relation in § 12, between the central extension $\hat{G}_k^\lambda$ and the tame symbol, and the implications these last results have for extending our results to the non-split case. We are also indebted to J. Tits for a long and detailed correspondence and many suggestions. We extend to both, our hearty thanks. Also, we wish to thank D. Belli for her patient and painstaking efforts in typing the manuscript.

1. General remarks on central extensions of Lie Algebras.

In this section we describe Lie algebra analogues of some of the results in Moore [18], Chapter I. At the outset we make the notational convention that $\mathbf{C}$, $\mathbf{R}$, $\mathbf{Q}$, and $\mathbf{Z}$ denote the fields of complex numbers, real numbers, rational numbers and the ring of rational integers, respectively. We let $\mathfrak{a}$ denote a Lie algebra over a field k. By a *central extension* of $\mathfrak{a}$ by a Lie algebra $\mathfrak{b}$, we mean an exact sequence of Lie algebras (over k):

$$(1.1) \qquad 0 \rightarrow \mathfrak{b} \rightarrow \mathfrak{e} \xrightarrow{\pi} \mathfrak{a} \rightarrow 0,$$

such that $\mathfrak{b}$ is in the center of $\mathfrak{e}$. If

$$0 \to \mathfrak{b}' \to \mathfrak{e}' \xrightarrow{\pi'} \mathfrak{a} \to 0 \tag{1.2}$$

is a second central extension of $\mathfrak{a}$, then by a morphism from the central extension (1.1) to the central extension (1.2), we mean a pair of Lie algebra homomorphisms (φ, ψ), $\varphi : \mathfrak{e} \to \mathfrak{e}'$, $\psi : \mathfrak{b} \to \mathfrak{b}'$, such that the diagram

$$\begin{array}{ccccc} \mathfrak{b} & \longrightarrow & \mathfrak{e} & & \\ & & & \searrow^{\pi} & \\ {\scriptstyle\psi}\downarrow & & {\scriptstyle\varphi}\downarrow & & \mathfrak{a} \\ & & & \nearrow_{\pi'} & \\ \mathfrak{b}' & \longrightarrow & \mathfrak{e}' & & \end{array} \tag{1.3}$$

is commutative.

We say a Lie algebra is *perfect*, if it is equal to its own commutator subalgebra.

Definition **(1.4)**. — *We call the central extension* (1.1) *a* covering *of* $\mathfrak{a}$, *in case* $\mathfrak{e}$ *is perfect. In this case, we will also call* $\mathfrak{e}$ *or* π *(or the pair* $(\mathfrak{e}, \pi)$*) a covering of* $\mathfrak{a}$, *and we will say that* $\mathfrak{e}$ *(or the pair* $(\mathfrak{e}, \pi)$*) covers* $\mathfrak{a}$.

Remark. — If the Lie algebra $\mathfrak{a}$ admits a covering, then $\mathfrak{a}$ is perfect.

Lemma **(1.5)**. — *If* (1.1) *is a covering of* $\mathfrak{a}$, *then there is at most one morphism from the central extension* (1.1) *to a second central extension of* $\mathfrak{a}$.

Proof. — Assume (φ, ψ) and (φ', ψ') are morphisms from the central extension (1.1) to the central extension (1.2). Thus, in particular, we also have a commutative diagram (1.3) with (φ', ψ') in place of (φ, ψ). For $x, y \in \mathfrak{e}$, we consider

$$\begin{aligned}(\varphi - \varphi')([x,y]) &= \varphi([x,y]) - \varphi'([x,y]) \\ &= [\varphi(x), \varphi(y)] - [\varphi'(x), \varphi'(y)] \\ &= [\varphi(x) - \varphi'(x), \varphi(y)] + [\varphi'(x), \varphi(y) - \varphi'(y)] \\ &= 0,\end{aligned}$$

where the last equality follows from the fact that $\varphi(z) - \varphi'(z) \in \mathfrak{b}'$, for all $z \in \mathfrak{e}$, this last assertion following from our assumption that (1.3) is commutative both for (φ, ψ) and for (φ', ψ').

Definition **(1.6)**. — *We say that a covering of* $\mathfrak{a}$ *is* universal, *if for every central extension of* $\mathfrak{a}$, *there is a* unique *morphism (in the sense of central extensions) from the covering to the central extension.*

Remark. — In view of Lemma (1.5), it suffices, in order to verify that a given covering is universal, to show that for every central extension of $\mathfrak{a}$, *there exists* a morphism from the given covering to the central extension. Uniqueness then follows automatically from Lemma (1.5).

The following proposition is an immediate consequence of Definition (1.6):

Proposition **(1.7).** — *Any two universal coverings of* $\mathfrak{a}$ *are isomorphic as central extensions.*

Of course, by Lemma (1.5), the isomorphism is unique.

Now let V be a vector space over k, and let $Z^2(\mathfrak{a}, V)$ denote the vector space over k, of all skew-symmetric, bilinear maps

$$f : \mathfrak{a} \times \mathfrak{a} \to V,$$

such that

(1.8) $$f([x,y], z) - f([x, z], y) + f([y, z], x) = 0 \qquad x, y, z \in \mathfrak{a}.$$

We let $B^2(\mathfrak{a}, V)$ denote the vector space of all f, as above, such that there exists a linear map $g : \mathfrak{a} \to V$, with

$$f(x,y) = g([x,y]), \qquad x, y \in \mathfrak{a}.$$

Then the Jacobi identity implies that $B^2(\mathfrak{a}, V) \subset Z^2(\mathfrak{a}, V)$, and we set

$$H^2(\mathfrak{a}, V) = Z^2(\mathfrak{a}, V)/B^2(\mathfrak{a}, V).$$

We call $Z^2(\mathfrak{a}, V)$ (resp. $B^2(\mathfrak{a}, V)$, resp. $H^2(\mathfrak{a}, V)$) the *space of* 2-*cocycles* (resp. 2-*coboundaries*; resp. the *second cohomology group* of $\mathfrak{a}$) with respect to V (regarded as a trivial $\mathfrak{a}$-module). It is well known that $H^2(\mathfrak{a}, V)$ parametrizes (suitably defined) equivalence classes of central extensions

$$0 \to V \to \mathfrak{e} \to \mathfrak{a} \to 0.$$

We briefly recall how one constructs such a central extension, given an element f in $Z^2(\mathfrak{a}, V)$. We let

$$\mathfrak{a}_f = \mathfrak{a} \oplus V,$$

and if $\xi = (x, v)$, $\eta = (y, w) \in \mathfrak{a}_f$ (so $x, y \in \mathfrak{a}$, $v, w \in V$) we define the bracket $[\xi, \eta] \in \mathfrak{a}_f$ by

(1.9) $$[\xi, \eta] = ([x,y], f(x,y)).$$

It follows directly from the cocycle identity (1.8) that this bracket satisfies the Jacobi identity, and so with this bracket, $\mathfrak{a}_f$ is a Lie algebra, and the exact sequence

$$0 \to V \xrightarrow{\iota} \mathfrak{a}_f \xrightarrow{\tilde{\omega}} \mathfrak{a} \to 0,$$

where ι is the inclusion, and $\tilde{\omega}$ the projection from $\mathfrak{a}_f$ onto its first factor, is a central extension of $\mathfrak{a}$.

In the next section we shall give an explicit construction of the universal covering, in a special case. We have already noted that if the Lie algebra $\mathfrak{a}$ admits a covering, then $\mathfrak{a}$ is perfect. Conversely, we have:

Lemma **(1.10)**. — *If* $\mathfrak{a}$ *is perfect, then* $\mathfrak{a}$ *has a universal covering.*

Proof. — We let $W'=\Lambda^2\mathfrak{a}$ denote the second exterior power of $\mathfrak{a}$, and we let $I\subset W'$ denote the subspace spanned by all elements

$$[x,y]\wedge z-[x,z]\wedge y+[y,z]\wedge x, \qquad x,y,z\in\mathfrak{a}.$$

We then set $W=W'/I$, and let $\alpha(x,y)\in W$ denote the image of $x\wedge y\in W'$. From the definition of I, one immediately sees that α satisfies the cocycle identity (1.8); i.e., $\alpha\in Z^2(\mathfrak{a}, W)$.

We let

(1.11) $$0\to W\to\mathfrak{a}_\alpha\to\mathfrak{a}\to 0$$

denote the corresponding central extension.

If $f\in Z^2(\mathfrak{a}, V)$ and

(1.12) $$0\to V\to\mathfrak{a}_f\to\mathfrak{a}\to 0$$

is another central extension of $\mathfrak{a}$, we consider the map $\psi': W\to V$ defined by $\psi'(\alpha(x,y))=f(x,y)$, $x,y\in\mathfrak{a}$. We then define $\varphi':\mathfrak{a}_\alpha\to\mathfrak{a}_f$ by $\varphi'(x,u)=(x,\psi(u))$, $x\in\mathfrak{a}$, $u\in W$.

It is easily checked that (φ',ψ') is a morphism from the central extension (1.11) to the central extension (1.12).

Now let $\hat{a}=[\mathfrak{a}_\alpha,\mathfrak{a}_\alpha]$ denote the commutator subalgebra of $\mathfrak{a}_\alpha$. Since $\mathfrak{a}$ is perfect, we have

$$\hat{\mathfrak{a}}+W=\mathfrak{a}_\alpha,$$

and hence $\hat{\mathfrak{a}}=[\mathfrak{a}_\alpha,\mathfrak{a}_\alpha]=[\hat{\mathfrak{a}},\hat{\mathfrak{a}}]$.

Thus, if $\mathfrak{c}=W\cap\hat{\mathfrak{a}}$, then the central extension

(1.13) $$0\to\mathfrak{c}\to\hat{\mathfrak{a}}\to\mathfrak{a}\to 0$$

is a covering of $\mathfrak{a}$, and if

$$\psi=\psi' \text{ restricted to } \mathfrak{c}$$
$$\varphi=\varphi' \text{ restricted to } \mathfrak{a}$$

then (φ,ψ) is a morphism from this central extension to the central extension (1.12). This proves that the central extension (1.13) is a universal covering of $\mathfrak{a}$. ■

The remainder of this section will not be needed in the sequel, but is included to complete the analogy with some of the results in Moore [18]. Thus:

Definition **(1.14)**. — *We will say a Lie algebra* $\mathfrak{a}$ *is* simply connected, *in case it is true that for every central extension*

$$0\to\mathfrak{b}\to\mathfrak{e}\xrightarrow{\pi}\mathfrak{a}\to 0,$$

of $\mathfrak{a}$, *there is a* unique *homomorphism* $\varphi:\mathfrak{a}\to\mathfrak{e}$, *such that* $\pi\circ\varphi=$ *identity.*

Definition **(1.15)**. — *If* $\mathfrak{a}$ *is a Lie algebra and*

$$0 \to \mathfrak{b} \to \mathfrak{e} \xrightarrow{\pi} \mathfrak{a} \to 0$$

a covering of $\mathfrak{a}$, *with* $\mathfrak{e}$ *simply connected, then we call this a* simply connected covering *of* $\mathfrak{a}$.

Theorem **(1.16)**. — *A covering of* $\mathfrak{a}$ *is universal if and only if it is simply connected.*

Proof. — Let

$$\textbf{(1.17)} \qquad 0 \to \mathfrak{c} \to \hat{\mathfrak{a}} \xrightarrow{\tilde{\omega}} \mathfrak{a} \to 0$$

be a simply connected covering of $\mathfrak{a}$, and let us consider a central extension (1.1) of $\mathfrak{a}$. We may "lift" this central extension to $\hat{\mathfrak{a}}$ as follows: We let $\mathfrak{e}(a) \subset \mathfrak{e} \times \hat{\mathfrak{a}}$ denote the subalgebra of all $(x, y) \in \mathfrak{e} \times \hat{\mathfrak{a}}$, such that $\pi(x) = \tilde{\omega}(y)$. The projection of $\mathfrak{e} \times \hat{\mathfrak{a}}$ onto the second factor, induces a surjective homomorphism

$$\mathfrak{e}(\mathfrak{a}) \xrightarrow{\rho} \hat{\mathfrak{a}} \to 0,$$

and it is easily checked that kernel ρ is contained in the center of $\mathfrak{e}(\mathfrak{a})$. Thus, since $\hat{\mathfrak{a}}$ is simply connected, there is a unique homomorphism $\varphi : \hat{\mathfrak{a}} \to \mathfrak{e}(\mathfrak{a})$, such that $\varphi \circ \rho = \text{identity}$. We let $\lambda : \hat{\mathfrak{a}} \to \mathfrak{e}$, denote the composition of φ with the projection of $\mathfrak{e}(\mathfrak{a})$ onto the first factor of $\mathfrak{e} \times \mathfrak{a}$. It is easily checked that $\lambda(\mathfrak{c}) \subset \mathfrak{b}$, and hence, if we let μ denote the restriction of λ to $\mathfrak{c}$, then (λ, μ) is a morphism from (1.17) to (1.1); i.e., the covering (1.17) is universal (since the morphism is automatically unique, thanks to Lemma (1.5).

Conversely, assume (1.17) is a universal covering of $\mathfrak{a}$. Let

$$\textbf{(1.18)} \qquad 0 \to \mathfrak{d} \to \tilde{\mathfrak{e}} \xrightarrow{\vee} \hat{\mathfrak{a}} \to 0$$

be a central extension. Since $\hat{\mathfrak{a}}$ is perfect, Lemma (1.5) implies there is at most one morphism from the covering

$$0 \to 0 \to \hat{\mathfrak{a}} \to \hat{\mathfrak{a}} \to 0$$

to the central extension (1.18), and hence, to prove (1.17) is a simply connected covering, it suffices to prove (1.18) is a split exact sequence. Again using the assumption that $\hat{\mathfrak{a}}$ is perfect, and arguing as in the proof of Lemma (1.10), we can show that $\tilde{\mathfrak{e}}_0 = [\tilde{\mathfrak{e}}, \tilde{\mathfrak{e}}]$ is perfect. But if $\mathfrak{d}_0 = \tilde{\mathfrak{e}}_0 \cap \mathfrak{d}$, then the exact sequence (1.18) will split if the exact sequence

$$0 \to \mathfrak{d}_0 \to \tilde{\mathfrak{e}}_0 \to \hat{\mathfrak{a}} \to 0$$

is split; i.e., we can assume the exact sequence (1.18) is a covering. We now consider the diagram

$$\tilde{\mathfrak{e}} \xrightarrow{\vee} \hat{\mathfrak{a}} \xrightarrow{\tilde{\omega}} \mathfrak{a} \to 0,$$

and we set $\gamma=\tilde{\omega}\circ\nu$. We wish to show that kernel γ is contained in the center of $\tilde{\mathfrak{e}}$. But if $x\in\text{kernel}\,\gamma$, then $\nu(x)\in\text{center}\,\hat{\mathfrak{a}}$. Hence

$$(1.19) \qquad [x,y]\in\text{kernel}\,\nu, \quad y\in\tilde{\mathfrak{e}}.$$

Hence $\qquad [x,[y_1,y_2]]=0, \quad y_1,y_2\in\tilde{\mathfrak{e}},$

by the Jacobi identity, (1.19), and the fact that $\text{kernel}\,\nu\subset\text{center}\,\tilde{\mathfrak{e}}$. But then $x\in\text{center}\,\tilde{\mathfrak{e}}$, since we are assuming $\tilde{\mathfrak{e}}$ is perfect. Thus

$$(1.20) \qquad 0\to\text{kernel}\,\gamma\to\tilde{\mathfrak{e}}\xrightarrow{\gamma}\mathfrak{a}\to 0$$

is a central extension. Since we are assuming (1.17) is universal, there is a morphism (φ,ψ) from (1.17) to (1.20); i.e., we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{c} & \longrightarrow & \hat{\mathfrak{a}} & \xrightarrow{\tilde{\omega}} & \mathfrak{a} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\varphi} & \nearrow{\scriptstyle\gamma} & & & \\ 0 & \to & \text{kernel}\,\gamma & \to & \tilde{\mathfrak{e}} & & & & \end{array}$$

We will be done, if we show that $\nu\circ\varphi$ is the identity.

But by the commutativity of the above diagram

$$\tilde{\omega}\circ(\nu\circ\varphi)=\gamma\circ\varphi=\tilde{\omega},$$

and hence for $x\in\hat{\mathfrak{a}}$,

$$g(x)=\nu(\varphi(x))-x\in\mathfrak{c}.$$

If $x_1,x_2\in\hat{\mathfrak{a}}$, then

$$g([x_1,x_2])=\nu\circ\varphi([x_1,x_2])-[x_1,x_2]=[g(x_1),\nu\circ\varphi(x_2)]+[x_1,g(x_2)]=0,$$

since g takes values in $\mathfrak{c}$. But then g is identically zero, since $\hat{\mathfrak{a}}$ is perfect. Hence $\nu\circ\varphi=\text{identity}$. ■

Remarks. — (i) If $\mathfrak{a}$ is simply connected, then $\mathfrak{a}$ is perfect. Indeed, if $\mathfrak{a}$ is not perfect, there is a non-trivial linear map $f:\mathfrak{a}/[\mathfrak{a},\mathfrak{a}]\to k$, and hence, if we consider the trivial central extension

$$0\to k\to\mathfrak{a}\oplus k\to\mathfrak{a}\to 0,$$

with $k\to\mathfrak{a}\oplus k$ denoting the inclusion, and $\mathfrak{a}\oplus k\to\mathfrak{a}$ the projection, then, using f, there are two splitting homomorphisms, and this nonuniqueness contradicts the simple connectivity of $\mathfrak{a}$. (ii) If $\mathfrak{a}$ has a simply connected covering,

$$0\to\mathfrak{c}\to\hat{\mathfrak{a}}\to\mathfrak{a}\to 0,$$

then $\hat{\mathfrak{a}}$ covers every covering of $\mathfrak{a}$. Indeed, if

$$0\to\mathfrak{b}\to\mathfrak{e}\to\mathfrak{a}\to 0$$

is a covering of $\mathfrak{a}$, we can, since a simply connected covering is universal, find a morphism (φ, ψ) from the first central extension:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{c} & \longrightarrow & \hat{\mathfrak{a}} & \longrightarrow & \mathfrak{a} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\varphi} & \nearrow & & & \\ 0 & \longrightarrow & \mathfrak{b} & \longrightarrow & \mathfrak{e} & & & & \end{array}$$

We then need only check φ is surjective. But if $\mathfrak{e}_0 = \varphi(\hat{\mathfrak{a}})$, then the commutativity of the above diagram implies $\mathfrak{e} = \mathfrak{e}_0 + \mathfrak{b}$, and hence

$$\mathfrak{e} = [\mathfrak{e}, \mathfrak{e}] = [\mathfrak{e}_0, \mathfrak{e}_0] \subset \mathfrak{e}_0,$$

and so $\mathfrak{e} = \mathfrak{e}_0$ and φ is surjective.

2. An explicit construction for the universal covering of the loop Algebra.

We let $\mathfrak{g}$ denote a split simple Lie algebra over a field k of characteristic zero. Let $k[t, t^{-1}]$ (with t an indeterminate) denote the ring of polynomials in t and t^{-1}, with coefficients in k. We let

$$\tilde{\mathfrak{g}} = k[t, t^{-1}] \otimes_k \mathfrak{g},$$

we write ux for $u \otimes x$ $(u \in k[t, t^{-1}],\ x \in \mathfrak{g})$ and we define a Lie bracket on $\tilde{\mathfrak{g}}$ by

(2.1) $$[ux_1, vx_2] = uv \otimes [x_1, x_2],$$

where $u, v \in k[t, t^{-1}]$, $x_1, x_2 \in \mathfrak{g}$, and $[x_1, x_2]$ denotes the Lie bracket of x_1, x_2 in $\mathfrak{g}$. We regard $\tilde{\mathfrak{g}}$ with the bracket (2.1) as a Lie algebra over k (which of course is infinite-dimensional). We call $\tilde{\mathfrak{g}}$ the *loop algebra* of $\mathfrak{g}$. It is easy to verify that $\tilde{\mathfrak{g}}$ is perfect (since $\mathfrak{g}$ is) and hence $\tilde{\mathfrak{g}}$ has a universal covering. In this section, we shall give an explicit construction for the universal covering

$$0 \to \mathfrak{c} \to \hat{\mathfrak{g}} \to \tilde{\mathfrak{g}} \to 0$$

of $\tilde{\mathfrak{g}}$, and in particular, find $\dim_k \mathfrak{c} = 1$. To determine the covering, it suffices to give a corresponding cocycle $\tau \in Z^2(\tilde{\mathfrak{g}}, k)$. We must then show that the central extension constructed from τ, as in § 1, is the universal covering of $\tilde{\mathfrak{g}}$.

In order to define τ, we first define a k-bilinear pairing

$$\tau_0 : k[t, t^{-1}] \times k[t, t^{-1}] \to k.$$

Namely, we let

$$\tau_0(u, v) = \text{residue}\ (udv),$$

for $u, v \in k[t, t^{-1}]$. For example,

$$\tau_0(t^n, t^m) = \begin{cases} -n & \text{if } n + m = 0 \\ 0 & \text{otherwise.} \end{cases}$$

We have

(2.2) $$0 = \tau_0(u_1u_2, u_3) + \tau_0(u_3u_1, u_2) + \tau_0(u_2u_3, u_1), \qquad u_1, u_2, u_3 \in k[t, t^{-1}].$$

We let (,) denote the Killing form on $\mathfrak{g}$, and we define τ by

$$\tau(ux, vy) = -\tau_0(u, v)(x, y), \qquad u, v \in k[t, t^{-1}], \qquad x, y \in \mathfrak{g}.$$

A direct verification, using (2.1) and the invariance of the Killing form, shows that τ satisfies the cocycle identity (1.8), and hence $\tau \in Z^2(\tilde{\mathfrak{g}}, k)$. We let

$$0 \to k \to \hat{\mathfrak{g}}_\tau \xrightarrow{\tilde{\omega}} \tilde{\mathfrak{g}} \to 0$$

denote the corresponding central extension, constructed as in § 1. We shall now prove that this central extension is a universal cover of $\tilde{\mathfrak{g}}$.

First note that if

$$0 \to \mathfrak{b} \to \mathfrak{e} \xrightarrow{\pi} \mathfrak{a} \to 0$$

is a central extension of the Lie algebra $\mathfrak{a}$, if $x, y \in \mathfrak{a}$ and if $x', y' \in \mathfrak{e}$ are inverse images of x, y, respectively (so $\pi(x') = x$, $\pi(y') = y$) then $[x', y']$ depends only on x and y, and not on our choice of inverse images x', y'. We therefore denote $[x', y']$ by $[x, y]'$.

Now, let $\mathfrak{g} = \mathfrak{sl}_2(k)$, the Lie algebra of all 2×2 trace zero matrices, with coefficients in k. Let H, E_+, E_- be the Chevalley basis of $\mathfrak{sl}_2(k)$:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Let $$0 \to Z \to \tilde{\mathfrak{g}}' \xrightarrow{\rho} \tilde{\mathfrak{g}} \to 0$$

be an arbitrary central extension of $\tilde{\mathfrak{g}} = \widetilde{\mathfrak{sl}}_2$. For $u \in k[t, t^{-1}]$, we define elements $uE'_\pm$, uH' in $\tilde{\mathfrak{g}}'$ by

$$uE'_\pm = \pm \frac{1}{2}[H, uE_\pm]'$$

$$uH' = [E_+, uE_-]'.$$

When $u = 1$, we write $E'_\pm$, H' for $1E'_\pm$, $1H'$, respectively. We note that

(2.3) $$\rho(uH') = uH$$
$$\rho(uE'_\pm) = uE_\pm.$$

Thus for $u, v \in k[t, t^{-1}]$, we have

(2.4) $$[uH', vH'] = [uH, vH]' \in Z.$$

We set

(2.5) $$\{u, v\} = [uH', vH'] = [uH, vH]' \in Z.$$

Also, we have from (2.3) that for $u \in k[t, t^{-1}]$,

(2.6) $$uE'_\pm = \pm \frac{1}{2}[H', uE'_\pm].$$

Thus, substituting the above expression for $uE'_{\pm}$, and using the Jacobi identity, we have

$$[vH', uE'_{\pm}] = \pm 1/2([[vH', H'], uE'_{\pm}] + [H', [vH', uE'_{\pm}]])$$
$$= \pm 1/2[H', [vH', uE'_{\pm}]], \quad \text{by (2.4)}.$$

On the other hand, $[vH', uE'_{\pm}] = \pm 2uvE'_{\pm} \bmod Z$, and hence the last expression in the above computation equals

$$\pm 1/2[H', \pm 2uvE'_{\pm}].$$

But (2.6) implies that this equals $\pm 2uvE'_{\pm}$. Thus, we have shown

(2.7) $$[vH', uE'_{\pm}] = \pm 2uvE'_{\pm}, \qquad u, v \in k[t, t^{-1}].$$

Next, we consider (for $u, v \in k[t, t^{-1}]$):

$$[uE'_{+}, vE'_{-}] = \left[\left[\frac{1}{2}uH', E'_{+}\right], vE'_{-}\right], \quad \text{by (2.7)},$$
$$= \left[\left[\frac{1}{2}uH', vE'_{-}\right], E'_{+}\right] + \left[\frac{1}{2}uH', \left[E'_{+}, vE'_{-}\right]\right]$$
$$= [-uvE'_{-}, E'_{+}] + \left[\frac{1}{2}uH', vH'\right],$$
by (2.7) and the definition of vH',
$$= uvH' + \frac{1}{2}\{u, v\}, \quad \text{by the definition of } uvH', \text{(2.3) and (2.5)}.$$

Thus, we have shown

(2.8) $$[uE'_{+}, vE'_{-}] = uvH' + \frac{1}{2}\{u, v\}, \qquad u, v \in k[t, t^{-1}].$$

Next, for $u, v \in k[t, t^{-1}]$, we consider

$$[uE'_{\pm}, vE'_{\pm}] = \pm \frac{1}{2}[[H', uE'_{\pm}], vE'_{\pm}], \quad \text{by (2.7)},$$
$$= \pm \frac{1}{2}([[H', vE'_{\pm}], uE'_{\pm}] + [H', [uE'_{\pm}, vE'_{\pm}]]).$$

The first summand equals $[\pm 2vE'_{\pm}, uE'_{\pm}]$ by (2.7), and the second summand equals zero, since clearly $[uE'_{\pm}, vE'_{\pm}] \in Z$. Hence the last expression in the above computation equals $[vE'_{\pm}, uE'_{\pm}]$; i.e.,

$$[uE'_{\pm}, vE'_{\pm}] = -[uE'_{\pm}, vE'_{\pm}],$$

and hence, since $\operatorname{char} k = 0$,

(2.9) $$[uE'_{\pm}, vE'_{\pm}] = 0, \qquad u, v \in k[t, t^{-1}].$$

We summarize the multiplication rules, (2.5), (2.7), (2.8), and (2.9) in

(2.10) a) $[u\mathrm{H}', v\mathrm{H}'] = \{u, v\}$,

b) $[v\mathrm{H}', u\mathrm{E}'_{\pm}] = \pm 2uv\mathrm{E}'_{\pm}$,

c) $[u\mathrm{E}'_{+}, v\mathrm{E}'_{-}] = uv\mathrm{H}' + \frac{1}{2}\{u, v\}$,

d) $[u\mathrm{E}'_{\pm}, v\mathrm{E}'_{\pm}] = 0, \quad u, v \in k[t, t^{-1}]$.

Lemma **(2.11)**. — *The pairing* $\{\ ,\ \}$ *defines a skew symmetric, k-bilinear mapping from* $k[t, t^{-1}] \times k[t, t^{-1}]$ *to* k. *Moreover, relations* (2.10) *define a Lie algebra structure on* $\tilde{\mathfrak{g}}'$, *if and only if* $\{\ ,\ \}$ *satisfies the relations*

(2.12)
$$\{u, u\} = 0$$
$$\{uv, w\} + \{wu, v\} + \{vw, u\} = 0, \quad u, v, w \in k[t, t^{-1}].$$

Proof. — The bilinearity and skew symmetry of $\{\ ,\ \}$, and the relation $\{u, u\} = 0$, all follow from (2.5). To understand the second identity in (2.12), we consider $(u, v, w \in k[t, t^{-1}])$

$$[u\mathrm{H}', [v\mathrm{E}'_{\pm}, w\mathrm{E}'_{\mp}]] + [w\mathrm{E}'_{\mp}, [u\mathrm{H}', v\mathrm{E}'_{\pm}]] + [v\mathrm{E}'_{\pm}, [w\mathrm{E}'_{\mp}, u\mathrm{H}']]$$
$$= \pm\{u, vw\} \pm \{w, uv\} \pm \{v, uw\},$$

and hence the second relation of (2.12) is equivalent to the Jacobi identity for $u\mathrm{H}'$, $v\mathrm{E}'_{\pm}$, $w\mathrm{E}'_{\mp}$. As the remaining Jacobi identities follow from (2.10), we obtain the Lemma. ■

We now wish to study a skew symmetric, bilinear map

$$\{\ ,\ \} : k[t, t^{-1}] \times k[t, t^{-1}] \to k,$$

satisfying (2.12). First, for $u \in k[t, t^{-1}]$, we let

$$\mathrm{M}_u : k[t, t^{-1}] \to k[t, t^{-1}],$$

denote the multiplication operator

$$\mathrm{M}_u(v) = uv, \quad v \in k[t, t^{-1}].$$

We let $f_u : k[t, t^{-1}] \to k$ denote the k-linear transformation, defined by

$$f_u(v) = \{u, v\}, \quad v \in k[t, t^{-1}].$$

We easily deduce from (2.12) that

(2.13)
$$f_u \circ \mathrm{M}_v + f_v \circ \mathrm{M}_u = f_{uv}, \quad u, v \in k[t, t^{-1}].$$

Then, applying (2.13) when $u = 1$, we see that $f_1 = 0$. But if $u, u^{-1} \in k[t, t^{-1}]$, and if we set $v = u^{-1}$ in (2.13), we obtain

(2.14)
$$f_1 = 0$$
$$f_{u^{-1}} = -f_u \circ \mathrm{M}_{u^{-2}}, \quad u, u^{-1} \in k[t, t^{-1}].$$

Then, by (2.13), (2.14), and a straightforward induction, we obtain

(2.15) $f_{t^n} = f_t \circ M_{nt^{n-1}}, \quad n \in \mathbf{Z}.$

Thus, for $r, s \in \mathbf{Z}$, we have

$$\{t^r, t^s\} = f_{t^r}(t^s) = f_t \circ M_{rt^{r-1}}(t^s), \quad \text{by (2.15)},$$
$$= r\{t, t^{r+s-1}\}.$$

But, interchanging r and s, and using skew symmetry of $\{\ ,\ \}$, we then also have

$$\{t^r, t^s\} = -s\{t, t^{r+s-1}\}.$$

Thus, we have:

Lemma **(2.16)**. — *Let*

$$\{\ ,\ \} : k[t, t^{-1}] \times k[t, t^{-1}] \to k$$

be a skew symmetric, bilinear pairing, satisfying the relations (2.12). *Then we have*

(2.17) $\{t^r, t^s\} = \delta_{r,-s}\, r\{t, t^{-1}\},$

or equivalently

(2.18) $\{u, v\} = -\text{Residue}\,(udv)\{t, t^{-1}\}, \quad u, v \in k[t, t^{-1}].$

Remark. — In essence, Lemmas (2.11) and (2.16) imply that the central extension introduced earlier,

(2.19) $0 \to k \to \widehat{\mathfrak{g}}_\tau \to \widetilde{\mathfrak{g}} \to 0,$

is a universal cover of $\widetilde{\mathfrak{g}}$, in the case when $\mathfrak{g} = \mathfrak{sl}_2(k)$. Indeed, one need only check $\widehat{\mathfrak{g}}_\tau$ is perfect. We shall check this in general later on (in any case it is easy). We also remark that this central extension is therefore non-trivial.

We shall now assume $\mathfrak{g}$ split semi-simple over k, and we let

$$0 \to Z \to \widetilde{\mathfrak{g}}' \xrightarrow{\rho} \widetilde{\mathfrak{g}} \to 0$$

be an arbitrary central extension of $\widetilde{\mathfrak{g}}$. We consider the Lie algebra $\mathfrak{g}$, and fix a splitting Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. We let Δ denote the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$, and we let

$$\mathfrak{g} = \mathfrak{h} \oplus \coprod_{\alpha \in \Delta} \mathfrak{g}^\alpha$$

denote the root space decomposition (relative to $\mathfrak{h}$) of $\mathfrak{g}$. Thus $\Delta \subset \mathfrak{h}^*$, the dual space of $\mathfrak{h}$, and

$$\mathfrak{g}^\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x, \quad h \in \mathfrak{h}\}.$$

For each $\alpha \in \Delta$ we pick an element $E_\alpha \in \mathfrak{g}^\alpha$ and an element $H_\alpha \in \mathfrak{h}$, so that for all $\alpha \in \Delta$, we have

$$[E_\alpha, E_{-\alpha}] = H_\alpha$$
$$[H_\alpha, E_{\pm\alpha}] = \pm 2E_{\pm\alpha},$$

(*We will further normalize our choice of the* E_α's *in* § 4). We let $\mathfrak{g}(\alpha)$ denote the three dimensional subalgebra spanned by the elements $E_{\pm\alpha}$ and H_α, and we let $\tilde{\mathfrak{g}}(\alpha)$ be the subalgebra spanned by the elements ux, $u \in k[t, t^{-1}]$, $x \in \mathfrak{g}(\alpha)$. We let

$$\tilde{\mathfrak{g}}'(\alpha) = \rho^{-1}(\tilde{\mathfrak{g}}(\alpha)),$$

so that if we let ρ_α denote the restriction of ρ to $\tilde{\mathfrak{g}}'(\alpha)$, we have a central extension

$$0 \to Z \to \tilde{\mathfrak{g}}'(\alpha) \xrightarrow{\rho_\alpha} \tilde{\mathfrak{g}}(\alpha) \to 0. \qquad (2.20)$$

Now for $\alpha \in \Delta$, we have an isomorphism from $\mathfrak{sl}_2(k)$ onto $\mathfrak{g}(\alpha)$ defined by the conditions

$$\begin{aligned} E_\pm &\mapsto E_{\pm\alpha} \\ H &\mapsto H_\alpha. \end{aligned}$$

We then have a corresponding isomorphism Φ_α from $\tilde{\mathfrak{sl}}_2(k)$ onto $\tilde{\mathfrak{g}}(\alpha)$, defined by the conditions

$$\begin{aligned} uE_\pm &\mapsto uE_{\pm\alpha} \\ uH &\mapsto uH_\alpha \end{aligned} \qquad u \in k[t, t^{-1}].$$

We may thus regard the central extension (2.20) as a central extension of $\tilde{\mathfrak{sl}}_2(k)$. Then the elements $uE'_\pm$, uH' defined earlier, correspond to the elements

$$\begin{aligned} uE'_{\pm\alpha} &= \pm\frac{1}{2}[H_\alpha, uE_{\pm\alpha}]' \\ uH'_\alpha &= [E_\alpha, uE_{-\alpha}]' \end{aligned} \qquad u \in k[t, t^{-1}], \qquad (2.21)$$

in $\tilde{\mathfrak{g}}'(\alpha)$, and if we set

$$\{u, v\}_\alpha = [uH'_\alpha, vH'_\alpha], \qquad u, v \in k[t, t^{-1}],$$

then we have that the relations (2.10) are valid in $\tilde{\mathfrak{g}}'(\alpha)$, with H', $E_\pm$, $\{\ ,\ \}$, replaced by H_α, $E_{\pm\alpha}$, $\{\ ,\ \}_\alpha$, respectively. Of course, we may also regard these as being bracket relations in $\tilde{\mathfrak{g}}'$.

We now wish to consider the brackets of elements (2.21) (regarded as elements in $\tilde{\mathfrak{g}}'$) as α varies. First, we consider $[uH'_\alpha, vE'_\beta]$, $u, v \in k[t, t^{-1}]$, $\alpha, \beta \in \Delta$. Now

$$\begin{aligned} \rho(uH'_\alpha) &= uH_\alpha \\ \rho(vE'_\beta) &= vE_\beta \end{aligned} \qquad u, v \in k[t, t^{-1}], \quad \alpha, \beta \in \Delta. \qquad (2.22)$$

Thus, we have

$$\begin{aligned} [uH'_\alpha, vE'_\beta] &= \left[uH'_\alpha, \frac{1}{2}[H'_\beta, vE'_\beta]\right], \quad \text{by (2.21) and (2.22)} \\ &= \frac{1}{2}[[uH'_\alpha, H'_\beta], vE'_\beta] + \frac{1}{2}[H'_\beta, [uH'_\alpha, vE'_\beta]]. \end{aligned}$$

Now we have

$$\begin{aligned} [uH'_\alpha, H'_\beta] &= 0 \mod Z \\ [uH'_\alpha, vE'_\beta] &= \beta(H_\alpha) uvE'_\beta \mod Z, \end{aligned}$$

and hence the last expression in the above computation, is equal to

$$\frac{1}{2}[H'_\beta, \beta(H_\alpha)uvE'_\beta]=\beta(H_\alpha)uvE'_\beta, \quad \text{by (2.10) } b).$$

Thus we have shown

(2.23) $$[uH'_\alpha, vE'_\beta]=\beta(H_\alpha)uvE'_\beta, \quad \alpha, \beta\in\Delta, \ u, v\in k[t, t^{-1}].$$

Next, consider $(\alpha, \beta\in\Delta, u, v\in k[t, t^{-1}])$:

$$\begin{aligned}[uH'_\alpha, vH'_\beta] &= [[E'_\alpha, uE'_{-\alpha}], vH'_\beta], \quad \text{by (2.21) and (2.22)},\\ &= [[E'_\alpha, vH'_\beta], uE'_{-\alpha}]+[E'_\alpha, [uE'_{-\alpha}, vH'_\beta]], \\ &\qquad\qquad \text{by the Jacobi identity},\\ &= [-\alpha(H_\beta)vE'_\alpha, uE'_{-\alpha}]+[E'_\alpha, \alpha(H_\beta)uvE'_{-\alpha}], \quad \text{by (2.23)},\\ &= -\alpha(H_\beta)(uvH'_\alpha+\frac{1}{2}\{v, u\}_\alpha)+\alpha(H_\beta)\left(uvH'_\alpha+\frac{1}{2}\{1, uv\}_\alpha\right),\\ &\qquad\qquad \text{by (2.10), } c),\\ &= \frac{-\alpha(H_\beta)}{2}\{v, u\}_\alpha, \quad \text{by (2.17)}.\end{aligned}$$

On the other hand, we may interchange the roles of α and β in the above computation, and we thus obtain:

(2.24) $$[uH'_\alpha, vH'_\beta]=\frac{-\alpha(H_\beta)}{2}\{v, u\}_\alpha=\frac{\beta(H_\alpha)}{2}\{u, v\}_\beta.$$

As a Corollary of (2.24), we obtain:

(2.25) $$\text{If } (\alpha, \beta)\neq 0, \quad \text{then } \{u, v\}_\alpha=\frac{(\beta, \beta)}{(\alpha, \alpha)}\{u, v\}_\beta,$$

(where (,) is also used to denote the inner product on $\mathfrak{h}^*$ induced by the Killing form).

Next, for $\alpha, \beta\in\Delta$, H' a linear combination of the H'_δ, $\delta\in\Delta$, $u, v\in k[t, t^{-1}]$, we have from (2.22) and (2.23):

(2.26) $$[[H', uE'_\alpha], vE'_\beta]=\alpha(\rho(H'))[uE'_\alpha, vE'_\beta].$$

On the other hand, by the Jacobi identity, the left side of the above equality equals

(2.27) $$[[H', vE'_\beta], uE'_\alpha]+[H', [uE'_\alpha, vE'_\beta]].$$

Now $$[H', vE'_\beta]=\beta(\rho(H'))vE'_\beta, \quad \text{by (2.23)},$$

and

(2.28) $$[uE'_\alpha, vE'_\beta]=N_{\alpha,\beta}uvE'_{\alpha+\beta} \bmod Z,$$

where $N_{\alpha,\beta}\in k$ is defined by

$$[E_\alpha, E_\beta]=N_{\alpha,\beta}E_{\alpha+\beta}.$$

Hence, we have

$$[H', [uE'_\alpha, vE'_\beta]] = (\alpha+\beta)(\rho(H'))N_{\alpha,\beta}uvE'_{\alpha+\beta}, \quad \text{by } (2.23),\ (2.28),$$

and hence (2.27) equals

$$\beta(\rho(H'))[vE'_\beta, uE'_\alpha] + (\alpha+\beta)(\rho(H'))N_{\alpha,\beta}uvE'_{\alpha+\beta}.$$

Comparing with the right side of (2.26), we obtain

$$(\alpha+\beta)(\rho(H'))[uE'_\alpha, vE'_\beta] = (\alpha+\beta)(\rho(H'))N_{\alpha,\beta}uvE'_{\alpha+\beta}.$$

If $\alpha \neq -\beta$, then we may choose H' so that $(\alpha+\beta)(\rho(H')) \neq 0$, and thus conclude

(2.29) $$[uE'_\alpha, vE'_\beta] = N_{\alpha,\beta}uvE'_{\alpha+\beta}, \quad \alpha+\beta \neq 0.$$

We collect (2.23), (2.24), (2.29), and (2.10), *c)* and *d)* (with $E'_{\pm\alpha}$ in place of $E'_\pm$, H'_α in place of H', and $\{\ ,\ \}_\alpha$ in place of $\{\ ,\ \}$) into:

(2.30) *a)* $[uH'_\alpha, vE'_\beta] = \beta(H_\alpha)uvE'_\beta$,

b) $[uH'_\alpha, vH'_\beta] = \dfrac{-\alpha(H_\beta)}{2}\{v, u\}_\alpha = \dfrac{\beta(H_\alpha)}{2}\{u, v\}_\beta$,

c) $[uE'_\alpha, vE'_\beta] = N_{\alpha,\beta}uvE'_{\alpha+\beta}$, if $\alpha+\beta \neq 0$,

d) $[uE'_\alpha, vE'_{-\alpha}] = uvH'_\alpha + \frac{1}{2}\{u, v\}_\alpha$, $\quad u, v \in k[t, t^{-1}]$, $\alpha, \beta \in \Delta$.

We will need

Lemma **(2.31)**. — *If* $\alpha \in \Delta$, if $\alpha = \beta_1 + \beta_2$, $\beta_1, \beta_2 \in \Delta$, *and if* $c_1, c_2 \in k$ *are such that*

$$H_\alpha = c_1 H_{\beta_1} + c_2 H_{\beta_2},$$

then for all $u \in k[t, t^{-1}]$, *we have*

$$uH'_\alpha = c_1 uH'_{\beta_1} + c_2 uH'_{\beta_2}.$$

Proof. — We have (in $\mathfrak{g}$):

$$H_\alpha = [E_\alpha, E_{-\alpha}],$$
$$N^{-1}E_\alpha = [E_{\beta_1}, E_{\beta_2}],$$

for some $N \in k^*$, the multiplicative group of k. We then have

(*) $$\begin{aligned} N^{-1}H_\alpha &= [[E_{\beta_1}, E_{\beta_2}], E_{-\alpha}] \\ &= [[E_{\beta_1}, E_{-\alpha}], E_{\beta_2}] + [E_{\beta_1}[E_{\beta_2}, E_{-\alpha}]]. \end{aligned}$$

If we define N_i $(i = 1, 2)$ in k by

$$[E_{\beta_i}, E_{-\alpha}] = N_i E_{-\beta_{i'}} \quad (i' = 1 \text{ if } i = 2, \quad \text{and} \quad i' = 2 \text{ if } i = 1),$$

then the last expression in (*) equals:

$$N_1[E_{-\beta_2}, E_{\beta_2}] + N_2[E_{\beta_1}, E_{-\beta_1}] = N_2 H_{\beta_1} - N_1 H_{\beta_2},$$

and hence

(2.32) $$H_\alpha = NN_2H_{\beta_1} - NN_1H_{\beta_2}.$$

On the other hand (see (2.21))

$$uH'_\alpha = [E_\alpha, uE_{-\alpha}]',$$
$$uH'_{\beta_i} = [E_{\beta_i}, uE_{-\beta_i}]', \quad i=1, 2.$$

Hence $$N^{-1}uH'_\alpha = [[E'_{\beta_1}, E'_{\beta_2}], uE'_{-\alpha}], \text{ by (2.30) } c),$$
$$= [[E'_{\beta_1}, uE'_{-\alpha}], E'_{\beta_2}] + [E'_{\beta_1}[E'_{\beta_2}, uE'_{-\alpha}]]$$
$$= N_1[uE'_{-\beta_2}, E'_{\beta_2}] + N_2[E'_{\beta_1}, uE'_{-\beta_1}], \text{ by (2.30) } c),$$
$$= -N_1uH'_{\beta_2} + N_2uH'_{\beta_1}, \text{ by (2.21)},$$

and hence $$uH'_\alpha = NN_2uH'_{\beta_1} - NN_1uH'_{\beta_2},$$

and comparing with (2.32), we obtain the lemma. ■

We now fix an order on the roots Δ, and let $\alpha_1, \ldots, \alpha_\ell$ ($\ell = \dim \mathfrak{h}$) denote the corresponding set of simple roots. We define $\varphi : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}'$ by the conditions

$$\varphi(uE_\alpha) = uE'_\alpha, \quad u \in k[t, t^{-1}], \ \alpha \in \Delta,$$
$$\varphi(uH_{\alpha_i}) = uH'_{\alpha_i}, \quad u \in k[t, t^{-1}], \ i = 1, \ldots, \ell.$$

Then, φ is a section of our central extension

$$0 \to Z \to \tilde{\mathfrak{g}}' \xrightarrow{\rho} \tilde{\mathfrak{g}} \to 0$$

(i.e., $\varphi : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}'$ is a linear transformation such that $\rho \circ \varphi = \text{identity}$), and, thanks to Lemma (2.31), we have

(2.33) $$\varphi(uH_\alpha) = uH'_\alpha, \quad u \in k[t, t^{-1}], \ \alpha \in \Delta.$$

One knows that if

$$\tau_\varphi(x, y) = \varphi([x, y]) - [\varphi(x), \varphi(y)], \quad x, y \in \tilde{\mathfrak{g}},$$

then $\tau_\varphi \in Z^2(\tilde{\mathfrak{g}}, Z)$ and our given central extension is equivalent to the central extension (see (1.12))

(2.34) $$0 \to Z \to \tilde{\mathfrak{g}}_{\tau_\varphi} \to \tilde{\mathfrak{g}} \to 0$$

defined by τ_φ.

Moreover, thanks to Lemmas (2.11) and (2.16) and also thanks to (2.25), we have that if β is any long root, then, in view of (2.33), we may reformulate (2.30) as the simple assertion:

(2.35) $$\tau_\varphi(u \otimes x, v \otimes y) = -\operatorname{Res}(udv)(x, y)h'_\iota,$$

where

$$h'_\iota = \{t, t^{-1}\}_\beta \frac{(\beta, \beta)}{4}.$$

From (2.35) we easily deduce:

Theorem **(2.36)**. — *Let* $\tau \in Z^2(\tilde{\mathfrak{g}}, k)$ *be defined by*:

$$\tau(u\otimes x, v\otimes y) = -\operatorname{Res}(udv)(x,y), \qquad x, y \in k[t, t^{-1}],\ u, v \in \mathfrak{g}.$$

Then the corresponding central extension

(2.37) $$0 \to k \to \hat{\mathfrak{g}}_\tau \overset{\tilde{\omega}}{\to} \tilde{\mathfrak{g}} \to 0$$

is the universal covering of $\tilde{\mathfrak{g}}$.

Proof. — Thanks to (2.35), the map

$$(\xi, s) \to (\xi, sh'_\iota)$$

from $\hat{\mathfrak{g}}_\tau = \tilde{\mathfrak{g}} \oplus k$ to $\hat{\mathfrak{g}}_{\tau_\varphi} = \tilde{\mathfrak{g}} \oplus Z$ (where $\xi \in \tilde{\mathfrak{g}}$, $s \in k$) defines a morphism from the central extension (2.37) to the central extension (2.34). Thus, to prove the theorem, we need only prove that $\hat{\mathfrak{g}}_\tau$ is perfect. Since $\tilde{\mathfrak{g}}$ is perfect, it suffices to prove that $[\hat{\mathfrak{g}}_\tau, \hat{\mathfrak{g}}_\tau]$ contains k. But (see (1.9)) this follows from

$$\tau(t\otimes H_\alpha, t^{-1}\otimes H_\alpha) = \frac{4}{(\alpha, \alpha)} \neq 0,$$

where we may take $\alpha \in \Delta$ to be any root. ■

3. Kac-Moody Lie algebras.

In this section we indicate a different method for constructing the central extension (2.37). In fact, this alternate approach is part of the very general theory of a Kac-Moody Lie algebra associated with a symmetrizable Cartan matrix (for details the reader may consult the papers of Kac and Moody [10], [15] and [16], and Garland-Lepowsky [8]).

Thus if ℓ is a positive integer, then we say that an $\ell\times\ell$ matrix $B = (B_{ij})_{i,j=1,\dots,\ell}$ is a *symmetrizable Cartan matrix* in case $B_{ij} \in \mathbf{Z}$ for all i and j, $B_{ii} = 2$ for all i, $B_{ij} \leq 0$ whenever $i \neq j$, and finally, there exist positive rational numbers $q_1, \dots, q_\ell$ such that

$$\operatorname{diag}(q_1, \dots, q_\ell)B$$

is a symmetric matrix. (If B satisfies all but the last condition it is called a Cartan matrix, and the last condition is called the symmetrizability condition. In this paper we shall only be concerned with symmetrizable Cartan matrices.)

Now given a field k of characteristic zero, and given a symmetrizable Cartan matrix B, Kac and Moody have constructed a certain Lie algebra $\mathfrak{g}(B)$ over k, and we now proceed to describe $\mathfrak{g}(B)$ (also see Garland-Lepowsky, [8]).

One lets $\mathfrak{g}_1 = \mathfrak{g}_1(B)$ denote the Lie algebra on 3ℓ generators e_i, f_i, h_i $(i = 1, \dots, \ell)$ with relations

(3.1) $$[h_i, h_j] = 0, \quad [e_i, f_j] = \delta_{ij} h_i,$$
$$[h_i, e_j] = B_{ij} e_j, \quad [h_i, f_j] = -B_{ij} f_j,$$

for $i, j = 1, \ldots, \ell$, and with the relations

$$(3.2)\qquad \begin{aligned}(\operatorname{ad} e_i)^{-B_{ij}+1}(e_j) &= 0,\\ (\operatorname{ad} f_i)^{-B_{ij}+1}(f_j) &= 0,\end{aligned}$$

for $i \neq j$, $i, j = 1, \ldots, \ell$.

For each ℓ-tuple $(n_1, \ldots, n_\ell)$ of nonnegative (resp. nonpositive) integers not all zero, define $\mathfrak{g}_1(n_1, \ldots, n_\ell)$ to be the subspace of $\mathfrak{g}_1$ spanned by the elements

$$[e_{i_1}, [e_{i_2}, \ldots, [e_{i_{r-1}}, e_{i_r}] \ldots]],$$

$$(\text{resp.}\qquad [f_{i_1}, [f_{i_2}, \ldots, [f_{i_{r-1}}, f_{i_r}] \ldots]]),$$

where e_j (resp. f_j) occurs $|n_j|$ times. Also, define $\mathfrak{g}_1(0, \ldots, 0) = \mathfrak{h}_1(B)$, the linear span of $h_1, \ldots, h_\ell$, and $\mathfrak{g}_1(n_1, \ldots, n_\ell) = 0$ for any other ℓ-tuple of integers. Then

$$\mathfrak{g}_1(B) = \coprod_{(n_1, \ldots, n_\ell) \in \mathbf{Z}^\ell} \mathfrak{g}_1(n_1, \ldots, n_\ell),$$

and this is a Lie algebra gradation of $\mathfrak{g}_1(B)$. The elements $h_1, \ldots, h_\ell, e_1, \ldots, e_\ell, f_1, \ldots, f_\ell$ are linearly independent in $\mathfrak{g}_1$ (see [10], [15]). In particular, $\dim \mathfrak{h}_1(B) = \ell$. The space $\mathfrak{g}_1(0, \ldots, 0, 1, 0, \ldots, 0)$ (resp. $\mathfrak{g}_1(0, \ldots, 0, -1, 0, \ldots, 0)$) is nonzero and is spanned by e_i (resp. f_i); here ± 1 is in the i-th position. Also, each space $\mathfrak{g}_1(n_1, \ldots, n_\ell)$ is finite-dimensional. There is clearly a Lie algebra involution η of $\mathfrak{g}_1$ interchanging e_i and f_i and taking h_i to $-h_i$ for all $i = 1, \ldots, \ell$. The involution η takes each space $\mathfrak{g}_1(n_1, \ldots, n_\ell)$ onto $\mathfrak{g}_1(-n_1, \ldots, -n_\ell)$.

The $\mathbf{Z}^\ell$-graded Lie algebra $\mathfrak{g}_1$ contains a unique graded ideal $\mathfrak{r}_1$, maximal among those graded ideals not intersecting the span of h_i, e_i, and f_i $(1 \leq i \leq \ell)$ (see [10], [15]). We let $\mathfrak{g}(B)$ be the $\mathbf{Z}^\ell$-graded Lie algebra $\mathfrak{g}_1(B)/\mathfrak{r}_1$. The images in $\mathfrak{g}(B)$ of h_i, e_i, f_i, $\mathfrak{g}_1(n_1, \ldots, n_\ell)$ and $\mathfrak{h}_1(B)$ will be denoted by h_i, e_i, f_i, $\mathfrak{g}(n_1, \ldots, n_\ell)$ and $\mathfrak{h}(B)$, respectively.

We let $D_i (1 \leq i \leq \ell)$ be the i-th degree derivation of $\mathfrak{g}(B)$; that is, the derivation which acts on $\mathfrak{g}(n_1, \ldots, n_\ell)$ as scalar multiplication by n_i. Then $D_1, \ldots, D_\ell$ span an ℓ-dimensional subspace $\mathfrak{d}_0$ of commuting derivations of $\mathfrak{g}(B)$. Let $\mathfrak{d}$ be a subspace of $\mathfrak{d}_0$. Since $\mathfrak{d}$ may be regarded as an abelian Lie algebra acting on the $\mathfrak{d}$-module $\mathfrak{g}(B)$ by derivations, we may form the semidirect product $\mathfrak{g}^e(B) = \mathfrak{d} \times \mathfrak{g}$ (e for "extended") with respect to this action. We note that $\mathfrak{h}^e(B) = \mathfrak{d} \oplus \mathfrak{h}(B)$ is then an abelian Lie subalgebra of $\mathfrak{g}^e(B)$, and $\mathfrak{h}^e(B)$ acts on each space $\mathfrak{g}_1(n_1, \ldots, n_\ell)$ via scalar multiplication. We define $a_1, \ldots, a_\ell \in \mathfrak{h}^e(B)^*$, the dual space of $\mathfrak{h}^e(B)$, by the conditions $[h, e_i] = a_i(h) e_i$, for all $h \in \mathfrak{h}^e(B)$, and all $i = 1, \ldots, \ell$. We note that $a_j(h_i) = B_{ij}$ for $i, j = 1, \ldots, \ell$ (see (3.1)).

We now make the basic assumption that $\mathfrak{d}$ *is chosen so that* $a_1, \ldots, a_\ell$ *are linearly independent.* This is always possible, as we may take $\mathfrak{d} = \mathfrak{d}_0$. (In this case, we have $a_i(D_j) = \delta_{ij}$ for all $i, j = 1, \ldots, \ell$). However, we may wish to choose $\mathfrak{d}$ smaller than $\mathfrak{d}_0$; e.g., when B is nonsingular, then $\mathfrak{d} = 0$ is a natural choice.

For $a \in \mathfrak{h}^e(\mathrm{B})^*$, define

$$\mathfrak{g}^a = \{x \in \mathfrak{g}(\mathrm{B}) \mid [h, x] = a(h)x, \quad \text{for all} \quad h \in \mathfrak{h}^e(\mathrm{B})\}.$$

Note that $[\mathfrak{g}^a, \mathfrak{g}^b] \subset \mathfrak{g}^{a+b}$, for $a, b \in \mathfrak{h}^e(\mathrm{B})^*$. Also, it is clear that e_i (resp. f_i) spans $\mathfrak{g}^{a_i}$ (resp. $\mathfrak{g}^{-a_i}$) for each $i = 1, \ldots, \ell$ and that for all $(n_1, \ldots, n_\ell) \in \mathbf{Z}^\ell$,

$$\mathfrak{g}(n_1, \ldots, n_\ell) \subset \mathfrak{g}^{n_1 a_1 + \cdots + n_\ell a_\ell}.$$

Indeed, since $a_1, \ldots, a_\ell$ are linearly independent, the inclusion is an equality, and the decomposition

$$\mathfrak{g}(\mathrm{B}) = \coprod_{(n_1, \ldots, n_\ell) \in \mathbf{Z}^\ell} \mathfrak{g}(n_1, \ldots, n_\ell)$$

coincides with the decomposition

$$\mathfrak{g}(\mathrm{B}) = \coprod_{a \in \mathfrak{h}^e(\mathrm{B})^*} \mathfrak{g}^a.$$

We define the *roots* of $\mathfrak{g}(\mathrm{B})$ (with respect to $\mathfrak{h}^e(\mathrm{B})$) to be the nonzero elements $a \in \mathfrak{h}^e(\mathrm{B})^*$ such that $\mathfrak{g}^a \neq 0$. We let $\Delta(\mathrm{B})$ denote the set of roots, $\Delta_+(\mathrm{B})$ (the set of *positive* roots) the set of roots which are nonnegative integral linear combinations of $a_1, \ldots, a_\ell$, and $\Delta_-(\mathrm{B}) = -\Delta_+(\mathrm{B})$ (the set of *negative* roots). Then

$$\Delta(\mathrm{B}) = \Delta_+(\mathrm{B}) \cup \Delta_-(\mathrm{B})$$

(disjoint union), $\mathfrak{g}^0 = \mathfrak{h}(\mathrm{B})$,

$$\mathfrak{g}(\mathrm{B}) = \mathfrak{h}(\mathrm{B}) \oplus \coprod_{a \in \Delta_+(\mathrm{B})} \mathfrak{g}^a \oplus \coprod_{a \in \Delta_-(\mathrm{B})} \mathfrak{g}^a,$$

and $\dim \mathfrak{g}^{-a} = \dim \mathfrak{g}^a$, for all $a \in \Delta(\mathrm{B})$. We call the elements $a_1, \ldots, a_\ell$ *simple roots* (this being relative to our choice of $\Delta_+(\mathrm{B})$).

We let $\mathrm{R} \subset \mathfrak{h}^e(\mathrm{B})^*$ be the subspace spanned by $\Delta(\mathrm{B})$ (so $a_1, \ldots, a_\ell$ is a basis for R). Then the restriction map $\mathrm{R} \to \mathfrak{h}(\mathrm{B})^*$ is an isomorphism if and only if B is nonsingular.

Now, since B is symmetrizable, there are positive rational numbers $q_1, \ldots, q_\ell$ such that $\operatorname{diag}(q_1, \ldots, q_\ell)\,\mathrm{B}$ is a symmetric matrix. We then define a symmetric bilinear form σ on R by the conditions $\sigma(a_i, a_j) = q_i \mathrm{B}_{ij}$, $i, j = 1, \ldots, \ell$. Note that $q_i = \sigma(a_i, a_i)/2$ for each i. Set $h'_{a_i} = q_i h_i = \sigma(a_i, a_i) h_i/2$ in $\mathfrak{h}(\mathrm{B})$, for $i = 1, \ldots, \ell$. Then for $a \in \mathrm{R}$, with $a = \sum_{i=1}^{\ell} \mu_i a_i$, $\mu_i \in k$, define $h'_a = \sum_{i=1}^{\ell} \mu_i h'_{a_i}$ in $\mathfrak{h}(\mathrm{B})$. Transfer σ to a symmetric bilinear form (again denoted by σ) on $\mathfrak{h}(\mathrm{B})$, determined by the conditions $\sigma(h'_{a_i}, h'_{a_j}) = \sigma(a_i, a_j)$, for all $i, j = 1, \ldots, \ell$. Then $\sigma(h'_a, h'_b) = \sigma(a, b)$ for all $a, b \in \mathrm{R}$. Also, $\sigma(a_i, a_j) = a_j(h'_{a_i})$, for all $i, j = 1, \ldots, \ell$, so that

$$\sigma(h'_a, h'_b) = \sigma(a, b) = a(h'_b) = b(h'_a),$$

for all $a, b \in \mathrm{R}$. The form σ extends to a symmetric, $\mathfrak{g}(\mathrm{B})$-invariant bilinear form (again denoted by σ) on $\mathfrak{g}(\mathrm{B})$, such that $[x, y] = \sigma(x, y) h'_a$, for $a \in \Delta(\mathrm{B})$, $x \in \mathfrak{g}^a$, $y \in \mathfrak{g}^{-a}$ (see [10], [15]). In particular $\sigma(e_i, f_i) = 2/\sigma(a_i, a_i)$, for $i = 1, \ldots, \ell$. Also, for

$a \in \Delta(\mathrm{B})$, $b \in \Delta(\mathrm{B}) \cup \{0\}$, one has $\sigma(\mathfrak{g}^a, \mathfrak{g}^b) = 0$, unless $a = -b$, and then σ induces a nonsingular pairing between $\mathfrak{g}^a$ and $\mathfrak{g}^{-a}$ (see [10], [15]).

It is clearly possible to extend the symmetric form σ on R to a symmetric form σ on $\mathfrak{h}^e(\mathrm{B})^*$ satisfying the following condition: For all $a \in \mathrm{R}$ and $\lambda \in \mathfrak{h}^e(\mathrm{B})^*$, $\sigma(\lambda, a) = \lambda(h'_a)$. Fix such a form σ on $\mathfrak{h}^e(\mathrm{B})^*$. For each $i = 1, \ldots, \ell$, define the linear transformation $r_i : \mathfrak{h}^e(\mathrm{B})^* \to \mathfrak{h}^e(\mathrm{B})^*$, by

(3.3) $$r_i(\lambda) = \lambda - \lambda(h_i) a_i.$$

We let $\mathrm{W} = \mathrm{W}(\mathrm{B})$ (the Weyl group) be the group of linear automorphisms of $\mathfrak{h}^e(\mathrm{B})^*$ generated by the reflections $r_i, i = 1, \ldots, \ell$. The following is proved in [8], Proposition (2.10) (and is due to Kac and Moody).

Proposition **(3.3)**. — *The form σ on $\mathfrak{h}^e(\mathrm{B})^*$ is* W-*invariant.*

Examples: (i) We let A be the $\ell \times \ell$ Cartan matrix associated in the usual way, with a k-split, *simple* Lie algebra $\mathfrak{g}$ of rank ℓ. (We will call such an A a *classical* Cartan matrix—we note that a classical Cartan matrix is always symmetrizable: see below.) Then it is a theorem of Serre (see [19], Chapitre VI, p. 19) that

$$\mathfrak{g} \simeq \mathfrak{g}_1(\mathrm{A}) \simeq \mathfrak{g}(\mathrm{A}).$$

For a classical Cartan matrix A, when there is no danger of confusion, we write Δ, $\Delta_\pm$, $\mathfrak{h}$, and $\mathfrak{g}$ for $\Delta(\mathrm{A})$, $\Delta_\pm(\mathrm{A})$, $\mathfrak{h}(\mathrm{A})$, and $\mathfrak{g}(\mathrm{A})$, respectively. This notation is consistent with that in § 2. Since A is nonsingular we may (and *do*) take $\mathfrak{d} = 0$, so we have $\mathfrak{h}^e(\mathrm{A}) = \mathfrak{h}$, $\mathfrak{g}^e(\mathrm{A}) = \mathfrak{g}$. Also, we write $\mathrm{E}_1, \ldots, \mathrm{E}_\ell$, $\mathrm{F}_1, \ldots, \mathrm{F}_\ell$, $\mathrm{H}_1, \ldots, \mathrm{H}_\ell$, for the generators $e_1, \ldots, e_\ell$, $f_1, \ldots, f_\ell$, $h_1, \ldots, h_\ell$, respectively. The Kac-Moody construction of a Lie algebra corresponding to a symmetrizable Cartan matrix B, automatically gives a choice of simple roots. In the case of a classical Cartan matrix A, we denote these simple roots by $\alpha_1, \ldots, \alpha_\ell$, and we denote roots in Δ by Greek letters $\alpha, \beta, \gamma, \ldots$. If (,) denotes the Killing form (on $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{h}^*$), and if $q_i = (\alpha_i, \alpha_i)/2$, then the matrix

$$\mathrm{diag}(q_1, \ldots, q_\ell)\mathrm{A}$$

is symmetric. Letting σ denote the corresponding inner product on R ($= \mathfrak{h}^*$, in this case) we see that σ and (,) agree as forms on $\mathfrak{h}^*$, $\mathfrak{h}$, and $\mathfrak{g}$. We let α_0 denote the highest root of $\mathfrak{g}$ (with respect to our choice $\alpha_1, \ldots, \alpha_\ell$ of simple roots), and we set $\alpha_{\ell+1} = -\alpha_0$. *Throughout this paper, we will use* A *to denote a classical Cartan matrix.*

(ii) We let $\widetilde{\mathrm{A}} = (\widetilde{\mathrm{A}}_{ij})_{i,j=1,\ldots,\ell+1}$ denote the symmetrizable Cartan matrix, defined by the conditions

$$\widetilde{\mathrm{A}}_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}, \quad i, j = 1, \ldots, \ell + 1.$$

We call $\widetilde{\mathrm{A}}$ the *affine* Cartan matrix (associated with A) and $\mathfrak{g}(\widetilde{\mathrm{A}})$ an *affine* Lie algebra. The matrix A has rank ℓ, and if we take $\mathfrak{d}$ to be the k-span of $\mathrm{D} = \mathrm{D}_{\ell+1}$, the

$\ell+1$-st degree derivation, then our basic assumption, that the simple roots $a_1, \ldots, a_{\ell+1}$ are linearly independent on $\mathfrak{h}^e(A)$, is satisfied. We therefore make this choice for $\mathfrak{d}$, and define $\mathfrak{g}^e(\widetilde{A})$, $\mathfrak{h}^e(\widetilde{A})$ accordingly.

Our next goal is to show, using a result of Kac and Moody, that $\mathfrak{g}(\widetilde{A})$ is exactly our central extension $\hat{\mathfrak{g}}_\tau$ of the loop algebra $\widetilde{\mathfrak{g}}$. Recall that $\tau \in Z^2(\widetilde{\mathfrak{g}}, k)$ is the cocycle defined at the beginning of § 2, and

$$0 \to k \to \hat{\mathfrak{g}}_\tau \overset{\widetilde{\omega}}{\to} \widetilde{\mathfrak{g}} \to 0$$

is the corresponding central extension, constructed as in § 1. In particular, $\hat{\mathfrak{g}}_\tau = \widetilde{\mathfrak{g}} \oplus k$, and we let $\varphi : \widetilde{\mathfrak{g}} \to \hat{\mathfrak{g}}_\tau$ be the injective linear map defined by $\varphi(\xi) = (\xi, 0)$, $\xi \in \widetilde{\mathfrak{g}}$. Then φ is a section in the sense that

$$\widetilde{\omega} \circ \varphi = \text{identity}. \tag{3.4}$$

We let $\widetilde{\mathfrak{u}}^{\pm} \subset \widetilde{\mathfrak{g}}$ denote the subalgebra

$$\widetilde{\mathfrak{u}}^{\pm} = \coprod_{\alpha \in \Delta_{\pm}(A)} \mathfrak{g}^{\alpha} \oplus \coprod_{\pm n \in \mathbf{Z}_+} t^n \otimes \mathfrak{g},$$

where $\mathbf{Z}_+$ denotes the nonzero, positive integers. Since τ restricted to $\widetilde{\mathfrak{u}}^{\pm} \times \widetilde{\mathfrak{u}}^{\pm}$ is zero, the map φ restricted to $\widetilde{\mathfrak{u}}^{\pm}$ is a Lie algebra monomorphism.

We fix elements $E_{\pm\alpha_0} \in \mathfrak{g}^{\pm\alpha_0} \subset \mathfrak{g}$, so that if $H_{\alpha_0} = [E_{\alpha_0}, E_{-\alpha_0}]$, then

$$\beta(H_{\alpha_0}) = 2(\beta, \alpha_0)(\alpha_0, \alpha_0)^{-1},$$

for all $\beta \in \Delta$ (our choice of $E_{\pm\alpha_0}$ is consistent with our choice of E_α in § 2; these choices will be further normalized in § 4). We set

$$\hat{e}_i = \varphi(E_i), \qquad \hat{f}_i = \varphi(F_i), \qquad \hat{h}_i = \varphi(H_i), \qquad i = 1, \ldots, \ell, \tag{3.5}$$

and we set

$$\hat{e}_{\ell+1} = \varphi(t \otimes E_{-\alpha_0}), \qquad \hat{f}_{\ell+1} = \varphi(t^{-1} \otimes E_{\alpha_0}), \qquad \hat{h}_{\ell+1} = (-H_{\alpha_0}, 2(\alpha_0, \alpha_0)^{-1}). \tag{3.6}$$

We now state a theorem of Kac and Moody (see [10], [16]):

Theorem **(3.7)**. — *There is a surjective Lie algebra homomorphism* $\pi : \mathfrak{g}(\widetilde{A}) \to \widetilde{\mathfrak{g}}$, *and* π *is uniquely determined by the conditions:* $\pi(e_i) = E_i$, $\pi(f_i) = F_i$, $\pi(h_i) = H_i$, *for* $i = 1, \ldots, \ell$ *and* $\pi(e_{\ell+1}) = t \otimes E_{-\alpha_0}$, $\pi(f_{\ell+1}) = t^{-1} \otimes E_{\alpha_0}$, and $\pi(h_{\ell+1}) = -H_{\alpha_0}$. *Moreover, the kernel* $\mathfrak{c}$ *of* π *is a one dimensional subspace of* $\mathfrak{h}(\widetilde{A})$. *Indeed, if* $\alpha_0 = \sum_{i=1}^{\ell} m_i \alpha_i$ *(the* m_i *being nonnegative integers), and if we set* $h'_\iota = (\sum_{i=1}^{\ell} m_i h'_{a_i}) + h'_{a_{\ell+1}} \in \mathfrak{h}(A)$, *then* h'_ι *spans* $\mathfrak{c}$. *Finally,* $\mathfrak{c}$ *is the center of* $\mathfrak{g}(\widetilde{A})$.

Of course the elements e_i, f_i, h_i in $\mathfrak{g}(\widetilde{A})$, denote the generators given by the Kac-Moody construction described earlier.

Now, in particular, Theorem (3.7) implies that we have a central extension

$$0 \to \mathfrak{c} \to \mathfrak{g}(\widetilde{A}) \overset{\pi}{\to} \widetilde{\mathfrak{g}} \to 0. \tag{3.8}$$

Hence, by the universal property of the central extension (2.37), we have a unique morphism from the central extension (2.37) to the central extension (3.8); i.e., we have a commutative diagram

$$\textbf{(3.9)}\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & k & \longrightarrow & \hat{\mathfrak{g}}_\tau & \xrightarrow{\widetilde{\omega}} & \widetilde{\mathfrak{g}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \mathrm{I}} & & \\ 0 & \longrightarrow & \mathfrak{c} & \longrightarrow & \mathfrak{g}(\widetilde{\mathrm{A}}) & \xrightarrow{\pi} & \widetilde{\mathfrak{g}} & \longrightarrow & 0, \end{array}$$

where I denotes the identity, ψ is a Lie algebra homomorphism, and of course the map $k\to\mathfrak{c}$ is just the restriction of ψ to k.

Lemma **(3.10)**. — *We have*

$$\psi(\hat{e}_i)=e_i, \qquad \psi(\hat{f}_i)=f_i, \qquad \textit{and} \qquad \psi(\hat{h}_i)=h_i,$$

for $i=1, \ldots, \ell+1$.

Proof. — By the commutativity of the diagram (3.9), we have

$$\textbf{(3.11)}\qquad \begin{aligned} \pi(\psi(\hat{e}_i)) &= \widetilde{\omega}(\hat{e}_i) \\ \pi(\psi(\hat{f}_i)) &= \widetilde{\omega}(\hat{f}_i) \\ \pi(\psi(\hat{h}_i)) &= \widetilde{\omega}(\hat{h}_i), \quad i=1, \ldots, \ell+1. \end{aligned}$$

On the other hand, from the definition (3.5) and (3.6) of $\hat{e}_i, \hat{f}_i, \hat{h}_i$, the fact that $\widetilde{\omega}\circ\varphi=$ identity, and from the conditions $\pi(e_i)=\mathrm{E}_i$, $\pi(f_i)=\mathrm{F}_i$, $\pi(h_i)=\mathrm{H}_i$, $i=1, \ldots, \ell$, and $\pi(e_{\ell+1})=t\otimes\mathrm{E}_{-\alpha_0}$, $\pi(f_{\ell+1})=t^{-1}\otimes\mathrm{E}_{\alpha_0}$, $\pi(h_{\ell+1})=-\mathrm{H}_{\alpha_0}$, of Theorem (3.5), we see that

$$\textbf{(3.12)}\qquad \begin{aligned} \pi(e_i) &= \widetilde{\omega}(\hat{e}_i) \\ \pi(f_i) &= \widetilde{\omega}(\hat{f}_i) \\ \pi(h_i) &= \widetilde{\omega}(\hat{h}_i), \quad i=1, \ldots, \ell+1. \end{aligned}$$

Hence, from the exactness of (3.8), we have, comparing (3.11) with (3.12):

$$\textbf{(3.13)}\qquad \begin{aligned} e_i &= \psi(\hat{e}_i) \mod \mathfrak{c} \\ f_i &= \psi(\hat{f}_i) \mod \mathfrak{c} \\ h_i &= \psi(\hat{h}_i) \mod \mathfrak{c}. \end{aligned}$$

But then a direct computation shows:

$$\hat{e}_i=1/2[\hat{h}_i, \hat{e}_i], \qquad \hat{f}_i=-\frac{1}{2}[\hat{h}_i,\hat{f}_i], \qquad \hat{h}_i=[\hat{e}_i,\hat{f}_i], \qquad i=1, \ldots, \ell+1.$$

(the crucial computation is to show that $\hat{h}_{\ell+1}=[\hat{e}_{\ell+1},\hat{f}_{\ell+1}]$). Hence, since $\mathfrak{c}$ is contained in the center of $\mathfrak{g}(\widetilde{\mathrm{A}})$, we have:

$$\psi(\hat{e}_i)=\frac{1}{2}[\psi(\hat{h}_i), \psi(\hat{e}_i)]=\frac{1}{2}[h_i, e_i]=e_i,$$

and similarly $\psi(\hat{f}_i)=f_i$, for $i=1, \ldots, \ell+1$. But then $\psi(\hat{h}_i)=\psi([\hat{e}_i,\hat{f}_i])=[e_i,f_i]=h_i$, $i=1, \ldots, \ell+1$, and this completes the proof of Lemma (3.10). ∎

We can now prove:

Theorem **(3.14)**. — *The map* ψ *is a Lie algebra isomorphism. Thus the central extension* (3.8) *is the universal covering of* $\tilde{\mathfrak{g}}$.

Proof. — Since, by construction, the algebra $\tilde{\mathfrak{g}}(A)$ is generated by the elements e_i, f_i, and h_i, $i=1, \dots, \ell+1$, Lemma (3.10) immediately implies that ψ is surjective.

On the other hand, by the commutativity of (3.9), kernel $\psi \subset$ kernel $\tilde{\omega} = k$. If kernel $\psi \neq 0$, then $\psi((0, 1)) = 0$, and hence (see (3.6)) we have

$$\psi(\hat{h}_{\ell+1}) = \psi((-H_{\alpha_0}, 2(\alpha_0, \alpha_0)^{-1})) = \psi((-H_{\alpha_0}, 0)).$$

But $-H_{\alpha_0}$ is a linear combination of $H_1, \dots, H_\ell$, and hence $\psi((-H_{\alpha_0}, 0))$ is a linear combination of the elements $\psi((H_1, 0)), \dots, \psi((H_\ell, 0))$; i.e., $h_{\ell+1} = \psi(\hat{h}_{\ell+1})$ is a linear combination of the elements $h_1 = \psi(\hat{h}_1), \dots, h_\ell = \psi(\hat{h}_\ell)$. But since the ideal $\mathfrak{r}_1 \subset \mathfrak{g}_1(\tilde{A})$ does not intersect the span of the e_i's, f_i's, and h_i's in $\mathfrak{g}_1(\tilde{A})$, and since these elements are linearly independent in $\mathfrak{g}_1(\tilde{A})$ (as we noted earlier—see [10], [15]), we have obtained a contradiction. Thus ψ is an isomorphism. ∎

4. The Chevalley basis in the universal covering.

In [7], § 4, we introduced a Chevalley basis in the algebra $\mathfrak{g}(\tilde{A})$. By means of the isomorphism $\psi : \hat{\mathfrak{g}}_\tau \to \mathfrak{g}(\tilde{A})$, we may then pull this basis back to a basis in $\hat{\mathfrak{g}}_\tau$. In fact it is quite easy to describe the "pull back" basis in $\hat{\mathfrak{g}}_\tau$, and then to compute the bracket relations using the cocycle τ. We wish to give the description in the present section.

First, we begin with a simple Lie algebra $\mathfrak{g}$ over the field **C** of complex numbers. Indeed, let $\mathfrak{g}$ have rank ℓ, and let A be the $\ell \times \ell$ (classical) Cartan matrix corresponding to $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{g}(A)$ is the Kac-Moody algebra constructed from A, as described in § 3. From the Kac-Moody construction (in this case, the Serre construction), we have a Cartan subalgebra $\mathfrak{h} = \mathfrak{h}(A)$, the set of roots Δ (relative to $\mathfrak{h}$) and a given set of simple roots $\alpha_1, \dots, \alpha_\ell$. We then know that $\mathfrak{g}(A)$ has a Chevalley basis (see Steinberg, [21]). Thus, for each $\alpha \in \Delta$, we have nonzero elements $E_\alpha \in \mathfrak{g}^\alpha$, $H_\alpha \in \mathfrak{h}$, and these elements have the bracket relations:

(4.1)
$$[E_\alpha, E_{-\alpha}] = H_\alpha, \quad \alpha \in \Delta$$
$$[E_\alpha, E_\beta] = \pm(r+1)E_{\alpha+\beta}, \quad \text{if } \alpha, \beta \text{ and } \alpha+\beta \in \Delta, \text{ and}$$
$$\alpha - r\beta, \dots, \alpha, \dots, \alpha + q\beta \text{ is the } \beta\text{-string through } \alpha,$$
$$[E_\alpha, E_\beta] = 0, \quad \text{otherwise},$$
$$[H_\alpha, E_\beta] = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} E_\beta, \quad \alpha, \beta \in \Delta.$$

For the Chevalley basis of $\mathfrak{g}$, we take the set:

(4.2) $$\Psi = \{E_\alpha\}_{\alpha\in\Delta} \cup \{H_1, \ldots, H_\ell\}$$

(where $H_i = H_{\alpha_i}$, $i = 1, \ldots, \ell$). We note that, in particular, this basis has integral structure constants.

Now, since the isomorphism ψ sends $\hat{e}_i$ to e_i, and $\hat{f}_i$ to f_i, it is easy to determine the subspaces of $\hat{\mathfrak{g}}_\tau$ which correspond to the root spaces of $\mathfrak{g}(\widetilde{A})$. Moreover, one can easily compute the derivation of $\hat{\mathfrak{g}}_\tau$ which corresponds to the $\ell+1$-st degree derivation D of $\mathfrak{g}(\widetilde{A})$.

If $a_1, \ldots, a_{\ell+1} \in \mathfrak{h}^e(\widetilde{A})^*$ are the simple roots and if $\alpha_0 = \sum_{i=1}^{\ell} m_i\alpha_i$ is the highest root of $\mathfrak{g}$ (relative to the choice $\alpha_1, \ldots, \alpha_\ell$ of simple roots) then we let $\iota \in \mathfrak{h}^e(\widetilde{A})^*$ be the element

$$\iota = \left(\sum_{i=1}^{\ell} m_i a_i\right) + a_{\ell+1}.$$

Also, if $\alpha = \sum_{i=1}^{\ell} n_i\alpha_i \in \Delta$, we let $a(\alpha) \in \mathfrak{h}^e(\widetilde{A})^*$ be the element

$$a(\alpha) = \sum_{i=1}^{\ell} n_i a_i.$$

We then have (see [7], § 2):

(4.3) $$\Delta_+(\widetilde{A}) = \{a(\alpha)\}_{\alpha\in\Delta_\pm(A)} \cup \{a(\alpha) + n\iota\}_{\alpha\in\Delta(A), n\in\mathbf{Z}_+} \cup \{n\iota\}_{n\in\mathbf{Z}_+},$$

where $\mathbf{Z}_+$ denotes the set of all (strictly) positive integers. We let

$$\Delta_W(\widetilde{A}) = \{a(\alpha) + n\iota\}_{\alpha\in\Delta(A), n\in\mathbf{Z}},$$
$$\Delta_I(\widetilde{A}) = \{n\iota\}_{n\in\mathbf{Z}, n\neq 0},$$

and call the elements of $\Delta_W(\widetilde{A})$ *Weyl* (or *real*) *roots*, and the elements of $\Delta_I(\widetilde{A})$ *imaginary roots*. We let $\Delta_{W,\pm}(\widetilde{A})$ (resp. $\Delta_{I,\pm}(\widetilde{A})$) denote the set of $\pm$ Weyl (resp. $\pm$ imaginary) roots.

Then, if we identify $\mathfrak{g}(\widetilde{A})$ with $\hat{\mathfrak{g}}_\tau$ by means of ψ, we have

(4.4) $$\mathfrak{g}^{a(\alpha)+n\iota} = (t^n \otimes \mathfrak{g}^\alpha, 0), \quad \alpha\in\Delta(A),\ n\in\mathbf{Z},$$
$$\mathfrak{g}^{n\iota} = (t^n \otimes \mathfrak{h}, 0), \quad n\in\mathbf{Z},\ n\neq 0.$$

Also, we have

$$\iota(h) = \begin{cases} 0, & h\in\mathfrak{h}(\widetilde{A}) \\ 1, & h = D, \end{cases}$$

and thus

(4.5) $$D = \begin{cases} \text{scalar mult. by } n, \text{ on } \mathfrak{g}^{a(\alpha)+n\iota} \\ \text{scalar mult. by } n, \text{ on } \mathfrak{g}^{n\iota}. \end{cases}$$

Indeed, (4.3), (4.4), (4.5) follow easily from Theorem (3.7), Theorem (3.14), and the commutativity of the diagram (3.9) (also, see [7]).

Remark. — One can deduce the existence of D on $\hat{\mathfrak{g}}_\tau$ directly from the universality of the central extension $0 \to k \to \hat{\mathfrak{g}}_\tau \to \tilde{\mathfrak{g}} \to 0$, and from the existence of the derivation $D_0 : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$, defined by $D_0(t^n \otimes x) = nt^n \otimes x$, $n \in \mathbf{Z}$, $x \in \mathfrak{g}$. Thus D on $\hat{\mathfrak{g}}_\tau$ is the lift of D_0 on $\tilde{\mathfrak{g}}$.

We define elements ξ_a in $\mathfrak{g}^a$, $a \in \Delta_W(\tilde{A})$, by

$$\xi_a = (t^n \otimes E_\alpha, 0), \qquad a = a(\alpha) + n\iota.$$

For $n \in \mathbf{Z}$, $n \neq 0$, we then define elements $\xi_i(n)$, $i = 1, \ldots, \ell$, in $\mathfrak{g}^{n\iota}$ by

$$\xi_i(n) = t^n \otimes H_i, \qquad n \in \mathbf{Z},\ n \neq 0,\ i = 1, \ldots, \ell.$$

Finally, we let

$$h_i = (H_i, 0), \qquad i = 1, \ldots, \ell$$
$$h_{\ell+1} = (-H_{\alpha_0}, 2(\alpha_0, \alpha_0)^{-1})$$

(so $h_i = \hat{h}_i$, $i = 1, \ldots, \ell + 1$). Then the set

(4.6) $$\Phi = \{h_i\}_{i=1,\ldots,\ell+1} \cup \{\xi_a\}_{a \in \Delta_W(\tilde{A})} \cup \{\xi_i(n)\}_{n \in \mathbf{Z}, n \neq 0, i=1,\ldots,\ell}$$

is a basis of $\hat{\mathfrak{g}}_\tau = \mathfrak{g}(\tilde{A})$, and is called a Chevalley basis.

One can of course compute the structure constants directly, using the cocycle τ. In any case these structure constants are computed in [7], § 4, and are seen to be integral. Indeed, the integrality of the structure constants boils down to showing that $\frac{2h'_\iota}{(\alpha_i, \alpha_i)}$ is an integral linear combination of $h_1, \ldots, h_{\ell+1}$, when α_i is a long root. This is part of the proof of Lemma (4.10) of [7], and is derived by examining root diagrams.

Remark. — The element (0, 1) in $\hat{\mathfrak{g}}_\tau = \tilde{\mathfrak{g}} \oplus k$ corresponds to $h'_\iota = \mathfrak{g}(\tilde{A})$. However, if we replace the Killing form (,) by the form $\langle\ ,\ \rangle = \nu(\ ,\ )$, where $\nu > 0$ is chosen so that $\langle H_\alpha, H_\alpha \rangle = 2$ for α a long root, if we define the cocycle τ' by

$$\tau'(u \otimes x, v \otimes y) = -\tau_0(u, v)\langle x, y \rangle, \qquad u, v \in k[t, t^{-1}],\ x, y \in \mathfrak{g},$$

(compare with § 2, after (2.2)), and if we consider $\hat{\mathfrak{g}}_{\tau'} = \tilde{\mathfrak{g}} \oplus k$, in place of $\hat{\mathfrak{g}}_\tau$, then $(0, 1) \in \hat{\mathfrak{g}}_{\tau'}$ corresponds to $2h'_\iota/(\alpha, \alpha)$, α a long root. Thus it is natural to construct our Chevalley basis in $\mathfrak{g}(\tilde{A})$, by using $\hat{\mathfrak{g}}_{\tau'}$ and $\langle\ ,\ \rangle$ in place of $\hat{\mathfrak{g}}_\tau$ and (,), respectively. Roughly speaking, a Chevalley basis in $\hat{\mathfrak{g}}_{\tau'}$ is constructed by lifting an obvious choice of Chevalley basis in $\tilde{\mathfrak{g}}$, and then adjoining the element $h_{\ell+1} = (-H_{\alpha_0}, 1)$.

Notational Remark **(4.7)**. — *From now on* we use $\mathfrak{g}(\tilde{A})$, and not $\hat{\mathfrak{g}}_\tau$, to denote the universal covering of $\tilde{\mathfrak{g}}$. Indeed, identifying $\mathfrak{g}(\tilde{A})$ with $\hat{\mathfrak{g}}_\tau$ by means of ψ, and using the commutativity of (3.9), we see that $\pi : \mathfrak{g}(\tilde{A}) \to \tilde{\mathfrak{g}}$ corresponds to $\tilde{\omega} : \hat{\mathfrak{g}}_\tau \to \tilde{\mathfrak{g}}$. We thus write $\tilde{\omega}$ in place of π; i.e., in sum, we let

(4.8) $$0 \to k \to \mathfrak{g}(\tilde{A}) \xrightarrow{\tilde{\omega}} \tilde{\mathfrak{g}} \to 0$$

now denote the universal (*and* Kac-Moody) central extension of $\tilde{\mathfrak{g}}$.

We now define subalgebras $\mathfrak{u}^{\pm}(\widetilde{A}) \subset \mathfrak{g}(\widetilde{A})$ by

$$\mathfrak{u}^{\pm}(\widetilde{A}) = \coprod_{a \in \Delta_{\pm}(\widetilde{A})} \mathfrak{g}^{a}.$$

On the other hand, in § 3 (just after (3.4)), we defined subalgebras $\widetilde{\mathfrak{u}}^{\pm} \subset \widetilde{\mathfrak{g}}$. Since kernel $\widetilde{\omega} \subset \mathfrak{h}(A)$, by Theorem (3.7) (we have set $\widetilde{\omega} = \pi$), we know that the restriction $\widetilde{\omega}^{\pm}$ of $\widetilde{\omega}$ to $\mathfrak{u}^{\pm}(\widetilde{A})$, is injective. Moreover, from our description of the root spaces in (4.4), we see that Image $\widetilde{\omega}^{\pm} = \widetilde{\mathfrak{u}}^{\pm}$; i.e., $\widetilde{\omega}^{\pm}$ defines an isomorphism

$$\widetilde{\omega}^{\pm} : \mathfrak{u}^{\pm}(\widetilde{A}) \to \widetilde{\mathfrak{u}}^{\pm}.$$

We will write $\mathfrak{u}(\widetilde{A})$ (resp. $\widetilde{\mathfrak{u}}$) for $\mathfrak{u}^{+}(\widetilde{A})$ (resp. $\widetilde{\mathfrak{u}}^{+}$), whenever convenient, and we will identify $\mathfrak{u}^{\pm}(A)$ with $\widetilde{\mathfrak{u}}^{\pm}$ by means of $\widetilde{\omega}^{\pm}$, whenever convenient. Thus, we may, for example, regard $t^{n} \otimes \mathfrak{g}^{\alpha}$, $n > 0$, $\alpha \in \Delta(A)$, as the root subspace $\mathfrak{g}^{a}$, $a = a(\alpha) + n\iota$, of $\mathfrak{u}(A)$, and we may regard ξ_a as an element of $\widetilde{\mathfrak{u}}^{+}$ (namely, $\xi_a = t^{n} \otimes E_{\alpha}$).

We let $\mathfrak{g}_{\mathbf{Z}}(\widetilde{A})$ (resp. $\mathfrak{g}_{\mathbf{Z}}(A)$) denote the **Z**-span of the Chevalley basis Φ (resp. of the Chevalley basis Ψ). Then $\mathfrak{g}_{\mathbf{Z}}(\widetilde{A})$ (resp. $\mathfrak{g}_{\mathbf{Z}}(A)$) is an integral subalgebra of $\mathfrak{g}(A)$ (resp. $\mathfrak{g}(A)$) by [7], Theorem (4.12) (resp. by [21], Theorem 1, p. 6). If R is a commutative ring with unit, we set

$$\mathfrak{g}_{R}(\widetilde{A}) = R \otimes_{\mathbf{Z}} \mathfrak{g}_{\mathbf{Z}}(\widetilde{A}),$$

$$\mathfrak{g}_{R}(A) = R \otimes_{\mathbf{Z}} \mathfrak{g}_{\mathbf{Z}}(A).$$

We let $R[t, t^{-1}]$ denote the ring of Laurent polynomials $\sum_{i_0 \leq i \leq i_1} q_i t^{i}$ (finite sum, with i_0 and i_1 allowed to take negative values) with coefficients q_i in R. We let

$$\widetilde{\mathfrak{g}}_{R} = R[t, t^{-1}] \otimes_{\mathbf{Z}} \mathfrak{g}_{\mathbf{Z}}(A),$$

and observe that

$$\widetilde{\mathfrak{g}}_{R} = R \otimes_{\mathbf{Z}} \widetilde{\mathfrak{g}}_{\mathbf{Z}}.$$

We note that $\widetilde{\omega}$ induces by restriction, a surjective **Z**-Lie algebra homomorphism

$$\widetilde{\omega}_{\mathbf{Z}} : \mathfrak{g}_{\mathbf{Z}}(\widetilde{A}) \to \widetilde{\mathfrak{g}}_{\mathbf{Z}}.$$

By (4.5) and the remark following, we then see that

$$D(\mathfrak{g}_{\mathbf{Z}}(\widetilde{A})) \subset \mathfrak{g}_{\mathbf{Z}}(\widetilde{A}),$$

$$D_0(\widetilde{\mathfrak{g}}_{\mathbf{Z}}) \subset \widetilde{\mathfrak{g}}_{\mathbf{Z}}.$$

Hence, for any commutative ring R with unit, the operator D (resp. D_0) on $\mathfrak{g}_{\mathbf{Z}}(\widetilde{A})$ (resp., on $\widetilde{\mathfrak{g}}_{\mathbf{Z}}$) induces an operator on $\mathfrak{g}_{R}(\widetilde{A})$ (resp., on $\widetilde{\mathfrak{g}}_{R}$) which we also denote by D (resp. D_0).

We let $\mathfrak{u}^{\pm}_{\mathbf{Z}}(\widetilde{A}) = \mathfrak{g}_{\mathbf{Z}}(\widetilde{A}) \cap \mathfrak{u}^{\pm}(\widetilde{A})$, and $\widetilde{\mathfrak{u}}^{\pm}_{\mathbf{Z}} = \widetilde{\mathfrak{g}}_{\mathbf{Z}} \cap \widetilde{\mathfrak{u}}^{\pm}$. We then have

$$\widetilde{\omega}_{\mathbf{Z}}(\mathfrak{u}^{\pm}_{\mathbf{Z}}(\widetilde{A})) = \widetilde{\mathfrak{u}}^{\pm}_{\mathbf{Z}}.$$

We let $\mathfrak{h}_{\mathbf{Z}}(\widetilde{A})$ denote the $\mathbf{Z}$-span of $h_1, \ldots, h_{\ell+1}$ and $\mathfrak{h}_{\mathbf{Z}}(A)$ the $\mathbf{Z}$-span of $H_1, \ldots, H_\ell$. We have

$$\mathfrak{g}_{\mathbf{Z}}(\widetilde{A}) = \mathfrak{u}_{\mathbf{Z}}^{+}(\widetilde{A}) \oplus \mathfrak{h}_{\mathbf{Z}}(\widetilde{A}) \oplus \mathfrak{u}_{\mathbf{Z}}^{-}(\widetilde{A}),$$

$$\widetilde{\mathfrak{g}}_{\mathbf{Z}} = \widetilde{\mathfrak{u}}_{\mathbf{Z}}^{+} \oplus \mathfrak{h}_{\mathbf{Z}}(A) \oplus \widetilde{\mathfrak{u}}_{\mathbf{Z}}^{-}.$$

Moreover, the algebra $\mathfrak{u}_{\mathbf{Z}}^{\pm}(\widetilde{A})$ (resp. $\widetilde{\mathfrak{u}}_{\mathbf{Z}}^{\pm}$; resp. $\mathfrak{h}_{\mathbf{Z}}(\widetilde{A})$; resp. $\mathfrak{h}_{\mathbf{Z}}(A)$) is a $\mathbf{Z}$-form of $\mathfrak{u}^{\pm}(\widetilde{A})$ (resp. $\widetilde{\mathfrak{u}}^{\pm}$; resp. $\mathfrak{h}(\widetilde{A})$; resp. $\mathfrak{h}(A)$). We let

$$\mathfrak{u}_{R}^{\pm}(\widetilde{A}) = R \otimes_{\mathbf{Z}} \mathfrak{u}_{\mathbf{Z}}^{\pm}(\widetilde{A})$$

$$\widetilde{\mathfrak{u}}_{R}^{\pm} = R \otimes_{\mathbf{Z}} \widetilde{\mathfrak{u}}_{\mathbf{Z}}^{\pm}$$

$$\mathfrak{h}_{R}(\widetilde{A}) = R \otimes_{\mathbf{Z}} \mathfrak{h}_{\mathbf{Z}}(\widetilde{A})$$

$$\mathfrak{h}_{R}(A) = R \otimes_{\mathbf{Z}} \mathfrak{h}_{\mathbf{Z}}(A).$$

We then have direct sum decompositions

$$\mathfrak{g}_{R}(\widetilde{A}) = \mathfrak{u}_{R}^{+}(\widetilde{A}) \oplus \mathfrak{h}_{R}(\widetilde{A}) \oplus \mathfrak{u}_{R}^{-}(\widetilde{A}),$$

$$\widetilde{\mathfrak{g}}_{R} = \widetilde{\mathfrak{u}}_{R}^{+} \oplus \mathfrak{h}_{R}(A) \oplus \widetilde{\mathfrak{u}}_{R}^{-}.$$

Moreover, $\widetilde{\omega}_{\mathbf{Z}} : \mathfrak{g}_{\mathbf{Z}}(\widetilde{A}) \to \widetilde{\mathfrak{g}}_{\mathbf{Z}}$ induces a surjective Lie algebra homomorphism

$$\widetilde{\omega}_{R} = 1 \otimes \widetilde{\omega}_{\mathbf{Z}},$$

$$\widetilde{\omega}_{R} : \mathfrak{g}_{R}(\widetilde{A}) \to \widetilde{\mathfrak{g}}_{R}$$

(recall that $\mathfrak{g}_{R}(\widetilde{A}) = R \otimes_{\mathbf{Z}} \widetilde{\mathfrak{g}}_{\mathbf{Z}}(\widetilde{A})$, $\widetilde{\mathfrak{g}}_{R} = R \otimes_{\mathbf{Z}} \widetilde{\mathfrak{g}}_{\mathbf{Z}}$), where

$$\widetilde{\omega}_{R}(\mathfrak{h}_{R}(\widetilde{A})) = \mathfrak{h}_{R}(A)$$

$$\widetilde{\omega}_{R}(\mathfrak{u}_{R}^{\pm}(\widetilde{A})) = \widetilde{\mathfrak{u}}_{R}^{\pm}$$

$$\text{kernel } \widetilde{\omega}_{R} \subset \mathfrak{h}_{R}(\widetilde{A}).$$

Thus $\widetilde{\omega}_{R}$ restricted to $\mathfrak{u}_{R}^{\pm}(\widetilde{A})$ is an isomorphism which we denote by $\widetilde{\omega}_{R}^{\pm}$. We identify $\mathfrak{u}_{R}^{\pm}(\widetilde{A})$ with $\widetilde{\mathfrak{u}}^{\pm}$, by means of $\widetilde{\omega}_{R}^{\pm}$.

Finally, *when there is no danger of confusion*, we will write $\mathfrak{u}_{R}^{\pm}$ for $\mathfrak{u}_{R}^{\pm}(\widetilde{A})$, and (more simply) $\mathfrak{u}_{R}$ for $\mathfrak{u}_{R}^{+}(\widetilde{A})$, and $\widetilde{\mathfrak{u}}_{R}$ for $\widetilde{\mathfrak{u}}_{R}^{+}$.

5. Completions.

For $i \in \mathbf{Z}$, we define

$$\mathfrak{g}_{R}(\widetilde{A})_i = \{x \in \mathfrak{g}_{R}(\widetilde{A}) \mid D(x) = ix\},$$

$$\widetilde{\mathfrak{g}}_{R,i} = \{x \in \mathfrak{g}_{R} \mid D_0(x) = ix\}.$$

Of course, $\widetilde{\mathfrak{g}}_{R,i} = t^i \otimes \mathfrak{g}_{R}(A), \quad i \in \mathbf{Z},$

and $\mathfrak{g}_{R}(\widetilde{A})_i = t^i \otimes \mathfrak{g}_{R}(A), \quad i \in \mathbf{Z},\ i \neq 0$

(recall we have identified $\mathfrak{u}_{\mathrm{R}}^{\pm}(\widetilde{\mathrm{A}})$ with $\widetilde{\mathfrak{U}}^{\pm}$, by means of $\widetilde{\omega}_{\mathrm{R}}^{\pm}$). Of course, also, we have the direct sum decompositions

$$(5.1)\qquad \mathfrak{g}_{\mathrm{R}}(\widetilde{\mathrm{A}}) = \coprod_{i\in\mathbf{Z}} \mathfrak{g}_{\mathrm{R}}(\widetilde{\mathrm{A}})_i,$$

$$\widetilde{\mathfrak{g}}_{\mathrm{R}} = \coprod_{i\in\mathbf{Z}} \widetilde{\mathfrak{g}}_{\mathrm{R},i},$$

and $\widetilde{\omega}_{\mathrm{R}}(\mathfrak{g}_{\mathrm{R}}(\widetilde{\mathrm{A}})_i) = \widetilde{\mathfrak{g}}_{\mathrm{R},i}$. If $x\in\mathfrak{g}_{\mathrm{R}}(\widetilde{\mathrm{A}})$ (resp. $x\in\widetilde{\mathfrak{g}}_{\mathrm{R}}$), if we write $x=\sum_{i\in\mathbf{Z}} x_i$ (finite sum), with $x_i\in\mathfrak{g}_{\mathrm{R}}(\widetilde{\mathrm{A}})_i$ (resp. $x_i\in\widetilde{\mathfrak{g}}_{\mathrm{R},i}$), if we set $i_0=\inf\{j\,|\,x_j\neq 0\}$, and

$$(5.2)\qquad |x| = 2^{-i_0},$$

(if $x=0$, we set $|x|=0$), then we call $|x|$ the *norm* of x. We remark that the choice of 2 in (5.2) is arbitrary—any $p>1$ would do. We have

$$(5.3)\qquad |x+y|\leq \sup(|x|,|y|) \quad\text{and}\quad |[x,y]|\leq |x||y|$$
$$\text{for}\quad x,y\in\mathfrak{g}_{\mathrm{R}}(\widetilde{\mathrm{A}}) \text{ (resp. } \widetilde{\mathfrak{g}}_{\mathrm{R}}).$$

If R has no zero divisors, we have

$$(5.4)\qquad |r.x| = \begin{cases} |x|, & r\in\mathrm{R}-\{0\} \\ 0, & r=0. \end{cases}$$

We note that

$$(5.5)\qquad |\widetilde{\omega}_{\mathrm{R}}(x)|\leq |x|, \quad x\in\mathfrak{g}_{\mathrm{R}}(\widetilde{\mathrm{A}}),$$

so that $\widetilde{\omega}_{\mathrm{R}}$ is norm decreasing, and hence *uniformly* continuous. Moreover,

$$(5.6)\qquad |\widetilde{\omega}_{\mathrm{R}}(x)| = |x|, \quad x\in\mathfrak{u}_{\mathrm{R}}^{\pm}.$$

We let $\mathfrak{g}_{\mathrm{R}}^{c}(\widetilde{\mathrm{A}})$ (resp. $\widetilde{\mathfrak{g}}_{\mathrm{R}}^{c}$) denote the completion of $\mathfrak{g}_{\mathrm{R}}(\widetilde{\mathrm{A}})$ (resp. $\widetilde{\mathfrak{g}}_{\mathrm{R}}$) with respect to the norm $|\ |$. Then $\mathfrak{g}_{\mathrm{R}}^{c}(\widetilde{\mathrm{A}})$ and $\widetilde{\mathfrak{g}}_{\mathrm{R}}^{c}$ have induced Lie algebra structures and the norms $|\ |$ extend to these completions. Moreover, the extended norms also satisfy (5.3) and (5.4) (the latter, *provided* R has no divisors of zero). Also, by (5.5), $\widetilde{\omega}_{\mathrm{R}}$ has an extension to a Lie algebra homomorphism

$$(5.7)\qquad \widetilde{\omega}_{\mathrm{R}} : \mathfrak{g}_{\mathrm{R}}^{c}(\widetilde{\mathrm{A}}) \to \widetilde{\mathfrak{g}}_{\mathrm{R}}^{c},$$

and (5.5) is still valid for this extended homomorphism (so the extended $\widetilde{\omega}_{\mathrm{R}}$ is continuous). We let $\mathfrak{u}_{\mathrm{R}}^{c}$ (resp. $\widetilde{\mathfrak{U}}_{\mathrm{R}}^{c}$) denote the closure of $\mathfrak{u}_{\mathrm{R}}$ (resp. $\widetilde{\mathfrak{U}}_{\mathrm{R}}$) in $\mathfrak{g}_{\mathrm{R}}^{c}(\widetilde{\mathrm{A}})$ (resp. $\widetilde{\mathfrak{g}}^{c}$). We then have the direct sum decompositions

$$\mathfrak{g}_{\mathrm{R}}^{c}(\widetilde{\mathrm{A}}) = \mathfrak{u}_{\mathrm{R}}^{c}\oplus\mathfrak{h}_{\mathrm{R}}(\widetilde{\mathrm{A}})\oplus\mathfrak{u}_{\mathrm{R}}^{-},$$

$$\widetilde{\mathfrak{g}}_{\mathrm{R}}^{c} = \widetilde{\mathfrak{U}}_{\mathrm{R}}^{c}\oplus\mathfrak{h}_{\mathrm{R}}(\mathrm{A})\oplus\widetilde{\mathfrak{U}}_{\mathrm{R}}^{-},$$

and the extended homomorphism $\widetilde{\omega}_{\mathrm{R}}$ (in (5.7)), when restricted to $\mathfrak{u}_{\mathrm{R}}^{c}$, defines an *isomorphism*

$$\widetilde{\omega}_{\mathrm{R}}^{+} : \mathfrak{u}_{\mathrm{R}}^{c} \to \widetilde{\mathfrak{U}}_{\mathrm{R}}^{c}.$$

This last assertion follows from (5.6). It then follows that $\widetilde{\omega}_R$ in (5.7) is surjective and

$$\mathrm{kernel}(\widetilde{\omega}_R : \mathfrak{g}^c_R(\widetilde{A}) \to \widetilde{\mathfrak{g}}^c_R)$$

is contained in $\mathfrak{h}_R(\widetilde{A})$, and is equal to $\mathrm{kernel}(\widetilde{\omega}_R : \mathfrak{g}_R(\widetilde{A}) \to \widetilde{\mathfrak{g}}_R)$.

Remarks **(5.8)**. — (i) We let $\mathscr{L}_R = R[[t, t^{-1}]$ denote the t-adic completion of $R[t, t^{-1}]$; i.e., $\mathscr{L}_R$ is the ring of all formal Laurent series

(5.9) $$\sigma = \sum_{i \geq i_0} q_i t^i, \qquad q_i \in R,$$

where the sum on the right is allowed to be infinite. We note that $\widetilde{\mathfrak{g}}^c_R$ is isomorphic to $\mathfrak{g}_{\mathscr{L}_R}(A) = \mathscr{L}_R \otimes_R \mathfrak{g}_R(A) \cong \mathscr{L}_R \otimes_{\mathbf{Z}} \mathfrak{g}_{\mathbf{Z}}(A)$. Moreover, if for σ as in (5.9), we have $q_{i_0} \neq 0$, we define $|\sigma| = 2^{-i_0}$ (we set $|0| = 0$). Then $\mathscr{L}_R$ is the completion of $R[t, t^{-1}]$ with respect to this norm, and this norm on $\mathscr{L}_R$ induces our norm $|\ |$ on $\widetilde{\mathfrak{g}}^c_R$. (ii) From the proof of Lemma (4.10) in [7], we have

$$\frac{2h'_\iota}{(\alpha_0, \alpha_0)} = h_{\ell+1} + \sum_{i=1}^{\ell} k_i h_i, \qquad k_i \in \mathbf{Z}.$$

Therefore, the set

$$\{h_1, \ldots, h_\ell, 2h'_\iota/(\alpha_0, \alpha_0)\}$$

is a $\mathbf{Z}$-basis for $\mathfrak{h}_{\mathbf{Z}}(\widetilde{A})$. It then follows that $1 \otimes (2h'_\iota/(\alpha_0, \alpha_0))$ in $\mathfrak{h}_R(A) = R \otimes_{\mathbf{Z}} \mathfrak{h}_{\mathbf{Z}}(\widetilde{A})$, spans $\mathrm{kernel}(\widetilde{\omega}_R : \mathfrak{g}^c_R(\widetilde{A}) \to \widetilde{\mathfrak{g}}^c_R) = \mathrm{kernel}(\widetilde{\omega}_R : \mathfrak{g}_R(A) \to \widetilde{\mathfrak{g}}_R)$. If we call this kernel $\mathfrak{c}_R$, we have the two central extensions

(5.10) $$a)\quad 0 \to \mathfrak{c}_R \to \mathfrak{g}_R(\widetilde{A}) \xrightarrow{\widetilde{\omega}_R} \widetilde{\mathfrak{g}}_R \to 0,$$

$$b)\quad 0 \to \mathfrak{c}_R \to \mathfrak{g}^c_R(\widetilde{A}) \xrightarrow{\widetilde{\omega}_R} \widetilde{\mathfrak{g}}^c_R \to 0.$$

Remarks **(5.11)**. — (i) It follows from Remarks (5.8) (ii), that (identifying $\mathfrak{g}(\widetilde{A})$ with $\hat{\mathfrak{g}}_\tau = \widetilde{\mathfrak{g}} \oplus k$, over a field k of characteristic zero) we have $\mathfrak{g}_{\mathbf{Z}}(\widetilde{A}) = \widetilde{\mathfrak{g}}_{\mathbf{Z}} \oplus \mathbf{Z}$, where $(0, 1)$ in $\widetilde{\mathfrak{g}}_{\mathbf{Z}} \oplus \mathbf{Z}$ corresponds to the element $2h'_\iota/(\alpha_0, \alpha_0)$ in $\mathfrak{g}_{\mathbf{Z}}(\widetilde{A})$. Thus $\mathfrak{g}_R(\widetilde{A}) = \widetilde{\mathfrak{g}}_R \oplus R$, $\mathfrak{g}^c_R(\widetilde{A}) = \widetilde{\mathfrak{g}}^c_R \oplus R$, and we have naturally defined sections $\varphi : \widetilde{\mathfrak{g}}_R \to \mathfrak{g}_R(\widetilde{A})$, $\varphi : \widetilde{\mathfrak{g}}^c_R \to \mathfrak{g}^c_R(\widetilde{A})$, each defined by $\varphi(x) = (x, 0)$ ($x \in \widetilde{\mathfrak{g}}_R, \widetilde{\mathfrak{g}}^c_R$, respectively). The first section is just the restriction of the second, and hence the cocycle corresponding to φ on $\widetilde{\mathfrak{g}}_R$ is just the restriction of the cocycle corresponding to φ on $\widetilde{\mathfrak{g}}^c_R$. The latter is the cocycle τ_φ defined by

$$\tau_\varphi(u \otimes x, v \otimes y) = -\mathrm{Res}(u\, dv)(x, y)\frac{(\alpha_0, \alpha_0)}{2}, \qquad u, v \in \mathscr{L}_R,\ x, y \in \mathfrak{g}_{\mathbf{Z}}(A).$$

(ii) If $R = k$ is a field of characteristic zero, then the central extension (5.10) *b)* is universal in the category of central extensions

(5.12) $$0 \to Z \to \widetilde{\mathfrak{g}}' \xrightarrow{\rho} \widetilde{\mathfrak{g}}^c_R \to 0,$$

which split over a sufficiently deep congruence subalgebra of $\tilde{\mathfrak{g}}_k^c$. More precisely, we set $\mathcal{O} \subset \mathcal{L}_k$ equal to the ring of formal power series

$$\sigma = \sum_{i \geq 0} q_i t^i, \qquad q_i \in k,$$

and consider $\mathfrak{g}_{\mathcal{O}}(A) \subset \mathfrak{g}_{\mathcal{L}_k}(A) = \tilde{\mathfrak{g}}_k^c$.

By the *congruence subalgebra* of $\tilde{\mathfrak{g}}_k^c$ of *level n*, we will mean the subalgebra of all $x \in \mathfrak{g}_{\mathcal{O}}(A)$, such that $x \equiv 0 \bmod t^n$. We then consider the category of central extensions (5.12) which split over the congruence subalgebra of level n, for some n, and we claim the central extension (5.10) *b)* (for $R = k$) is universal in this category. The argument is essentially the same as in § 2. There is just one additional point. Thus, define $\{u, v\}$, $u, v \in \mathcal{L}_k$, exactly as in (2.5). Then in order to prove (2.18), one can argue exactly as in § 2, once one proves that $\{\ ,\ \}$ is continuous on $\mathcal{L}_k \times \mathcal{L}_k$ in the sense that if $u_i \to u$, $v_i \to v$, are convergent sequences in $\mathcal{L}_k$ (relative to the t-adic topology) then $\{u_i, v_i\} = \{u, v\}$, for i sufficiently large. To see this, one first notes that there exists an integer $n > 0$ so that $\{u, v\} = 0$, whenever $u, v \in \mathcal{O}$ and $u, v \equiv 0 \bmod t^n$. But then the desired continuity is an easy consequence of this and the second identity of (2.12), for $u, v, w \in \mathcal{L}_k$.

6. Representation theory.

In [11], Kac introduced highest weight modules for a Kac-Moody Lie algebra $\mathfrak{g}(B)$ associated with a generalized $\ell \times \ell$ Cartan matrix B. Kac's modules are described in [8] and [7], § 10. We refer to the latter as a general reference. Briefly, the construction is as follows: We let

$$\mathfrak{u}^{\pm}(B) = \coprod_{a \in \Delta_{\pm}(B)} \mathfrak{g}^a$$
$$\mathfrak{p}^e = \mathfrak{h}^e(B) \oplus \mathfrak{u}^+(B),$$

so $\mathfrak{u}^{\pm}(B)$, and $\mathfrak{p}^e$ are subalgebras of $\mathfrak{g}^e(B)$. For any Lie algebra $\mathfrak{a}$ over a field k, we let $\mathcal{U}(\mathfrak{a})$ denote the universal enveloping algebra of $\mathfrak{a}$. We say $\lambda \in \mathfrak{h}^e(B)^*$ is *dominant integral*, in case, (i) $\lambda(h_i) \in \mathbf{Z}$, $i = 1, \ldots, \ell$, and (ii) $\lambda(h_i) \geq 0$, $i = 1, \ldots, \ell$. If λ only satisfies the first condition, we say λ is *integral*. We let $\mathbf{D} = \mathbf{D}(B)$ denote the family of dominant integral linear functionals in $\mathfrak{h}^e(B)^*$. For $\lambda \in \mathbf{D}$, we let $M(\lambda)$ denote the one-dimensional $\mathfrak{p}^e$-module, with $\mathfrak{p}^e$-action defined by

$$h.v = \lambda(h)v, \qquad h \in \mathfrak{h}^e(B), \qquad v \in M(\lambda),$$
$$\xi.v = 0, \qquad \xi \in \mathfrak{u}^+(B), \qquad v \in M(\lambda).$$

We let $\mathcal{P}^e = \mathcal{U}(\mathfrak{p}^e)$, and set

$$V^{M(\lambda)} = \mathcal{U}(\mathfrak{g}^e(B)) \otimes_{\mathcal{P}^e} M(\lambda).$$

Left multiplication gives $V^{M(\lambda)}$ a $\mathcal{U}(\mathfrak{g}^e(B))$ (or equivalently, $\mathfrak{g}^e(B)$) module structure. As a $\mathfrak{g}^e(B)$-module, $V^{M(\lambda)}$ contains a largest submodule not intersectiong the subspace $1\otimes M(\lambda)\subset V^{M(\lambda)}$ nontrivially. We let V^λ denote the quotient module of $V^{M(\lambda)}$ by this submodule. We call V^λ the highest weight module with (dominant integral) highest weight λ.

We now focus on the case when $B=\widetilde{A}$, the affine Cartan matrix associated with the classical Cartan matrix A. *From now on*, we take $\mathfrak{g}(\widetilde{A})$ to be the corresponding Lie algebra over **C**. We also fix a *highest weight vector* $v_0\neq 0$ (by definition, a highest weight vector is non zero) in the highest weight module V^λ. Thus we fix $v\neq 0$ in $M(\lambda)$, and let v_0 denote the image of $1\otimes v$ in V^λ.

In [7], we introduced a **Z**-form $\mathcal{U}_{\mathbf{Z}}(\widetilde{A})$ of $\mathcal{U}(\mathfrak{g}(\widetilde{A}))$, and the **Z**-form

$$V^\lambda_{\mathbf{Z}}=\mathcal{U}_{\mathbf{Z}}(\widetilde{A}).v_0$$

of V^λ. For $\mu\in\mathfrak{h}^e(\widetilde{A})^*$, we let V^λ_μ denote the subspace

$$V^\lambda_\mu=\{v\in V^\lambda\,|\,h.v=\mu(h)v,\ h\in\mathfrak{h}^e(\widetilde{A})\}.$$

If $V^\lambda_\mu\neq 0$, we call μ a weight of V^λ and we call V^λ_μ the weight space (for the weight μ). We call nonzero elements of V^λ_μ weight vectors (of weight μ). Then (see [8]), V^λ is a direct sum

$$V^\lambda=\coprod_{\mu\in\mathfrak{h}^e(\widetilde{A})^*}V^\lambda_\mu,$$

of its weight spaces. We know (see [7], Theorem (11.3) and its proof) that $V^\lambda_{\mathbf{Z}}$ is the **Z**-span of an admissible basis, i.e., of a basis Ω of V^λ which is the union of its intersections $\Omega_\mu=\Omega\cap V^\lambda_\mu$ with the weight spaces of V^λ (see [7], § 11, Definition (11.2)). We note that if we set $V^\lambda_{\mu,\mathbf{Z}}=V^\lambda_\mu\cap V^\lambda_{\mathbf{Z}}$, then it is a consequence of our last assertion that

(6.1) $$V^\lambda_{\mathbf{Z}}=\coprod_{\mu\in\mathfrak{h}^e(\widetilde{A})^*}V^\lambda_{\mu,\mathbf{Z}}.$$

For a commutative ring R with unit, we set $V^\lambda_R=R\otimes_{\mathbf{Z}}V^\lambda_{\mathbf{Z}}$ and

(6.2) $$V^\lambda_{\mu,R}=R\otimes_{\mathbf{Z}}V^\lambda_{\mu,\mathbf{Z}}.$$

From (6.1) we then have

(6.3) $$V^\lambda_R=\coprod_{\mu\in\mathfrak{h}^e(\widetilde{A})^*}V^\lambda_{\mu,R}.$$

Of course, in this direct sum, it suffices to let μ vary over the weights of V^λ.

Similarly, the **Z**-module $\mathfrak{h}_{\mathbf{Z}}(\widetilde{A})=_{\mathrm{df}}\mathfrak{g}_{\mathbf{Z}}(\widetilde{A})\cap\mathfrak{h}(\widetilde{A})$ spans $\mathfrak{h}(\widetilde{A})$, and for each root $a\in\Delta(\widetilde{A})$ the **Z**-module $\mathfrak{g}^a_{\mathbf{Z}}=_{\mathrm{df}}\mathfrak{g}_{\mathbf{Z}}(\widetilde{A})\cap\mathfrak{g}^a$ spans $\mathfrak{g}^a$. Moreover, we have

$$\mathfrak{h}(\widetilde{A})=\mathbf{C}\otimes_{\mathbf{Z}}\mathfrak{h}_{\mathbf{Z}}(\widetilde{A})$$

$$\mathfrak{g}^a=\mathbf{C}\otimes_{\mathbf{Z}}\mathfrak{g}^a_{\mathbf{Z}},\quad a\in\Delta(\widetilde{A}),$$

and $$\mathfrak{g}_{\mathbf{Z}}(\widetilde{A})=\mathfrak{h}_{\mathbf{Z}}(\widetilde{A})\oplus\coprod_{a\in\Delta(A)}\mathfrak{g}^a_{\mathbf{Z}}.$$

If R is a commutative ring with unit, and if we set

$$\mathfrak{h}_{\mathrm{R}}(\mathrm{A}) = \mathrm{R} \otimes_{\mathbf{Z}} \mathfrak{h}_{\mathbf{Z}}(\mathrm{A}) \qquad \text{and} \qquad \mathfrak{g}_{\mathrm{R}}^{a} = \mathrm{R} \otimes_{\mathbf{Z}} \mathfrak{g}_{\mathbf{Z}}^{a},$$

then

$$\mathfrak{g}_{\mathrm{R}}(\widetilde{\mathrm{A}}) = \mathfrak{h}_{\mathrm{R}}(\widetilde{\mathrm{A}}) \oplus \coprod_{a \in \Delta(\widetilde{\mathrm{A}})} \mathfrak{g}_{\mathrm{R}}^{a}.$$

We also note that if a is a Weyl root then $\mathfrak{g}_{\mathrm{R}}^{a}$, as an R-module, is isomorphic to R, while if a is an imaginary root, then $\mathfrak{g}_{\mathrm{R}}^{a}$ is a free R-module of rank ℓ. When considered as an R-module, $\mathfrak{h}_{\mathrm{R}}(\widetilde{\mathrm{A}})$ is free, of rank $\ell+1$. We also note that

$$\textbf{(6.4)} \qquad \mathfrak{g}_{\mathrm{R}}(\widetilde{\mathrm{A}})_j = \coprod_{\substack{a = a(\alpha) + j\iota \\ \alpha \in \Delta(\widetilde{\mathrm{A}})}}, \qquad \mathfrak{g}_{\mathrm{R}}^{a} \oplus \coprod_{a = j\iota} \mathfrak{g}_{\mathrm{R}}^{a}, \qquad \text{if} \qquad j \neq 0,$$

while

$$\textbf{(6.5)} \qquad \mathfrak{g}_{\mathrm{R}}(\widetilde{\mathrm{A}})_0 = \mathfrak{h}_{\mathrm{R}}(\widetilde{\mathrm{A}}) \oplus \coprod_{\substack{a = a(\alpha) \\ \alpha \in \Delta(\mathrm{A})}} \mathfrak{g}_{\mathrm{R}}^{a}.$$

Alternatively, we have

$$\mathfrak{g}_{\mathrm{R}}(\widetilde{\mathrm{A}})_0 = \widetilde{\omega}_{\mathrm{R}}^{-1}(\mathfrak{g}_{\mathrm{R}}(\mathrm{A})).$$

We note that $\mathfrak{g}_{\mathrm{R}}(\widetilde{\mathrm{A}})$ acts on $\mathrm{V}_{\mathrm{R}}^{\lambda}$.

Proposition **(6.6)**. — *Let* $\{x_n\}_{n=1,2,3,\dots}$, *be a Cauchy sequence in* $\mathfrak{g}_{\mathrm{R}}(\widetilde{\mathrm{A}})$ *(relative to the norm* $|\ |$ *of* (5.2)*). Then for each* $v \in \mathrm{V}_{\mathrm{R}}^{\lambda}$, *the sequence of elements* $x_n . v$ *in* $\mathrm{V}_{\mathrm{R}}^{\lambda}$ *is eventually constant.*

Proof. — As we noted in [8], § 10 (and as is easily seen from the definitions), each weight μ of V^{λ} is of the form

$$\textbf{(6.7)} \qquad \mu = \lambda - \sum_{i=1}^{\ell+1} k_i a_i, \qquad k_i \geq 0,\ k_i \in \mathbf{Z}.$$

Also, we set

$$\mathfrak{g}_{\mathbf{Z}}^{e}(\widetilde{\mathrm{A}}) = \mathfrak{g}_{\mathbf{Z}}(\widetilde{\mathrm{A}}) \oplus \mathbf{Z}\mathrm{D}$$
$$\mathfrak{g}_{\mathrm{R}}^{e}(\widetilde{\mathrm{A}}) = \mathrm{R} \otimes_{\mathbf{Z}} \mathfrak{g}_{\mathbf{Z}}^{e}(\widetilde{\mathrm{A}})$$
$$\mathfrak{h}_{\mathbf{Z}}^{e}(\widetilde{\mathrm{A}}) = \mathfrak{h}_{\mathbf{Z}}(\widetilde{\mathrm{A}}) \oplus \mathbf{Z}\mathrm{D}$$
$$\mathfrak{h}_{\mathrm{R}}^{e}(\widetilde{\mathrm{A}}) = \mathrm{R} \otimes_{\mathbf{Z}} \mathfrak{h}_{\mathbf{Z}}^{e}(\widetilde{\mathrm{A}}),$$

and note that if $\lambda(\mathrm{D}) \in \mathbf{Z}$, then $\mathrm{V}_{\mathrm{R}}^{\lambda}$ is a module for the R-Lie algebra $\mathfrak{g}_{\mathrm{R}}^{e}(\widetilde{\mathrm{A}})$. *From now on, we assume* $\lambda(\mathrm{D}) \in \mathbf{Z}$ (so D acts on $\mathrm{V}_{\mathbf{Z}}^{\lambda}$). We set

$$\mathrm{V}_j^{\lambda} = \{v \in \mathrm{V}^{\lambda} \mid \mathrm{D}(v) = (\lambda(\mathrm{D}) + j)v\},$$
$$\mathrm{V}_{j,\mathbf{Z}}^{\lambda} = \mathrm{V}_j^{\lambda} \cap \mathrm{V}_{\mathbf{Z}}^{\lambda},$$
$$\mathrm{V}_{j,\mathrm{R}}^{\lambda} = \mathrm{R} \otimes_{\mathbf{Z}} \mathrm{V}_{j,\mathbf{Z}}^{\lambda}, \qquad j \geq 0,\ j \in \mathbf{Z}.$$

We say that the weight μ of V^{λ} has D-level j, in case $\mu = \lambda - \sum_{i=1}^{\ell+1} k_i a_i$, and $-k_{\ell+1} = j$; then, thanks to (6.7),

$$\textbf{(6.8)} \qquad \mathrm{V}_j^{\lambda} = \coprod_{\substack{\mu = \text{weight} \\ \text{of D-level } j}} \mathrm{V}_{\mu}^{\lambda},$$

and therefore

$$\textbf{(6.9)}\qquad \begin{aligned} V^\lambda &= \coprod_{j\leq 0} V^\lambda_j,\\ V^\lambda_j &= \mathbf{C}\otimes_{\mathbf{Z}} V^\lambda_{j,\mathbf{Z}},\\ V^\lambda_{\mathbf{Z}} &= \coprod_{j\leq 0} V^\lambda_{j,\mathbf{Z}},\\ V^\lambda_{R} &= \coprod_{j\leq 0} V^\lambda_{j,R}. \end{aligned}$$

Indeed, the first equality of (6.9) follows from (6.8) and the fact that V^λ is a direct sum of its weight spaces. The second equality follows from (6.8), (6.1), and the fact that $V^\lambda_\mu = \mathbf{C}\otimes_{\mathbf{Z}} V^\lambda_{\mu,\mathbf{Z}}$. The third equality follows from (6.1) and (6.8), and the fourth from the third. Also, we have

$$\textbf{(6.10)}\qquad \mathfrak{g}_R(\widetilde{A})_i \,.\, V^\lambda_{j,R} \subset V^\lambda_{i+j,R},$$

as one can check directly.

To prove the proposition, we may, thanks to the fourth equality of (6.9), assume $v\in V^\lambda_{j,R}$ for some $j\leq 0$ (in $\mathbf{Z}$). Then, since $\{x_n\}_{n=1,2,3,\dots}$ is a Cauchy sequence, we may choose n_0 so that for all $n\geq n_0$ we have:

$$x_n - x_{n_0} \in \coprod_{i+j>0} \mathfrak{g}_R(\widetilde{A})_i.$$

But from the last equality of (6.9), and from (6.10), we have

$$\mathfrak{g}_R(\widetilde{A})_i \,.\, V^\lambda_{j,R} = 0 \quad \text{if} \quad i+j>0.$$

Thus, for $n\geq n_0$,

$$x_n \,.\, v = x_{n_0} \,.\, v,$$

and this proves the proposition. ∎

If $x\in \mathfrak{g}^c_R(\widetilde{A})$, let $\{x_n\}_{n=1,2,3,\dots}$, be a Cauchy sequence in $\mathfrak{g}_R(\widetilde{A})$ such that $\lim_{n\to\infty} x_n = x$. Then for $v\in V^\lambda_R$ the proposition implies that $x_n . v$ is eventually constant. It is easy to see that this constant value is independent of our choice of Cauchy sequence converging to x. We set $x . v$ equal to this constant value, and in this way we obtain a $\mathfrak{g}^c_R(\widetilde{A})$-module structure on V^λ_R. We let

$$\pi^\lambda_R : \mathfrak{g}^c_R(\widetilde{A}) \to \operatorname{End} V^\lambda_R$$

denote the corresponding representation of $\mathfrak{g}^c_R(\widetilde{A})$.

Notational Remark. — We also denote the restriction of π^λ_R to $\mathfrak{g}_R(\widetilde{A})$ by π^λ_R. When $R=\mathbf{C}$, we write π^λ for π^λ_R and $\mathfrak{g}^c(\widetilde{A})$ for $\mathfrak{g}^c_R(\widetilde{A})$.

Proposition **(6.11)**. — *If* $\lambda\in\mathbf{D}$ *and* $\lambda(h_i)\neq 0$ *for some* $i=1,\dots,\ell+1$, *then* $\pi^\lambda : \mathfrak{g}^c(\widetilde{A}) \to \operatorname{End} V^\lambda$ *is injective.*

Proof. — Choose a basis of V^λ consisting of weight vectors and order this basis so that the vectors in any given weight space appear in succession. If $x \in \mathfrak{g}^c(\widetilde{A})$, we may write x as

$$x = x_{\mathfrak{h}} + \sum_{a \in \Delta(\widetilde{A})} x^a,$$

where $x_{\mathfrak{h}} \in \mathfrak{h}(\widetilde{A})$, and $x^a \in \mathfrak{g}^a$, $a \in \Delta(\widetilde{A})$, and all but finitely many x^a with $a \in \Delta_-(\widetilde{A})$ are non-zero (however, infinitely many x^a with $a \in \Delta_+(\widetilde{A})$ may be non-zero). Relative to the basis of V^λ which we have just constructed, $\pi^\lambda(x)$ is represented by an infinite matrix with a natural block decomposition, each block corresponding to a pair of weights. Then the transformations $\pi^\lambda(x_{\mathfrak{h}})$, $\pi^\lambda(x^a)$, $a \in \Delta(\widetilde{A})$, correspond to various mutually distinct blocks of the matrix representing $\pi^\lambda(x)$. Thus, if $\pi^\lambda(x) = 0$, then $\pi^\lambda(x_{\mathfrak{h}}) = 0$ and $\pi^\lambda(x^a) = 0$, for all $a \in \Delta(\widetilde{A})$. Thus, if π^λ is not injective, we may assume $\pi^\lambda(x) = 0$ where either $x \in \mathfrak{h}(\widetilde{A})$ or $x \in \mathfrak{g}^a$ for some $a \in \Delta(\widetilde{A})$, and $x \neq 0$. We now show:

(6.12) If π^λ is not injective, then $\pi^\lambda(x) = 0$ for some non-zero element x in $\mathfrak{g}^a$, $a \in \Delta_W(\widetilde{A})$.

By the above remark, we may assume $\pi^\lambda(y) = 0$ for some $y \neq 0$ and either $y \in \mathfrak{h}(\widetilde{A})$ or $y \in \mathfrak{g}^a$, $a \in \Delta_I(A)$. Also, h_i' is a nonnegative linear combination of $h_1, \ldots, h_{\ell+1}$. Hence $h_i' \notin \text{kernel}\, \pi^\lambda$, since $\lambda(h_i) \geq 0$ for all $i = 1, \ldots, \ell+1$, and $\lambda(h_i) > 0$ for some $i = 1, \ldots, \ell+1$. It follows that if either $y \in \mathfrak{h}(\widetilde{A})$ or $y \in \mathfrak{g}^a$, $a \in \Delta_I(\widetilde{A})$, $y \neq 0$, and $\pi^\lambda(y) = 0$, then one of the elements $[y, e_i]$, $i = 1, \ldots, \ell+1$, is non-zero. Since $(\text{kernel}\, \pi^\lambda) \cap \mathfrak{g}(\widetilde{A})$ is an ideal, we obtain (6.12).

Thus, assume x is a non-zero element in $\mathfrak{g}^a$, $a \in \Delta_W(\widetilde{A})$, and $\pi^\lambda(x) = 0$. We will show π^λ is then identically zero, and this contradiction will prove the proposition. First, we will show we may take a to be a simple root $a = a_i$, for some $i = 1, \ldots, \ell+1$. Toward this end, we first recall that in [7], § 6, we defined for each $b \in \Delta_W(\widetilde{A})$, an automorphism $\widetilde{r}_b$ of $\mathfrak{g}_Z(\widetilde{A})$, by

$$\widetilde{r}_b = \exp(\text{ad}\, \xi_b)\exp(-\text{ad}\, \xi_b)\exp(\text{ad}\, \xi_b).$$

Of course we may also regard $\widetilde{r}_b$ as an automorphism of $\mathfrak{g}(\widetilde{A})$, and then by Lemma (6.9) in [7], we have

(6.13) $$\widetilde{r}_b(h) = h - b(h) h_b, \quad b \in \Delta_W(\widetilde{A}), \quad h \in \mathfrak{h}(\widetilde{A}),$$

where $h_b = 2h_b'/\sigma(b, b)$, for $b \in \Delta_W(\widetilde{A})$. On the other hand, for each $i = 1, \ldots, \ell+1$, we define a linear automorphism

$$r_i : \mathfrak{h}^e(\widetilde{A}) \to \mathfrak{h}^e(\widetilde{A}),$$

by

(6.14) $$\lambda(r_i(h)) = (r_i\lambda)(h), \quad h \in \mathfrak{h}^e(\widetilde{A}), \quad \lambda \in \mathfrak{h}^e(\widetilde{A})^*.$$

A direct computation, using (3.3) (i.e., the definition of r_i on $\mathfrak{h}^e(\widetilde{A})^*$) then shows that

$$r_i(h) = h - a_i(h) h, \quad i = 1, \ldots, \ell+1, \quad h \in \mathfrak{h}^e(\widetilde{A}).$$

Comparing with (6.13), and setting $\widetilde{r}_i = \widetilde{r}_{a_i}$, $i=1, \ldots, \ell+1$, we see that

$$\widetilde{r}_i(h) = r_i(h), \quad i=1, \ldots, \ell+1, \quad h \in \mathfrak{h}^e(\widetilde{A}).$$

Thus, if $w = r_{i_1} \ldots r_{i_k} \in W$, and if we set $\widetilde{w} = \widetilde{r}_{i_1} \ldots \widetilde{r}_{i_k}$, then

(6.15) $$\widetilde{w}(h) = w(h), \quad h \in \mathfrak{h}^e(\widetilde{A}).$$

From this, from the fact that $\widetilde{w}$ preserves $\mathfrak{g}_{\mathbf{Z}}(\widetilde{A})$, and from the fact that ξ_b is primitive in $\mathfrak{g}_{\mathbf{Z}}(\widetilde{A})$, for each $b \in \Delta_W(\widetilde{A})$, we easily obtain

(6.16) $$\widetilde{w}(\xi_b) = \pm \xi_{w(b)}, \quad b \in \Delta_W(\widetilde{A})$$

(here, we only need $\widetilde{w}(\mathfrak{g}^b) = \mathfrak{g}^{w(b)}$, but we will use the stronger assertion (6.16) later on).

Now we are assuming $\pi^\lambda(x) = 0$ for some non-zero $x \in \mathfrak{g}^a$, $a \in \Delta_W(\widetilde{A})$. We choose $w \in W$ such that $w(a)$ equals the simple root a_i. Then, by (6.16), we have $\widetilde{w}(x) \in \mathfrak{g}^{a_i}$. On the other hand, $\pi^\lambda(x) = 0$ implies $\pi^\lambda(\widetilde{w}(x)) = 0$, since kernel π^λ is an ideal in $\mathfrak{g}(\widetilde{A})$; i.e., we have shown that if kernel $\pi^\lambda \neq 0$, then for some $i = 1, \ldots, \ell+1$, we have $\mathfrak{g}^{a_i} \subset \text{kernel } \pi^\lambda$.

We let $\theta \subset \{1, \ldots, \ell+1\}$ be the subset of all j such that $\mathfrak{g}^{a_j} \not\subset \text{kernel } \pi^\lambda$. We let θ' denote the complement of θ. We shall now prove that θ is empty. If not, we may choose $i \in \theta$, $j \in \theta'$, such that

(6.17) $$\widetilde{A}_{ij} \neq 0$$

(we have assumed $\mathfrak{g} = \mathfrak{g}(A)$ is simple). We let $[\theta]$ denote the set of all roots $a \in \Delta(\widetilde{A})$ which are linear combinations of the a_m, with $m \in \theta$. We set

$$\mathscr{I} = \coprod_{a \in \Delta_+(\widetilde{A}) - [\theta]} \mathfrak{g}^a,$$

and we note that $\mathscr{I} \subset \text{kernel } \pi^\lambda$. But by (6.16) and (6.17) we have

$$\widetilde{r}_j(\mathfrak{g}^{a_i}) \subset \mathscr{I}.$$

Hence $\mathfrak{g}^{a_i} \subset \text{kernel } \pi^\lambda$, and this is a contradiction. Hence θ must be empty.

Thus $\mathfrak{g}^{a_i} \subset \text{kernel } \pi^\lambda$, for all $i = 1, \ldots, \ell+1$. But then $\mathfrak{g}^{-a_i} = \widetilde{r}_i(\mathfrak{g}^{a_i})$ (by (6.16)) is contained in kernel π^λ, for all $i = 1, \ldots, \ell+1$. Since the elements e_i, f_i, $i = 1, \ldots, \ell+1$ generate $\mathfrak{g}(\widetilde{A})$, all of $\mathfrak{g}(\widetilde{A})$ is contained in kernel π^λ; i.e., we have shown that if π^λ is not injective, then π^λ is identically zero on $\mathfrak{g}(\widetilde{A})$. But then $\pi^\lambda(h_i) = 0$, for $i = 1, \ldots, \ell+1$, and hence $\lambda(h_i) = 0$, $i = 1, \ldots, \ell+1$. This is a contradiction, and hence π^λ must be injective. ■

7. Chevalley groups.

From now on we make the following assumptions and notational conventions: We let $\mathfrak{g}(\widetilde{A})$ denote the Kac-Moody Lie algebra over **C**, corresponding to the affine Cartan matrix A. Until § 16, no restriction will be made on k, except in the first part of § 9. We assume $\lambda \in \mathbf{D}$ satisfies $\lambda(h_i) \neq 0$ for some $i = 1, \ldots, \ell+1$. Otherwise we continue with our earlier notation; e.g., $\mathfrak{g}_k(\widetilde{A}) = k \otimes_{\mathbf{Z}} \mathfrak{g}_{\mathbf{Z}}(\widetilde{A})$.

For each $a \in \Delta_W(\widetilde{A})$ we have defined (see § 4) $\xi_a \in \mathfrak{g}_{\mathbf{Z}}^a$, and we now also let ξ_a denote $1 \otimes \xi_a \in \mathfrak{g}_k^a = k \otimes_{\mathbf{Z}} \mathfrak{g}_{\mathbf{Z}}^a$. Now let $\lambda \in \mathbf{D}$. Then we have seen ([7], Lemma (10.4)) that for all $v \in V^\lambda$, there exists a positive integer r such that

$$(7.1) \qquad \xi_{\pm a_i}^r . v = 0, \qquad i = 1, \ldots, \ell+1.$$

In particular, this holds for $v \in V_{\mathbf{Z}}^\lambda$, and hence for $v \in V_k^\lambda$. Now, for $a \in \Delta_W(\widetilde{A})$, $\xi_a^n/n! \in \mathscr{U}_{\mathbf{Z}}(\widetilde{A})$ maps $V_{\mathbf{Z}}^\lambda$ to itself (see [7], § 11), and hence, for any commutative ring R with unit, $\xi_a^n/n!$ defines an endomorphism $\pi_R^\lambda(\xi_a^n/n!)$ of V_R^λ. Indeed, we obtain a representation π_R^λ of $\mathscr{U}_R(\widetilde{A}) =_{df} R \otimes_{\mathbf{Z}} \mathscr{U}_{\mathbf{Z}}(\widetilde{A})$, in V_R^λ. In particular, this is true for $R = k$.

We note also that a similar situation holds for the adjoint representation. As usual, for $\xi, \eta \in \mathfrak{g}(\widetilde{A})$, we set $\mathrm{ad}(\xi)(\eta) = [\xi, \eta]$. We then obtain a corresponding representation of $\mathscr{U}(\mathfrak{g}(\widetilde{A}))$ which we still denote by ad. For example, if $\xi, \eta \in \mathfrak{g}(\widetilde{A})$, then

$$\mathrm{ad}(\xi^n)(\eta) = \overbrace{[\xi[\xi \ldots [\xi}^{n\text{-times}}, \eta] \ldots]].$$

Then $\mathrm{ad}(\xi_a^n/n!)$ maps $\mathfrak{g}_{\mathbf{Z}}(\widetilde{A})$ to itself, and hence for any commutative ring R with unit we define an endomorphism $\mathrm{ad}_R(\xi_a^n/n!)$ of $\mathfrak{g}_R(\widetilde{A})$. Indeed, we obtain a representation ad_R of $\mathscr{U}_R(\widetilde{A})$ on $\mathfrak{g}_R(\widetilde{A})$.

For $s \in k$, we then have from (7.1) that

$$\chi_{\pm a_i}(s) =_{df} \sum_{n \geq 0} s^n \pi_k^\lambda(\xi_{\pm a_i}^n/n!), \qquad i = 1, \ldots, \ell+1,$$

is a well-defined endomorphism of V_k^λ. Moreover, we have

$$(7.2) \qquad \chi_{\pm a_i}(s_1 + s_2) = \chi_{\pm a_i}(s_1)\chi_{\pm a_i}(s_2), \qquad s_1, s_2 \in k, \; i = 1, \ldots, \ell+1.$$

Thus each $\chi_{\pm a_i}(s)$ has an inverse (namely $\chi_{\pm a_i}(-s)$) and hence $\chi_{\pm a_i}(s)$ is a k-vector space automorphism of V_k^λ.

We let $\mathrm{End}_k(V_k^\lambda)$ (resp. $\mathrm{Aut}_k(V_k^\lambda)$) denote the k-vector space endomorphisms (resp. automorphisms) of V_k^λ. For $\chi \in \mathrm{Aut}_k(V_k^\lambda)$ and $B \in \mathrm{End}_k(V_k^\lambda)$ we let $\mathrm{Ad}(\chi)(B) = \chi B \chi^{-1}$. We shall show that, for $\xi \in \mathfrak{g}_k(\widetilde{A})$,

$$(7.3) \qquad \mathrm{Ad}(\chi_{\pm a_i}(s))(\pi_k^\lambda(\xi)) = \sum_{n \geq 0} s^n \pi_k^\lambda(\mathrm{ad}(\xi_{\pm a_i}^n/n!)\xi) \text{ (finite sum)}, \qquad s \in k, \; i = 1, \ldots, \ell+1.$$

We note that only finitely many of the expressions $\mathrm{ad}(\xi_{\pm a_i}^n/n!)(\xi)$ are non-zero. Thus, for $b \in \Delta_W(\widetilde{A})$, we may define an endomorphism $\mathscr{Y}_b(s)$ of $\mathfrak{g}_k(\widetilde{A})$, by

$$\mathscr{Y}_b(s)(\xi) = \sum_{n \geq 0} s^n \, \mathrm{ad}(\xi_b^n/n!)(\xi), \qquad s \in k, \; i = 1, \ldots, \ell+1.$$

Analogous to (7.2) we have

$$(7.4) \qquad \mathscr{Y}_b(s_1 + s_2) = \mathscr{Y}_b(s_1)\mathscr{Y}_b(s_2), \qquad s_1, s_2 \in k, \; i = 1, \ldots, \ell+1,$$

and so $\mathscr{Y}_b(s)^{-1}=\mathscr{Y}_b(-s)$, and $\mathscr{Y}_b(s)$ is an automorphism of $\mathfrak{g}_k(\widetilde{A})$. We may rewrite (7.3) as

(7.3') $$\mathrm{Ad}(\chi_{\pm a_i}(s))(\pi_k^\lambda(\xi))=\pi_k^\lambda(\mathscr{Y}_{\pm a_i}(s)(\xi)), \quad \xi\in\mathfrak{g}_k(\widetilde{A}),\ s\in k,\ i=1,\ldots,\ell+1.$$

We prove (7.3') by regarding it as a formal identity in s, and computing the derivatives of both sides with respect to s. The (formal) nth derivative of the left side is

(7.5) $$\mathrm{Ad}(\chi_{\pm a_i}(s))(\pi_k^\lambda((\mathrm{ad}\,\xi_{\pm a_i})^n(\xi))),$$

and the (formal) nth derivative of the right side is

(7.6) $$\pi_k^\lambda(\mathscr{Y}_{\pm a_i}(s)((\mathrm{ad}\,\xi_{\pm a_i})^n(\xi))).$$

We see that the expressions (7.5) and (7.6) are equal when $s=0$. This implies (7.3'), and hence (7.3).

Now for each $i=1,\ldots,\ell+1$, we define $r_i^\lambda\in\mathrm{Aut}_k(V_k^\lambda)$ by

$$r_i^\lambda=\chi_{a_i}(1)\chi_{-a_i}(-1)\chi_{a_i}(1).$$

On the other hand, in § 6, we defined automorphisms $\widetilde{r}_b$, $b\in\Delta_W(\widetilde{A})$, of $\mathfrak{g}_{\mathbf{Z}}(\widetilde{A})$. Tensoring with k we then obtain automorphisms of $\mathfrak{g}_k(\widetilde{A})$, which we again denote by $\widetilde{r}_b$. As in § 6, we set $\widetilde{r}_i=\widetilde{r}_{a_i}$, $i=1,\ldots,\ell+1$. Clearly

$$\widetilde{r}_b=\mathscr{Y}_b(1)\mathscr{Y}_{-b}(-1)\mathscr{Y}_b(1),$$

and hence (7.3') implies

(7.7) $$\mathrm{Ad}(r_i^\lambda)(\pi_k^\lambda(\xi))=\pi_k^\lambda(\widetilde{r}_i(\xi)), \quad \xi\in\mathfrak{g}_k(\widetilde{A}),\ i=1,\ldots,\ell+1.$$

From this and from (6.16) we have

(7.8) $$\mathrm{Ad}(r_i^\lambda)(\pi_k^\lambda(\xi_a))=\pm\pi_k^\lambda(\xi_{r_i(a)}), \quad i=1,\ldots,\ell+1,\ a\in\Delta_W(\widetilde{A}).$$

If

(7.9) $$w=r_{i_1}\ldots r_{i_j}\in W,$$

we set

(7.9') $$w^\lambda=r_{i_1}^\lambda\ldots r_{i_j}^\lambda.$$

A priori, w^λ depends on the expression (7.9) for w in terms of the r_i's. In any case, we have from (7.8) that

(7.10) $$\mathrm{Ad}(w^\lambda)(\pi_k^\lambda(\xi_a))=\pm\pi_k^\lambda(\xi_{w(a)}), \quad a\in\Delta_W(\widetilde{A}).$$

Now if $a\in\Delta_W(\widetilde{A})$, then $a=w(a_i)$ for some $w\in W$, and $i=1,\ldots,\ell+1$. Thus, if we write w as in (7.9), and then consider the corresponding w^λ given in (7.9'), we have from (7.10) that

(7.11) $$\pi_k^\lambda(\xi_a)=\pm\mathrm{Ad}(w^\lambda)(\pi_k^\lambda(\xi_{a_i})).$$

It then follows from (7.1), that we have the

Lemma **(7.12)**. — *For all* $v \in V_k^\lambda$, $a \in \Delta_W(\widetilde{A})$, *there exists an integer* $r>0$, *such that*

(7.13) $$\xi_a^r . v = 0.$$

Thus, if we set

(7.14) $$\chi_a(s) = \sum_{n \geq 0} s^n \pi_k^\lambda(\xi_a^n / n!), \quad s \in k, \ a \in \Delta_W(\widetilde{A}),$$

then the sum on the right is a well-defined element of $\mathrm{End}_k(V_k^\lambda)$. Moreover, (7.11) implies

$$\chi_a(s) = w^\lambda \chi_{a_i}(\pm s)(w^\lambda)^{-1},$$

and it follows from (7.2) that

(7.15) $$\chi_a(s_1+s_2) = \chi_a(s_1)\chi_a(s_2), \quad a \in \Delta_W(\widetilde{A}), \ s_1, s_2 \in k,$$

and of course this implies, as before, that $\chi_a(s) \in \mathrm{Aut}_k(V_k^\lambda)$, for each $s \in k$.

We have:

Lemma **(7.16)**. — *Let* $\alpha \in \Delta(A)$. *For all* $v \in V_k^\lambda$, *there exists an integer* j_0, *so that if* $j \geq j_0$ *and* $a = a(\alpha) + j\iota$, *then* $\xi_a . v = 0$.

Proof. — This lemma is a consequence of (6.9), (6.10), and the fact that if $a = a(\alpha) + j\iota$, then $\xi_a \in \mathfrak{g}_R(\widetilde{A})_j$. ■

Corollary. — *Let* $\alpha \in \Delta(A)$. *For all* $v \in V_k^\lambda$, *there exists an integer* j_0, *so that if* $j \geq j_0$ *and* $a = a(\alpha) + j\iota$, *then*

(7.17) $$\chi_a(s) . v = v,$$

for all $s \in k$.

In § 5, we have set $\mathscr{L}_R = R[[t, t^{-1}]]$ equal to the ring of all formal Laurent series with coefficients in the commutative ring R with unit (see (5.9)). We have also, in § 5, defined a norm $|\ |$ on $\mathscr{L}_R$. We let

$$\mathcal{O} = \mathcal{O}_R = \{x \in \mathscr{L}_R \mid |x| \leq 1\}$$
$$\mathscr{P} = \{x \in \mathcal{O} \mid |x| < 1\}$$
$$\mathcal{O}^* = \mathcal{O} - \mathscr{P}.$$

Then $\mathcal{O}$ is the ring of formal power series in $\mathscr{L}_R$ (the "integers" of $\mathscr{L}_R$), the set $\mathscr{P}$ is the (unique) maximal ideal of $\mathcal{O}$, and $\mathcal{O}^*$ is the group of units in $\mathcal{O}$.

We will now consider $\mathscr{L}_k$; i.e., we take R to be the field k (which is *not*, we recall, necessarily of characteristic zero). If we are given $\sigma(t) \in \mathscr{L}_k$, where

(7.18) $$\sigma(t) = \sum_{j \geq j_0} q_j t^j, \quad q_j \in k,$$

and if $\alpha \in \Delta(A)$, we define $\chi_\alpha^\lambda(\sigma(t)) = \chi_\alpha(\sigma(t)) \in \mathrm{Aut}_k(V_k^\lambda)$ by

(7.19) $$\chi_\alpha^\lambda(\sigma(t)) = \chi_\alpha(\sigma(t)) = \prod_{j \geq j_0} \chi_{a(\alpha)+j\iota}(q_j).$$

We drop the superscript λ when there is no danger of confusion.

By the Corollary to Lemma (7.16), $\chi_\alpha(\sigma(t))$ is a well defined automorphism of V_k^λ, and using the fact that

$$[\xi_{a(\alpha)+j\iota}, \xi_{a(\alpha)+j'\iota}]=0, \quad j,j'\in\mathbf{Z},$$

and using (7.15), we obtain

(7.20) $$\chi_\alpha(\sigma_1(t))\chi_\alpha(\sigma_2(t))=\chi_\alpha(\sigma_1(t)+\sigma_2(t)), \quad \sigma_i(t)\in\mathcal{L}_k,\ i=1,2,\ \alpha\in\Delta(A).$$

The following definition is fundamental in this paper:

Definition **(7.21)**. — *We let* $\hat{G}=\hat{G}_k^\lambda=\hat{G}_k^\lambda(\tilde{A})\subset \operatorname{Aut} V_k^\lambda$ *denote the subgroup generated by the elements* $\chi_\alpha(\sigma(t))$, $\alpha\in\Delta(A)$, $\sigma(t)\in\mathcal{L}_k$. *We call* $\hat{G}$ *the (complete)* Chevalley group *over* k *(with respect to* π_k^λ*).*

For $\alpha\in\Delta(A)$, and $\sigma(t)\in\mathcal{L}_k$ with $\sigma(t)\neq 0$, we set

(7.22) $$w_\alpha^\lambda(\sigma(t))=w_\alpha(\sigma(t))=\chi_\alpha(\sigma(t))\chi_{-\alpha}(-\sigma(t)^{-1})\chi_\alpha(\sigma(t)),$$
$$h_\alpha^\lambda(\sigma(t))=h_\alpha(\sigma(t))=w_\alpha(\sigma(t))w_\alpha(1)^{-1}.$$

We drop the superscript λ, when there is no danger of confusion. Also, for $a\in\Delta_W(\tilde{A})$, $s\in k$, $s\neq 0$, we set

(7.23) $$w_a^\lambda(s)=w_a(s)=\chi_a(s)\chi_{-a}(-s^{-1})\chi_a(s),$$
$$h_a^\lambda(s)=h_a(s)=w_a(s)w_a(1)^{-1}.$$

Again, we omit the superscript λ when there is no danger of confusion.

We then have the following important

Definition **(7.24)**. — *We let* $\mathcal{I}\subset\hat{G}$ *denote the subgroup generated by the elements* $\chi_\alpha(\sigma(t))$ *where either* $\alpha\in\Delta_+(A)$, $\sigma(t)\in\mathcal{O}$ *or* $\alpha\in\Delta_-(A)$, $\sigma(t)\in\mathcal{P}$, *by the elements* $h_\alpha(\sigma(t))$, $\sigma(t)\in\mathcal{O}^*$, $\alpha\in\Delta_+(A)$, *and by the elements* $h_{a_{\ell+1}}(s)$, $s\in k^*$ $(k^*=_{\mathrm{df}}k-\{0\})$. *We call* $\mathcal{I}$ *the* Iwahori subgroup *of* $\hat{G}$.

Remark. — Obviously our definition of Iwahori subgroup is analogous to, and motivated by the corresponding notion introduced by Iwahori, Bruhat and Tits (see e.g., [5]). We shall make the relationship more precise later on (see §§ 13, 14, below).

8. The adjoint representation.

In § 7 we defined, for each $a\in\Delta_W(\tilde{A})$, a one-parameter group of automorphisms $\chi_a(s)$, $s\in k$, of V_k^λ, and a one-parameter group of automorphisms $\mathcal{Y}_a(s)$, $s\in k$, of $\mathfrak{g}_k(\tilde{A})$. In analogy with the way we defined the automorphisms $\chi_\alpha(\sigma(t))$, $\alpha\in\Delta(A)$, $\sigma(t)\in\mathcal{L}_k$ in terms of the $\chi_a(s)$, we now wish to define automorphisms $\mathcal{Y}_\alpha(\sigma(t))$ of $\mathfrak{g}_k^c(\tilde{A})$, in terms of the $\mathcal{Y}_a(s)$. We also wish to show that the $\chi_\alpha(\sigma(t))$ and $\mathcal{Y}_\alpha(\sigma(t))$ are related by an appropriate analogue of (7.3').

First we note the following generalization of (7.3′) (the proof being the same as for (7.3′)):

(8.1) $$\mathrm{Ad}(\chi_a(s))(\pi_k^\lambda(\xi)) = \pi_k^\lambda(\mathscr{Y}_a(s)(\xi)), \quad \xi \in \mathfrak{g}_k(\widetilde{\mathrm{A}}),\ a \in \Delta_{\mathrm{W}}(\widetilde{\mathrm{A}}),\ s \in k.$$

Next, we note that relative to the norm $|\ |$ on $\mathfrak{g}_k(\mathrm{A})$, defined in § 5, the automorphism $\mathscr{Y}_a(s)$, $a \in \Delta_{\mathrm{W}}(\widetilde{\mathrm{A}})$, $s \in k$, of $\mathfrak{g}_k(\widetilde{\mathrm{A}})$, is bounded; i.e., there exists $\mathrm{C} > 0$ so that

$$|\mathscr{Y}_a(s)(\xi)| \leq \mathrm{C}|\xi|, \quad \text{for all} \quad \xi \in \mathfrak{g}_k(\widetilde{\mathrm{A}}).$$

Hence $\mathscr{Y}_a(s)$ has a unique extension to a bounded operator on $\mathfrak{g}_k^c(\widetilde{\mathrm{A}})$, and we denote this extension also by $\mathscr{Y}_a(s)$.

Now let $\sigma(t) \in \mathscr{L}_k$ be given as in (7.18), and let $\alpha \in \Delta(\mathrm{A})$. We define the endomorphism $\mathscr{Y}_\alpha(\sigma(t))$ of $\mathfrak{g}_k^c(\widetilde{\mathrm{A}})$ by

(8.2) $$\mathscr{Y}_\alpha(\sigma(t)) = \prod_{j \geq j_0} \mathscr{Y}_{a(\alpha)+j\iota}(q_j).$$

Of course we must show that $\mathscr{Y}_\alpha(\sigma(t))$ is well defined. But if $\eta \in \mathfrak{g}_k^c(\widetilde{\mathrm{A}})$, and if for $i \geq j_0$, we set

$$\eta_i = \Big(\prod_{i \geq j \geq j_0} \mathscr{Y}_{a(\alpha)+j\iota}(q_j)\Big)(\eta),$$

then the sequence η_i, $i = j_0, j_0 + 1, \dots$ is convergent (relative to $|\ |$). Moreover, from (7.4) we have

(8.3) $$\mathscr{Y}_\alpha(\sigma_1(t))\mathscr{Y}_\alpha(\sigma_2(t)) = \mathscr{Y}_\alpha(\sigma_1(t) + \sigma_2(t)), \quad \sigma_i(t) \in \mathscr{L}_k,\ i = 1, 2,\ \alpha \in \Delta(\mathrm{A}).$$

Thus $\mathscr{Y}_\alpha(\sigma(t))^{-1} = \mathscr{Y}_\alpha(-\sigma(t))$, and $\mathscr{Y}_\alpha(\sigma(t))$ is an automorphism of $\mathfrak{g}_k^c(\widetilde{\mathrm{A}})$, for $\alpha \in \Delta(\mathrm{A})$, $\sigma(t) \in \mathscr{L}_k$.

Actually, for the same reason that $\mathscr{Y}_\alpha(\sigma(t))$ is well defined, we have:

(8.4) Let $\alpha \in \Delta(\mathrm{A})$. If $\sigma_i(t)$, $i = 1, 2, \dots$, is a sequence in $\mathscr{L}_k$, and if
$$\lim_i \sigma_i(t) = \sigma(t) \in \mathscr{L}_k$$
(relative to the norm $|\ |$ on $\mathscr{L}_k$), then for all $\eta \in \mathfrak{g}_k^c(\widetilde{\mathrm{A}})$, we have
$$\lim_i (\mathscr{Y}_\alpha(\sigma_i(t)))(\eta) = (\mathscr{Y}_\alpha(\sigma(t)))(\eta)$$
(relative to the norm $|\ |$ on $\mathfrak{g}_k^c(\widetilde{\mathrm{A}})$).

Similarly, from the Corollary to Lemma (7.16), we have

(8.4′) Let $\alpha \in \Delta(\mathrm{A})$. If $\sigma_i(t)$, $i = 1, 2, \dots$ is a sequence in $\mathscr{L}_k$, and if
$$\lim_i \sigma_i(t) = \sigma(t) \in \mathscr{L}_k$$
(relative to the norm $|\ |$ on $\mathscr{L}_k$), then for all $v \in \mathrm{V}_k^\lambda$, we have
$$\lim_i \chi_\alpha(\sigma_i(t))(v) = \chi_\alpha(\sigma(t))(v),$$
in the sense that, for i sufficiently large,
$$\chi_\alpha(\sigma_i(t))(v) = \chi_\alpha(\sigma(t))(v).$$

We note that Lemma (7.16) implies that the action of $\mathfrak{g}_k^c(\widetilde{A})$ on V_k^λ is continuous in the sense that

(8.5) If ξ_i, $i=1, 2, \ldots$, is a sequence in $\mathfrak{g}_k^c(\widetilde{A})$, if $\lim_i \xi_i = \xi \in \mathfrak{g}_k^c(A)$ (relative to the norm $|\ |$), and if $v \in V_k^\lambda$, then for i sufficiently large, $\xi_i . v = \xi . v$.

Now let $\alpha \in \Delta(A)$, $\sigma(t) \in \mathscr{L}_k$, and choose a sequence $\sigma_i(t)$, $i=1, 2, \ldots$, in $k[t, t^{-1}]$, such that $\lim_i \sigma_i(t) = \sigma(t)$ (relative to $|\ |$). Then for $\xi \in \mathfrak{g}_k^c(A)$, $v \in V_k^\lambda$,

$$\begin{aligned} \mathrm{Ad}\,\chi_\alpha(\sigma(t))(\pi_k^\lambda(\xi)).v &= \lim_i \mathrm{Ad}\,\chi_\alpha(\sigma_i(t))(\pi_k^\lambda(\xi)).v, \quad \text{by } (8.4') \\ &= \lim_i \pi_k^\lambda(\mathscr{Y}_\alpha(\sigma_i(t))\xi).v, \quad \text{by } (8.1) \\ &= \pi_k^\lambda(\lim_i(\mathscr{Y}_\alpha(\sigma_i(t))\xi)).v, \quad \text{by } (8.5) \\ &= \pi_k^\lambda(\mathscr{Y}_\alpha(\sigma(t))\xi).v, \quad \text{by } (8.4), \end{aligned}$$

and we thus have proved:

Lemma **(8.6)**. — *If* $\alpha \in \Delta(A)$, $\sigma(t) \in \mathscr{L}_k$, *and* $\xi \in \mathfrak{g}_k^c(\widetilde{A})$, *then:*

(8.7) $$\mathrm{Ad}\,\chi_\alpha(\sigma(t))(\pi_k^\lambda(\xi)) = \pi_k^\lambda(\mathscr{Y}_\alpha(\sigma(t))\xi).$$

When there is no danger of confusion, we will write $\mathscr{L}$ for $\mathscr{L}_k$. We then have

$$\widetilde{\mathfrak{g}}_k^c = \mathfrak{g}_{\mathscr{L}}(A).$$

For each $\alpha \in \Delta(A)$ and $\sigma(t) \in \mathscr{L}$, we define an automorphism $\mathscr{X}_\alpha(\sigma(t))$ of $\mathfrak{g}_{\mathscr{L}}(A)$ by

$$\mathscr{X}_\alpha(\sigma(t)) = \exp(\mathrm{ad}\ \sigma(t)E_\alpha).$$

On the other hand, as we observed in Remark (5.8) (see (5.10)) we have the central extensions (for $R=k$):

(8.8) $$a)\ 0 \to \mathfrak{c}_k \to \mathfrak{g}_k(\widetilde{A}) \xrightarrow{\widetilde{\omega}_k} \widetilde{\mathfrak{g}}_k \to 0,$$

$$b)\ 0 \to \mathfrak{c}_k \to \mathfrak{g}_k^c(\widetilde{A}) \xrightarrow{\widetilde{\omega}_k} \mathfrak{g}_{\mathscr{L}}(A) \to 0.$$

Moreover, as we noted in Remark (5.11), we have

$$\mathfrak{g}_k^c(\widetilde{A}) = \widetilde{\mathfrak{g}}_k^c \oplus k = \mathfrak{g}_{\mathscr{L}}(A) \oplus k,$$

and $\widetilde{\omega}_k$ is just the projection onto the first factor. From this, one easily sees that

$$\widetilde{\omega}_k(\exp q\ \mathrm{ad}\ \xi_{a(\alpha)+j\iota}(\eta)) = (\exp q\ \mathrm{ad}\ t^j \otimes E_\alpha)(\widetilde{\omega}_k(\eta)),$$
$$\eta \in \mathfrak{g}_k^c(\widetilde{A}),\ j \in \mathbf{Z},\ \alpha \in \Delta(A),\ q \in k.$$

Then, from this, from (8.4) and its obvious analogue for $\mathscr{X}_\alpha(\sigma(t))$, and from the continuity of $\widetilde{\omega}_k$, we obtain:

(8.9) $$\widetilde{\omega}_k(\mathscr{Y}_\alpha(\sigma(t))(\eta)) = \mathscr{X}_\alpha(\sigma(t))\widetilde{\omega}_k(\eta), \quad \alpha \in \Delta(A),\ \sigma(t) \in \mathscr{L},\ \eta \in \mathfrak{g}_k^c(\widetilde{A}).$$

Recall from § 1, that $\mathfrak{g}_k^c(\widetilde{A})$ is said to be perfect if

$$\textbf{(8.10)} \qquad [\mathfrak{g}_k^c(\widetilde{A}), \mathfrak{g}_k^c(\widetilde{A})] = \mathfrak{g}_k^c(\widetilde{A}).$$

Lemma **(8.11)**. — *If* $\varkappa : \mathfrak{g}_{\mathscr{L}}(A) \to \mathfrak{g}_{\mathscr{L}}(A)$ *is a Lie algebra automorphism (regarding* $\mathfrak{g}_{\mathscr{L}}(A)$ *as a Lie algebra over k), and if* $\mathfrak{g}_k^c(\widetilde{A})$ *is perfect, then there is at most one Lie algebra automorphism* $\varkappa' : \mathfrak{g}_k^c(\widetilde{A}) \to \mathfrak{g}_k^c(\widetilde{A})$, *such that*

$$\textbf{(8.12)} \qquad \widetilde{\omega}_k \circ \varkappa' = \varkappa \circ \widetilde{\omega}_k.$$

Proof. — Assume $\varkappa'' : \mathfrak{g}_k^c(\widetilde{A}) \to \mathfrak{g}_k^c(\widetilde{A})$ is a second automorphism satisfying (8.12) (with $\varkappa''$ in place of $\varkappa'$). Then set $\varkappa_0 = \varkappa'' \circ (\varkappa')^{-1}$, and note that

$$\widetilde{\omega}_k \circ \varkappa_0 = \widetilde{\omega}_k.$$

It then follows from Lemma (1.5) and our assumption that $\mathfrak{g}_k^c(\widetilde{A})$ is perfect ((8.10)), that $\varkappa_0 = \text{identity}$. ∎

We let $G_{ad,\mathscr{L}} = G_{ad,\mathscr{L}}(A) \subset \mathrm{Aut}(\mathfrak{g}_{\mathscr{L}}(A))$ denote the group of automorphisms generated by the automorphisms $\mathscr{X}_\alpha(\sigma(t))$, $\sigma \in \Delta(A)$, $\sigma(t) \in \mathscr{L}$. We let

$$G_{ad}(\widetilde{A}) \subset \mathrm{Aut}(\mathfrak{g}_k^c(\widetilde{A}))$$

denote the subgroup generated by the automorphisms $\mathscr{Y}_\alpha(\sigma(t))$, $\alpha \in \Delta(A)$, $\sigma(t) \in \mathscr{L}$. For any automorphism $\mathscr{Y} \in G_{ad}(\widetilde{A})$, we have that $\mathscr{Y}$ maps $\mathfrak{c}_k$ into itself (and in fact is the identity on $\mathfrak{c}_k$, since $\mathfrak{c}_k$ is in the center of $\mathfrak{g}_k^c(\widetilde{A})$), and hence $\mathscr{Y}$ induces an automorphism $\mathscr{X}$ of $\mathfrak{g}_{\mathscr{L}}(A)$; i.e., there is a unique automorphism $\mathscr{X}$ of $\mathfrak{g}_{\mathscr{L}}(A)$ such that

$$\textbf{(8.13)} \qquad \widetilde{\omega}_k(\mathscr{Y}(\eta)) = \mathscr{X}(\widetilde{\omega}_k(\eta)), \qquad \eta \in \mathfrak{g}_k^c(\widetilde{A}).$$

In this way we obtain a homomorphism

$$\Phi' : G_{ad}(\widetilde{A}) \to G_{ad,\mathscr{L}}(A),$$

where for $\mathscr{Y} \in G_{ad}(\widetilde{A})$, the image $\mathscr{X} = \Phi'(\mathscr{Y})$ is defined by (8.13). But then by (8.9) and the fact that (by definition of $G_{ad}(\widetilde{A})$) the $\mathscr{Y}_\alpha(\sigma(t))$ generate $G_{ad}(\widetilde{A})$, we have that Φ' is the unique homomorphism from $G_{ad}(\widetilde{A})$ to $G_{ad,\mathscr{L}}(A)$, such that

$$\Phi'(\mathscr{Y}_\alpha(\sigma(t))) = \mathscr{X}_\alpha(\sigma(t)), \qquad \alpha \in \Delta(A),\ \sigma(t) \in \mathscr{L}.$$

Then, by Lemma (8.11), Φ' is injective. Thanks to the fact that (by definition of $G_{ad,\mathscr{L}}(A)$) the $\mathscr{X}_\alpha(\sigma(t))$ generate $G_{ad,\mathscr{L}}(A)$, it is also surjective. Thus *if* $\mathfrak{g}_k^c(\widetilde{A})$ *is perfect*, we have:

Lemma **(8.14)**. — *There is a unique group isomorphism* $\Phi' : G_{ad}(\widetilde{A}) \to G_{ad,\mathscr{L}}(A)$ *of* $G_{ad}(\widetilde{A})$ *onto* $G_{ad,\mathscr{L}}$, *such that*

$$\Phi'(\mathscr{Y}_\alpha(\sigma(t))) = \mathscr{X}_\alpha(\sigma(t)), \qquad \alpha \in \Delta(A),\ \sigma(t) \in \mathscr{L}.$$

Remark. — We have given the proof except to note that the uniqueness follows from the fact that the $\mathscr{Y}_\alpha(\sigma(t))$ generate $G_{ad}(\widetilde{A})$.

9. Schur's Lemma and an important identity.

For the first part of this section we consider the special case when $k = \mathbf{C}$. We fix $\lambda \in \mathbf{D}$ and, as specified at the beginning of § 7, we always assume that $\lambda(h_i) \neq 0$ for some $i = 1, \ldots, \ell + 1$. From Proposition (6.11), we then have that

$$\pi^\lambda : \mathfrak{g}^e(\widetilde{A}) \to \operatorname{End} V^\lambda$$

is injective, and we therefore identify $\mathfrak{g}^e(\widetilde{A})$ with its image $\pi^\lambda(\mathfrak{g}^e(\widetilde{A}))$. In Definition (7.21), we introduced the group $\widehat{G} = \widehat{G}^\lambda_{\mathbf{C}} \subset \operatorname{Aut} V^\lambda$. We now prove:

Lemma **(9.1)** *(Schur's Lemma). — If $g \in \operatorname{End} V^\lambda$ commutes with x, for each $x \in \mathfrak{g}^e(\widetilde{A})$, then g is a scalar multiple of the identity.*

Proof. — In [7], § 12, we proved the existence of a positive-definite, Hermitian inner product $\{\ ,\ \}$ on V^λ, and of an involutive, conjugate-linear, anti-automorphism * of $\mathfrak{g}(\widetilde{A})$ (where for $x \in \mathfrak{g}(\widetilde{A})$, we let x^* denote the image of x under *) such that the weight spaces of V^λ are mutually orthogonal, such that

(9.2) $$\{x.v_1, v_2\} = \{v_1, x^*.v_2\}, \quad v_1, v_2 \in V^\lambda,\ x \in \mathfrak{g}(\widetilde{A}),$$

and such that

(9.3) $$\mathfrak{u}^\pm(\widetilde{A})^* = \mathfrak{u}^\mp(\widetilde{A}).$$

Moreover, we saw in [7], § 12, that * has an extension to an involutive, conjugate-linear, anti-automorphism (again denoted by *) of $\mathscr{U}(\mathfrak{g}(\widetilde{A}))$. Again, if $u \in \mathscr{U}(\mathfrak{g}(\widetilde{A}))$, we let u^* denote the image of u under *.

Now,

(9.4) $$V^\lambda = \mathscr{U}(\mathfrak{g}(\widetilde{A})).v_0,$$

since, e.g., V^λ is a quotient of $V^{M(\lambda)}$. Thus, to prove Lemma (9.1), it suffices to prove that $g.v_0$ is a scalar multiple of v_0. But for this, it suffices to prove that for any weight $\mu \neq \lambda$ of V^λ, and for $v_\mu \in V^\lambda_\mu$, we have

(9.5) $$\{g.v_0, v_\mu\} = 0.$$

However, we may write v_μ as

$$v_\mu = u.v_0,$$

where $u \in \mathscr{U}(\mathfrak{u}^-(\widetilde{A}))$ is a linear combination of products of elements of $\mathfrak{u}^-(\widetilde{A})$. Thanks to (9.3), u^* is a linear combination of products of elements of $\mathfrak{u}^+(\widetilde{A})$, and hence

(9.6) $$u^*.v_0 = 0.$$

But then

$$\{g.v_0, v_\mu\} = \{g.v_0, u.v_0\} = \{u^*.(g.v_0), v_0\}, \quad \text{by (9.2)}$$
$$= \{g.(u^*.v_0), v_0\}, \quad \text{since } g \text{ commutes with } \mathfrak{g}(\widetilde{A}),$$
$$= 0, \quad \text{by (9.6)},$$

and this proves (9.5), and hence the Lemma. ■

In § 7 (formula (7.20)), we showed that $\chi_\alpha(\sigma(t))$ is additive in $\sigma(t)$. We shall now use Lemma (9.1) to prove a second important identity for the $\chi_\alpha(\sigma(t))$. Thus, let $\alpha, \beta \in \Delta(A)$, with $\alpha+\beta \neq 0$, and let $\sigma_i = \sigma_i(t)$, $i = 1, 2$, be elements of $\mathcal{L}_k$, where we now allow k to be an arbitrary field. For automorphisms A, B of V_k^λ, we let (A, B) denote the commutator $ABA^{-1}B^{-1}$.

Lemma **(9.7)**. — *We have*

(9.8) $$(\chi_\alpha(\sigma_1), \chi_\beta(\sigma_2)) = \Pi \chi_{i\alpha+j\beta}(c_{ij}\sigma_1^i \sigma_2^j),$$

where the product on the right is taken over all roots $i\alpha+j\beta$, $i, j \in \mathbf{Z}$, $i, j > 0$, arranged in some fixed order, and the c_{ij}'s are integers which depend on α, β, and the fixed order, but not on k, σ_1, σ_2. Furthermore c_{11} satisfies

$$[E_\alpha, E_\beta] = c_{11} E_{\alpha+\beta};$$

i.e., the integer c_{11} coincides with $\pm(r+1)$, in (4.1).

Proof. — First, assume σ_1, σ_2 are Laurent polynomials in t and t^{-1},

(9.9) $$\sigma_1 = \Sigma q_j t^j$$
$$\sigma_2 = \Sigma p_j t^j,$$

where the sums on the right are finite, and the p_j's and q_j's are complex numbers. We let $R = \mathbf{Z}[p_j, q_j]$, denote the ring obtained by adjoining the p_j's and q_j's to $\mathbf{Z}$. Then $V_R^\lambda \subset V^\lambda$, and the automorphisms $\chi_\alpha(\sigma_1)$ and $\chi_\beta(\sigma_2)$, defined in (7.19), leave V_R^λ invariant.

Now thanks to (8.7) and Lemma (8.14), we have

$$\Phi' \circ \mathrm{Ad}(\chi_\alpha(\sigma_1)) = \mathcal{X}_\alpha(\sigma_1),$$
$$\Phi' \circ \mathrm{Ad}(\chi_\beta(\sigma_2)) = \mathcal{X}_\beta(\sigma_2).$$

Also, thanks to [21], Lemma (15), page 22, we have that Lemma (9.7) holds for $\mathcal{X}_\alpha(\sigma_1)$, $\mathcal{X}_\beta(\sigma_2)$ in place of $\chi_\alpha(\sigma_1)$, $\chi_\beta(\sigma_2)$, respectively. It then follows from Lemma (8.14), that Lemma (9.7) holds for $\mathcal{Y}_\alpha(\sigma_1)$, $\mathcal{Y}_\beta(\sigma_2)$ in place of $\chi_\alpha(\sigma_1)$, $\chi_\beta(\sigma_2)$, respectively (note that $\mathfrak{g}_k^c(\widetilde{A})$ is perfect, since $k = \mathbf{C}$). Then it follows from Lemma (9.1) that Lemma (9.7) holds for $\chi_\alpha(\sigma_1)$ and $\chi_\beta(\sigma_2)$ modulo scalars (given our special choice of σ_1, σ_2). Thus there is a complex numbers $\gamma = \gamma(\sigma_1, \sigma_2)$, such that

$$(\chi_\alpha(\sigma_1), \chi_\beta(\sigma_2))(\Pi \chi_{i\alpha+j\beta}(-c_{ij}\sigma_1^i \sigma_2^j)) = \gamma(\sigma_1, \sigma_2) I,$$

where I denotes the identity operator. However, the left side of this equation leaves V_R^λ invariant, and defines an automorphism of V_R^λ. It follows that $\gamma(\sigma_1, \sigma_2)$ is a unit of R. Now choose the p_j's and q_j's to be algebraically independent over **Z**. Then we must have $\gamma(\sigma_1, \sigma_2) = \pm 1$. Specializing the p_j's and q_j's to be zero, we find $\gamma(\sigma_1, \sigma_2) = 1$. Again, by specializing, we see that in fact Lemma (9.7) holds over any field k, provided σ_1, σ_2 are of the form (9.9), with the expressions on the right being finite sums. But then, thanks to (8.4′), we obtain Lemma (9.7). ■

10. Relations with the work of Matsumoto, Moore, and Steinberg.

In this section we wish to apply (7.20) and Lemma (9.7) to show that $\hat{G}_k^\lambda$ is a central extension of a classical Chevalley group. More precisely, if A is a classical Cartan matrix, we let $G_{\mathscr{L}_k} = G_{\mathscr{L}_k}(A)$ be the abstract group generated by symbols $\mathscr{X}'_\alpha(\sigma)$, $\alpha \in \Delta(A)$, $\sigma \in \mathscr{L}_k$ so that one has the defining relations:

A) $$\mathscr{X}'_\alpha(\sigma+\tau) = \mathscr{X}'_\alpha(\sigma)\mathscr{X}'_\alpha(\tau), \quad \alpha \in \Delta(A), \ \sigma, \tau \in \mathscr{L}_k,$$

B) $$(\mathscr{X}'_\alpha(\sigma_1), \mathscr{X}'_\beta(\sigma_2)) = \prod_{\substack{i,j>0 \\ i\alpha+j\beta \in \Delta(A)}} \mathscr{X}'_{i\alpha+j\beta}(c_{ij}\sigma_1^i\sigma_2^j), \quad \alpha \neq -\beta,$$

(where the order of the right-hand product, and the c_{ij} are as in Lemma (9.7)), provided A is *not* the 1×1 matrix $A = (2)$, in which case (B) is vacuous. But, if for any $\alpha \in \Delta(A)$ and non-zero $\sigma \in \mathscr{L}_k$, we set $w'_\alpha(\sigma) = \mathscr{X}'_\alpha(\sigma)\mathscr{X}'_{-\alpha}(-\sigma^{-1})\mathscr{X}'_\alpha(\sigma)$, then for $A = (2)$, we have:

B′) $$w'_\alpha(\sigma)\mathscr{X}'_\alpha(\tau)w'_\alpha(\sigma)^{-1} = \mathscr{X}'_{-\alpha}(-\tau\sigma^{-2}),$$

and finally, if we set $h'_\alpha(\sigma) = w'_\alpha(\sigma)w'_\alpha(1)^{-1}$, then

C) $$h'_\alpha(\sigma\tau) = h'_\alpha(\sigma)h'_\alpha(\tau), \quad \sigma, \tau \in \mathscr{L}_k.$$

Alternatively, if G denotes the Chevalley group scheme over **Z**, such that $G_{\mathbf{C}}$ is the simply connected topological group corresponding to A, then $G_{\mathscr{L}_k}$, defined above, is isomorphic to the group of $\mathscr{L}_k$ rational points of G (see [20]). Moreover, if we define $E(G_{\mathscr{L}_k})$ to be the group generated by objects $\chi_\alpha^e(\sigma)$, $\alpha \in \Delta(A)$, $\sigma \in \mathscr{L}_k$, which, for $\chi_\alpha^e(\sigma)$ in place of $\mathscr{X}'_\alpha(\sigma)$, satisfy the relations A) and B) (or B′), if $A = (2)$), but *not* C), then clearly there is a unique homomorphism

$$\varphi^e : E(G_{\mathscr{L}_k}) \to G_{\mathscr{L}_k},$$

such that

$$\varphi^e(\chi_\alpha^e(\sigma)) = \mathscr{X}'_\alpha(\sigma), \quad \sigma \in \mathscr{L}_k, \ \alpha \in \Delta(A).$$

Moreover, Steinberg in [20], shows that $E(G_{\mathscr{L}_k})$ is the universal covering, in the sense of Moore [18], of $G_{\mathscr{L}_k}$. We also have

Lemma (**10.1**). — *There is a unique homomorphism*

$$\Psi^e : E(G_{\mathscr{L}_k}) \to \hat{G}_k^\lambda,$$

such that

$$\Psi^e(\chi^e_\alpha(\sigma)) = \chi_\alpha(\sigma), \quad \alpha \in \Delta(A), \ \sigma \in \mathscr{L}_k.$$

Proof. — If $A \neq (2)$, then the Lemma follows from Lemma (9.7) and from (7.20). We are thus left to prove the Lemma in the case when $A = (2)$. We let A_1 denote the 2×2 classical Cartan matrix

$$A_1 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

and, as in § 3, we let E_1, E_2, F_1, F_2, H_1, H_2 denote the generators of $\mathfrak{g}(A_1)$. Similarly, we let E, F, H denote the generators of $\mathfrak{g}(A)$. We then have an injection $\iota : \mathfrak{g}(A) \hookrightarrow \mathfrak{g}(A_1)$, defined by the conditions

$$\iota(E) = E_1, \qquad \iota(F) = F_1, \qquad \iota(H) = H_1;$$

then ι defines an injection

$$\tilde{\iota} : \mathbf{C}[t, t^{-1}] \otimes \mathfrak{g}(A) \hookrightarrow \mathbf{C}[t, t^{-1}] \otimes \mathfrak{g}(A_1),$$

where $\tilde{\iota} = I \otimes \iota$, and I denotes the identity map of $\mathbf{C}[t, t^{-1}]$. In turn, $\tilde{\iota}$ induces an injective Lie algebra homomorphism

$$\hat{\iota} : \mathfrak{g}(\tilde{A}) \hookrightarrow \mathfrak{g}(\tilde{A}_1).$$

Using the identification of $\mathfrak{g}(\tilde{A})$ (resp. of $\mathfrak{g}(\tilde{A}_1)$) with the universal covering of $\tilde{\mathfrak{g}}(A) =_{\mathrm{df}} \mathbf{C}[t, t^{-1}] \otimes \mathfrak{g}(A)$ (resp. of $\tilde{\mathfrak{g}}(A_1) =_{\mathrm{df}} \mathbf{C}[t, t^{-1}] \otimes \mathfrak{g}(A_1)$) given by Theorem (3.14), we can give a simple, explicit description of $\hat{\iota}$. Thus let $c > 0$ be defined by:

$$c(X, Y) = (X, Y)_1, \ X, Y \in \mathfrak{g}(A),$$

where $(\ , \ )$ (resp. $(\ , \ )_1$) denotes the Killing form of $\mathfrak{g}(A)$ (resp. of $\mathfrak{g}(A_1)$). As in § 2, we identify the universal covering of $\tilde{\mathfrak{g}}(A)$ (resp. of $\tilde{\mathfrak{g}}(A_1)$) with $\tilde{\mathfrak{g}}(A) \oplus \mathbf{C}$ (resp. with $\tilde{\mathfrak{g}}(A_1) \oplus \mathbf{C}$). We then define $\hat{\iota}$ by:

(10.2) $$\hat{\iota}(\xi, d) = (\tilde{\iota}(\xi), cd), \quad \xi \in \tilde{\mathfrak{g}}(A), \ d \in \mathbf{C}.$$

One checks directly that

$$\hat{\iota}(\mathfrak{g}_{\mathbf{Z}}(\tilde{A})) \subset \mathfrak{g}_{\mathbf{Z}}(\tilde{A}_1).$$

On the other hand, if $\lambda_1 \in \mathbf{D}(\tilde{A}_1)$, then λ, the restriction of λ_1 to $\mathfrak{h}^e(\tilde{A})$, is in $\mathbf{D}(\tilde{A})$, and every element of $\mathbf{D}(\tilde{A})$ can be obtained as such a restriction. As in § 6, we fix the "highest weight" vector v_0 in V^{λ_1}. It follows from a result of Kac (see [11] and [8], Remark on page 61) that $\mathscr{U}(\mathfrak{g}(\tilde{A})).v_0$ is the $\mathfrak{g}(\tilde{A})$ module V^λ. Moreover, from [7], §§ 11, 12, one sees that

$$V^\lambda_{\mathbf{Z}} = V^{\lambda_1}_{\mathbf{Z}} \cap V^\lambda.$$

Thus, for any field k, we obtain an imbedding

$$\iota(\tilde{A}, \tilde{A}_1) : \hat{G}^\lambda_k(\tilde{A}) \hookrightarrow \hat{G}^{\lambda_1}_k(\tilde{A}_1).$$

To prove the Lemma for $\hat{G}_k^{\lambda}(\tilde{A})$, we must prove relation B'). But this now follows from the existence of the imbedding $\iota(\tilde{A}, \tilde{A}_1)$, from Lemma (10.1) applied to $\hat{G}_k^{\lambda_1}(\tilde{A}_1)$ (which we have proved) and from Matsumoto [13], Lemma (5.1) (or Steinberg [20]). ■

11. Relations with the work of Bruhat and Tits.

We now consider the group $E(G_{\mathscr{L}_k})$ defined in § 10. Following Steinberg [20], we can introduce a BN-pair structure on $E(G_{\mathscr{L}_k})$. However, here we also follow a suggestion of Tits and first introduce a donnée radicielle ([4], § 6.1) in $E(G_{\mathscr{L}_k})$.

Thus, for each $\alpha \in \Delta(A)$, we let U_α^e denote the subgroup of all elements $\chi_\alpha^e(\sigma)$, $\sigma \in \mathscr{L}_k$. We set

$$w_\alpha^e(\sigma) = \chi_\alpha^e(\sigma)\chi_{-\alpha}^e(-\sigma^{-1})\chi_\alpha^e(\sigma), \qquad \alpha \in \Delta(A),\ \sigma \in \mathscr{L}_k - \{0\} = \mathscr{L}_k^*,$$

and we set

$$h_\alpha^e(\sigma) = w_\alpha^e(\sigma) w_\alpha^e(1)^{-1}, \qquad \alpha \in \Delta(A),\ \sigma \in \mathscr{L}_k^*.$$

We let T^e be the subgroup of $E(G_{\mathscr{L}_k})$ generated by the elements $h_\alpha^e(\sigma)$, $\alpha \in \Delta(A)$, $\sigma \in \mathscr{L}_k - \{0\} = \mathscr{L}_k^*$. We let M_α^e, $\alpha \in \Delta(A)$, denote the right coset of T^e

$$M_\alpha^e = T^e w_\alpha^e(1).$$

Proposition **(11.1)**. — *The system*

$$(T^e, (U_\alpha^e, M_\alpha^e)_{\alpha \in \Delta(A)})$$

is a donnée radicielle *of type* $\Delta(A)$, *as defined in* [4], § 6.1.

Thus this system has the following properties (from [4], 6.1.1):

(DR 1) T^e is a subgroup of $E(G_{\mathscr{L}_k})$, and for each $\alpha \in \Delta(A)$, U_α^e is a subgroup of $E(G_{\mathscr{L}_k})$, and U_α^e contains more than one element.

(DR 2) For $\alpha, \beta \in \Delta(A)$, the group of commutators (U_α^e, U_β^e) is contained in the group generated by the $U_{p\alpha+q\beta}^e$ for $p, q \in \mathbf{Z}$, $p > 0$, $q > 0$ and $p\alpha + q\beta \in \Delta(A)$.

(DR 4) For $\alpha \in \Delta(A)$, M_α^e is a right coset of T^e, and one has

$$U_{-\alpha}^e - \{1\} \subset U_\alpha^e M_\alpha^e U_\alpha^e,$$

where $1 \in E(G_{\mathscr{L}_k})$ denotes the identity.

(DR 5) For $\alpha, \beta \in \Delta(A)$, and $n \in M_\alpha^e$, one has

$$n U_\beta^e n^{-1} = U_{r_\alpha(\beta)}^e,$$

where $r_\alpha(\beta) = \beta - 2(\alpha, \beta)(\alpha, \alpha)^{-1}\alpha$.

(DR 6) If U_+^e (resp. U_-^e) denotes the group generated by the U_α^e, $\alpha \in \Delta_+(A)$ (resp. $\Delta_-(A)$) one has $T^e U_+^e \cap U_-^e = \{1\}$.

Remarks. — We have omitted (DR 3) of [4] since this property only applies to nonreduced root systems. Also, we note that the donnée radicielle $(T^e, (U_\alpha^e, M_\alpha^e)_{\alpha \in \Delta(A)})$

is *generating* in the sense of Bruhat and Tits; i.e., T^e and the subgroups U^e_α generate $E(G_{\mathscr{L}_k})$. Indeed, the subgroups U^e_α generate $E(G_{\mathscr{L}_k})$.

Proof (of Proposition (11.1)). — (DR 1) This is clear from the definitions. (DR 2) This follows from the definition of the U^e_α and from B), at the beginning of § 10, as applied to the $\chi^e_\alpha(\sigma)$ in place of the $\mathscr{X}'_\alpha(\sigma)$. Incidentally, though at first glance B) only applies to the case $\alpha \neq -\beta$, we note that (DR 2) is in fact a tautology if $\alpha = -\beta$. (DR 4) The first assertion just comes from the definition of M^e_α. For the second assertion of (DR 4), we apply Matsumoto [13], Lemma (5.2) (*h*): $\chi^e_{-\alpha}(-\sigma) = \chi^e_\alpha(-\sigma^{-1}) w^e_\alpha(\sigma^{-1}) \chi^e_\alpha(-\sigma^{-1})$, $\sigma \neq 0$, which follows from the definition of $w^e_\alpha(\sigma^{-1})$. (DR 5) This is just a consequence of (7.2) and of (7.3), (*c*) in Steinberg [20] (also, see Matsumoto [13]). (DR 6) We define U_α, $\alpha \in \Delta(A)$, to be the subgroup of $G_{\mathscr{L}_k}$ consisting of the elements $\mathscr{X}'_\alpha(\sigma)$, $\sigma \in \mathscr{L}_k$. We let U_+ (resp. U_-) denote the group generated by the U_α, $\alpha \in \Delta_+(A)$ (resp. $\Delta_-(A)$) and we let T be the subgroup of $G_{\mathscr{L}_k}$ generated by the elements $h'_\alpha(\sigma)$, $\alpha \in \Delta(A)$, $\sigma \in \mathscr{L}_k$. We then consider the homomorphism $\varphi^e : E(G_{\mathscr{L}_k}) \to G_{\mathscr{L}_k}$, defined in § 10. We have

$$\varphi^e(U^e_\alpha) = U_\alpha, \quad \alpha \in \Delta(A),$$
$$\varphi^e(U^e_\pm) = U_\pm,$$
$$\varphi^e(T^e) = T.$$

Moreover, φ^e restricted to U^e_- is injective (see Steinberg [20], (7.1) and Matsumoto [13]). Hence, in order to prove (DR 6), it suffices to prove $TU_+ \cap U_- = \{1\}$, where 1 now denotes the identity in $G_{\mathscr{L}_k}$. But this is an easy consequence of the representation theory of $G_{\mathscr{L}_k}$ (we can assume TU_+ represented by upper, and U_- by *strictly* lower triangular matrices), and is well known. ∎

For $a \in \Delta_W(\widetilde{A})$, we let

$$r_a : \mathfrak{h}^e(\widetilde{A})^* \to \mathfrak{h}^e(\widetilde{A})^*$$

($\mathfrak{h}^e(\widetilde{A})^*$ = dual space of $\mathfrak{h}^e(\widetilde{A})$) be defined by

$$r_a(\mu) = \mu - \mu(h_a) a,$$

where, as in § 6, after (6.13), we set

$$h_a = 2h'_a / \sigma(a, a), \quad a \in \Delta_W(\widetilde{A}).$$

In Appendix I, at the end of this paper, we shall prove:

Lemma **(11.2)**. — *Let* $a \in \Delta_W(\widetilde{A})$, *and let* μ *be a weight of* V^λ, *then:*

(i) *If* $v \in V^\lambda_{\mu, k}$ *(see* (6.2)*), there exists* $v' \in V^\lambda_{r_a(\mu), k}$, *such that*

$$w_a(s) . v = s^{-\mu(h_a)} v', \quad s \in k^*.$$

(ii) $h_a(s)$, $s \in k^*$ *acts diagonally on* $V^\lambda_{\mu, k}$ *as multiplication by* $s^{\mu(h_a)}$.

(See (7.23) for the definition of $w_a(s)$, $h_a(s)$, with $a \in \Delta_W(\widetilde{A})$, $s \in k^*$.)

Recall, that at the beginning of § 7, we assumed $\lambda(h_i)\neq 0$ for some $i=1, \ldots, \ell+1$. It then follows from Lemma (11.2) that $w_{a_i}(1)$ is not the identify automorphism of V_k^λ (just apply $w_{a_i}(1)$ to a highest weight vector and use (i)). Hence $\hat{G}_k^\lambda$ has more than one element. But $\hat{G}_k^\lambda$ is generated by the elements $\chi_\alpha(\sigma)$, $\alpha\in\Delta(A)$, $\sigma\in\mathscr{L}_k$ (see Definition (7.21)). Hence the homomorphism

$$\Psi^e : E(G_{\mathscr{L}_k}) \to \hat{G}_k^\lambda,$$

of Lemma (10.1) is surjective. Since $\hat{G}_k^\lambda \neq 1$, as we just noted, we have

(11.3) $\qquad \text{kernel}\ \Psi^e \neq E(G_{\mathscr{L}_k}).$

In $\hat{G}_k^\lambda$ we let T^λ denote the subgroup generated by the elements $h_\alpha(\sigma)$, $\alpha\in\Delta(A)$, $\sigma\in\mathscr{L}_k^*$. We let U_α^λ, $\alpha\in\Delta(A)$ denote the subgroup consisting of the elements $\chi_\alpha(\sigma)$, $\sigma\in\mathscr{L}_k$, and we let U_+^λ (resp. U_-^λ) denote the subgroup generated by the U_α^λ, $\alpha\in\Delta_+(A)$ (resp. $\Delta_-(A)$). The homomorphism $\Psi^e : E(G_{\mathscr{L}_k}) \to \hat{G}_k^\lambda$ satisfied (and was uniquely determined by) the conditions

$$\Psi^e(\chi_\alpha^e(\sigma)) = \chi_\alpha(\sigma), \qquad \alpha\in\Delta(A),\ \sigma\in\mathscr{L}_k.$$

It follows that

(11.4)
$$\begin{aligned} \Psi^e(U_\alpha^e) &= U_\alpha^\lambda, \\ \Psi^e(U_\pm^e) &= U_\pm^\lambda, \\ \Psi^e(T^e) &= T^\lambda. \end{aligned}$$

Moreover, thanks to Proposition (11.1), and to Bruhat-Tits [4], Proposition (6.1.12), we obtain a BN-pair in $E(G_{\mathscr{L}_k})$, given by the pair of subgroups (B^e, N^e), where $B^e = T^e U_+^e$ and N^e is the subgroup generated by the $w_\alpha^e(\sigma)$, $\alpha\in\Delta(A)$, $\sigma\in\mathscr{L}_k$. But then from Bourbaki [3], Theorem 5, p. 30, and from (11.3), above, we have

(11.5) $\qquad \text{kernel}\ \Psi^e \subset T^e.$

For $\alpha\in\Delta(A)$, we let M_α^λ denote the right coset of T^λ

$$M_\alpha^\lambda = T^\lambda w_\alpha(1).$$

The following proposition is then essentially a corollary of our above remarks and of Proposition (11.1):

Proposition **(11.6)**. — *The system*

$$(T^\lambda, (U_\alpha^\lambda, M_\alpha^\lambda)_{\alpha\in\Delta(A)})$$

is a donnée radicielle of type $\Delta(A)$.

Proof. — Properties (DR 2), (DR 4) and (DR 5) follow from the corresponding properties for $(T^e, (U_\alpha^e, M_\alpha^e)_{\alpha\in\Delta(A)})$, and then applying the homomorphism Ψ^e. For property (DR 6), we note that if $g\in T^\lambda U_+^\lambda \cap U_-^\lambda$, then we may choose $g'\in U_-^e$ and $g''\in T^e U_+^e$ such that

$$\Psi^e(g') = \Psi^e(g'') = g.$$

Then, thanks to (11.5), we have

$$g' = g''t, \qquad t \in \text{kernel } \Psi'^{e} \subset T^{e}.$$

But $U^e_- \cap T^e U^e_+ = \{1\}$, and hence $g' = 1$. Therefore $g = \Psi'^e(g') = 1$, where, in this last equality, 1 denotes the identity element of $\hat{G}^\lambda_k$. This proves (DR 6).

It only remains to prove (DR 1), and for this, it suffices to show that U^λ_α contains more than one element. Indeed, if $\sigma \in \mathscr{L}_k$ and $\sigma \neq 0$, then $\chi_\alpha(\sigma) \neq 1$. For if $\chi_\alpha(\sigma) = 1$, then $\chi^e_\alpha(\sigma) \in \text{kernel } \Psi'^e$. But then, by (11.5), we would have $\chi^e_\alpha(\sigma) \in T^e$, hence $\mathscr{X}'_\alpha(\sigma) = \varphi^e(\chi^e_\alpha(\sigma)) \in T = \varphi^e(T^e)$, and this is only possible if $\mathscr{X}'_\alpha(\sigma) = 1$ (with 1 now denoting the identity in $G_{\mathscr{L}_k}$). This, in turn, implies $\sigma = 0$, a contradiction (see e.g. Steinberg [21], Corollary 1, p. 26). ∎

(11.7). *Remark.* — It follows from the observations in the proof of Proposition (11.6), that each of the maps

$$\sigma \mapsto \chi^e_\alpha(\sigma)$$
$$\sigma \mapsto \chi_\alpha(\sigma)$$
$$\sigma \mapsto \mathscr{X}'_\alpha(\sigma) \qquad \sigma \in \mathscr{L}_k,$$

is injective.

We let $\quad \nu : \mathscr{L}_k \to \mathbf{R} \cup \{\infty\}$

be the t-adic valuation. Thus, if

$$\sigma = \sum_{i \geq i_0} a_i t^i \in \mathscr{L}_k, \qquad a_{i_0} \neq 0,$$

then $\quad \nu(\sigma) = i_0,$

while $\quad \nu(0) = \infty.$

For each $\alpha \in \Delta(A)$, we define functions

$$\nu^e_\alpha : U^e_\alpha \to \mathbf{R} \cup \{\infty\},$$
$$\nu^\lambda_\alpha : U^\lambda_\alpha \to \mathbf{R} \cup \{\infty\},$$
$$\nu_\alpha : U_\alpha \to \mathbf{R} \cup \{\infty\},$$

by $\quad \nu^e_\alpha(\chi^e_\alpha(\sigma)) = \nu^\lambda_\alpha(\chi_\alpha(\sigma)) = \nu_\alpha(\mathscr{X}'_\alpha(\sigma)) = \nu(\sigma), \qquad \sigma \in \mathscr{L}_k.$

We let M_α denote the right coset of T,

$$M_\alpha = T w'_\alpha(1).$$

It is then known that the system

$$(T, (U_\alpha, M_\alpha)_{\alpha \in \Delta(A)})$$

is a donnée radicielle of type $\Delta(A)$ (see Bruhat-Tits [4], § 6.1.3). We make the notational convention that ν^*_α (resp. U^*_α, resp. M^*_α) may denote any one of ν^e_α, ν^λ_α or ν_α (resp. U^e_α, U^λ_α or U_α, resp. M^e_α, M^λ_α or M_α). Similarly, we let T^* denote either T^e, T^λ or T.

We now prove:

Lemma **(11.8)**. — *The family* $(\nu^*_\alpha)_{\alpha\in\Delta(A)}$ *is a* valuation *in the sense of Bruhat-Tits* [4], § 6.2, *of the donnée radicielle* $(T^*, (U^*_\alpha, M^*_\alpha)_{\alpha\in\Delta(A)})$.

That is, the $(\nu^*_\alpha)_{\alpha\in\Delta(A)}$ satisfy the following conditions (each ν^*_α mapping U^*_α into $\mathbf{R}\cup\{\infty\}$):

(V0) For each $\alpha\in\Delta(A)$, the image of ν^*_α contains at least three elements.

(V1) For each $\alpha\in\Delta(A)$ and $i\in\mathbf{R}\cup\{\infty\}$, the set $U^*_{\alpha,i}=\nu^{*-1}_\alpha([i,\infty])$ is a subgroup of U^*_α and one has $U^*_{\alpha,\infty}=\{1\}$.

(V2) For each $\alpha\in\Delta(A)$ and $m\in M^*_\alpha$, the function

$$u\to\nu^*_{-\alpha}(u)-\nu^*_\alpha(mum^{-1})$$

is constant on $U^*_{-\alpha}-\{1\}$.

(V3) Let $\alpha,\beta\in\Delta(A)$ and $i,j\in\mathbf{R}$; if $\beta\notin-\mathbf{R}_+\alpha$, then the group of commutators $(U^*_{\alpha,i}, U^*_{\beta,j})$ is contained in the group generated by the

$$U^*_{p\alpha+q\beta,pi+qj},$$

where p, q are strictly positive integers and $p\alpha+q\beta\in\Delta(A)$.

(V5) If $\alpha\in\Delta(A)$, $u\in U^*_\alpha$, and $u', u''\in U^*_{-\alpha}$, and if $u'uu''\in M^*_\alpha$, then one has

$$\nu^*_{-\alpha}(u')=-\nu^*_\alpha(u).$$

Remark. — We have omitted (V4) of [4] since that condition only applies to root systems which are not reduced.

Proof. — For $G_{\mathscr{L}_k}$ and the donnée radicielle $(T, (U_\alpha, M_\alpha)_{\alpha\in\Delta(A)})$, with valuation $(\nu_\alpha)_{\alpha\in\Delta(A)}$, the Lemma is proved in Bruhat-Tits [4], (6.2.3) (*b*). But then consider the homomorphism

$$\varphi^e: E(G_{\mathscr{L}_k})\to G_{\mathscr{L}_k}.$$

As we noted in the proof of Proposition (11.1), we have $\varphi^e(U^e_\alpha)=U_\alpha$, $\varphi^e(T^e)=T$, and similarly, we have $\varphi^e(M^e_\alpha)=M_\alpha$. Moreover, we clearly have

$$\textbf{(11.9)}\qquad \nu^e_\alpha(x)=\nu_\alpha(\varphi^e(x)),\quad x\in U^e_\alpha.$$

Thus, (V0) for $E(G_{\mathscr{L}_k})$ follows from the corresponding assertion for $G_{\mathscr{L}_k}$. By (11.7), φ^e restricted to U^e_α is an isomorphism onto U_α. Hence (V1) for $E(G_{\mathscr{L}_k})$ follows from (11.9) and the corresponding fact for $G_{\mathscr{L}_k}$. We now consider (V2) for $E(G_{\mathscr{L}_k})$. Thus, let $m\in M^e_\alpha$; then the function

$$u\mapsto\nu^e_{-\alpha}(u)-\nu^e_\alpha(mum^{-1}),\quad u\in U^e_{-\alpha}-\{1\},$$

is the same as the function

$$u\mapsto\nu_{-\alpha}(\varphi^e(u))-\nu_\alpha(\varphi^e(m)\varphi^e(u)\varphi^e(m)^{-1}),$$

which is constant as $\varphi^e(u)$ varies over $U_{-\alpha}-\{1\}$, and hence as u varies over $U^e_{-\alpha}-\{1\}$. Property (V3) for $E(G_{\mathscr{L}_k})$ follows from (B), at the beginning of § 10, but applied to the $\chi^e_\alpha(\sigma)$ in place of the $\mathscr{X}'_\alpha(\sigma)$, $\sigma\in\mathscr{L}_k$, $\alpha\in\Delta(A)$. Finally, (V5) follows for $E(G_{\mathscr{L}_k})$ from the corresponding property for $G_{\mathscr{L}_k}$, and an argument analogous to the one used to prove (V2) for $E(G_{\mathscr{L}_k})$ (assuming (V2) for $G_{\mathscr{L}_k}$).

Finally, we come to the proof of Lemma (11.8) for the $(\nu^\lambda_\alpha)_{\alpha\in\Delta(A)}$ and $\hat{G}^\lambda_k$. Here we use the homomorphism $\Psi^e : E(G_{\mathscr{L}_k})\to\hat{G}^\lambda_k$. Thanks to (11.4) and (11.7) we have that for each $\alpha\in\Delta(A)$, Ψ^e induces an isomorphism from U^e_α onto U^λ_α, where, recall,

$$\Psi^e(\chi^e_\alpha(\sigma))=\chi_\alpha(\sigma), \quad \sigma\in\mathscr{L}_k.$$

Since we clearly have

(11.10) $$\nu^\lambda_\alpha(\Psi^e(x))=\nu^e_\alpha(x), \quad x\in U^e_\alpha,$$

we obtain (V0) and (V1) for $\hat{G}^\lambda_k$ from the corresponding properties for $E(G_{\mathscr{L}_k})$. Property (V3) for $\hat{G}^\lambda_k$ follows from (9.8). Property (V2) for $\hat{G}^\lambda_k$ follows from (11.4), (11.10), from the observation, made above, that Ψ^e induces an isomorphism on U^e_α (see (11.7)), and from (V2) for $E(G_{\mathscr{L}_k})$. Similarly, (V5) for $\hat{G}^\lambda_k$ follows from (V5) for $E(G_{\mathscr{L}_k})$, from the facts just noted, and from (11.5), which implies

$$(\Psi^e)^{-1}(M^\lambda_\alpha)=M^e_\alpha.$$

12. Computation of the symbol.

We fix $\lambda\in\mathbf{D}$ (= the family of dominant integral linear functionals in $\mathfrak{h}^e(\tilde{A})^*$, as in § 6). As specified at the beginning of § 7, we assume $\lambda(h_i)\neq 0$, for some $i=1, \ldots, \ell+1$. We consider the highest weight module V^λ, and fix a highest weight vector $v_0\in V^\lambda_{\mathbf{Z}}$. We also let v_0 denote the element $1\otimes v_0\in V^\lambda_k=k\otimes V^\lambda_{\mathbf{Z}}$ (where k continues to denote a field, *not* necessarily of characteristic 0). As we observed in (6.7), every weight μ of V^λ is of the form

$$\mu=\lambda-\sum_{i=1}^{\ell+1} n_i a_i, \quad n_i\geq 0, \ n_i\in\mathbf{Z}.$$

We define $dp(\mu)$ (the depth of μ) by

$$dp(\mu)=\sum_{i=1}^{\ell+1} n_i$$

(see [7], proof of Lemma (10.4)).

We let $v_0, v_1, v_2, \ldots, v_m, \ldots$ be a basis of V^λ_k consisting of vectors in the weight subspaces $V^\lambda_{\mu,k}$, μ a weight of V^λ. We assume our basis ordered so that if $v_i\in V^\lambda_{\mu,k}$, $v_j\in V^\lambda_{\mu',k'}$, and $i<j$, then $dp(\mu)\leq dp(\mu')$. Moreover, for each weight μ of V^λ, we assume that the basis vectors in $V^\lambda_{\mu,k}$ appear consecutively. We will call such a basis *coherently ordered*. We note that, with respect to a coherently ordered basis, the

elements of $\mathfrak{u}_k^+$ (resp. $\mathfrak{u}_k^-$, resp. $\mathfrak{h}_k(\widetilde{A})$) are represented by upper triangular matrices with zeroes on the diagonal (resp. by lower triangular matrices with zeroes on the diagonal, resp. by diagonal matrices).

We let $\mathcal{I}_U \subset \mathcal{I}$ denote the subgroup generated by the elements $\chi_\alpha(\sigma(t))$, where either $\alpha \in \Delta_+(A)$, $\sigma(t) \in \mathcal{O}$, or $\alpha \in \Delta_-(A)$, $\sigma(t) \in \mathcal{P}$. From the observations of the previous paragraph, and the definition of $\chi_\alpha(\sigma(t))$ (see (7.19)) we have:

(12.1) With respect to a coherently ordered basis, the elements of $\mathcal{I}_U$ are represented by upper triangular matrices with ones on the diagonal.

Next, we have

Lemma **(12.2)**. — *Let*

$$\sigma(t) = q_0 + q_1 t + \ldots + q_j t^j + \ldots, \tag{12.3}$$

$q_j \in k$, $q_0 \neq 0$, *be an element of* $\mathcal{O}^*$. *Then, for* $\alpha \in \Delta(A)$, $h_\alpha(\sigma(t))$ *can be written as a product*

$$h_\alpha(\sigma(t)) = \chi h_\alpha(q_0), \quad \chi \in \mathcal{I}_U.$$

Proof. — We set

$$\sigma(t)^{-1} = q'_0 + q'_1 t + \ldots + q'_j t^j + \ldots,$$

where $q'_j \in k$, $q'_0 \neq 0$, and in fact,

$$q'_0 = q_0^{-1}.$$

Also, we set

$$p(t) = \sigma(t) - q_0$$
$$q(t) = \sigma^{-1}(t) - q'_0.$$

We then have:

$$\begin{aligned} h_\alpha(\sigma(t)) &= w_\alpha(\sigma(t)) w_\alpha(1)^{-1} \quad \text{(see (7.22))}, \\ &= \chi_\alpha(\sigma(t)) \chi_{-\alpha}(-\sigma(t)^{-1}) \chi_\alpha(\sigma(t)) w_\alpha(1)^{-1} \quad \text{(also by (7.22))}, \\ &= \chi_\alpha(p(t)) \chi_\alpha(q_0) \chi_{-\alpha}(-q(t)) \chi_{-\alpha}(-q_0^{-1}) \chi_\alpha(p(t)) \chi_\alpha(q_0) w_\alpha(1)^{-1} \\ &= \chi_\alpha(p(t)) \chi_{-\alpha}(-q(t))^{\chi_\alpha(q_0)} \chi_\alpha(q_0) \chi_{-\alpha}(-q_0^{-1}) \chi_\alpha(p(t)) \chi_\alpha(q_0) w_\alpha(1)^{-1} \\ &= \chi_\alpha(p(t)) \chi_{-\alpha}(-q(t))^{\chi_\alpha(q_0)} \chi_\alpha(p(t))^{w_\alpha(q_0)} h_\alpha(q_0), \quad \text{by (7.22)}, \end{aligned} \tag{12.4}$$

where for h, g in a group, we set $g^h = hgh^{-1}$. Also, a direct computation using (7.3), (7.11), and the following consequence of (7.11) already noted in § 7, preceeding (7.15):

$$\chi_a(s) = w^\lambda \chi_{a_i}(\pm s) (w^\lambda)^{-1},$$

shows that

$$\operatorname{Ad} \chi_a(s) (\pi_k^\lambda(\xi)) = \sum_{n \geq 0} s^n \pi_k^\lambda(\operatorname{ad}(\xi_a^n / n!)(\xi)), \quad \text{(finite sum)}, \tag{12.5}$$
$$s \in k, \ a \in \Delta_W(\widetilde{A}), \ \xi \in \mathfrak{g}_k(\widetilde{A}).$$

But then, another computation, using (12.5), yields

$$\operatorname{Ad} w_\alpha(q) (\pi_k^\lambda(\xi_a)) = -q^{-2} \pi_k^\lambda(\xi_{-a(\alpha)+j\iota}),$$

for $q \in k^* = k - \{0\}$, $a = a(\alpha) + j\iota \in \Delta(\widetilde{A})$, $j \neq 0$ in $\mathbf{Z}$. From this last equality, we have

$$\chi_\alpha(p(t))^{w_\alpha(q_0)} = \chi_{-\alpha}(-q_0^{-2}p(t)),$$

Similarly, $\chi_{-\alpha}(-q(t))^{w_{-\alpha}(-q_0^{-1})} = \chi_\alpha(q_0^2 q(t))$, and it thus follows that

(i) $$\chi_\alpha(p(t))\chi_{-\alpha}(-q(t))^{\chi_\alpha(q_0)}\chi_\alpha(p(t))^{w_\alpha(q_0)} =$$

(ii) $$\chi_\alpha(p(t))(\chi_{-\alpha}(-q(t))^{\chi_{-\alpha}(q_0^{-1})w_{-\alpha}(-q_0^{-1})})\chi_\alpha(p(t))^{w_\alpha(q_0)},$$

where if $\alpha \in \Delta_+(A)$, the expression (i) is clearly in $\mathscr{I}_U$, and if $\alpha \in \Delta_-(A)$, the expression (ii) is clearly in $\mathscr{I}_U$. Calling χ the common value of (i) and (ii) and using the expression (12.4) for $h_\alpha(\sigma(t))$, we have

$$h_\alpha(\sigma(t)) = \chi h_\alpha(q_0),$$

and this proves Lemma (12.2). ■

Corollary. — *For* $\sigma(t) \in \mathcal{O}^*$, *with*

$$\sigma(t) = q_0 + q_1 t + \ldots + q_j t^j + \ldots,$$

and for $\alpha \in \Delta(A)$, *we have*

(12.6) $$h_\alpha(\sigma(t)) . v_0 = q_0^{\lambda(h_{a(\alpha)})} v_0.$$

Proof. — We note that $h_\alpha(q_0) = h_{a(\alpha)}(q_0)$, and apply Lemma (11.2), Lemma (12.2), and (12.1). ■

We now wish to compute

$$h_\alpha(\sigma(t)t^{-1})h_\alpha(t^{-1})^{-1}h_\alpha(\sigma(t))^{-1}.v_0, \quad \alpha \in \Delta_+(A), \ \sigma(t) \in \mathcal{O}^*.$$

Thanks to (12.6), it suffices to compute

$$h_\alpha(\sigma(t)t^{-1})h_\alpha(t^{-1})^{-1}.v_0.$$

But we write $\sigma(t)$ as in (12.3), and then, as we noted in the proof of Lemma (12.2), we have

$$\sigma(t)^{-1} = q_0' + q_1' t + \ldots + q_j' t^j + \ldots, \quad q_0' = q_0^{-1}, \ q_j' \in k.$$

We write σ for $\sigma(t)$, and recall that we defined (in the proof of Lemma (12.2)) $p(t)$, $q(t)$ by

$$\sigma = q_0 + p(t),$$
$$\sigma^{-1} = q_0^{-1} + q(t).$$

We then have

$$\sigma t^{-1} = q_0 t^{-1} + \widetilde{p}(t)$$
$$\sigma^{-1} t = q_0^{-1} t + \widetilde{q}(t),$$

where $$\widetilde{p}(t) = p(t)t^{-1} \in \mathcal{O},$$
$$\widetilde{q}(t) = q(t)t \in \mathcal{O}t^2.$$

We have

(12.7) $$w_\alpha(\sigma t^{-1})=\chi_\alpha(\sigma t^{-1})\chi_{-\alpha}(-\sigma^{-1}t)\chi_\alpha(\sigma t^{-1}),\quad \text{by (7.22)},$$
$$=\chi_\alpha(\widetilde{p}(t))\chi_\alpha(q_0t^{-1})\chi_{-\alpha}(-\widetilde{q}(t))\chi_{-\alpha}(-q_0^{-1}t)\chi_\alpha(\widetilde{p}(t))\chi_\alpha(q_0t^{-1})$$
$$=\chi_\alpha(\widetilde{p}(t))\chi_{-\alpha}(-\widetilde{q}(t))^{\chi_\alpha(q_0t^{-1})}w_\alpha(q_0t^{-1})\chi_\alpha(\widetilde{p}(t))$$
$$=\chi_\alpha(\widetilde{p}(t))\chi_{-\alpha}(-\widetilde{q}(t))^{\chi_\alpha(q_0t^{-1})}\chi_\alpha(\widetilde{p}(t))^{w_\alpha(q_0t^{-1})}w_\alpha(q_0t^{-1}).$$

Now of course

$$\chi_\alpha(\widetilde{p}(t))\in\mathscr{I}_{\mathrm{U}}\quad \text{(recall } \alpha\in\Delta_+(\mathrm{A})).$$

We wish to also show:

(12.8) $$\chi_{-\alpha}(-\widetilde{q}(t))^{\chi_\alpha(q_0t^{-1})},\ \chi_\alpha(\widetilde{p}(t))^{w_\alpha(q_0t^{-1})}\in\mathscr{I}_{\mathrm{U}}.$$

However, thanks to Lemma (10.1), and to Matsumoto [13], Lemma (5.1) (*b*):

(12.9) $$w_\beta(\sigma_1)\chi_{\pm\beta}(\sigma_2)w_\beta(\sigma_1)^{-1}=\chi_{\mp\beta}(\eta\sigma_1^{\mp 2}\sigma_2),$$

where $\eta=+1$ or -1, $\sigma_1,\sigma_2\in\mathscr{L}_k$, $\beta\in\Delta(\mathrm{A})$. From (12.9), it follows that

(12.10) $$\chi_\alpha(\widetilde{p}(t))^{w_\alpha(q_0t^{-1})}=\chi_{-\alpha}(\pm q_0^{-2}t^2p(t))\in\mathscr{I}_{\mathrm{U}}.$$

On the other hand, also by (12.9),

$$\chi_{-\alpha}(-\widetilde{q}(t))^{w_{-\alpha}(ct)}=\chi_\alpha(-\eta(ct)^{-2}\widetilde{q}(t)),\qquad c\in k^*,$$

where again, $\eta=+1$ or -1. But, since $\widetilde{q}(t)\in\mathcal{O}t^2$, $\alpha\in\Delta_+(\mathrm{A})$, we see that this last element is again in $\mathscr{I}_{\mathrm{U}}$. But, on the other hand, we have:

$$\chi_{-\alpha}(-\widetilde{q}(t))^{w_{-\alpha}(ct)}=\chi_{-\alpha}(ct)\chi_\alpha(-c^{-1}t^{-1})\chi_{-\alpha}(-\widetilde{q}(t))\chi_\alpha(c^{-1}t^{-1})\chi_{-\alpha}(-ct).$$

Since $\chi_{-\alpha}(ct)\in\mathscr{I}_{\mathrm{U}}$, we therefore obtain:

$$\chi_{-\alpha}(-\widetilde{q}(t))^{\chi_\alpha(-c^{-1}t^{-1})}\in\mathscr{I}_{\mathrm{U}},\qquad c\in k^*,$$

and setting $c=-q_0^{-1}$, and recalling (12.10), we get (12.8).

But then $h_\alpha(\sigma(t)t^{-1})h_\alpha(t^{-1})^{-1}.v_0$

$$=w_\alpha(\sigma(t)t^{-1})w_\alpha(1)^{-1}w_\alpha(1)w_\alpha(t^{-1})^{-1}.v_0,\quad \text{by (7.22)},$$
$$=w_\alpha(\sigma(t)t^{-1})w_\alpha(t^{-1})^{-1}.v_0$$
$$=\mathscr{X}w_\alpha(q_0t^{-1})w_\alpha(t^{-1})^{-1}.v_0,\qquad \mathscr{X}\in\mathscr{I}_{\mathrm{U}},$$

thanks to the expression (12.7) for $w_\alpha(\sigma(t)t^{-1})$ and to (12.8). But for $c\in k^*$

$$w_\alpha(ct^{-1})=\chi_\alpha(ct^{-1})\chi_{-\alpha}(-c^{-1}t)\chi_\alpha(ct^{-1})$$
$$=\chi_a(c)\chi_{-a}(-c^{-1})\chi_a(c)=w_a(c),\qquad a=a(\alpha)-\iota.$$

Hence $w_\alpha(q_0t^{-1})w_\alpha(t^{-1})^{-1}=w_a(q_0)w_a(1)^{-1}=h_a(q_0)$, and hence

$$h_\alpha(\sigma(t)t^{-1})h_\alpha(t^{-1})^{-1}.v_0=\mathscr{X}h_a(q_0).v_0=q_0^{\lambda(h_a)}\mathscr{X}.v_0,\quad \text{by Lemma (11.2)}$$
$$=q_0^{\lambda(h_a)}v_0,$$

since $\mathscr{X}\in\mathscr{I}_U$, so we can apply (12.1). Combining this last computation with (12.6), we obtain

$$h_\alpha(\sigma(t)t^{-1})h_\alpha(t^{-1})^{-1}h_\alpha(\sigma(t))^{-1}.v_0=q_0^{-\lambda(h_{a(\alpha)})}q_0^{\lambda(h_a)}v_0=q_0^\omega v_0,$$

where

$$a=a(\alpha)-\iota \qquad \text{and} \qquad \omega=-\frac{2\lambda(h'_\iota)}{(\alpha,\alpha)};$$

i.e., *we have for* $\alpha\in\Delta_+(A)$, $\sigma(t)\in\mathscr{O}^*$,

(12.11) $$h_\alpha(\sigma(t)t^{-1})h_\alpha(t^{-1})^{-1}h_\alpha(\sigma(t))^{-1}.v_0=q_0^\omega v_0, \qquad \omega=-\frac{2\lambda(h'_\iota)}{(\alpha,\alpha)}.$$

We also note that

(12.12) $$h_{a(\alpha)}(q_0^{-1})h_a(q_0).v_0=q_0^\omega v_0.$$

On the other hand, if $\mathscr{U}(\mathfrak{g}_k(\widetilde{A}))=k\otimes\mathscr{U}_{\mathbf{Z}}(\widetilde{A})$, then

$$V_k^\lambda=k\otimes V_{\mathbf{Z}}^\lambda=k\otimes\mathscr{U}_{\mathbf{Z}}(\widetilde{A})v_0=\mathscr{U}(\mathfrak{g}_k(\widetilde{A}))v_0 \quad (\text{see § 6}),$$

and hence, thanks to Lemma (8.6) and to Lemma (8.11) we have from (12.11) and (12.12) (note that $h_{a(\alpha)}(q_0^{-1})h_a(q_0)=h_\alpha(q_0^{-1})h_\alpha(q_0t^{-1})h_\alpha(t^{-1})^{-1}$)

(12.13) $$\begin{aligned}h_\alpha(\sigma(t)t^{-1})h_\alpha(t^{-1})^{-1}h_\alpha(\sigma(t))^{-1}&=h_{a(\alpha)}(q_0^{-1})h_a(q_0)\\&=q_0^\omega I, \quad \alpha\in\Delta_+(A),\ \sigma(t)=\sum_{j\geq 0}q_jt^j,\ q_0\neq 0,\end{aligned}$$

where I denotes the identity operator on V_k^λ, and ω is defined as in (12.11).

We now consider the group $\hat{G}_k^\lambda$ from the point of view taken in § 10. Thus, we recall from § 10, that the $\chi_\alpha(\sigma)$ in place of the $\mathscr{X}'_\alpha(\sigma)$, $\alpha\in\Delta(A)$, $\sigma\in\mathscr{L}_k$, satisfy the relations (A), (B) ((B′) if A=(2)) at the beginning of § 10. But then, if we divide $\hat{G}_k^\lambda$ by the subgroup C generated by the elements $h_\alpha(\sigma_1\sigma_2)h_\alpha(\sigma_2)^{-1}h_\alpha(\sigma_1)^{-1}$, $\alpha\in\Delta(A)$, $\sigma_1,\sigma_2\in\mathscr{L}_k$ (this subgroup is central by Lemma (10.1) and by Steinberg [20]), the resulting quotient group, $G_{\mathscr{L}_k}^\lambda$ say, is also a quotient group of $G_{\mathscr{L}_k}$. Indeed, we have surjective group homomorphisms

$$\pi_1:\hat{G}_k^\lambda\to G_{\mathscr{L}_k}^\lambda,$$
$$\pi_2:G_{\mathscr{L}_k}\to G_{\mathscr{L}_k}^\lambda,$$

defined by the conditions

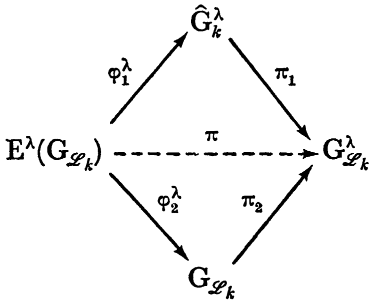
$$\pi_1(\chi_\alpha(\sigma))=\chi_\alpha(\sigma)C$$
$$\pi_2(\mathscr{X}'_\alpha(\sigma))=\chi_\alpha(\sigma)C,$$

for $\alpha\in\Delta(A)$, $\sigma\in\mathscr{L}_k$. By Steinberg [20], Theorem (3.2), we have that kernel(π_2) is finite and central. We have already noted that C = kernel π_1 is central in $\hat{G}_k^\lambda$.

We let $E^\lambda(G_{\mathscr{L}_k}) \subset \hat{G}_k^\lambda \times G_{\mathscr{L}_k}$ denote the subgroup of all elements (g_1, g_2), $g_1 \in \hat{G}_k^\lambda$, $g_2 \in G_{\mathscr{L}_k}$, such that:

(12.14) $\pi_1(g_1) = \pi_2(g_2)$.

Then, thanks to (12.14), we have a commutative diagram



where $\varphi_1 = \varphi_1^\lambda$, $\varphi_2 = \varphi_2^\lambda$ (we drop the superscript λ when there is no danger of confusion) are induced by the projections of $\hat{G}_k^\lambda \times G_{\mathscr{L}_k}$ onto the first and second factors, respectively. We let $\pi = \pi_1 \circ \varphi_1 = \pi_2 \circ \varphi_2$; then since kernel$(\pi_1)$ and kernel(π_2) are central in $\hat{G}_k^\lambda$ and in $G_{\mathscr{L}_k}$, respectively, we have

$$\text{kernel}(\pi) \subset \text{center}(E^\lambda(G_{\mathscr{L}_k})).$$

We now fix a *long* root $\alpha \in \Delta_+(A)$, and we let

$$C^\lambda = \text{kernel } \varphi_2.$$

Then, by Moore [18], Lemma (8.4), the central extension

$$1 \to C^\lambda \to E^\lambda(G_{\mathscr{L}_k}) \overset{\varphi_2}{\to} G_{\mathscr{L}_k} \to 1$$

is determined by the function $b_\alpha^\lambda(\ ,\ )$ from $\mathscr{L}_k^* \times \mathscr{L}_k^*$ to C^λ, defined by

(12.15) $b_\alpha^\lambda(\sigma, \tau) = h_\alpha''(\sigma) h_\alpha''(\tau) h_\alpha''(\sigma\tau)^{-1}$, $\quad \sigma, \tau \in \mathscr{L}_k^* = \mathscr{L}_k - \{0\}$,

where for $\sigma \in \mathscr{L}_k^*$

$$h_\alpha''(\sigma) = (h_\alpha(\sigma), h_\alpha'(\sigma)) \in E^\lambda(G_{\mathscr{L}_k}).$$

We note that

$$h_\alpha'(\sigma) h_\alpha'(\tau) = h_\alpha'(\sigma\tau), \qquad \sigma, \tau \in \mathscr{L}_k^*,$$

thanks to (C), at the beginning of § 10. Hence, we have

(12.16) $b_\alpha^\lambda(\sigma, \tau) = (b_\alpha(\sigma, \tau), 1)$,

where

(12.17) $b_\alpha(\sigma, \tau) = h_\alpha(\sigma) h_\alpha(\tau) h_\alpha(\sigma\tau)^{-1}$, $\quad \sigma, \tau \in \mathscr{L}_k^*$.

We let M denote an abelian group (in which we write the group operation multiplicatively) and we let k denote a field. Following Matsumoto [13], we let $S(k^*, M)$ $(k^* = k - \{0\})$ denote the group of all mappings

$$c : k^* \times k^* \to M$$

satisfying the following relations for $x, y, z \in k^*$:

(12.18) (S1) $c(x,y)c(xy,z)=c(x,yz)c(y,z)$;

(S2) $c(1,1)=1$; $\quad c(x,y)=c(x^{-1},y^{-1})$;

(S3) $c(x,y)=c(x,(1-x)y)$, $\quad$ if $\quad x \neq 1$.

We let $S^0(k^*, M) \subset S(k^*, M)$ denote the subgroup of those $c \in S(k^*, M)$ which are bilinear; i.e., $c(xy,z)=c(x,z)c(y,z)$, $c(z,xy)=c(z,x)c(z,y)$, $x, y, z \in k^*$.

Lemma **(12.19)**. — *Let* $\alpha \in \Delta_+(A)$ *be a long root; then* $C^\lambda = \text{kernel}\ \varphi_2$ *is generated by the elements* $b_\alpha^\lambda(\sigma, \tau)$, $\sigma, \tau \in \mathscr{L}_k^*$, *and we have* $b_\alpha^\lambda(\ ,\ ) \in S^0(\mathscr{L}_k^*, C^\lambda)$.

Proof. — The first assertion follows from Lemma (8.2) in Moore [18]. When $G_{\mathscr{L}_k}$ is not of symplectic type; i.e., when A is not the classical Cartan matrix corresponding to the symplectic group, then the second assertion follows from Matsumoto [13], Theorem (5.10). Also, Theorem (5.10) of [13] asserts that in any case, $b_\alpha^\lambda(\ ,\ ) \in S(\mathscr{L}_k^*, C^\lambda)$. Thus, to complete the proof of the lemma, we need only prove that $b_\alpha^\lambda(\ ,\ )$ is bilinear in the symplectic case. By (12.16), this is equivalent to showing that $b_\alpha(\ ,\ )$ is bilinear in the symplectic case (note that, also by (12.16), we know that $b_\alpha(\ ,\ )$ is bilinear in the non-symplectic case).

In the symplectic case we consider the subgroup $\hat{H}$ of $\hat{G}_k^\lambda$ generated by the elements $\chi_\alpha(\sigma_1(t))$, $\chi_{-\alpha}(\sigma_2(t))$, $\sigma_1(t)$, $\sigma_2(t) \in \mathscr{L}_k$. Passing to a suitable $\hat{H}$ subrepresentation of V_k^λ (generated, over **Z**, by the highest weight vector and the subalgebra corresponding to the subgroup $\hat{H}$), we may in fact assume that $G_{\mathscr{L}_k} = SL_2(\mathscr{L}_k)$ and that A is the 1×1 Cartan matrix $A=(2)$. But then, arguing as in § 10, we may imbed $\mathfrak{g}_k(\widetilde{A})$ into $\mathfrak{g}_k(\widetilde{A}_1)$ where

$$A_1 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

is the classical Cartan matrix corresponding to SL_3. We may also assume V_k^λ is obtained from a highest weight module for $\mathfrak{g}_k(\widetilde{A}_1)$, as the $\mathfrak{g}_k(\widetilde{A})$-submodule generated by a highest weight vector. We may then use the known bilinearity assertion for SL_3, to obtain the desired assertion for SL_2. ■

We let

$$c_T : \mathscr{L}_k^* \times \mathscr{L}_k^* \to k^*$$

denote the tame-symbol, defined by

$$\textbf{(12.20)} \qquad c_T(\sigma_1, \sigma_2) = (-1)^{\nu(\sigma_1)\nu(\sigma_2)}((\sigma_1^{\nu(\sigma_2)}\sigma_2^{-\nu(\sigma_1)})(0)),$$

where we recall that ν denotes the t-adic valuation on $\mathscr{L}_k$. It is known that $c_T(\ ,\ ) \in S^0(\mathscr{L}_k^*, k^*)$ (see Milnor [14], Lemma (11.5), p. 98). Indeed, one can easily check bilinearity (which then implies (S1) of (12.18)) and property (S2) of (12.18), directly. Property (S3) is also easily checked by noting, thanks to bilinearity, that it suffices to check $c_T(\sigma, 1-\sigma)=1$, for $\sigma\in\mathcal{L}_k^*$, $\sigma\neq 1$. One can in turn check this by setting $\sigma=\sigma' t^m$, $\sigma'\in\mathcal{O}^*$, and separately considering the cases $m>0$; $m<0$; $m=0$, $\sigma(0)=1$; $m=0$, $\sigma(0)\neq 1$.

Now we have observed that $b_\alpha(\ ,\ )$ is bilinear (by (12.16) and Lemma 12.19), and thanks to (12.13), we know that

(12.21) $b_\alpha(\sigma(t), t^{-1})$ is a non-zero scalar multiple of the identity operator on V_k^λ, for all $\sigma(t)\in\mathcal{O}^*$.

In fact, we have:

Lemma **(12.22)**. — *For all* $\sigma_1, \sigma_2\in\mathcal{L}_k^*$, $b_\alpha(\sigma_1, \sigma_2)$ *is a scalar multiple of the identity on* V_k^λ.

Proof. — If we write

$$\sigma_i=\sigma_i' t^{m_i}, \quad \sigma_i'\in\mathcal{O}^*, \quad i=1, 2,$$

then we see from the bilinearity of $b_\alpha(\ ,\ )$, from property (S2) of (12.18), and from (12.21), that it suffices to prove the lemma for

(i) $b_\alpha(\sigma_1, \sigma_2)$, $\sigma_1, \sigma_2\in\mathcal{O}^*$,

(ii) $b_\alpha(t, t^{-1})$, $b_\alpha(t, t)$.

But, in case (i), each of the operators $h_\alpha(\sigma_1)$, $h_\alpha(\sigma_2)$, $h_\alpha(\sigma_1\sigma_2)$ is upper triangular with respect to a coherently ordered basis (Lemma (12.2)). Then, we can use (12.6) to show

(12.23) $b_\alpha(\sigma_1, \sigma_2)=1, \quad \sigma_1, \sigma_2\in\mathcal{O}^*.$

We note that since for $c\in S^0(k^*, M)$,

$$c(1-x, x)=c(1-x, 1-(1-x)),$$

for $x\in\mathcal{L}_k^*$, $x\neq 1$, the relation (S3) of (12.18) implies

(S3') $c(1-x, x)=1, \quad x\in\mathcal{L}_k^*, \ x\neq 1.$

But then

$$b_\alpha(t, t^{-1})=b_\alpha(t(1-t^{-1}), t^{-1}), \text{ by (S3') and bilinearity,}$$
$$=b_\alpha(t-1, t^{-1})=\text{non-zero scalar multiple of the identity,}$$

by (12.21).

On the other hand

$$\begin{aligned} b_\alpha(t, t) &= b_\alpha(t, t^2t^{-1}) \\ &= b_\alpha(t, t^2)b_\alpha(t, t^{-1}), \text{ by bilinearity} \\ &= b_\alpha(t, t)^2 b_\alpha(t, t^{-1}), \text{ by bilinearity.} \end{aligned}$$

Hence $\quad b_\alpha(t, t)=b_\alpha(t, t^{-1})^{-1},$

and hence $b_\alpha(t, t)$ is also a scalar multiple of the identity. This concludes the proof of Lemma (12.22). ■

By Lemma (12.22), we may identify $b_\alpha(\ , \ )$ with an element of $S^0(\mathscr{L}_k^*, k^*)$. By (12.13), we have

$$b_\alpha(\sigma(t), t^{-1}) = q_0^{-\omega}, \qquad \omega = -\frac{2\lambda(h'_\iota)}{(\alpha, \alpha)}.$$

for $\sigma(t) = q_0 + q_1 t + \ldots + q_j t^j + \ldots$, $q_0 \in k^*$. On the other hand,

$$c_T(\sigma(t), t^{-1}) = \sigma(t)^{-1}(0) = q_0^{-1},$$

and so $$b_\alpha(\sigma(t), t^{-1}) = c_T(\sigma(t), t^{-1})^\omega, \qquad \sigma(t) \in \mathcal{O}^*.$$

Similarly, $c_T(\sigma_1, \sigma_2) = 1$, for $\sigma_1, \sigma_2 \in \mathcal{O}^*$, as one sees from the definition. Hence

$$b_\alpha(\sigma_1, \sigma_2) = c_T(\sigma_1, \sigma_2)^\omega, \qquad \sigma_1, \sigma_2 \in \mathcal{O}^*.$$

But then, arguing exactly as in the proof of Lemma (12.22), we see:

Theorem **(12.24)**. — *If* $\omega = -2\lambda(h'_\iota)(\alpha, \alpha)^{-1}$, *for* $\alpha \in \Delta_+(A)$ *a long root, then*:

$$b_\alpha(\sigma_1, \sigma_2) = c_T(\sigma_1, \sigma_2)^\omega, \qquad \sigma_1, \sigma_2 \in \mathscr{L}_k^*.$$

In particular, we may take $\lambda = \lambda_0$, where

(12.25) $$\lambda_0(h_i) = \begin{cases} 0, & i = 1, \ldots, \ell, \\ 1, & i = \ell + 1. \end{cases}$$

Then $\omega = -1$, and

$$b_\alpha(\sigma_1, \sigma_2) = c_T(\sigma_1, \sigma_2)^{-1}.$$

We thus have:

Theorem **(12.26)**. — *If* $\lambda \in \mathbf{D}$, and $\lambda(h_i) \neq 0$ *for some* $i = 1, \ldots, \ell + 1$, *then there is a unique homomorphism*

$$\Phi^\lambda : E^{\lambda_0}(G_{\mathscr{L}_k}) \to E^\lambda(G_{\mathscr{L}_k}),$$

such that

$$\Phi^\lambda((\chi_\beta^{\lambda_0}(\sigma(t)), \mathscr{X}'_\beta(\sigma(t))) = (\chi_\beta^\lambda(\sigma(t)), \mathscr{X}'_\beta(\sigma(t))), \qquad \beta \in \Delta(A), \ \sigma(t) \in \mathscr{L}_k^*.$$

Proof. — For any λ, we set

$$\mathscr{X}_\beta^\lambda(\sigma(t)) = (\chi_\beta^\lambda(\sigma(t)), \mathscr{X}'_\beta(\sigma(t))), \qquad \sigma(t) \in \mathscr{L}_k^*, \ \beta \in \Delta(A).$$

Now let $\mathscr{Y}^{\lambda_0} = \mathscr{Y}^{\lambda_0}(\mathscr{X}_\beta^{\lambda_0}(\sigma(t)))$ be a word in the $\mathscr{X}_\beta^{\lambda_0}(\sigma(t))$, such that $\mathscr{Y}^{\lambda_0} = 1$ in $E^{\lambda_0}(G_{\mathscr{L}_k})$. To prove the theorem, it then suffices to show that the corresponding word $\mathscr{Y}^\lambda$ in $E^\lambda(G_{\mathscr{L}_k})$, is equal to the identity.

Indeed, let $\mathscr{Y}^e$ be the corresponding word in the $\chi_\beta^e(\sigma(t))$, in $E(G_{\mathscr{L}_k})$. We have the commutative diagram

$$\begin{array}{ccc} & & E^{\lambda_0}(G_{\mathscr{L}_k}) \\ & \overset{\rho^{\lambda_0}}{\nearrow} & \\ E(G_{\mathscr{L}_k}) & & \downarrow \varphi_2^{\lambda_0} \\ & \underset{\varphi^e}{\searrow} & \\ & & G_{\mathscr{L}_k} \end{array}$$

where for $\lambda \in \mathbf{D}$ $(\lambda(h_i) \neq 0$, some $i = 1, \ldots, \ell+1)$, we let ρ^λ be defined by the conditions

$$\rho^\lambda(\chi_\beta^e(\sigma(t))) = \mathscr{X}_\beta^\lambda(\sigma(t)), \qquad \beta \in \Delta(A),\ \sigma(t) \in \mathscr{L}_k^*,$$

noting that ρ^λ is well defined since the $\mathscr{X}_\beta^\lambda(\sigma(t))$ (in place of $\mathscr{X}_\beta'(\sigma(t))$) satisfy the relations (A), (B) ((B′) when $G_{\mathscr{L}_k} = SL_2(\mathscr{L}_k)$) of § 10. Then

$$\varphi^e(\mathscr{Y}^e) = \varphi_2^{\lambda_0} \circ \rho^{\lambda_0}(\mathscr{Y}^e) = \varphi_2^{\lambda_0}(\mathscr{Y}^{\lambda_0}) = 1,$$

and so $\mathscr{Y}^e \in \operatorname{kernel} \varphi^e$. But then $\mathscr{Y}^e$ is a product of elements $h_\alpha^e(\sigma)h_\alpha^e(\tau)h_\alpha^e(\sigma\tau)^{-1}$, $\sigma, \tau \in \mathscr{L}_k^*$ (and inverses of such elements) in $E(G_{\mathscr{L}_k})$ (see Moore [13], Lemma (8.2)). Hence $\mathscr{Y}^{\lambda_0} = \rho^{\lambda_0}(\mathscr{Y}^e)$ (resp. $\mathscr{Y}^\lambda = \rho^\lambda(\mathscr{Y}^e)$) is the corresponding product of elements $b_\alpha^{\lambda_0}(\sigma, \tau)$ (and inverses of such elements) (resp. of elements $b_\alpha^\lambda(\sigma, \tau)$ (and inverses of such elements)). But $\mathscr{Y}^{\lambda_0} = (I, 1)$, where I denotes the identity operator on $V_k^{\lambda_0}$, and 1 denotes the identity in $G_{\mathscr{L}_k}$. It follows from Theorem (12.24), and from our choice of λ_0, that $\mathscr{Y}^\lambda$ is the identity in $E^\lambda(G_{\mathscr{L}_k})$. ∎

13. The Tits system.

We begin by summarizing some of the material in Bruhat-Tits [4], § 6. Recall that in Lemma (11.8) of the present paper, we proved that the family $(\nu_\alpha^*)_{\alpha \in \Delta(A)}$ is a valuation of the donnée radicielle $(T^*, (U_\alpha^*, M_\alpha^*)_{\alpha \in \Delta(A)})$. The elements of $\Delta(A)$ may be identified with elements of $\mathfrak{h}_{\mathbf{R}}(A)^*$, the real dual space of $\mathfrak{h}_{\mathbf{R}}(A)$. For each $v \in \mathfrak{h}_{\mathbf{R}}(A)$, for $s \in \mathbf{R}_+^*$ (= the set of non-zero, positive real numbers), and for each valuation $(\mu_\alpha^*)_{\alpha \in \Delta(A)}$ (so $\mu_\alpha^* = \mu_\alpha^e$, μ_α^λ, or μ_α) of $(T^*, (U_\alpha^*, M_\alpha^*)_{\alpha \in \Delta(A)})$, we define a new valuation $(\psi_\alpha^*)_{\alpha \in \Delta(A)}$ by

(13.1) $\qquad \psi_\alpha^*(u) = s\mu_\alpha^*(u) + \alpha(v) \qquad \alpha \in \Delta(A),\ u \in U_\alpha^*.$

We will denote a valuation $(\mu_\alpha^*)_{\alpha \in \Delta(A)}$ simply by μ^* (e.g. we will denote $(\nu_\alpha^*)_{\alpha \in \Delta(A)}$ by ν^*). We then write (13.1) more simply as

(13.1′) $\qquad \psi^* = s\mu^* + v.$

Thus the additive group $\mathfrak{h}_{\mathbf{R}}(A)$ acts on the valuations of $(T^*, (U_\alpha^*, M_\alpha^*)_{\alpha \in \Delta(A)})$, via (13.1′) (with $s = 1$). We call two valuations in the same orbit of this $\mathfrak{h}_{\mathbf{R}}(A)$ action, *équipollentes*.

On the other hand, consider the group $\mathbf{N}^*$ generated by T^* and by the M^*_α, $\alpha\in\Delta(A)$. We note there exists a homomorphism

$$^v p : \mathbf{N}^* \to W(A),$$

such that for all $\alpha\in\Delta(A)$, $n\in\mathbf{N}^*$, one has

$$(13.2) \qquad nU_\alpha n^{-1} = U_\beta, \qquad \beta = {}^v p(n)(\alpha).$$

Moreover, we have ${}^v p(M^*_\alpha) = r_\alpha$, where r_α is the reflection on $\mathfrak{h}_{\mathbf{R}}(A)^*$ defined by

$$(13.3) \qquad r_\alpha(\mu) = \mu - \frac{2(\alpha, \mu)}{(\alpha, \alpha)}\alpha, \qquad \mu\in\mathfrak{h}_{\mathbf{R}}(A)^*,$$

(see Bruhat-Tits [4], § 6.1.2 (10)).

If $n\in\mathbf{N}^*$, and if $w = {}^v p(n)\in W(A)$, then, as noted in (13.2), we have

$$n^{-1}un\in U^*_{w^{-1}(\alpha)}, \qquad \text{for} \qquad u\in U^*_\alpha, \quad \alpha\in\Delta(A).$$

Given a valuation $\mu^* = (\mu_\alpha)_{\alpha\in\Delta(A)}$ of $(T^*, (U^*_\alpha, M^*_\alpha)_{\alpha\in\Delta(A)})$, and $n\in\mathbf{N}^*$, we define a new valuation $n.\mu^* = \psi^*$, by

$$(13.4) \qquad \psi^*_\alpha(u) = \mu^*_{w^{-1}(\alpha)}(n^{-1}un), \qquad \alpha\in\Delta(A),\ u\in U^*_\alpha.$$

As noted in Bruhat-Tits [4], (6.2.5), (1), one easily checks that

$$(13.5) \qquad n.(s\mu^* + v) = s(n.\mu^*) + {}^v p(n)(v), \qquad s\in\mathbf{R}^*_+,\ v\in\mathfrak{h}_{\mathbf{R}}(A),\ n\in\mathbf{N}^*.$$

We let $\mathbf{A}$ denote the set of valuations équipollentes to ν^*. But of course we may identify $\mathbf{A}$ with $\mathfrak{h}_{\mathbf{R}}(A)$, and then furnish $\mathbf{A}$ with the inner product structure $(\ ,\ )$ on $\mathfrak{h}_{\mathbf{R}}(A)$, given by the Killing form. We note that if $\mu^*\in\mathbf{A}$, then it is natural to let $\mu^* - \nu^*$ denote the corresponding element of $\mathfrak{h}_{\mathbf{R}}(A)$.

For $\alpha\in\mathfrak{h}_{\mathbf{R}}(A)^*$, and $m\in\mathbf{R}$, we let $a_{\alpha,m}$ (or ${}^\nu a_{\alpha,m}$, if we wish to make the dependence on the valuation ν^* explicit) denote the half space

$$a_{\alpha,m} = \{x\in\mathbf{A} \mid \alpha(x-\nu^*) + m \geq 0\}.$$

We define the *affine roots* of $\mathbf{A}$ to be the half spaces $a_{\alpha,m}$, $\alpha\in\Delta(A)$, $m\in\mathbf{Z}$, and we let Σ denote the set of affine roots. We let

$$\mathscr{E} \subset \Sigma\times\Delta(A)$$

denote the subset of all pairs (a, α), where $a = a_{\alpha,m}$, for some $m\in\mathbf{Z}$. We let $\partial a_{\alpha,m}$, the boundary of the half space $a_{\alpha,m}$, $\alpha\in\Delta(A)$, $m\in\mathbf{Z}$, be defined by

$$\partial a_{\alpha,m} = \{x\in\mathbf{A} \mid \alpha(x-\nu^*) + m = 0\}.$$

Recalling that $\mathbf{A}$ inherits the inner product $(\ ,\ )$ from $\mathfrak{h}_{\mathbf{R}}(A)$, we let

$$r_{\alpha,m} : \mathbf{A}\to\mathbf{A},$$

denote the orthogonal reflection with respect to the hyperplane $\partial a_{\alpha,m}$.

Quoting Proposition 6.2.10 in [4], we have:

Proposition **(13.6).** — *The space* **A** *is stable under the action of* $\mathbf{N}^*$ *defined in* (13.4). *For* $n \in \mathbf{N}^*$, *we let* $p(n) : \mathbf{A} \to \mathbf{A}$ *denote the mapping* $p(n) : x \to n.x$, $x \in \mathbf{A}$. *Then* $p(n)$ *is an automorphism of the Euclidean space* **A**, *and the canonical image of* $p(n)$ *in* $\mathrm{Aut}(\mathfrak{h}_{\mathbf{R}}(\mathrm{A}))$ *(the group of linear automorphisms of* $\mathfrak{h}_{\mathbf{R}}(\mathrm{A})$*) is* ${}^v p(n)$. *Moreover:*

(i) *For each* $m \in \mathbf{Z}$, $\alpha \in \Delta(\mathrm{A})$ *and* $n \in \mathrm{M}^*_{\alpha,m} = \mathrm{M}^*_\alpha \cap \mathrm{U}^*_{-\alpha} v^{*-1}_\alpha(m) \mathrm{U}^*_{-\alpha}$, *the mapping* $p(n)$ *is the orthogonal reflection* $r_{\alpha,m}$.

(ii) *For* $a = a_{\alpha,m}$, $\alpha \in \Delta(\mathrm{A})$, $m \in \mathbf{Z}$, *we let* $\mathrm{U}^*_a = \mathrm{U}^*_{\alpha,m}$. *Then, if* $n \in \mathbf{N}^*$, *one has*

$$p(n)(a) \in \Sigma,$$

and

$$n\mathrm{U}^*_a n^{-1} = \mathrm{U}^*_{p(n)(a)}.$$

We let $\mathrm{T}^*_{\mathcal{O}} \subset \mathbf{N}^*$ denote the subgroup

$$\mathrm{T}^*_{\mathcal{O}} = p^{-1}(1),$$

where 1 denotes the identity automorphism of **A**. We have:

Lemma **(13.7).** — *The group* $\mathrm{T}^\lambda_{\mathcal{O}}$, $\lambda \in \mathbf{D}$, $\lambda(h_i) \neq 0$ *for some* $i = 1, \ldots, \ell+1$, *is the group generated by the elements* $h_\alpha(\sigma(t))$, $\alpha \in \Delta(\mathrm{A})$, $\sigma(t) \in \mathcal{O}^*$, *and by the elements* $h_{a_{\ell+1}}(s)$, $s \in k^*$.

Proof. — Denote by $\widetilde{\mathrm{T}}^\lambda$ the group generated by the elements $h_\alpha(\sigma(t))$, $\alpha \in \Delta(\mathrm{A})$, $\sigma(t) \in \mathcal{O}^*$, and the elements $h_{a_{\ell+1}}(s)$, $s \in k^*$. Using the identity

(13.8) $$h_{a_{\ell+1}}(s) = w_{-\alpha_0}(ts) w_{-\alpha_0}(t)^{-1}, \quad s \in k^*,$$

(which may be checked directly from the definitions) we see that $h_{a_{\ell+1}}(s) \in \mathbf{N}^\lambda$, $s \in k^*$. Also, the relation

(13.9) $$w_\alpha(\sigma)\chi_\beta(\tau)w_\alpha(\sigma)^{-1} = \chi_{r_\alpha(\beta)}(\eta\sigma^{-c}\tau),$$
$$\alpha,\ \beta \in \Delta(\mathrm{A}), \quad \sigma,\ \tau \in \mathscr{L}^*_k, \quad \eta = \eta(\alpha, \beta) = \pm 1,$$
$$c = 2(\beta, \alpha)(\alpha, \alpha)^{-1}, \quad \eta(\alpha, -\beta) = \eta(\alpha, \beta),$$

(see Matsumoto [13], Lemma (5.1), Steinberg [20], (7.2), and apply our Lemma (10.1), implies the relation

(13.10) $$h_\alpha(\sigma)\chi_\beta(\tau)h_\alpha(\sigma)^{-1} = \chi_\beta(\sigma^d \tau), \quad d = 2(\alpha, \beta)(\alpha, \alpha)^{-1},$$
$$\alpha,\ \tau \in \mathscr{L}^*_k,\ \alpha,\ \beta \in \Delta(\mathrm{A}).$$

But (13.8) implies

$$h_{a_{\ell+1}}(s) = h_{-\alpha_0}(st)h_{-\alpha_0}(t)^{-1}, \quad s \in k^*,$$

and (13.10) then implies $h_{\alpha_{\ell+1}}(s)\in T^\lambda_{\mathcal{O}}$. On the other hand, (13.10) also clearly implies that the elements $h_\alpha(\sigma(t))$, $\alpha\in\Delta(A)$, $\sigma(t)\in\mathcal{O}^*$, are in $T^\lambda_{\mathcal{O}}$, and hence we have proved

$$\widetilde{T}^\lambda \subset T^\lambda_{\mathcal{O}}.$$

Conversely, assume $n\in T^\lambda_{\mathcal{O}}$, so we have $p(n)=1$. Then ${}^v p(n)=1$, and hence (by Bruhat-Tits [4], (6.1.11)) we have $n\in T^\lambda$. Now T^λ is the subgroup of $\hat{G}^\lambda_k$ generated by the elements $h_\alpha(\sigma(t))$, $\sigma(t)\in\mathcal{L}^*_k$, $\alpha\in\Delta(A)$. Thanks to (13.10) and to Remark (11.7), for each $\alpha\in\Delta(A)$ we may define a character $\alpha: u\to u^\alpha$, $u\in T^\lambda$, $u^\alpha\in\mathcal{L}^*_k$, on T^λ, by

(13.11) $$u\chi_\alpha(\sigma)u^{-1}=\chi_\alpha(u^\alpha\sigma),\qquad \sigma\in\mathcal{L}_k,\ u\in T^\lambda.$$

Thus, if $p(u)=1$, then

$$\nu(u^\alpha\sigma)=\nu(\sigma),\qquad \text{for all}\qquad \sigma\in\mathcal{L}_k,\quad \alpha\in\Delta(A).$$

Taking $\sigma=1$, it follows, still assuming $p(u)=1$, that

(13.12) $$\nu(u^\alpha)=0,\qquad \text{for all}\qquad \alpha\in\Delta(A).$$

Now u may be expressed as a word in the $h_\alpha(\sigma)$'s and their inverses ($\alpha\in\Delta(A)$, $\sigma\in\mathcal{L}^*_k$), $u=u(h_\alpha(\sigma))$. Let $u^e=u^e(h^e_\alpha(\sigma))$ be the corresponding word in the $h^e_\alpha(\sigma)$'s and their inverses, in $E(G_{\mathcal{L}_k})$. Then of course

$$\Psi^e(u^e)=u$$

(see § 10 for the definition of $\Psi^e: E(G_{\mathcal{L}_k})\to\hat{G}^\lambda_k$). Also, we let $u'\in G_{\mathcal{L}_k}$ denote the corresponding word in the $h'_\alpha(\sigma)$'s and their inverses. Then

$$\varphi^e(u^e)=u'$$

(see § 10 for the definition of $\varphi^e: E(G_{\mathcal{L}_k})\to G_{\mathcal{L}_k}$). Now, thanks to standard results in the theory of Chevalley groups (see Steinberg [20], (8.2)), u' is a product

$$u'=h'_{\alpha_1}(\sigma_1)\ \dots\ h'_{\alpha_\ell}(\sigma_\ell),\qquad \sigma_i\in\mathcal{L}^*_k,\ i=1,\dots,\ell.$$

Hence,

$$u^e=h^e_{\alpha_1}(\sigma_1)\ \dots\ h^e_{\alpha_\ell}(\sigma_\ell)\ \text{mod kernel}\ \Psi^e.$$

But from Moore [18], Lemma (8.2), we have that if $\beta\in\Delta(A)$ is a fixed long root, then kernel Ψ^e is generated by the elements $h^e_\beta(\sigma)h^e_\beta(\tau)h^e_\beta(\sigma\tau)^{-1}$, $\sigma,\tau\in\mathcal{L}^*_k$. But then

(13.13) $$u=\Psi^e(u^e)=h_{\alpha_1}(\sigma_1)\ \dots\ h_{\alpha_\ell}(\sigma_\ell)\ \text{mod}\ \Lambda,$$

where Λ denotes the subgroup of $\hat{G}^\lambda_k$ generated by the elements $b_\beta(\sigma,\tau)$, $\sigma,\tau\in\mathcal{L}^*_k$.

On the one hand, each $b_\beta(\sigma,\tau)\in\text{Aut}\,V^\lambda_k$, is a scalar multiple of the identity operator of V^λ_k (Lemma (12.22)). Thus we have:

(13.14) Every element of $\Lambda=\varphi^e(\text{kernel}\ \Psi^e)$ is a scalar multiple of the identity operator of V^λ_k.

We let $u_1 = h_{\alpha_1}(\sigma_1) \dots h_{\alpha_\ell}(\sigma_\ell)$. Then from (13.11), (13.13) and (13.14), we have

(13.15) $\quad u^\alpha = u_1^\alpha, \quad$ all $\quad \alpha \in \Delta(\mathrm{A})$.

But if $p(u) = 1$, (13.12) and (13.10) imply

$$\nu\Big(\prod_{i=1}^{\ell} \sigma_i^{\langle \alpha, \alpha_i \rangle}\Big) = 0, \qquad \alpha \in \Delta(\mathrm{A}), \qquad \langle \alpha, \alpha_i \rangle = 2(\alpha, \alpha_i)(\alpha_i, \alpha_i)^{-1};$$

i.e., we have

$$\sum_{i=1}^{\ell} \langle \alpha, \alpha_i \rangle \nu(\sigma_i) = 0, \qquad \alpha \in \Delta(\mathrm{A}).$$

Taking $\alpha = \alpha_1, \dots, \alpha_\ell$, and using the fact that the Cartan matrix A is invertible, we see that

$$\nu(\sigma_i) = 0, \qquad i = 1, \dots, \ell;$$

therefore

$$u_1 \in \widetilde{\mathrm{T}}^\lambda.$$

Thus, to prove Lemma (13.7), it suffices to prove:

(13.16) $\quad \Lambda \subset \widetilde{\mathrm{T}}^\lambda.$

Of course, (13.16) is equivalent to showing

(13.16′) $\quad b_\beta(\sigma, \tau) \in \widetilde{\mathrm{T}}^\lambda, \qquad \sigma, \tau \in \mathscr{L}_k^*.$

First, if $\alpha \in \mathcal{O}^*$, then $b_\beta(\sigma, t^{-1}) \in \widetilde{\mathrm{T}}^\lambda$, by (12.13). If $\sigma_1, \sigma_2 \in \mathcal{O}^*$, we have $b_\beta(\sigma_1, \sigma_2) = 1$, by (12.23). We can then argue as in Lemma (12.22), to prove (13.16′), and hence Lemma (13.7). ■

Let $\Omega \subset \mathbf{A}$ be a bounded region; then following Bruhat-Tits [4], (6.4.2), we define an integral valued function f_Ω on $\Delta(\mathrm{A})$ by

$$f_\Omega(\alpha) = \inf\{m \in \mathbf{Z} \mid a_{\alpha,m} \supset \Omega\}.$$

We let

$$\mathrm{U}_\Omega = \mathrm{U}_\Omega^\lambda \subset \widehat{\mathrm{G}}_k^\lambda$$

denote the subgroup generated by the union of the subgroups $\mathrm{U}^\lambda_{\alpha, f_\Omega(\alpha)}$, $\alpha \in \Delta(\mathrm{A})$, and we set

$$\mathrm{U}^\lambda_{\pm,\Omega} = \mathrm{U}_{\pm,\Omega} = \mathrm{U}^\lambda_\pm \cap \mathrm{U}_\Omega.$$

Remark **(13.17)**. — Let C denote the subset of all $x \in \mathbf{A}$ satisfying the conditions

$$\alpha_i(x - \nu^\lambda) \geq 0, \qquad i = 1, \dots, \ell,$$
$$\alpha_0(x - \nu^\lambda) \leq 1.$$

Then $$f_{\mathrm{C}}(\alpha) = \begin{cases} 0, & \alpha \in \Delta_+(\mathrm{A}) \\ 1, & \alpha \in \Delta_-(\mathrm{A}). \end{cases}$$

From Remark (13.17), we see that U_C is the subgroup generated by the elements $\chi_\alpha(\sigma)$, $\sigma\in\mathcal{O}$, $\alpha\in\Delta_+(A)$, and by the elements $\chi_\alpha(\sigma)$, $\sigma\in\mathcal{P}$, $\alpha\in\Delta_-(A)$; i.e., we have

(13.18) $\qquad \mathcal{I}_U = U_C.$

Also, if we let P_{fc} denote the subgroup generated by U_C and by $T^\lambda_{\mathcal{O}}$, then we see that from (13.18) and Lemma (13.7),

(13.19) $\qquad \mathcal{I} = P_{fc}.$

We let $\mathbf{W}$ be the group of affine automorphisms of $\mathbf{A}$ generated by the orthogonal reflections $r_{\alpha,m}$, $\alpha\in\Delta(A)$, $m\in\mathbf{Z}$. Thanks to Proposition (13.6), we note that

$$\mathbf{W}\subset p(\mathbf{N}^\lambda),$$

and we let $\widetilde{\mathbf{N}}^\lambda = p^{-1}(\mathbf{W})$. We let $\mathbf{S}\subset\mathbf{W}$ be the set of reflections $r_{\alpha_1,0},\ldots,r_{\alpha_\ell,0}$, and $r_{-\alpha_0,1}$; i.e., $\mathbf{S}$ is the set of reflections with respect to the walls of C. The relation (13.19), and the theorem of § 6.5 in Bruhat-Tits [4] now imply:

Theorem **(13.20)**. — *We have* $\widetilde{\mathbf{N}}^\lambda\cap\mathcal{I} = T^\lambda_{\mathcal{O}}$. *The quadruplet* $(\hat{G}^\lambda_k, \mathcal{I}, \widetilde{\mathbf{N}}^\lambda, \mathbf{S})$ *is a Tits system, where we identify* $\mathbf{W}$ *with* $\widetilde{\mathbf{N}}^\lambda/T^\lambda_{\mathcal{O}}$.

We recall that a quadruplet (G, B, N, S) is a Tits system if

(13.21) (1) B, N are subgroups of G, $B\cup N$ generates G, and $B\cap N$ is a normal subgroup of N.
(2) $S\subset N/(B\cap N) =_{\mathrm{df}} W$, consists of elements of order 2.
(3) $sBw\subset BwB\cup BswB$, $s\in S$, $w\in W$.
(4) For all $s\in S$, we have $sBs\not\subset B$.

From Proposition (6.4.9) and Lemma (6.4.11) of Bruhat-Tits [4], and from the observation that (in the notation of Lemma (6.4.11) of [4]) Φ_{fc} is empty, it follows that

(13.22) $\qquad \mathcal{I} = U_{-,C}T^\lambda_{\mathcal{O}}U_{+,C}.$

We also have from Proposition (6.4.9) of [4], that if one is given an order on $\Delta_\pm(A)$ and forms the product

$$\prod_{\alpha\in\Delta_\pm(A)} U^\lambda_{\alpha,fc(\alpha)}$$

with respect to this order, then

(13.23) $\qquad$ The product map $\prod_{\alpha\in\Delta_\pm(A)} U^\lambda_{\alpha,fc(\alpha)}\to U_{\pm,C}$ is bijective.

14. The Tits system (continued).

Our first goal is to define a certain subgroup of $\widetilde{\mathbf{N}}^\lambda$. Toward this end we first show:

Lemma **(14.1)**. — *For* $m \in \mathbf{Z}$, $\alpha \in \Delta(A)$, *we have*

$$p(w_\alpha(t^m s)) = r_{\alpha,m},$$

for all $s \in k^*$.

Proof. — Let $v \in \mathfrak{h}(A)$ and set

$$\psi^\lambda = \nu^\lambda + v$$

(see (13.1) and (13.1′)). We then have, for $n \in \mathbf{N}^\lambda$,

$$n \, . \, \psi^\lambda = n \, . \, \nu^\lambda + {}^v p(n)(v) \quad \text{(by (13.5))}.$$

But, by (13.2) and (13.9),

(14.1′) $\qquad {}^v p(w_\alpha(t^m s)) = r_\alpha, \qquad \alpha \in \Delta(A),\ m \in \mathbf{Z},\ s \in k^*.$

In other words (see (13.3))

(14.2) $\qquad {}^v p(w_\alpha(t^m s))(v) = v - \alpha(v) H_\alpha, \qquad v \in \mathfrak{h}(A).$

On the other hand, by (13.4), and (14.1′), with $n = w_\alpha(t^m s)$, we have

(14.3) $\qquad (n \, . \, \nu^\lambda)_\beta(u) = \nu^\lambda_{r_\alpha(\beta)}(w_\alpha(t^m s)^{-1} u w_\alpha(t^m s)), \qquad \beta \in \Delta(A),\ u \in U^\lambda_\beta.$

But (13.9) implies:

(14.4) $\qquad w_\alpha(\sigma)^{-1} \chi_\beta(\tau) w_\alpha(\sigma) = \chi_{r_\alpha(\beta)}(\eta \sigma^{-c} \tau),$

with $\sigma \in \mathscr{L}_k^*$, $\tau \in \mathscr{L}_k$, $c = 2(\alpha, \beta)(\alpha, \alpha)^{-1}$, $\eta = \pm 1$. Indeed, by (13.9), we have

$$w_\alpha(\sigma) \chi_\beta(\tau) w_\alpha(\sigma)^{-1} = \chi_{r_\alpha(\beta)}(\eta \sigma^{-c} \tau), \qquad \sigma \in \mathscr{L}_k^*,\ \tau \in \mathscr{L}_k,$$

hence

$$w_\alpha(\sigma)^{-1} \chi_{\beta'}(\tau') w_\alpha(\sigma) = \chi_{r_\alpha(\beta')}(\eta \sigma^c \tau'),$$

where $\beta' = r_\alpha(\beta)$, $\tau' = \eta \sigma^{-c} \tau$. But

$$c = 2(\alpha, r_\alpha(\beta'))(\alpha, \alpha)^{-1} = -2(\alpha, \beta')(\alpha, \alpha)^{-1},$$

and we thus obtain (14.4).

But from (14.3) and (14.4) we have

(14.5) $\qquad w_\alpha(t^m s) \, . \, \nu^\lambda = \nu^\lambda - m H_\alpha, \qquad \alpha \in \Delta(A),\ m \in \mathbf{Z},\ s \in k^*.$

Hence, for $n = w_\alpha(t^m s)$,

$$n \, . \, \psi^\lambda = n \, . \, \nu^\lambda + {}^v p(n)(v) = \nu^\lambda - m H_\alpha + (v - \alpha(v) H_\alpha),$$

by (14.2) and (14.5). To prove Lemma (14.1), it thus suffices to show that the transformation

$$v \mapsto v - m H_\alpha - \alpha(v) H_\alpha$$

is $r_{\alpha,m}$, the orthogonal reflection with respect to the hyperplane $\alpha(v) = -m$. But this is clear. ∎

Now let $a \in \Delta_W(\widetilde{A})$, and assume

$$a = a(\alpha) + n\iota, \quad \alpha \in \Delta(A),\ n \in \mathbf{Z}.$$

(See (4.3) and the subsequent definition.)

Then

$$\begin{aligned} w_a(s) &= \chi_a(s)\chi_{-a}(-s^{-1})\chi_a(s), \quad \text{by } (7.23) \\ &= \chi_\alpha(t^n s)\chi_{-\alpha}(-t^{-n}s^{-1})\chi_\alpha(t^n s), \quad \text{by } (7.19) \\ &= w_\alpha(t^n s), \quad \text{by } (7.22). \end{aligned}$$

In particular, for $a_{\ell+1} = -\alpha_0 + \iota$, we have

(14.6) $$w_{a_{\ell+1}}(s) = w_{-\alpha_0}(ts), \quad s \in k^*,$$

while, for $i = 1, \ldots, \ell$,

(14.6′) $$w_{a_i}(s) = w_{\alpha_i}(s), \quad a_i = a(\alpha_i),\ s \in k^*.$$

It thus follows from Lemma (14.1) that, for $s \in k^*$,

(14.7) $$p(w_{a_i}(s)) = \begin{cases} r_{\alpha_i, 0}, & i = 1, \ldots, \ell \\ r_{-\alpha_0, 1}, & i = \ell + 1. \end{cases}$$

In fact, thanks to (14.6), (14.6′), and (14.7), we have proved:

Proposition **(14.8)**. — *Each of the elements* $w_a(s)$, $a \in \Delta_W(\widetilde{A})$, $s \in k^*$, *is in* $\widetilde{\mathbf{N}}^\lambda$, *and if* $N \subset \widetilde{\mathbf{N}}^\lambda$ *is the subgroup generated by the* $w_a(s)$, $a \in \Delta_W(\widetilde{A})$, $s \in k^*$, *then under* p, *the set* $\{w_{a_i}(1)\}_{i=1,\ldots,\ell+1}$ *maps bijectively onto* $\mathbf{S}$. *In particular* $\mathbf{W} = p(N)$.

The next lemma follows from the axioms (13.21) for a Tits system and from the Bruhat decomposition (see Bourbaki [3], Chapter 4, § 2.3, p. 25).

Lemma **(14.9)**. — *Let* (G, B, N', S) *be a Tits system, and let* $N_0 \subset N'$ *be a subgroup such that the projection*

$$\pi : N' \to N'/(N' \cap B) = W$$

maps N_0 *onto* W. *We then have*: (i) $N_0 \cap B$ *is a normal subgroup of* N_0. (ii) *The inclusion* $N_0 \hookrightarrow N'$ *induces an isomorphism from* $N_0/(N_0 \cap B)$ *onto* W. (iii) *If we identify* $N_0/(N_0 \cap B)$ *with* W, *by means of this isomorphism, then* (G, B, N_0, S) *is a Tits system.*

Proof. — Since $\pi(N_0) = W$, it follows from the Bruhat decomposition (Bourbaki [3], Chapter 4, § 2.3, p. 25)

$$G = \bigcup_{w \in W} BwB,$$

that $B \cup N_0$ generates G. The other axioms for (G, B, N_0, S) follow directly from the corresponding axioms for (G, B, N', S). ∎

From Proposition (14.8), Lemma (14.9), and Theorem (13.20), we have

Theorem **(14.10)**. — *The quadruplet*

$$(\hat{G}_k^\lambda, \mathscr{I}, N, \mathbf{S})$$

is a Tits system.

We now wish to study the intersection $N \cap \mathscr{I}$. Toward this end, we first study $\mathscr{I}$. Thus, we let $H_k^\lambda = H_k$ denote the subgroup of $\hat{G}_k^\lambda$ generated by the elements $h_\alpha(s)$, $\alpha \in \Delta_+(A)$, $s \in k^*$ and by the elements $h_{a_{\ell+1}}(s)$, $s \in k^*$. From (14.6) we have

(14.11) $$h_{a_{\ell+1}}(s) = w_{a_{\ell+1}}(s) w_{a_{\ell+1}}(1)^{-1} = h_{-\alpha_0}(ts) h_{-\alpha_0}(t)^{-1}, \quad s \in k^*.$$

But then, from (14.11) and (13.10) we deduce

Lemma **(14.12)**. — *The subgroup* H_k *normalizes the subgroup* $\mathscr{I}_U$.

On the other hand, Lemma (12.2) implies that $\mathscr{I}_U$ and H_k generate $\mathscr{I}$, and so

(14.13) $$\mathscr{I} = H_k \mathscr{I}_U = \mathscr{I}_U H_k.$$

Also, since $h_\alpha(s) = h_{a(\alpha)}(s)$, $\alpha \in \Delta(A)$, $s \in k^*$, we see by Lemma (11.2), (ii), that:

(14.14) Relative to a coherently ordered basis of V_k^λ, the elements of H_k are represented by diagonal matrices, which restricted to a weight space $V_{\mu,k}^\lambda$, are scalar multiples of the identity.

Also, by (12.1) we have that relative to a coherently ordered basis, the elements of $\mathscr{I}_U$ are represented by upper triangular matrices with ones on the diagonal. Indeed, the argument leading to (12.1) in fact shows:

(14.15) Relative to a coherently ordered basis, the elements of $\mathscr{I}_U$ are represented by upper triangular matrices with ones on the diagonal. Moreover, the diagonal blocks corresponding to weight spaces, are identity matrices.

We will use this stronger conclusion a little later on. But even from (12.1) and (14.14) we see that

(14.16) $$H_k \cap \mathscr{I}_U = \{1\},$$

and hence we have:

Proposition **(14.17)**. — $\mathscr{I}$ *is the semi-direct product of* H_k *and the normal subgroup* $\mathscr{I}_U$.

Corollary **(14.18)**. — *The group* H_k *is precisely the set of all diagonal elements (with respect to a coherently ordered basis of* V_k^λ*) in* $\mathscr{I}$. *Relative to a coherently ordered basis of* V_k^λ, *the group* $\mathscr{I}$ *consists of upper triangular matrices whose diagonal blocks corresponding to the weight spaces, are equal to scalar matrices.*

We are now ready to prove:

Proposition **(14.19)**. — *The intersection* $N \cap \mathscr{I}$ *is equal to* H_k.

Proof. — Thanks to Lemma (11.2), the elements of N permute the weight spaces of V_k^λ. Thus, by Corollary (14.18), the elements of $N \cap \mathscr{I}$ must have off diagonal blocks zero, hence must be diagonal. Since $H_k \subset N \cap \mathscr{I}$, the Lemma now follows from Corollary (14.18). ■

We conclude this section with:

Lemma **(14.20)**. — *The group* H_k *is the group generated by the elements* $h_{a_i}(s)$, $i = 1, \ldots, \ell + 1$, $s \in k^*$. *Moreover,* H_k *contains* $h_a(s)$ *for every* $a \in \Delta_W(\widetilde{A})$, $s \in k^*$.

The proof of this lemma depends on:

Lemma **(14.21)**. — *For every* $a \in \Delta_W(\widetilde{A})$, *the coroot* $2a/\sigma(a, a)$ *is an integral linear combination of the fundamental coroots* $2a_i/\sigma(a_i, a_i)$, $i = 1, \ldots, \ell + 1$.

The proof of Lemma (14.21) is *exactly* the same as the proof of Lemma (4.4) in [7] and is therefore omitted.

Proof of Lemma (14.20). — By definition, the group H_k is generated by the elements $h_{a(\alpha)}(s) = h_\alpha(s)$, $\alpha \in \Delta(A)$, $s \in k^*$, and by the elements $h_{a_{\ell+1}}(s)$, $s \in k^*$. But $a(\alpha_i) = a_i$, $i = 1, \ldots, \ell$, and hence H_k contains the elements $h_{a_i}(s)$, $i = 1, \ldots, \ell + 1$, $s \in k^*$. The rest follows from Lemma (14.21), and from Lemma (11.2), (ii). ■

15. Comparison of the $\hat{G}_k^\lambda$ for different λ.

As always since § 7, we let k denote a field of arbitrary characteristic, and $\lambda \in \mathbf{D}$ an element such that $\lambda(h_i) \neq 0$ for some $i = 1, \ldots, \ell + 1$. *From now on* we call such an element of $\mathbf{D}$ *normal.* Then for each normal $\lambda \in \mathbf{D}$ we have defined the group $\hat{G}_k^\lambda \subset \operatorname{Aut} V_k^\lambda$. For each $i = 1, \ldots, \ell + 1$, we let $\lambda_i \in \mathfrak{h}^e(\widetilde{A})^*$ denote the dominant, integral linear functional (i.e., element of $\mathbf{D}$), defined by the conditions

$$(15.1) \qquad \lambda_i(h_j) = \delta_{ij}, \qquad \lambda_i(D) = 0, \qquad i, j = 1, \ldots, \ell + 1.$$

By restriction we may regard each λ_i as an element of $\mathfrak{h}(\widetilde{A})^*$, the dual space of $\mathfrak{h}(\widetilde{A})$. We let $\Xi \subset \mathfrak{h}(\widetilde{A})^*$ denote the $\mathbf{Z}$-lattice of rank $\ell + 1$, spanned by $\lambda_1, \ldots, \lambda_{\ell+1}$. Also, by restriction, each $a \in \Delta(\widetilde{A})$ defines an element of $\mathfrak{h}(\widetilde{A})^*$, and we let $\Xi_r \subset \mathfrak{h}(\widetilde{A})^*$ denote the $\mathbf{Z}$-span of the restrictions of elements of $\Delta(\widetilde{A})$. Then Ξ_r is the free $\mathbf{Z}$-module of rank ℓ, generated by $a_1, \ldots, a_\ell$, restricted to $\mathfrak{h}(\widetilde{A})$, and since $a_i(h) \in \mathbf{Z}$, $i, j = 1, \ldots, \ell + 1$, we have $\Xi_r \subset \Xi$. For each normal, dominant integral $\lambda \in \mathfrak{h}^e(\widetilde{A})^*$, we let $\Xi_\lambda \subset \mathfrak{h}(\widetilde{A})^*$ denote the $\mathbf{Z}$-span of the restrictions to $\mathfrak{h}(\widetilde{A})$ of the weights of V^λ. From our previous observation that $\Xi_r \subset \Xi$, we immediately deduce that $\Xi_\lambda \subset \Xi$. In fact, we have:

Lemma **(15.2)**. — *Let* $\lambda \in \mathbf{D}$ *be normal. Then* $\Xi_r \subset \Xi_\lambda \subset \Xi$, *and* Ξ_λ *is of finite index in* Ξ.

Proof. — We have already noted that $\Xi_\lambda \subset \Xi$. Also, since over $\mathbf{Z}$, Ξ_r has rank ℓ, we note that if $\Xi_r \subset \Xi_\lambda$, then Ξ_λ must have rank $\ell+1$ over $\mathbf{Z}$, because λ restricted to $\mathfrak{h}(A)$ is clearly linearly independent from the elements of Ξ_r. Hence, since Ξ has rank $\ell+1$, it suffices, in order to prove the lemma, to show that $\Xi_r \subset \Xi_\lambda$.

We let $I = \{i \in \{1, \ldots, \ell+1\} \mid \lambda(h_i) \neq 0\}$. By our assumption that λ is normal, I is not equal to the empty set. Let $v_0 \in V^\lambda$ be a highest weight vector. We then have for $i \in I$

$$e_i \cdot (f_i \cdot v_0) = f_i \cdot (e_i \cdot v_0) + h_i \cdot v_0 = \lambda(h_i) v_0 \neq 0.$$

Thus $f_i \cdot v_0 \neq 0$, and since, as one checks directly,

(15.3) $$\mathfrak{g}^a . V^\lambda_\mu \subset V^\lambda_{\mu+a}, \quad a \in \Delta(\widetilde{A}) \quad \text{and} \quad \mu, \mu + a \text{ weights of } V^\lambda,$$

the difference $\lambda - a_i$ is a weight of V^λ, and hence a_i restricted to $\mathfrak{h}(\widetilde{A})$ is in Ξ_λ (for *each* $i \in I$). Now let $j \in \{1, \ldots, \ell+1\} - I$. Then we can find a sequence $j_1, \ldots, j_s$ in $\{1, \ldots, \ell+1\}$, so that

$$j_1 \in I, \qquad j_s = j,$$

and

(15.4) $$\sigma(a_{j_m}, a_{j_{m+1}}) \neq 0, \qquad m = 1, \ldots, s-1.$$

We *assume* we have chosen the above sequence $j_1, \ldots, j_s$ so that s is minimal. Then

(15.5) $$\lambda(h_{j_p}) = 0, \qquad p = 2, \ldots, s.$$

We define $v_q \in V^\lambda$, $q = 1, \ldots, s$, inductively by

$$v_q = f_{j_q} \cdot v_{q-1},$$

where v_0 is the fixed highest weight vector. We set $\lambda_q = \lambda - a_{j_1} - \ldots - a_{j_q}$. If $v_q \neq 0$, then $v_r \neq 0$ and $v_r \in V^\lambda_{\lambda_r}$, for $r = 1, \ldots, q$. Therefore $a_{j_1}, \ldots, a_{j_q}$ restricted to $\mathfrak{h}(\widetilde{A})$ are all in Ξ_λ. Thus, if we make the inductive hypothesis:

(H) for q with $1 \leq q < s$, we have $v_q \neq 0$;

then to prove the lemma, it will suffice to show that $v_{q+1} \neq 0$ (note that we have verified (H) for $q = 1$). But

(15.6) $$e_{j_{q+1}} \cdot v_q = 0,$$

thanks to (6.7), (15.3) and our minimality assumption on s (the last implying $a_{j_{q+1}} \neq a_m$, $1 \leq m \leq q$). Hence

$$\begin{aligned} e_{j_{q+1}} \cdot v_{q+1} &= e_{j_{q+1}} \cdot f_{j_{q+1}} \cdot v_q \\ &= h_{j_{q+1}} \cdot v_q, \quad \text{by (15.6)} \\ &= \lambda_q(h_{j_{q+1}}) v_q, \end{aligned}$$

where the last equality follows from our previous observation that if $v_q \neq 0$, then $v_q \in V^{\lambda}_{\lambda_q}$. But our minimality assumption on s implies

$$\sigma(a_{j_m}, a_{j_{q+1}}) = 0, \quad 1 \leq m < q,$$

and
$$\lambda(h_{j_{q+1}}) = 0.$$

Hence,
$$\lambda_q(h_{j_{q+1}}) = a_{j_q}(h_{j_{q+1}}),$$

and the latter is not equal to zero, by (15.4). Hence $e_{j_{q+1}} \cdot v_{q+1} \neq 0$, and v_{q+1} is different from zero. We have already noted that this proves the lemma. ■

Lemma **(15.7)**. — *If* $\lambda \in \mathbf{D}$ *is normal, then for each normal element* $\mu \in \mathbf{D}$, *there is a positive integer m such that* $\Xi_{m\mu} \subset \Xi_\lambda$.

Proof. — Since both Ξ_λ and Ξ_μ are of finite index in Ξ (by Lemma (15.2)), Ξ_μ is in the **Q**-span of Ξ_λ, and hence $m\mu \in \Xi_\lambda$ for some positive integer m. But $\Xi_r \subset \Xi_\lambda$ (by Lemma (15.2)); therefore $\Xi_{m\mu} \subset \Xi_\lambda$, by (6.7) (as applied to μ in place of λ, and to a weight μ' of V^μ, in place of the "μ" appearing in (6.7)). ■

Our next point is to note that:

Lemma **(15.8)**. — *An element* ξ *of the center of* $G_{\mathscr{L}_k}$ *is a product*

$$\xi = h'_{\alpha_1}(s_1) \ldots h'_{\alpha_\ell}(s_\ell), \quad s_1, \ldots, s_\ell \in k^*.$$

Proof. — By Steinberg [21], Lemma 28 (d), p. 43, we have

$$\xi = h'_{\alpha_1}(\sigma_1) \ldots h'_{\alpha_\ell}(\sigma_\ell), \quad \sigma_1, \ldots, \sigma_\ell \in \mathscr{L}_k^*,$$

and

$$(*) \qquad \prod_{i=1}^{\ell} \sigma_i^{\langle \alpha_j, \alpha_i \rangle} = 1, \quad j = 1, \ldots, \ell,$$

where $\langle \alpha, \beta \rangle =_{df} 2(\alpha, \beta)(\beta, \beta)^{-1}$, $\alpha, \beta \in \Delta(A)$. But then, applying the t-adic valuation ν to both sides of $(*)$, we have

$$\sum_{i=1}^{\ell} \langle \alpha_j, \alpha_i \rangle \nu(\sigma_i) = 0, \quad j = 1, \ldots, \ell.$$

Hence, since the Cartan matrix A is nonsingular, we have $\nu(\sigma_i) = 0$, $i = 1, \ldots, \ell$, and we may set $s_i = \sigma_i \in k^*$. ■

We are now ready to prove:

Theorem **(15.9)**. — *Let* λ_1, λ_2 *be two normal elements of* $\mathbf{D}$, *and assume* $\Xi_{\lambda_2} \subset \Xi_{\lambda_1}$. *Then there is a unique group homomorphism*

$$\pi(\lambda_1, \lambda_2) : \hat{G}_k^{\lambda_1} \to \hat{G}_k^{\lambda_2},$$

of $\hat{G}_k^{\lambda_1}$ *onto* $\hat{G}_k^{\lambda_2}$, *such that*

$$\pi(\lambda_1, \lambda_2)(\chi_\alpha^{\lambda_1}(\sigma(t))) = \chi_\alpha^{\lambda_2}(\sigma(t)), \quad \alpha \in \Delta(A), \ \sigma(t) \in \mathscr{L}_k.$$

Proof. — Let

$$w^{\lambda_1} = w(\chi_\alpha^{\lambda_1}(\sigma(t)))$$

be a word in the $\chi_\alpha^{\lambda_1}(\sigma(t))$, $\alpha \in \Delta(A)$, $\sigma(t) \in \mathscr{L}_k^*$, and assume $w^{\lambda_1} = 1$ in $\hat{G}_k^{\lambda_1}$. Let w^{λ_2} (resp. w^e; resp. w') denote the corresponding word in the $\chi_\alpha^{\lambda_2}(\sigma(t))$ in $\hat{G}_k^{\lambda_2}$ (resp. in the $\chi_\alpha^e(\sigma(t))$ in $E(G_{\mathscr{L}_k})$; resp. in the $\mathscr{X}'_\alpha(\sigma(t))$ in $G_{\mathscr{L}_k}$). We consider the diagram

$$\begin{array}{ccc} \hat{G}_k^{\lambda_1} & & G_{\mathscr{L}_k} \\ & \searrow^{\pi_1} \quad \swarrow_{\pi_2} & \\ & G_{\mathscr{L}_k}^{\lambda_1} & \end{array}$$

where π_1, π_2 are as in § 12. We have

$$1 = \pi_1(w^{\lambda_1}) = \pi_2(w'),$$

and hence w' is in the center of $G_{\mathscr{L}_k}$. But we also have the homomorphism

$$\varphi^e : E(G_{\mathscr{L}_k}) \to G_{\mathscr{L}_k}$$

of § 10, where $\varphi^e(\chi_\alpha^e(\sigma(t))) = \mathscr{X}'_\alpha(\sigma(t))$, $\alpha \in \Delta(A)$, $\sigma(t) \in \mathscr{L}_k^*$, and hence

$$\varphi^e(w^e) = w'.$$

Then, by Lemma (15.8), there exist $s_1, \ldots, s_\ell \in k^*$, so that

(15.10) $$w^e = h_{\alpha_1}^e(s_1) \ldots h_{\alpha_\ell}^e(s_\ell) \bmod (\text{kernel } \varphi^e).$$

If we fix a long root $\alpha \in \Delta(A)$, then kernel φ^e is generated by the elements

(15.11) $$c_\alpha(\sigma_1, \sigma_2) = h_\alpha^e(\sigma_1) h_\alpha^e(\sigma_2) h_\alpha^e(\sigma_1\sigma_2)^{-1}, \quad \sigma_1, \sigma_2 \in \mathscr{L}_k^*,$$

thanks to Moore [18], Lemma (8.2).

In Lemma (10.1), we defined the homomorphism $\Psi^e : E(G_{\mathscr{L}_k}) \to \hat{G}_k^\lambda$, such that

(15.12) $$\Psi^e(\chi_\alpha^e(\sigma(t))) = \chi_\alpha^\lambda(\sigma(t)),$$

$\alpha \in \Delta(A)$, $\sigma(t) \in \mathscr{L}_k^*$. We write $\Psi^{e,\lambda}$ for Ψ^e, to denote the λ-dependence. We then have from (15.12) that

$$\Psi^{e,\lambda_i}(c_\alpha(\sigma_1, \sigma_2)) = h_\alpha^{\lambda_i}(\sigma_1) h_\alpha^{\lambda_i}(\sigma_2) h_\alpha^{\lambda_i}(\sigma_1\sigma_2)^{-1}, \quad \sigma_1, \sigma_2 \in \mathscr{L}_k^*, \ i = 1, 2,$$

and, thanks to Theorem (12.24),

(15.13) $$\Psi^{e,\lambda_i}(c_\alpha(\sigma_1, \sigma_2)) = c_T(\sigma_1, \sigma_2)^{\omega(\lambda_i)} I, \quad \sigma_1, \sigma_2 \in \mathscr{L}_k^*, \ i = 1, 2,$$

where $\omega(\lambda_i) = -2\lambda_i(h'_\iota)(\alpha, \alpha)^{-1}$, $i = 1, 2$, and I denotes the identity operator on $V_k^{\lambda_i}$.

On the other hand, by (12.13) (second equality)

(15.14) $$h_{a(\alpha)}^{\lambda_i}(c^{-1}) h_{a(\alpha)-\iota}^{\lambda_i}(c) = c^{\omega(\lambda_i)} I;$$

i.e., taking into account (15.13), we have

(15.15) $$\Psi^{e,\lambda_i}(c_\alpha(\sigma_1, \sigma_2)) = h_{a(\alpha)}^{\lambda_i}(c_T(\sigma_1, \sigma_2)^{-1}) h_{a(\alpha)-\iota}^{\lambda_i}(c_T(\sigma_1, \sigma_2)).$$

Now we use (15.10) to write w^e as a product

$$w^e = h^e_{\alpha_1}(s_1) \ldots h^e_{\alpha_\ell}(s_\ell)p^e,$$

where p^e is a product of elements $c_\alpha(\sigma_1, \sigma_2)$, as defined in (15.11). We let p^{λ_i}, $i=1, 2$, be the corresponding product of elements

$$c^{\lambda_i}_\alpha(\sigma_1, \sigma_2) =_{\mathrm{df}} h^{\lambda_i}_{a(\alpha)}(c_{\mathrm{T}}(\sigma_1, \sigma_2)^{-1}) h^{\lambda_i}_{a(\alpha)-\iota}(c_{\mathrm{T}}(\sigma_1, \sigma_2))$$

in $\hat{\mathrm{G}}^{\lambda_i}_k$. We then have, from (15.15),

$$\Psi^{e, \lambda_i}(p^e) = p^{\lambda_i}, \quad i=1, 2.$$

Hence $$w^{\lambda_i} = \Psi^{e, \lambda_i}(w^e) = h^{\lambda_i}_{a_1}(s_1) \ldots h^{\lambda_i}_{a_\ell}(s_\ell)p^{\lambda_i}, \quad i=1, 2.$$

In other words, we have written w^{λ_1} as a product in elements $h^{\lambda_1}_a(s_a)$, $a \in \Delta_{\mathrm{W}}(\widetilde{\mathrm{A}})$, $s_a \in k^*$, and w^{λ_2} as the exactly corresponding product in elements $h^{\lambda_2}_a(s_a)$, $a \in \Delta_{\mathrm{W}}(\widetilde{\mathrm{A}})$, $s_a \in k^*$. Thus, if $w^{\lambda_1}=1$, as we assumed, then $w^{\lambda_2}=1$, thanks to Lemma (11.2) and our assumption that $\Xi_{\lambda_2} \subset \Xi_{\lambda_1}$.

It follows that if we set

$$\pi(\lambda_1, \lambda_2)(\chi^{\lambda_1}_\alpha(\sigma(t))) = \chi^{\lambda_2}_\alpha(\sigma(t)), \quad \alpha \in \Delta(\mathrm{A}), \ \sigma(t) \in \mathscr{L}^*_k,$$

then $\pi(\lambda_1, \lambda_2)$ defines a homomorphism from $\hat{\mathrm{G}}^{\lambda_1}_k$ onto $\hat{\mathrm{G}}^{\lambda_2}_k$. ∎

We conclude this section with a few observations. First, note that in the proof of Theorem (15.9), we showed:

(15.16) The kernel of the homomorphism

$$\pi_1 : \hat{\mathrm{G}}^\lambda_k \to \mathrm{G}^\lambda_{\mathscr{L}_k}$$

is contained in H_k.

It then follows from (15.16) and from (14.16), that

(15.17) The homomorphism π_1, restricted to the subgroup $\mathscr{I}_{\mathrm{U}} \subset \hat{\mathrm{G}}^\lambda_k$, is injective.

Also, we have, using Lemma (15.8):

(15.18) The kernel of the homomorphism

$$\pi_2 : \mathrm{G}_{\mathscr{L}_k} \to \mathrm{G}^\lambda_{\mathscr{L}_k}$$

is contained in the subgroup $\mathrm{H}'_k \subset \mathrm{G}_{\mathscr{L}_k}$, generated by the elements $h_\alpha(s)$, $\alpha \in \Delta(\mathrm{A})$, $s \in k^*$.

Indeed, (15.18) results from Lemma (15.8), and from the fact (see Steinberg [21], Corollary 5, after Lemma 28, page 44) that $\mathrm{kernel}\ \pi_2 \subset \mathrm{center}\ \mathrm{G}_{\mathscr{L}_k}$.

Finally, we note:

(15.19) If $\Xi_{\lambda_2} \subset \Xi_{\lambda_1}$, for λ_1, λ_2 normal elements in **D**, then

(*) $$\lambda_2 = m\lambda_1 + \text{integral linear combination of } a_1, \ldots, a_{\ell+1},$$

where m is a strictly positive integer, and one has $m\lambda_1 \in \Xi_{\lambda_2}$.

Proof. — Clearly λ_2 may be expressed as in (*) with m an integer, so we need only check $m>0$ ($m\lambda_1 \in \Xi_{\lambda_2}$ follows from the fact that $\Xi_1 \subset \Xi_{\lambda_2}$, by Lemma (15.2)). Evaluating both sides of (*) on h'_ι, we have:

$$\lambda_2(h'_\iota) = m\lambda_1(h'_\iota),$$

since $a(h'_\iota)=0$, for all $a \in \Delta(\widetilde{A})$. But $h'_\iota = \sum_{i=1}^{\ell+1} q_i h_i$, $q_i > 0$, and since $\lambda_1, \lambda_2 \in \mathbf{D}$ are normal, we must have $m>0$.

16. The Iwasawa decomposition.

With the exception of Lemma (16.3), in this section we take $k=\mathbf{R}$ or $\mathbf{C}$. Now it follows from § 12 in [7] (see the discussion following (12.6) in [7]) that

(16.1) $$\xi_a^* = \xi_{-a}, \quad a \in \Delta_W(\widetilde{A}).$$

As a result, * leaves $\mathfrak{g}_{\mathbf{R}}(\widetilde{A}) \subset \mathfrak{g}_{\mathbf{C}}(\widetilde{A})$ invariant, and hence we may regard * as an $\mathbf{R}$-linear, involutive, antiautomorphism on $\mathfrak{g}_k(\widetilde{A})$, $k=$ either $\mathbf{R}$ or $\mathbf{C}$. Of course, if $k=\mathbf{C}$, then * is conjugate-linear.

As we discussed in § 9 of the present paper, we also obtain from [7], § 12, that there exists a positive-definite, Hermitian inner product $\{\ ,\ \}$ on V_k^λ ($k=\mathbf{R}$ or $\mathbf{C}$) such that we have (by (9.2) and (16.1)):

(16.1′) For all $a \in \Delta_W(\widetilde{A})$, the element ξ_a of $\mathfrak{g}_k(\widetilde{A})$, regarded as an operator on V_k^λ, is the Hermitian conjugate of ξ_{-a}, with respect to $\{\ ,\ \}$.

We thus let * either denote the Hermitian conjugate with respect to $\{\ ,\ \}$, or the involutive, anti-automorphism of $\mathfrak{g}_k(\widetilde{A})$ introduced in § 9 (also see [7], § 12), and defined on $\mathfrak{g}_{\mathbf{R}}(\widetilde{A})$ by restriction from $\mathfrak{g}_{\mathbf{C}}(\widetilde{A})$. From (16.1) we then have

(16.2) $$\chi_a(s)^* = \chi_{-a}(\bar{s}), \quad s \in k,\ a \in \Delta_W(\widetilde{A}).$$

Remark. — In particular, $\chi_a(s)^*$ is defined!

We now prove:

Lemma **(16.3)**. — *Let k be an arbitrary field. For each $a \in \Delta_W(\widetilde{A})$, we can define a homomorphism*

$$\Psi_a : \mathrm{Sl}_2(k) \to \widehat{G}_k^\lambda$$

by the conditions

$$\Psi_a\left(\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\right) = \chi_a(s)$$

$$\Psi_a\left(\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}\right) = \chi_{-a}(s).$$

We also have

(16.4)
$$\Psi_a\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = w_a(1)$$
$$\Psi_a\left(\begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}\right) = h_a(q), \quad q \in k^*.$$

Proof. — Once we prove the existence of Ψ_a, the identities of (16.4) will follow by direct computation. In view of the well known presentation of $\mathrm{Sl}_2(k)$ (see Steinberg [21], Theorem 8, § 6) it suffices to check:

a) $\chi_{\pm a}(s)$ is additive in s $(s \in k)$;
b) $h_a(s)$ is multiplicative in s $(s \in k^*)$;
c) $w_a(s)\chi_a(\tilde{s})w_a(-s) = \chi_{-a}(-s^{-2}\tilde{s})$, $s \in k^*$, $\tilde{s} \in k$.

However, we observed *a)* in (7.15), and *b)* is an immediate consequence of (ii), of Lemma (11.2). As for *c)*, we first note that if $a = a(\alpha) + n\iota$, $\alpha \in \Delta(\mathrm{A})$, $n \in \mathbf{Z}$, then
$$\chi_a(\tilde{s}) = \chi_\alpha(t^n \tilde{s}),$$
$$w_a(s) = w_\alpha(t^n s).$$

Also, *a)* and a direct computation, imply that
$$w_a(-s) = w_a(s)^{-1}.$$
Thus
$$w_a(s)\chi_a(\tilde{s})w_a(-s) = w_\alpha(t^n s)\chi_\alpha(t^n \tilde{s})w_\alpha(t^n s)^{-1}$$
$$= \chi_{-\alpha}(\eta t^{-n} s^{-2}\tilde{s}), \quad \text{by (12.9)},$$
$$= \chi_{-a}(\eta s^{-2}\tilde{s}),$$

where $\eta = \pm 1$. Indeed, by Matsumoto [13], Lemma (5.1), *b)* and *c)*, we have in this case that $\eta = -1$. ■

We again assume $k = \mathbf{R}$ or $\mathbf{C}$, and for
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}_2(k),$$
we let
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$$
denote the conjugate transpose. We set
$$\chi_+(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \qquad \chi_-(s) = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}, \qquad s \in k,$$
and note that, thanks to (16.2),
$$\Psi_a(\chi_\pm(s)^*) = \Psi_a(\chi_\pm(s))^*, \quad s \in k.$$
Since the $\chi_\pm(s)$ generate $\mathrm{Sl}_2(k)$, it follows that

(16.5) for each $a \in \Delta_{\mathrm{W}}(\tilde{\mathrm{A}})$, we have $\Psi_a(g)^* = \Psi_a(g^*)$, $g \in \mathrm{Sl}_2(k)$.

But then, thanks to (16.4), we have from (16.5)

(16.6) $$w_a(1)^* = w_a(1)^{-1}, \quad a \in \Delta_W(\widetilde{A}).$$

Definition **(16.7)**. — *We let* $\hat{K} = \hat{K}^\lambda \subset \hat{G}_k^\lambda$ *denote the subgroup*

$$\hat{K} = \{g \in \hat{G}^\lambda \mid g^* \text{ is defined, and equals } g^{-1}\}.$$

Theorem **(16.8)** *(Iwasawa decomposition)*. — *We have*

$$\hat{G}_k^\lambda = \hat{K}\mathscr{I}.$$

The proof of this theorem rests on the following lemma for Tits systems, from [21]:

Lemma **(16.9)**. — *Let* (G, B, N, S) *be a Tits system, and let* $S = (r_i)_{i \in I}$, *with* I *a suitable index set. For each* $i \in I$, *let* Y_i *be a set of representatives for the family of cosets*

$$Br_iB/B.$$

If $w \in W = N/(N \cap B)$, *and if*

$$w = r_{j_1} \ldots r_{j_m}$$

is an expression of minimal length for w *in terms of the* r_j, *then*

(16.10) $$BwB = Y_{j_1} \ldots Y_{j_m} B.$$

Proof. — We make full use of the standard results on Tits systems (see Bourbaki [3], Chapter IV), and we argue by induction on m. By assumption, there is nothing to prove for $m = 1$. Assume we have proved (16.10) for some $m \geq 1$, and let

$$w = r_{j_1} \ldots r_{j_{m+1}}$$

be an expression of minimal length; then

$$\begin{aligned} BwB &= Br_{j_1} \ldots r_{j_m} Br_{j_{m+1}} B \\ &= Y_{j_1} \ldots Y_{j_m}(Br_{j_{m+1}}B) \\ &= Y_{j_1} \ldots Y_{j_m} Y_{j_{m+1}} B, \end{aligned}$$

where the first equality follows from the minimality of $m+1$ and a standard property of Tits systems, where the second equality follows from our induction hypothesis, and where the last equality follows from our initial assumption concerning the Y_i. ■

We now prove Theorem (16.8). — We let $K \subset Sl_2(k)$ denote the subgroup

$$K = \begin{cases} SU(2), & k = \mathbf{C} \\ SO(2), & k = \mathbf{R}, \end{cases}$$

and we let $B_1 \subset Sl_2(k)$ denote the subgroup of upper triangular matrices. For each $a \in \Delta_W(\widetilde{A}) \cap \Delta_+(A)$, we then have

$$\textbf{(16.11)} \qquad \Psi_a(K) \subset \hat{K}$$
$$\Psi_a(B_1) \subset \mathscr{I},$$

where the first inclusion follows from (16.5), and the second inclusion, from (16.4) and the definition of Ψ_a.

We now apply Lemma (16.9) to the Tits system

$$(\hat{G}_k^\lambda, \mathscr{I}, N, \mathbf{S})$$

of Theorem (14.10), where we take our index set I to be $\{1, \ldots, \ell+1\}$. First, by Proposition (14.8), we have $w_{a_i}(1) \in \widetilde{\mathbf{N}}^\lambda$, $i = 1, \ldots, \ell+1$; therefore each $w_{a_i}(1)$, $i = 1, \ldots, \ell+1$, normalizes $T_\mathscr{O}^\lambda$, since $T_\mathscr{O}^\lambda$ is, by definition, the kernel of the homomorphism

$$p : \widetilde{\mathbf{N}}^\lambda \to \mathbf{W},$$

of §§ 13-14. But then, from this, and from (13.22) and (13.23), we have

$$\textbf{(16.12)} \qquad \mathscr{I} w_{a_i}(1) \mathscr{I} = U_{a_i} w_{a_i}(1) \mathscr{I},$$

where $U_{a_i} \subset \hat{G}_k^\lambda$ is the subgroup

$$U_{a_i} = \{\chi_{a_i}(s)\}_{s \in k^*}, \qquad i = 1, \ldots, \ell+1$$

(we note that (16.12) holds for an arbitrary field k). Then, thanks to the Iwasawa decomposition

$$Sl_2(k) = KB_1,$$

to (16.4), (16.11), and (16.12), we may take the Y_i of Lemma (16.9) to be contained in $\Psi_{a_i}(K)$, $i = 1, \ldots, \ell+1$. Hence we obtain Theorem (16.8) from Lemma (16.9), and the Bruhat decomposition for a Tits system (see [3], Chapter IV, § 2, Theorem 1). ■

Next, we obtain a uniqueness result concerning our Iwasawa decomposition. Thus, we let $H_{k,+}$ denote the subgroup of H_k generated by all $h_{a_i}(s)$, where s is a positive real number, and we let $H_{k,\theta}$ denote the subgroup of H_k generated by all $h_{a_i}(s)$, where s has absolute value one. It follows from Lemma (14.20) that the elements $h_{a_i}(s)$, $s \in k^*$, $i = 1, \ldots, \ell+1$, generate H_k. We then have

$$\textbf{(16.13)} \qquad H_k = H_{k,\theta} H_{k,+},$$

and all the elements of $H_{k,+}$ (resp. of $H_{k,\theta}$) have eigenvalues, when considered as automorphisms of V_k^λ, which are real and positive (resp. of modulus one), thanks to Lemma (11.2). Since the elements of $H_{k,\theta}$ act on each weight space as a scalar operator of modulus one, and since the various weight spaces are orthogonal with respect to $\{\ ,\ \}$, we have that $H_{k,\theta} \subset \hat{K}$, and

$$\hat{G}_k^\lambda = \hat{K} H_{k,+} \mathscr{I}_U.$$

Thus each $g \in \hat{G}_k^\lambda$ has an expression

$$g = g_K g_H g_U, \quad g_K \in \hat{K}, \ g_H \in H_{k,+}, \ g_U \in \mathscr{I}_U.$$

We wish to show that g_K, g_H, g_U are uniquely determined by g. We fix a basis of V_k^λ which is both orthonormal and coherently ordered (we can do this since the weight spaces are mutually orthogonal, with respect to $\{\ ,\ \}$). Then, $\hat{K} \cap H_{k,+}\mathscr{I}_U$ consists of unitary matrices which are upper triangular. It follows that these matrices must be diagonal. By Corollary (14.18), H_k is the set of diagonal elements in $\mathscr{I}$. Hence $H_{k,+}$ is the set of diagonal elements in $H_{k,+}\mathscr{I}_U$ (since $H_k \cap \mathscr{I}_U = \{1\}$, by (14.16)). Thus

$$\hat{K} \cap H_{k,+}\mathscr{I}_U \subset H_{k,+}.$$

But the only unitary element of $H_{k,+}$ is the identity, and hence

$$\hat{K} \cap H_{k,+}\mathscr{I}_U = \{1\}.$$

Since $H_{k,+} \cap \mathscr{I}_U \subset H_k \cap \mathscr{I}_U$ is the identity, as we just observed, we have:

Lemma **(16.14)**. — *The group* $\hat{G}_k^\lambda$ *decomposes as*

$$\hat{G} = \hat{K} H_{k,+} \mathscr{I}_U,$$

with uniqueness of expression.

17. A fundamental estimate.

From now on, with the exception of the appendices, we take our field k to be $\mathbf{R}$ *or* $\mathbf{C}$. Recall from § 6 (following (6.7)) that we introduced the extended k-algebra $\mathfrak{g}_k^e(\widetilde{A})$, and the extended subalgebra $\mathfrak{h}_k^e(\widetilde{A})$. Indeed $\mathfrak{g}_k^e(\widetilde{A}) = k \otimes_{\mathbf{Z}} \mathfrak{g}_{\mathbf{Z}}^e(\widetilde{A})$, $\mathfrak{h}_k^e(\widetilde{A}) = k \otimes h_{\mathbf{Z}}^e(\widetilde{A})$, where $\mathfrak{g}_{\mathbf{Z}}^e(\widetilde{A}) = \mathfrak{g}_{\mathbf{Z}}(\widetilde{A}) \oplus \mathbf{Z}D$, $\mathfrak{h}_{\mathbf{Z}}^e(\widetilde{A}) = \mathfrak{h}_{\mathbf{Z}}(\widetilde{A}) \oplus \mathbf{Z}D$, are semi-direct products ($\mathfrak{h}_{\mathbf{Z}}^e(A)$ is a direct product). We should remark that the degree derivation $D = D_{\ell+1}$ leaves $\mathfrak{g}_{\mathbf{Z}}(\widetilde{A})$ invariant, so that $\mathfrak{g}_{\mathbf{Z}}^e(\widetilde{A})$ is well defined. Also we let D denote the induced derivation on $\mathfrak{g}_{\mathbf{Z}}(\widetilde{A})$, and on $\mathfrak{g}_k(A)$. We fix a normal element $\lambda \in \mathbf{D}$ (recall from § 15, that λ being normal means that $\lambda(h_i) \neq 0$ for some $i = 1, \ldots, \ell+1$), we consider the corresponding highest weight module V_k^λ, and the Chevalley group $\hat{G}_k^\lambda \subset \operatorname{Aut} V_k^\lambda$. Recall from Theorem (14.10), that we have the Tits system $(\hat{G}_k^\lambda, \mathscr{I}, N, \mathbf{S})$ with Weyl group $\mathbf{W}$ (more precisely, we saw that $N/(\mathscr{I} \cap N)$ was isomorphic to $\mathbf{W}$, which was defined in § 13, after (13.19), and we identified $N/(\mathscr{I} \cap N)$ with $\mathbf{W}$). Also recall from Proposition (14.8) that the set $\{w_{\alpha_i}(1)\}_{i=1,\ldots,\ell+1} \subset N$ is a set of coset representatives for the set of reflections $\mathbf{S} \subset \mathbf{W}$, where $\mathbf{S}$ is in fact the set of reflections

$$\{r_{\alpha_1,0}, \ldots, r_{\alpha_\ell,0}, r_{-\alpha_0,1}\}$$

(see § 13, preceeding Theorem (13.20)).

Since $k = \mathbf{R}$ or $\mathbf{C}$, we have from Proposition (6.11), that the representation π_k^λ is a faithful representation of $\mathfrak{g}_k(\widetilde{A})$ into $\operatorname{End} V_k^\lambda$. We let $\mathfrak{h}_k^e(\widetilde{A})^*$ denote the k-dual of $\mathfrak{h}_k^e(\widetilde{A})$. In (3.3) we defined the Weyl group $W \subset \mathrm{Aut}\, \mathfrak{h}_k^e(\widetilde{A})^*$, generated by reflections $r_1, \ldots, r_{\ell+1}$, and in § 6, (6.14), we introduced the contragredient action on $\mathfrak{h}_k^e(\widetilde{A})$. Identifying $\mathfrak{g}_k(\widetilde{A})$ with its image under π_k^λ, we have from Lemma (A.1) in Appendix I of this paper, and from (8.1) that

$$\text{(17.1)} \qquad \mathrm{Ad}\, w_{a_i}(1)(h) = r_i(h) = h - a_i(h) h_i, \qquad h \in \mathfrak{h}_k^e(\widetilde{A}),\ i = 1, \ldots, \ell+1.$$

Moreover, we have from Lemma (11.2) that for $a \in \Delta_W(\widetilde{A})$ and $s \in k^*$, the operator $h_a(s) \in \mathrm{Aut}\, V_k^\lambda$ is represented by a diagonal matrix with respect to a coherently ordered basis. The same is true for $h \in \mathfrak{h}_k^e(\widetilde{A})$. Thus H_k centralizes $\mathfrak{h}_k^e(\widetilde{A})$ (see Lemma (14.20)). But $H_k = N \cap \mathscr{I}$ (see Proposition (14.19)). Thus (17.1) allows us to define a surjection

$$\Phi_0 : \mathbf{W} \to W,$$

where

$$\Phi_0(w_{a_i}(1)(\mathscr{I} \cap N)) = r_i, \qquad i = 1, \ldots, \ell+1.$$

We examine the action of W on $\mathfrak{h}_k^e(\widetilde{A})$ further. We note that W leaves $\mathfrak{h}_k(\widetilde{A})$ invariant, and that

$$w.D = D \bmod \mathfrak{h}_k(\widetilde{A}), \qquad w \in W.$$

Thus for $w \in W$, $h \in \mathfrak{h}_k(\widetilde{A})$, we may define $w * h \in \mathfrak{h}_k(\widetilde{A})$ by

$$w(D+h) = D + w * h.$$

We also have the surjective Lie algebra homomorphism (see Notational Remark (4.7)) $\widetilde{\omega} : \mathfrak{g}(\widetilde{A}) \to \widetilde{\mathfrak{g}}$, given in Theorem (3.7). This homomorphism induces a surjective homomorphism (which we still denote by $\widetilde{\omega}$)

$$\widetilde{\omega} : \mathfrak{h}_k(\widetilde{A}) \to \mathfrak{h}_k(\widetilde{A}),$$

where

$$\widetilde{\omega}(h_i) = H_i, \qquad i = 1, \ldots, \ell$$

$$\widetilde{\omega}(h_{\ell+1}) = -H_{\alpha_0}$$

(see Theorem (3.7)).

We now compute $\widetilde{\omega}(w * h)$ for $w \in W$, $h \in \mathfrak{h}_k(\widetilde{A})$. It suffices to compute $\widetilde{\omega}(r_i * h)$ for $i = 1, \ldots, \ell+1$, and in this case we have from a direct computation

$$\text{(17.2)} \qquad \begin{aligned} \widetilde{\omega}(r_i * h) &= \widetilde{\omega}(h) - \alpha_i(\widetilde{\omega}(h)) H_i, \qquad i = 1, \ldots, \ell, \\ \widetilde{\omega}(r_{\ell+1} * h) &= w_{\alpha_0}(\widetilde{\omega}(h)) + H_{\alpha_0}, \end{aligned}$$

where for $\alpha \in \Delta(\widetilde{A})$, we let w_α denote the orthogonal reflection with respect to the hyperplane $\{H \in \mathfrak{h}_k(A) \mid \alpha(H) = 0\}$.

Now we have identified $\mathbf{A}$ with $\mathfrak{h}_{\mathbf{R}}(A)$ in § 13. Also, each of the reflections $r_{\alpha,k}$ has a natural extension to $\mathfrak{h}_{\mathbf{C}}(A)$ (still denoted $r_{\alpha,k}$). We may then reformulate (17.2) as

$$\text{(17.2')} \qquad \begin{aligned} \widetilde{\omega}(r_i * h) &= r_{\alpha_i,0}(\widetilde{\omega}(h)), \qquad i = 1, \ldots, \ell \\ \widetilde{\omega}(r_{\ell+1} * h) &= r_{-\alpha_0,1}(\widetilde{\omega}(h)). \end{aligned}$$

Thus, by means of $*$ and the projection $\widetilde{\omega}:\mathfrak{h}_k(\widetilde{A})\to\mathfrak{h}_k(A)$, we may define a homomorphism

$$\widetilde{\Phi}: W\to \mathbf{W}$$

such that
$$\widetilde{\Phi}(r_i)=r_{\alpha_i,0}, \quad i=1,\ldots,\ell,$$
$$\widetilde{\Phi}(r_{\ell+1})=r_{-\alpha_0,1}.$$

But then, thanks to (14.7), we see that $w_{a_i}(1)(\mathscr{I}\cap N)$, regarded as an element of $\mathbf{W}$, is $r_{\alpha_i,0}$ for $i=1,\ldots,\ell$, and $r_{-\alpha_0,1}$ for $i=\ell+1$, and hence Φ_0 and $\widetilde{\Phi}$ are inverses of each other. Hence, in particular we have

(17.3) The homomorphism $\Phi_0:\mathbf{W}\to W$ is an isomorphism of $\mathbf{W}$ onto W.

From the theory of affine Weyl groups (see for example [9], Proposition (1.2)), it follows that for all $h\in\mathfrak{h}_{\mathbf{R}}(A)$ we can find $w\in W$ so that

$$\alpha_i(\widetilde{\omega}(w*h))\geq 0, \quad i=1,\ldots,\ell,$$
$$1\geq\alpha_0(\widetilde{\omega}(w*h))\geq 0,$$

or equivalently

(17.4)
$$a_i(w(D+h))\geq 0, \quad i=1,\ldots,\ell,$$
$$1\geq a_{\ell+1}(w(D+h))\geq 0.$$

We now prove some technical lemmas concerning the structure of the $\mathfrak{g}_k(\widetilde{A})$-module V_k^λ.

Lemma **(17.5)**. — *Let* $\mu\in\mathfrak{h}_{\mathbf{R}}(A)^*$ *be dominant integral; i.e., assume* $\mu(H_i)\geq 0$ *for* $i=1,\ldots,\ell$. *Then the set of all* $\nu\in\mathfrak{h}_{\mathbf{R}}(A)^*$ *of the form*

(♮)
$$\nu=\mu-\sum_{i=1}^{\ell} n_i\alpha_i, \quad n_i\in\mathbf{Z},\ n_i\geq 0,$$

and such that ν *is dominant integral, is finite.*

Proof. — For ν of the form (♮) we have:

$$(\nu,\nu)=(\nu,-\sum_{i=1}^{\ell} n_i\alpha_i)+(\mu,\mu)-(\mu,\sum_{i=1}^{\ell} n_i\alpha_i)\leq(\mu,\mu),$$

since μ and ν are both dominant. But ν must vary over the dual lattice to the lattice generated (over $\mathbf{Z}$) by the coroots. Since the above inequality implies that ν must be of bounded norm, and since (,) is positive-definite on $\mathfrak{h}_{\mathbf{R}}(A)^*$, we obtain the desired finiteness assertion. ■

Lemma **(17.6)**. — *For each* $i=1, \ldots, \ell+1$, *and each integer* $n>0$, *the set of weights of* V^λ *of the form*

$$\lambda - \sum_{j=1}^{\ell+1} n_j a_j,$$

with $n_i \leq n$, *is finite* (1).

Proof. — Let $\theta_i = \{a_1, \ldots, \hat{a}_i, \ldots, a_{\ell+1}\}$; i.e., θ_i is the set of simple roots of $\Delta(\widetilde{A})$ with a_i omitted. We let $\mathfrak{m}_i \subset \mathfrak{g}(A)$ denote the subalgebra generated by the e_j, h_j, f_j, with $j \neq i$, and let $\mathfrak{h}_i$ denote the linear span of the h_j, $j \neq i$. We consider the subspace of V^λ

$$\coprod_{n_j \geq 0} V^\lambda_{\lambda' - \sum_{j \neq i} n_j a_j} =_{\mathrm{df}} \widetilde{V},$$

where $\lambda' = \lambda - \sigma a_i$, with σ some integer such that $0 \leq \sigma \leq n$. Then $\widetilde{V}$ is an $\mathfrak{m}_i$-submodule of V^λ. As we shall show in Lemma (17.7), below, V^λ is a direct sum of finite-dimensional, $\mathfrak{m}_i$-submodules, and hence is completely reducible. It follows that $\widetilde{V}$ is also a direct sum of finite-dimensional $\mathfrak{m}_i$-submodules. But thanks to Lemma (17.5), $\widetilde{V}$ must in fact be a direct sum of finitely many, finite-dimensional $\mathfrak{m}_i$-submodules, and hence $\widetilde{V}$ must be finite-dimensional. Lemma (17.6) now follows (letting σ vary between 0 and n). ∎

We now prove the following assertion, which was used in the proof of Lemma (17.6):

Lemma **(17.7)**. — V^λ *is a direct sum of finite-dimensional, irreducible* $\mathfrak{m}_i$*-submodules.*

Proof. — The highest weight vector $v_0 \in V^\lambda$ is a weight vector for $\mathfrak{m}_i$ (where we take the Cartan subalgebra spanned by the h_j, $j \neq i$), and clearly corresponds to a dominant, integral weight of $\mathfrak{m}_i$. Thanks to Lemma (7.12), v_0 must generate a finite-dimensional $\mathfrak{m}_i$-submodule W_0 of V^λ (see in [8] the remark following Proposition (6.2), p. 61), and W_0 is homogeneous with respect to the weight space decomposition of V^λ. Let $k \geq 0$ be an integer. Assume inductively there exists an $\mathfrak{m}_i$-submodule $W_q \subset V^\lambda$, which is a direct sum of finite-dimensional $\mathfrak{m}_i$-submodules, is homogeneous with respect to the weight space decomposition of V^λ, and contains V^λ_μ for every weight μ of V^λ, with $\mathrm{dp}(\mu) \leq q$. Since W_q is homogeneous with respect to the weight space decomposition of V^λ, the subspace W_q has a well defined orthocomplement $W_q^\perp$ with respect to the positive-definite, Hermitian form $\{\ ,\ \}$ (see § 9, and [7], § 12), and $W_q^\perp$ is homogeneous with respect to the weight space decomposition of V^λ. Thanks to (16.1), we have $\mathfrak{m}_i = \mathfrak{m}_i^*$, so $W_q^\perp$ is $\mathfrak{m}_i$-invariant. Let μ_0 be a weight of V^λ such that $\mathrm{dp}(\mu_0) = q+1$, and $V^\lambda_{\mu_0}$ intersects $W_q^\perp$ nontrivially. Fix a non-zero element $v' \in V^\lambda_{\mu_0} \cap W_q^\perp$. Then v' generates a finite-dimensional $\mathfrak{m}_i$-submodule W' of $W_q^\perp$.

---

(1) See [8], Lemma (5.3).

We then repeat this process with $W_q \oplus W'$ in place of W_q, and so on. We eventually construct in this way, an $\mathfrak{m}_i$-submodule $W_{q+1} \subset V^\lambda$ which is a direct sum of finite-dimensional irreducible submodules, and such that $W_{q+1} \supset V^\lambda_\mu$, for every weight μ of V^λ, with $\mathrm{dp}(\mu) \leq q+1$. This completes the induction, and hence the proof of Lemma (17.7). ■

We now consider $H_{k,+} \subset \hat{G}^\lambda_k$, the subgroup (defined in § 16) generated by all $h_{a_i}(s)$, with $s>0$ and $i=1, \ldots, \ell+1$. If $\varkappa \in H_{k,+}$, then $\varkappa$ has an expression

$$\varkappa = \prod_{i=1}^{\ell+1} h_{a_i}(s_i)^{n_i}, \qquad n_i>0,\ s_i>0,\ n_i \in \mathbf{Z}.$$

We define $\ln \varkappa \in \mathrm{End}\, V^\lambda_k$ by:

$$\ln \varkappa = \sum_{i=1}^{\ell+1} n_i \ln(s_i) h_i.$$

By (17.4), we have that for all $r>0$, there exists $w \in W$ such that

(17.8)
$$a_i(w(-r\mathrm{D} + \ln \varkappa)) \leq 0, \qquad i=1, \ldots, \ell,$$
$$-r \leq a_{\ell+1}(w(-r\mathrm{D} + \ln \varkappa)) \leq 0.$$

For each $w \in W$, we choose $w' \in N$ so that

(17.9) (i) w' is a product of elements $w_{a_i}(1)$, $i=1, \ldots, \ell+1$.
(ii) $\Phi_0(w'(\mathscr{I} \cap N)) = w$.

Since V^λ is a module for the extended Lie algebra $\mathfrak{g}^e(\widetilde{A}) = \mathfrak{g}(\widetilde{A}) \oplus \mathbf{C}\mathrm{D}$, and since D preserves $V^\lambda_{\mathbf{Z}}$ (we assumed in § 6 that $\lambda(\mathrm{D}) \in \mathbf{Z}$), we have that V^λ_k is a $\mathfrak{g}^e_k(\widetilde{A})$-module. For any $h \in \mathfrak{h}^e_k(A)$, we define $e^h \in \mathrm{Aut}\, V^\lambda_k$, by stipulating that e^h maps each weight space into itself and that on $V^\lambda_{\mu,k}$, e^h is the scalar operator $e^{\mu(h)}$. We note that for $\varkappa \in H_{k,+}$, we have

(17.10)
$$e^{\ln \varkappa} = \varkappa.$$

Since we have defined e^h for each $h \in \mathfrak{h}^e_k(\widetilde{A})$, we have, in particular, defined $e^{r\mathrm{D}}$ for each $r \in k$.

For each $w \in W$, we fix $w' \in N$ as in (17.9). Let $\varkappa \in H_{k,+}$, $r \in k$. We then have, by Lemma (11.2),

(17.11)
$$w' \varkappa (w')^{-1} = e^{w(\ln \varkappa)},$$
$$w'(e^{r\mathrm{D}})(w')^{-1} = e^{rw(\mathrm{D})}, \qquad r \in k.$$

Now H_k is normalized by N (recall from Proposition (14.19), that $H_k = N \cap \mathscr{I}$, so H_k is normal in N by the axioms for a Tits system, (13.21), and by Theorem (14.10), which asserts that $(\hat{G}^\lambda_k, \mathscr{I}, N, \mathbf{S})$ is a Tits system). Also, $H_{k,+}$ consists of those elements in H_k with positive eigenvalues (as follows from (16.13)). Hence the first equality of (17.11) implies that

$$w'(H_{k,+})(w')^{-1} = H_{k,+}.$$

For $\varkappa \in H_k$, $r \in k$, $\mu \in \mathfrak{h}^e(\widetilde{A})^*$, with $\mu \in \Xi_\lambda$, we set

$$(\varkappa e^{rD})^{\mu} = \varkappa^{\mu} e^{-r\mu(D)},$$

where $\varkappa^{\mu} \in k^*$ is defined by

$$\varkappa . v = \varkappa^{\mu} v, \qquad v \in V^{\lambda}_{\mu},$$

if μ is a weight, and $\varkappa^{\mu}$ is then defined for all $\mu \in \Xi_\lambda$. We note that if $\varkappa \in H_{k,+}$, then, thanks to (17.10), we have

$$\varkappa^{\mu} = e^{\mu(\ln \varkappa)}, \qquad \mu \in \Xi_\lambda.$$

Now given $\varkappa \in H_{k,+}$, $r \in \mathbf{R}$ with $r > 0$, we choose $w \in W$ as in (17.8), and then $w' \in N$ as in (17.9). From (17.8) and (17.11) (the second equality being applied to e^{-rD}), we have

(17.12)
$$(w'\varkappa e^{-rD}(w')^{-1})^{a_i} \leq 1, \qquad i = 1, \ldots \ell,$$
$$e^{-r} \leq (w'\varkappa e^{-rD}(w')^{-1})^{a_{\ell+1}} \leq 1.$$

Lemma **(17.13)**. — *Let* $\varkappa \in H_{k,+}$ *and* $r \in \mathbf{R}$ *with* $r > 0$. *Assume* $(\varkappa e^{-rD})^{a_i} \leq 1$, $i = 1, \ldots, \ell+1$. *Then for some* $j = 1, \ldots, \ell+1$, *we have*

$$(\varkappa e^{-rD})^{a_j} < 1.$$

Proof. — If $(\varkappa e^{-rD})^{a_i} < 1$ for some $i = 1, \ldots, \ell$, there is nothing to prove. Thus, assume $(\varkappa e^{-rD})^{a_i} = 1$ for each $i = 1, \ldots, \ell$. Then, since $a_i(D) = 0$ for $i = 1, \ldots, \ell$, we have

$$(*) \qquad \varkappa^{a_i} = 1, \qquad i = 1, \ldots, \ell.$$

Now $\varkappa$ has an expression as a product

$$\varkappa = \prod_{i=1}^{\ell+1} h_{a_i}(s_i)^{n_i}, \qquad s_i > 0, \ n_i \geq 0 \text{ in } \mathbf{Z}.$$

We have
$$\varkappa^{a_{\ell+1}} = \prod_{i=1}^{\ell+1} s_i^{n_i a_{\ell+1}(h_i)},$$

and thanks to $(*)$, this equals one, since when acting on $\mathfrak{h}(\widetilde{A})$, $a_{\ell+1}$ is equal to an integral linear combination of $a_1, \ldots, a_\ell$. But then

$$(\varkappa e^{-rD})^{a_{\ell+1}} = e^{-r} < 1,$$

(since $r > 0$) and Lemma (17.13) now follows. ∎

From Lemmas (17.6) and (17.13), we immediately have

Lemma **(17.14)**. — *Let* $\varkappa \in H_k$ *and let* $c \in k$ *with* $r = \mathcal{R}e(c) > 0$. *Assume* $\varkappa = \varkappa_1 \varkappa_2$ *with* $\varkappa_1 \in H_{k,\theta}$, $\varkappa_2 \in H_{k,+}$ *and*

$$(\varkappa_2 e^{-rD})^{a_i} \leq 1, \qquad i = 1, \ldots, \ell+1.$$

Then for all $A > 0$, *there is a finite subset of weights of* V^{λ}, *such that if* μ *is a weight of* V^{λ} *which is not in this finite subset, then*

$$|(\varkappa e^{-cD})^{\mu}| > A.$$

We define the ring $J \subset k$ as follows:

$$J = \begin{cases} \mathbf{Z}, & \text{if } k = \mathbf{R} \\ \text{ring of Gaussian} \\ \text{integers,} & \text{if } k = \mathbf{C}. \end{cases}$$

We let $\mathscr{L}_J \subset \mathscr{L}_{\mathbf{C}}$ denote the subring of all formal Laurent series in t, with coefficients in J. We fix a Chevalley lattice $V^\lambda_{\mathbf{Z}}$ in V^λ, and a coherent Hermitian structure $\{\ ,\ \}$ on V^λ, as in the proof of Theorem (12.1) of [7] (also, see §§ 6, 9 of the present paper). We may thus assume that we have fixed a highest weight vector $v_0 \neq 0$ in V^λ_J so that $\{v_0, v_0\} = 1$, and so that $\{\ ,\ \}$ also satisfies

$$\{m_1, m_1\} \in \mathbf{Z}$$
$$\{m_1, m_2\} \in J, \qquad \text{for} \qquad m_1, m_2 \in V^\lambda_J.$$

We let $\hat{\Gamma} = \hat{\Gamma}^\lambda_k \subset \hat{G} = \hat{G}^\lambda_k$ denote the subgroup

$$\hat{\Gamma} = \{\gamma \in \hat{G} \mid \gamma(V^\lambda_J) = V^\lambda_J\}.$$

The following is the central result of this section:

Lemma **(17.15)**. — *For* $g \in \hat{G}$, $c \in k$, *with* $r = \mathscr{R}e(c) > 0$, *we can find* $\gamma_0 \in \hat{\Gamma}$ *such that*

(17.16) $$\{ge^{-cD}\gamma_0 v_0, ge^{-cD}\gamma_0 v_0\} \leq \{ge^{-cD}\gamma v_0, ge^{-cD}\gamma v_0\},$$

for all $\gamma \in \hat{\Gamma}$.

Remark. — Lemma (17.15) remains valid if we replace $\hat{\Gamma}$ by the subgroup $\hat{\Gamma}_0$ generated by $\chi_\alpha(\sigma(t))$, $\alpha \in \Delta(A)$, $\sigma(t) \in \mathscr{L}_J$. The same proof (given below) applies.

Proof. — The argument is based on the Iwasawa decomposition

$$\hat{G} = \hat{K} H_{k,+} \mathscr{I}_U$$

of Lemma (16.14). Thus $g \in \hat{G}$ has a unique expression $g = g_K g_H g_U$, with $g_K \in \hat{K}$, $g_H \in H_{k,+}$, $g_U \in \mathscr{I}_U$.

We first note that for any positive number A, we may choose a finite set of weights $\Lambda = \Lambda_A$ of V^λ_k, such that for $\mu \notin \Lambda$, a weight of V^λ_k, we have

$$|(g_H e^{-cD})^\mu| > A.$$

To see this, we choose $w \in W$, and corresponding $w' \in N$ (as in (17.9)) so that $\Phi_0(w'(\mathscr{I} \cap N)) = w$ and so that

$$(w'(g_H e^{-rD})(w')^{-1})^{a_i} \leq 1, \qquad i = 1, \ldots, \ell,$$
$$e^{-r} \leq (w'(g_H e^{-rD})(w')^{-1})^{a_{\ell+1}} \leq 1$$

(we have applied (17.12)). Now

$$w'(g_H e^{-cD})(w')^{-1} = \varkappa e^{-cD},$$

for some $\varkappa \in H_k$, and thus it follows from Lemma (17.14) that for all weights μ of V^λ, but those in a certain finite subset, we have

$$|(w'(g_H e^{-cD})(w')^{-1})^\mu| > A.$$

The left side of this inequality is just $(g_H e^{-cD})^{w^{-1}(\mu)}$, so

$$|(g_H e^{-cD})^\mu| > A,$$

for all but finitely many weights of V^λ, since W permutes the weights of V^λ. We have thus found our set Λ_A. Enlarging Λ_A if necessary, we may assume that if $\mu \in \Lambda_A$, then all weights of V^λ, of depth $< \mathrm{dp}(\mu)$, are also in Λ_A.

For $v \in V_k^\lambda$, we set $\|v\|^2 =_{\mathrm{df}} \{v, v\}$, and for $\gamma \in \hat{\Gamma}$ we set $m_\gamma = \gamma . v_+ \in V_J^\lambda$. Fix $\gamma_1 \in \hat{\Gamma}$ arbitrarily, and let

(17.17) $$A = \|g e^{-rD} m_{\gamma_1}\| > 0.$$

We set $V_{\mu,J}^\lambda = J \otimes_{\mathbf{Z}} V_{\mu,\mathbf{Z}}^\lambda$, where $V_{\mu,\mathbf{Z}}^\lambda = V_\mu^\lambda \cap V_{\mathbf{Z}}^\lambda$, for every weight μ of V^λ. For $\gamma \in \hat{\Gamma}$, the vector m_γ may be written as a sum, with respect to the weight space decomposition

$$V_J^\lambda = \coprod V_{\mu,J}^\lambda,$$

where μ runs through the set of weights of V^λ.

If m_γ has a non-zero component $m_\gamma(\mu_0)$ in $V_{\mu_0,J}^\lambda$, with $\mu_0 \notin \Lambda_A$, and if we choose μ_0 of maximal depth (see § 12) among all μ such that m_γ has a non-zero $V_{\mu,J}^\lambda$ component, then

$$\begin{aligned}\|g e^{-cD} m_\gamma\| = \|g_H g_U e^{-cD} m_\gamma\| &\geq \|g_H e^{-cD} m_\gamma(\mu_0)\| \\ &= |(g_H e^{-cD})^{\mu_0}|\ \|m_\gamma(\mu_0)\| \geq |(g_H e^{-cD})^{\mu_0}|,\end{aligned}$$

since $\|m_\gamma(\mu_0)\|$ is the (positive) square root of a non-zero, positive integer, and is hence at least one. Summarizing:

$$\|g e^{-cD} m_\gamma\| \geq |(g_H e^{-cD})^{\mu_0}|.$$

But since $\mu_0 \notin \Lambda_A$ we have

(17.18) $$\|g e^{-cD} m_\gamma\| > A.$$

Recall that if $\mu \in \Lambda_A$, then all weights of V^λ, of depth $< \mathrm{dp}(\mu)$ are also in Λ_A. Now consider the set Ξ_Γ of those $\gamma \in \hat{\Gamma}$ such that $m_\gamma = \gamma . v_+$ satisfies

$$m_\gamma \in \coprod_{\mu \in \Lambda_A} V_\mu^\lambda =_{\mathrm{df}} V_A.$$

Then, thanks to (17.17) and (17.18), we have $\gamma_1 \in \Xi_\Gamma$. Also, (17.18) shows that if $\gamma \notin \Xi_\Gamma$, then $\|g e^{-cD} m_\gamma\| > A$. If we set $\varphi(\gamma) = \|g e^{-cD} m_\gamma\|$, $\gamma \in \hat{\Gamma}$, then for $\gamma \in \Xi_\Gamma$, we have

$$\varphi(\gamma) = \|g_H g_U e^{-cD} m_\gamma\|,$$

where $g_H g_U e^{-cD} m_\gamma \in V_A$. Since V_A is finite-dimensional, $\varphi(\gamma)$ achieves a minimum as γ varies over Ξ_Γ, say for $\gamma=\gamma_0$. Then, since $\gamma_1\in\Xi_\Gamma$, we have

$$\varphi(\gamma_0)\leq A,$$

and thus for $\gamma\notin\Xi_\Gamma$

$$\varphi(\gamma_0)\leq A<||ge^{-cD}m_\gamma||,$$

by (17.18). Thus, for all $\gamma\in\hat{\Gamma}$, we have

$$||ge^{-cD}m_{\gamma_0}||\leq||ge^{-cD}m_\gamma||.$$

This is obviously equivalent to (17.16), and so we have proved Lemma (17.15). ■

18. A fundamental domain for $\hat{\Gamma}\cap\mathscr{I}_U$ in $\mathscr{I}_U$.

We recall that beginning in the last section we have adopted the notational convention that in the remainder of this paper, with the exception of the appendices, we take our field k to be **R** or **C**.

In § 8 we defined the group homomorphism

$$\Phi' : G_{ad}(\widetilde{A})\to G_{ad,\mathscr{L}}(A).$$

Thanks to Lemma (8.14), and the fact that $\mathfrak{g}_k(\widetilde{A})$ is perfect since $\operatorname{char} k=0$, the homomorphism Φ' is injective. We now note that:

Lemma **(18.1)**. — *The homomorphism*

$$\Phi'\circ \mathrm{Ad} : \hat{G}^\lambda_k\to G_{ad,\mathscr{L}}(A),$$

is injective, when restricted to the subgroup $\mathscr{I}_U\subset\hat{G}^\lambda_k$.

Proof. — If $g\in\mathscr{I}_U$, and if $\Phi'\circ\mathrm{Ad}(g)=e$, the identity of $G_{ad,\mathscr{L}}(A)$, then $g\in\mathrm{kernel}(\mathrm{Ad})$, since Φ' is injective. But then $g\in\mathrm{Aut}(V^\lambda_k)$ is a scalar multiple of the identity, by Schur's lemma (Lemma (9.1)). However, relative to a coherently ordered basis, the element g is represented by an upper triangular matrix with ones on the diagonal (by (14.15)). Hence g must be the identity. ■

We fix a Chevalley basis of $\mathfrak{g}(A)$, and order this basis so that positive root vectors are represented in the adjoint representation, by upper triangular matrices. For any commutative ring R with unit, we may also regard this Chevalley basis as a basis of $\mathfrak{g}_R(A)$, and of course it is then still true that positive root vectors (in $\mathfrak{g}_R(A)$) are represented in the adjoint representation, by upper triangular matrices.

We then have:

Lemma **(18.2)** — *The subgroup* $\Phi'\circ\mathrm{Ad}(\mathscr{I}_U)$ *of* $G_{ad,\mathscr{L}}(A)$ *is the subgroup of all* $g\in G_{ad,\mathscr{L}}(A)$ *such that relative to the Chevalley basis of* $\mathfrak{g}(A)$, *ordered as above*, g *is represented by a matrix with coefficients in* $\mathcal{O}$, *and such that the reduction of* $g \bmod t$ *is an upper triangular matrix with ones on the diagonal.*

Proof. — From the definition of $\mathscr{I}_U$, at the beginning of § 12, we see that $\Phi' \circ \mathrm{Ad}(\mathscr{I}_U)$ is contained in the subgroup $\mathscr{I}'_U$ of all $g \in G_{ad,\mathscr{L}}(A)$, such that g is represented, relative to our fixed Chevalley basis of $\mathfrak{g}(A)$, by a matrix with coefficients in $\mathcal{O}$, and with a reduction mod t which is upper triangular with ones on the diagonal. To prove that $\mathscr{I}_U \subset \Phi' \circ \mathrm{Ad}(\mathscr{I}_U)$, we first note that, thanks to Theorems (2.5) and (2.24) in Iwahori-Matsumoto [9] (where one interchanges the roles of $\Delta_+(A)$ and $\Delta_-(A)$), it suffices to show $\mathscr{I}'_U \cap (\Phi' \circ \mathrm{Ad}(T^\lambda_{\mathcal{O}})) \subset \Phi' \circ \mathrm{Ad}(\mathscr{I}_U)$. This is in fact proved in the process of proving (18.14), below. ■

From now on we identify $\mathscr{I}_U$ with its image $\mathscr{I}'_U \subset G_{ad,\mathscr{L}}(A)$, where this identification is made possible by Lemma (18.1). For each $j \geq 0$ in $\mathbf{Z}$, we define the subgroup $\mathscr{I}_U^{(j)} \subset \mathscr{I}_U$ by

$$\mathscr{I}_U^{(j)} = \{g \in \mathscr{I}_U \mid g \equiv 1 \bmod t^j\}.$$

In particular, $\mathscr{I}_U^{(0)} = \mathscr{I}_U$. We let

$$\Gamma_U^{(j)} = \hat{\Gamma} \cap \mathscr{I}_U^{(j)}, \quad j \geq 1,\ j \in \mathbf{Z}.$$

We also set $\Gamma_U = \hat{\Gamma} \cap \mathscr{I}_U$. We shall show that, for each $j \geq 1$ in $\mathbf{Z}$, there is an isomorphism of abelian groups

$$\textbf{(18.3)} \qquad \Psi^{(j)} : \mathscr{I}_U^{(j)} / \mathscr{I}_U^{(j+1)} \xrightarrow{\sim} \mathfrak{g}_k(A), \quad (j \geq 1),$$

where $\mathfrak{g}_k(A)$ is a group with respect to the vector space addition, such that if we let

$$\pi^{(j)} : \mathscr{I}_U^{(j)} \to \mathscr{I}_U^{(j)} / \mathscr{I}_U^{(j+1)}, \quad j \geq 0,$$

denote the natural projection, then, for $r \in k$, $\alpha \in \Delta(A)$, $j \geq 1$, we have

$$\textbf{(18.4)} \qquad \begin{aligned} \Psi^{(j)} \circ \pi^{(j)}(\chi_\alpha(rt^j)) &= r E_\alpha, \\ \Psi^{(j)} \circ \pi^{(j)}(h_\alpha(1 + rt^j)) &= r H_\alpha, \end{aligned}$$

and hence

$$\textbf{(18.5)} \qquad \Psi^{(j)} \circ \pi^{(j)}(\Gamma_U^{(j)}) \supset \mathfrak{g}_J(A), \quad j \geq 1.$$

To prove the existence of the isomorphism $\Psi^{(j)}$ satisfying (18.4), we consider the ring $\mathcal{O}_j = \mathcal{O}/t^{j+1}\mathcal{O}$, which, as a vector space over k, has a direct sum decomposition

$$\mathcal{O}_j = \coprod_{0 \leq i \leq j} t^i k.$$

Then the algebra $\mathfrak{g}_{\mathcal{O}_j}(A)$ has a corresponding direct sum decomposition (as a vector space over k):

$$\textbf{(18.6)} \qquad \mathfrak{g}_{\mathcal{O}_j}(A) = \coprod_{0 \leq i \leq j} t^i \mathfrak{g}_k(A).$$

We fix an ordered basis of $\mathfrak{g}_{\mathcal{O}_j}(A)$ relative to this decomposition, so that the basis vectors of $\mathfrak{g}_{\mathcal{O}_j}(A)$ consist of a union of basis vectors of the subspaces $t^i \mathfrak{g}_k(A)$, so that the basis vectors in each $t^i \mathfrak{g}_k(A)$ appear consecutively, and so that for $i \leq i'$, the basis vectors in $t^{i'}\mathfrak{g}_k(A)$ appear after those in $t^i\mathfrak{g}_k(A)$. Then for $g\in\mathscr{I}_U^{(j)}$, $\mathrm{Ad}\,g$ determines a k-linear transformation of $\mathfrak{g}_{\mathcal{O}_j}(A)$, and relative to our basis, and the direct sum decomposition (18.6), the matrix of this linear transformation has a block decomposition

$$\begin{pmatrix} 1 & & & & \bigcirc \\ & \ddots & & & \\ 0 & & \ddots & & \\ & \ddots & & \ddots & \\ \Lambda_j(g) & & 0 & & 1 \end{pmatrix}$$

where $\Lambda_j(g)\in\mathrm{Hom}_k(\mathfrak{g}_k(A),\ t^j\mathfrak{g}_k(A))\cong\mathrm{End}_k\mathfrak{g}_k(A)$. One sees directly that

(18.7) $$\Lambda_j(\chi_\alpha(rt^j))=r\,\mathrm{ad}\,E_\alpha,$$
$$\Lambda_j(h_\alpha(1+rt^j))=r\,\mathrm{ad}\,H_\alpha,$$

where $r\in k$, $\alpha\in\Delta(A)$, $j\geq 1$, and thus, identifying $\mathfrak{g}_k(A)$ with its image under the adjoint representation, we see that we may define $\Psi^{(j)}$ by

$$\Psi^{(j)}\circ\pi^j=\Lambda_j,$$

and then note that $\Psi^{(j)}$ satisfies (18.4), thanks to (18.7).

We also wish to consider the projection $\pi^0:\mathscr{I}_U\to\mathscr{I}_U/\mathscr{I}_U^{(1)}$. With a slight abuse of notation, we let $\chi_\alpha(r)$, $\alpha\in\Delta(A)$, $r\in k$, denote the automorphism of $\mathfrak{g}_k(A)$ defined by

$$\chi_\alpha(r)=\sum_{j\geq 0}\frac{(\mathrm{ad}\,X_\alpha)^j}{j!}r^j.$$

We let $G_{\mathrm{ad},\mathcal{O}}\subset G_{\mathrm{ad},\mathscr{L}}$ denote the subgroup of all elements which, relative to our fixed Chevalley basis of $\mathfrak{g}(A)$, are represented by matrices with coefficients in $\mathcal{O}$. Then π^0 is the restriction of the projection (again denoted by π^0) from $G_{\mathrm{ad},\mathcal{O}}$ to $G_{\mathrm{ad},\mathcal{O}}/\mathscr{I}_U^{(1)}$, where $G_{\mathrm{ad},\mathcal{O}}/\mathscr{I}_U^{(1)}$ may be identified with the subgroup $G_k\subset\mathrm{Aut}\,\mathfrak{g}_k(A)$, generated by all $\chi_\alpha(r)$, $\alpha\in\Delta(A)$, $r\in k$. Moreover (identifying $\mathscr{I}_U$ with a subgroup of $\hat{G}_k^\lambda$), our notation is consistent, in the sense that if $\alpha\in\Delta_+(A)$, $r\in k$, then

$$\pi^0(\chi_\alpha(r))=\chi_\alpha(r).$$

We let $U_k\subset G_k$ denote the subgroup generated by all $\chi_\alpha(r)$, $\alpha\in\Delta_+(A)$, $r\in k$. Then

$$\pi^0(\mathscr{I}_U)=U_k,$$
$$\pi^0(\Gamma_U)\supset\text{the subgroup } \mathscr{A}_U \text{ of } U_k,$$
$$\text{generated by all } \chi_\alpha(r),\ \alpha\in\Delta_+(A),\ r\in J.$$

We define a fundamental domain $\mathscr{D}\subset k$, for the action of J (acting as translations) as follows:

$$\mathscr{D}=\begin{cases} x\in\mathbf{R}, & |x|\leq 1/2, \quad \text{if } k=\mathbf{R},\\ z=x+iy, & x,y\in\mathbf{R},\ |x|\leq 1/2,\ |y|\leq 1/2, \quad \text{if } k=\mathbf{C}.\end{cases}$$

We define $\mathfrak{g}_{\mathscr{D}}(A) \subset \mathfrak{g}_k(A)$ to be the subset of all $x \in \mathfrak{g}_k(A)$ such that each coordinate of x, relative to our fixed Chevalley basis is in $\mathscr{D}$.

We now fix an order on $\Delta_+(A)$. Then every element of U_k has a unique expression as a product $\prod_{\alpha \in \Delta_+(A)} \chi_\alpha(s_\alpha)$, $s_\alpha \in k$, where the product is taken with respect to our fixed order on $\Delta_+(A)$. We let $U_{\mathscr{D}} \subset U_k$ consist of all products $\prod_{\alpha \in \Delta_+(A)} \chi_\alpha(s_\alpha)$, $s_\alpha \in \mathscr{D}$ (the product again being taken with respect to our fixed order on $\Delta_+(A)$). A straightforward induction then shows:

(18.8) For all $\mathscr{Y} \in U_k$, there exists $\gamma \in \mathscr{A}_U$, such that $\mathscr{Y}\gamma \in U_{\mathscr{D}}$.

Now we have fixed an order on $\Delta_+(A)$. This order, in turn, determines an order on $\Delta_-(A) = -\Delta_+(A)$. We let

$$U_{\mathscr{O}} = \text{subgroup of } \hat{G}_k^\lambda \text{ generated by the elements } \chi_\alpha(\sigma(t)),$$
$$\alpha \in \Delta_+(A),\ \sigma(t) \in \mathscr{O},$$
$$U_{\mathscr{P}}^- = \text{subgroup of } \hat{G}_k^\lambda \text{ generated by the elements } \chi_\alpha(\sigma(t)),$$
$$\alpha \in \Delta_-(A),\ \sigma(t) \in \mathscr{P}.$$

We also recall, from Lemma (13.7), that $T_{\mathscr{O}}^\lambda \subset \hat{G}_k^\lambda$ is the subgroup generated by the elements $h_\alpha(\sigma(t))$, $\alpha \in \Delta(A)$, $\sigma(t) \in \mathscr{O}^*$, and by the elements $h_{a_{l+1}}(s)$, $s \in k^*$. We note that in the notation of § 13, we have

(18.9) $$U_{\mathscr{O}} = U_{+,C},$$
$$U_{\mathscr{P}}^- = U_{-,C}.$$

It thus follows from (13.22) that

(18.10) $$\mathscr{I} = U_{\mathscr{P}}^- T_{\mathscr{O}}^\lambda U_{\mathscr{O}},$$

and, from (13.23), that every element of $U_{\mathscr{O}}$ (resp. of $U_{\mathscr{P}}^-$) has a unique expression as a product

$$\prod_{\alpha \in \Delta_+(A)} \chi_\alpha(\sigma_\alpha(t)), \quad \sigma_\alpha(t) \in \mathscr{O}$$
$$\Big(\text{resp. } \prod_{\alpha \in \Delta_-(A)} \chi_\alpha(\sigma'_\alpha(t)), \quad \sigma'_\alpha(t) \in \mathscr{P}\Big),$$

where the product is taken with respect to our fixed order on $\Delta_+(A)$ (resp. on $\Delta_-(A)$). It then follows from (18.10) and the above remarks, that every element x of $\mathscr{I}$ has an expression

(18.11) $$x = \Big(\prod_{\alpha \in \Delta_+(A)} \chi_\alpha(\sigma_\alpha(t))\Big) h \Big(\prod_{\alpha \in \Delta_-(A)} \chi_\alpha(\sigma'_\alpha(t))\Big),$$

with $\sigma_\alpha(t) \in \mathscr{O}$, $\alpha \in \Delta_+(A)$, $\sigma'_\alpha(t) \in \mathscr{P}$, $\alpha \in \Delta_-(A)$, $h \in T_{\mathscr{O}}^\lambda$,

where products are taken with respect to our fixed orders on $\Delta_\pm(A)$. Moreover, we have

Lemma **(18.12)**. — *The subgroup* $\mathscr{I}_U \subset \mathscr{I}$ *consists of all* $x \in \mathscr{I}$ *such that in an expression of the form* (18.11) *for* x, *the element* h *may be written as a product*

(18.13) $$h = \prod_{i=1}^{\ell} h_{\alpha_i}(\sigma_i(t)),$$

with $\sigma_i(t) \equiv 1 \bmod t(\sigma_i(t) \in \mathcal{O}^*)$, $i = 1, \ldots, \ell$. *For* $j \geq 1$, *the subgroup* $\mathscr{I}_U^{(j)}$ *consists of all* $x \in \mathscr{I}_U$ *such that in an expression of the form* (18.11) *for* x, *we have* $\sigma_\alpha(t) \equiv 0 \bmod t^j$, $\alpha \in \Delta_+(A)$, *and* $\sigma'_\alpha(t) \equiv 0 \bmod t^j$, $\alpha \in \Delta_-(A)$, *and in the expression* (18.13) *for* h *we may assume* $\sigma_i(t) \equiv 1 \bmod t^j$, *for* $i = 1, \ldots, \ell$.

Proof. — Recall, we have identified $\mathscr{I}_U$ with its image under $\Phi' \circ \mathrm{Ad}$. Relative to a Chevalley basis of $\mathfrak{g}(A)$, ordered so that positive root vectors are represented by (strictly) upper triangular matrices, the elements of $\mathscr{I}_U$ acting on $\mathfrak{g}_{\mathscr{L}_k}(A)$, by means of $\Phi' \circ \mathrm{Ad}$, are represented mod t by unipotent matrices. Hence, the elements of $T^\lambda_\mathcal{O} \cap \mathscr{I}_U$ act on $\mathfrak{g}_{\mathscr{L}_k}(A)$ as the identity mod t. On the other hand, for any $h \in T^\lambda_\mathcal{O}$, we have

$$h = \prod_{i=1}^{\ell} h_{\alpha_i}(\tilde{\sigma}_i(t)) \bmod(\text{kernel } \Phi' \circ \mathrm{Ad}), \quad \tilde{\sigma}_i(t) \in \mathcal{O}^*, \ i = 1, \ldots, \ell.$$

Each $\tilde{\sigma}_i(t)$ has an expression

$$\tilde{\sigma}_i(t) = \sigma_i(t) s_i, \quad s_i \in k^*, \ \sigma_i(t) \in \mathcal{O}^*, \ \sigma_i(t) \equiv 1 \bmod t.$$

Then we have

$$h = h' h'' \bmod(\text{kernel } \Phi' \circ \mathrm{Ad}),$$

where $$h' = \prod_{i=1}^{\ell} h_{\alpha_i}(\sigma_i(t)),$$

$$h'' = \prod_{i=1}^{\ell} h_{\alpha_i}(s_i).$$

Now, if $h \in T^\lambda_\mathcal{O} \cap \mathscr{I}_U$, then

$$h'' \in T^\lambda_\mathcal{O} \cap \mathscr{I}_U \bmod(\text{kernel } \Phi' \circ \mathrm{Ad}),$$

since $h' \in T^\lambda_\mathcal{O} \cap \mathscr{I}_U$, by Lemma (12.2). But then, h'' acts on $\mathfrak{g}_{\mathscr{L}_k}(A)$ as the identity mod t. But $\Phi' \circ \mathrm{Ad}(h'')$ is equal to its reduction mod t. Hence $h'' \in \text{kernel}(\Phi \circ \mathrm{Ad})$. But $\Phi' \circ \mathrm{Ad}$ is injective when restricted to $\mathscr{I}_U$, by Lemma (18.1). Since, by Lemma (12.2), $h' \in T^\lambda_\mathcal{O} \cap \mathscr{I}_U$, we thus have $h = h'$; i.e., we have proved:

(18.14) If $h \in T^\lambda_\mathcal{O} \cap \mathscr{I}_U$, then h has an expression as in (18.13) with $\sigma_i(t) \equiv 1 \bmod t$, $i = 1, \ldots, \ell$.

But now if $x \in \mathscr{I}_U$, and if we express x as in (18.11), then $h \in \mathscr{I}_U \cap T^\lambda_\mathcal{O}$, and hence the first assertion of Lemma (18.12) follows from (18.14).

We now turn to the second assertion of Lemma (18.12). It is clear that if $x \in \mathscr{I}_U$ has an expression (18.11), with $\sigma_\alpha(t)$, $\sigma'_\alpha(t) \equiv 0 \bmod t^j$, $\alpha \in \Delta(A)$, and with h

expressed as in (18.13), with $\sigma_i(t) \equiv 1 \bmod t^j$, then $x \in \mathscr{I}_U^{(j)}$. The converse for $j \geq 2$ follows easily from (18.7) and the result for $j=1$. For $j=1$, we have that if $x \in \mathscr{I}_U$ and if x is expressed as in (18.11), with h as in (18.13), with $\sigma_i(t) \equiv 1 \bmod t$, $i=1, \ldots, \ell$, then

$$\pi^0(x) = \prod_{\alpha \in \Delta_+(A)} \chi_\alpha(q_\alpha),$$

where q_α is the constant term of $\sigma_\alpha(t)$, $\alpha \in \Delta_+(A)$. If $x \in \mathscr{I}_U^{(1)}$ then $\pi^0(x)=1$, and we must have $q_\alpha = 0$ for all $\alpha \in \Delta_+(A)$; i.e., $\sigma_\alpha(t) \equiv 0 \bmod t$, for $\alpha \in \Delta_+(A)$. This concludes the proof of Lemma (18.12). ∎

Definition **(18.15)**. — *We let* $\mathcal{O}_\mathscr{D}$ *denote the set of all* $\sigma(t) = \sum_{j \geq 0} q_j t^j$ *in* $\mathcal{O}$ *such that* $q_j \in \mathscr{D}$, *and* $\mathcal{O}^*_\mathscr{D}$ *the set of all* $\sigma(t) = \sum_{j \geq 0} q_j t^j$ *in* $\mathcal{O}$ *such that* $q_0 = 1$ *and* $q_j \in \mathscr{D}$, $j \geq 1$. *We let* $\mathscr{I}_{U,\mathscr{D}}$ *denote the set of all* $x \in \mathscr{I}_U$ *such that in the expression* (18.11) *for* x, *with* h *as in* (18.13) *with* $\sigma_i(t) \in \mathcal{O}^*$, $\sigma_i(t) \equiv 1 \bmod t$, $i=1, \ldots, \ell$, *we have* $\sigma_\alpha(t)$, $\sigma'_{-\alpha}(t) \in \mathcal{O}_\mathscr{D}$, *for all* $\alpha \in \Delta_+(A)$, *and* $\sigma_i(t) \in \mathcal{O}^*_\mathscr{D}$, *for* $i=1, \ldots, \ell$.

Lemma **(18.16)**. — *For all* $x \in \mathscr{I}_U$, *there exists* $\gamma \in \Gamma_U$ *such that*

$$x\gamma \in \mathscr{I}_{U,\mathscr{D}}.$$

Proof. — We have observed that $\pi^0(\Gamma_U)$ contains $\mathscr{A}_U \subset U_k$, and, thanks to (18.8), we can find $\gamma_0 \in \Gamma_U$ so that if we express $x\gamma_0$ as in (18.11), then the constant term of $\sigma_\alpha(t)$, $\alpha \in \Delta_+(A)$, is in $\mathscr{D}$. For an integer $j \geq 0$, we let $P(j)$ denote the following assertion:

$P(j)$: For each integer k with $0 \leq k \leq j$, there is an element $\gamma_k \in \Gamma_U$ such that

(i) $\gamma_k = \gamma_{k+1} \bmod \Gamma_U^{(k+1)}$,

(ii) There exist elements $\sigma_\alpha(t)$, $\sigma'_{-\alpha}(t) (\alpha \in \Delta_+(A))$ in $\mathcal{O}_\mathscr{D}$, of order j (i.e., whose coefficients of t^m are zero for $m > j$), and with $\sigma'_{-\alpha}(t) \in \mathscr{P}$, and there exist elements $\sigma_i(t)$ $(i=1, \ldots, \ell)$ in $\mathcal{O}^*_\mathscr{D}$ of order j, so that if x_j denotes the corresponding product given by (18.11), with h given by (18.13), then

$$x\gamma_k = x_j \bmod \mathscr{I}_U^{(k+1)}, \quad 0 \leq k \leq j.$$

We note that our initial comments in the proof serve to verify $P(0)$, with γ_0 chosen as above. Assume then for some $j \geq 0$ we have proved $P(j)$. Then

$$x\gamma_j = x_j y_j, \quad y_j \in \mathscr{I}_U^{(j+1)}.$$

Moreover, thanks to (18.4) and (18.5), we can find $\gamma' \in \Gamma_U^{(j+1)}$, such that

$$\Psi^{(j+1)} \circ \pi^{(j+1)}(y_j \gamma') \in \mathfrak{g}_\mathscr{D}(A).$$

We set $\gamma_{j+1} = \gamma_j \gamma'$, and note that with this choice of γ_{j+1}, we have that (i) of $P(j+1)$ is satisfied. We write $\Psi^{(j+1)} \circ \pi^{(j+1)}(y_j \gamma') \in \mathfrak{g}_\mathscr{D}(A)$ as

$$\sum_{\alpha \in \Delta(A)} q_\alpha E_\alpha + \sum_{i=1}^{\ell} q_i H_i, \quad q_\alpha, q_i \in \mathscr{D}.$$

Now in (ii) of $P(j)$ we have specified certain $\sigma_\alpha(t)$, $\sigma'_{-\alpha}(t)$, $\sigma_i(t)$. For $P(j+1)$ we replace these elements by

$$\begin{aligned} &\sigma_\alpha(t)+q_\alpha t^{j+1}, && \alpha\in\Delta_+(A)\\ &\sigma'_{-\alpha}(t)+q_{-\alpha}t^{j+1}, && \alpha\in\Delta_+(A)\\ &\sigma_i(t)+q_i t^{j+1}, && i=1,\ldots,\ell,\end{aligned}$$

respectively. We use these elements to define a corresponding x_{j+1}, by (18.11) and (18.13), and note that γ_k, $0\leq k\leq j+1$, and x_{j+1} satisfy (ii) of $P(j+1)$. We see that moreover, the sequences $\gamma_j\in\Gamma_U$, $x_j\in\mathscr{I}_{U,\mathscr{D}}$, have well defined t-adic limits $\gamma\in\Gamma_U$, $x_{\mathscr{D}}\in\mathscr{I}_{U,\mathscr{D}}$, respectively, and

$$x\gamma=x_{\mathscr{D}}.$$

This proves Lemma (18.16). ∎

Remark **(18.17)**. — Our proof of Lemma (18.16) shows more: If $\Gamma_{0,U}\subset\Gamma_U$ is the subgroup of Γ_U generated by all $\chi_\alpha(\sigma(t))$, $\alpha\in\Delta(A)$, $\sigma(t)\in J[[t]]$, the ring of power series in t, with coefficients in J, then for all $x\in\mathscr{I}_U$ we may find $\gamma\in\Gamma_{0,U}$, so that $x\gamma\in\mathscr{I}_{U,\mathscr{D}}$.

19. A fundamental domain from Siegel sets.

In accordance with the notational convention adopted in § 17, we continue to take $k=\mathbf{R}$ or $\mathbf{C}$. We define $\sigma_0>0$ by

$$\sigma_0=\begin{cases}2/\sqrt{3}, & k=\mathbf{R}\\ \sqrt{2}, & k=\mathbf{C}.\end{cases}$$

Definition **(19.1)**. — *For* $\sigma>0$ *we let* H_σ *consist of all* he^{-cD}, *where* $h\in H_{k,+}$, $r=\mathscr{R}e\ c>0$ *and*

$$(he^{-rD})^{a_i}<\sigma,\qquad i=1,\ldots,\ell+1.$$

Definition **(19.2)**. — *For* $\sigma>0$, *we let* $\mathfrak{S}_\sigma=\hat{K}H_\sigma\mathscr{I}_{U,\mathscr{D}}$, *and we call* $\mathfrak{S}_\sigma$ a Siegel set.

We set $\hat{\Gamma}_0$ equal to the subgroup of $\hat{\Gamma}$ generated by all $\chi_\alpha(\sigma(t))$, $\alpha\in\Delta(A)$, $\sigma(t)\in\mathscr{L}_J$. We have

Theorem **(19.3)**. — *Let* λ *be a dominant integral, linear functional in* $\mathfrak{h}^e_k(\widetilde{A})^*$, *and assume* $\lambda(h_i)=1$, *for* $i=1,\ldots,\ell+1$. *Then for* $g\in\hat{G}^\lambda_k$ *and* $c\in k$ *with* $r=\mathscr{R}e\ c>0$, *we can find* $\gamma\in\Gamma_0$ *such that*

$$ge^{-cD}\gamma\in\mathfrak{S}_{\sigma_0}.$$

Remarks. — (i) Later on we shall see that if $c\in\mathbf{R}$, then we can drop the restriction that $\lambda(h_i)=1$ for all $i=1,\ldots,\ell+1$ (see Theorem (20.14)). (ii) Of course

Theorem (19.3) also holds for $\hat{\Gamma}$ in place of $\hat{\Gamma}_0$. We do not know the exact relationship between $\hat{\Gamma}$ and $\hat{\Gamma}_0$. (iii) Needless to say, Theorem (19.3) is an analogue of Theorem (1.6) in [1]. Indeed our proof is modeled after that in [1], though the present case presents some new technical difficulties (e.g., we must introduce the operator D, and the space V^λ is infinite-dimensional). (iv) The appearence of D seems to us to be natural. Indeed, the operator D is naturally related to the Fourier expansion of automorphic forms on $Sl_2(\mathbf{R})$ (see e.g. [6]). The picture which emerges in Theorem (19.3), when $k=\mathbf{C}$, may be described as follows: Let $\mathscr{P}^*$ denote the Poincaré upper half plane. For each $z\in\mathscr{P}^*$, let $\hat{\Gamma}^{iz}=e^{iz\mathrm{D}}\hat{\Gamma}e^{-iz\mathrm{D}}\subset\hat{G}^\lambda_{\mathbf{C}}$, and let

$$\varphi_z:\hat{\Gamma}\to\hat{\Gamma}^{iz}$$

be the isomorphism defined by

$$\varphi_z(\gamma)=e^{iz\mathrm{D}}\gamma e^{-iz\mathrm{D}}.$$

We let $\hat{\Gamma}$ act on $\hat{G}^\lambda_{\mathbf{C}}\times\mathscr{P}^*$ by

$$(g,z)\cdot\gamma=(g\varphi_z(\gamma),z).$$

We let $\tilde{\mathfrak{S}}_\sigma=\mathrm{all}(g,z)\in\hat{G}^\lambda_{\mathbf{C}}\times\mathscr{P}^*$ such that $ge^{iz\mathrm{D}}\in\mathfrak{S}_\sigma$. Then Theorem (19.3) tells us that

$$\tilde{\mathfrak{S}}_\sigma\hat{\Gamma}_0=\hat{G}^\lambda_{\mathbf{C}}\times\mathscr{P}^*.$$

(v) When $k=\mathbf{C}$, our methods in fact prove Theorem (19.3), whenever J is the ring of integers in a Euclidean, imaginary quadratic field. Of course, for each such J, one must make an appropriate choice for σ_0. Also, one must utilize the following for such a J:

(*) There exists ε, with $0<\varepsilon<1$, such that for all $\xi\in\mathbf{C}$, there exists $\eta\in\mathrm{J}$, such that $|\xi-\eta|<\varepsilon$.

The proof of (*) follows from Hardy and Wright, *An Introduction to the Theory of Numbers*, p. 213, Thm. 246 (and accompanying remarks).

Proof of Theorem (19.3). — The proof rests upon the existence of minima (as in Lemma (17.15)). We recall that by the remark following the statement of Lemma (17.15), the lemma also holds for $\hat{\Gamma}_0$. Thus, given $g\in\hat{G}^\lambda_k$, we may choose $\gamma_0\in\hat{\Gamma}_0$ so that

$$\textbf{(19.4)}\qquad ||ge^{-c\mathrm{D}}\gamma_0 v_0||\leq||ge^{-c\mathrm{D}}\gamma v_0||,$$

for all $\gamma\in\hat{\Gamma}_0$. Now $ge^{-c\mathrm{D}}\gamma_0 e^{c\mathrm{D}}\in\hat{G}^\lambda_{\mathbf{C}}$, and relative to our Iwasawa decomposition of Lemma (16.14), we may write this element as

$$ge^{-c\mathrm{D}}\gamma_0 e^{c\mathrm{D}}=g'_{\mathrm{K}}g'_{\mathrm{H}}g'_{\mathrm{U}},$$

where $g'_{\mathrm{K}}\in\hat{\mathrm{K}}$, $g'_{\mathrm{H}}\in\mathrm{H}_{k,+}$, and $g'_{\mathrm{U}}\in\mathscr{I}_{\mathrm{U}}$. We note that $g''_{\mathrm{U}}=e^{c\mathrm{D}}g'_{\mathrm{U}}e^{-c\mathrm{D}}\in\mathscr{I}_{\mathrm{U}}$. We choose $\gamma_1\in\hat{\Gamma}_0$, so that $g_{\mathscr{D}}=_{\mathrm{df}}g''_{\mathrm{U}}\gamma_1\in\mathscr{I}_{\mathrm{U},\mathscr{D}}$ (we are using Remark (18.17), following the proof of Lemma (18.16)). We have

$$||ge^{-c\mathrm{D}}\gamma_0\gamma_1 v_0||=||ge^{-c\mathrm{D}}\gamma_0 v_0||,$$

since $\gamma_1 v_0 = v_0$. Hence (19.4) implies

(19.5) $$\|ge^{-cD}\gamma_0\gamma_1 v_0\| \leq \|ge^{-cD}\gamma_0\gamma_1\gamma v_0\|, \quad \text{for all} \quad \gamma\in\hat{\Gamma}_0.$$

On the other hand, we have

$$ge^{-cD}\gamma_0\gamma_1 = g'_K g'_H g'_U e^{-cD}\gamma_1 = g'_K g'_H e^{-cD} g_{\mathscr{D}}.$$

We set this last element equal to g'. We have from (19.5):

(19.6) $$\|g' v_0\|^2 \leq \|g'\gamma v_0\|^2, \quad \text{for all} \quad \gamma\in\hat{\Gamma}_0.$$

We will have proved Theorem (19.3), if we show that $g'\in\mathfrak{S}_{\sigma_0}$. To prove this, we take $\gamma = w_{a_i}(1)$, $i=1, \ldots, \ell+1$, in (19.6). The left side of (19.6) is, in any case, equal to

(19.7) $$|(g'_H e^{-cD})^\lambda|^2.$$

On the other hand, thanks to Lemma (11.2) and to our assumption that $\lambda(h_i)=1$, for all $i=1, \ldots, \ell+1$, we have

(19.8) $$w_{a_i}(1).v_0\in V^\lambda_{\lambda-a_i}, \quad i=1, \ldots, \ell+1.$$

Also, identifying $\mathfrak{g}_k(\widetilde{A})$ with $\pi^\lambda_k(\mathfrak{g}_k(\widetilde{A}))$ (recall from Proposition (6.11), that π^λ_k is faithful, for $k=\mathbf{R}$ or $\mathbf{C}$), we have from (7.8) that

$$w_{a_i}(1)^{-1}\xi_{a_i} w_{a_i}(1) = \pm\xi_{-a_i}, \quad \text{for all} \quad i=1, \ldots, \ell+1.$$

We note that $v'_i = w_{a_i}(1).v_0\in V^\lambda_{\lambda-a_i}$, by (19.8), and

$$\mathfrak{g}^a.V^\lambda_\mu \subset V^\lambda_{\mu+a}, \quad a\in\Delta(\widetilde{A}),\ \mu = \text{weight of } V^\lambda,$$

as one checks directly. Hence

$$\begin{aligned}\xi^2_{-a_i}.v_0 &= w_{a_i}(1)^{-1}\xi^2_{a_i} w_{a_i}(1).v_0\\ &= w_{a_i}(1)^{-1}\xi^2_{a_i}.v'_i = 0,\end{aligned}$$

since $\xi^2_{a_i}.v'_i\in V^\lambda_{\lambda+a_i}$, and $V^\lambda_{\lambda+a_i}=0$ because $\lambda+a_i$ is not a weight of V^λ (see (6.7)). But then, since $w_{a_i}(1).v_0\in V^\lambda_{\lambda-a_i}$, we have

$$\begin{aligned}w_{a_i}(1).v_0 &= (1+\xi_{a_i})(1-\xi_{-a_i}).v_0\\ &= (1-\xi_{-a_i}-\xi_{a_i}\xi_{-a_i}).v_0\\ &= -\xi_{-a_i}.v_0,\end{aligned}$$

and hence

(19.9) $$\xi_{a_i} w_{a_i}(1) v_0 = -h_i.v_0 = -v_0, \quad i=1, \ldots, \ell+1,$$

since $\lambda(h_i)=1$ for $i=1, \ldots, \ell+1$.

We can now compute the right side of (19.6) for $\gamma = w_{a_i}(1)$. Indeed, we get from (19.9):

(19.10) $$\|g' w_{a_i}(1).v_0\|^2 = \|g'_H e^{-cD} g_{\mathscr{D}}.v'_i\|^2 = |(g'_H e^{-cD})^{\lambda-a_i}|^2 + |\rho|^2 |(g'_H e^{-cD})^\lambda|^2,$$

where

$$\rho=\begin{cases}\text{the constant term of } \sigma_{\alpha_i}(t), \text{ in the expansion (18.11) for } g_{\mathscr{D}}, \\ \text{when } i=1, \ldots, \ell. \\ \text{the coefficient of } t \text{ in } \sigma'_{-\alpha_0}(t) \ (\alpha_0=\text{highest root of } \Delta(\mathrm{A}), \text{ relative} \\ \text{to } \alpha_1, \ldots, \alpha_\ell) \text{ in the expansion (18.11) for } g_{\mathscr{D}}, \text{ when } i=\ell+1.\end{cases}$$

Since $g_{\mathscr{D}}\in\mathscr{I}_{\mathrm{U},\mathscr{D}}$, we have $\rho\in\mathscr{D}$, so

$$|\rho|^2\leq\begin{cases}1/4, & k=\mathbf{R}\\ 1/2, & k=\mathbf{C},\end{cases}$$

and if we set $\rho_0^2=1/4$ when $k=\mathbf{R}$ and $1/2$ when $k=\mathbf{C}$, we obtain from the fact that (19.7) is at most equal to the last expression of (19.10) (by (19.6)) that

$$(1-\rho_0^2)^r\leq|(g'_{\mathrm{H}}e^{-c\mathrm{D}})^{-a_i}|^2,$$

or

$$(g'_{\mathrm{H}}e^{-r\mathrm{D}})^{a_i}\leq\begin{cases}\dfrac{2}{\sqrt{3}}, & k=\mathbf{R}\\ \sqrt{2}, & k=\mathbf{C}.\end{cases}$$

This proves that $g'\in\mathfrak{S}_{\sigma_0}$, which, as we noted earlier, is sufficient to prove Theorem (19.3). ■

20. The structure of arithmetic quotients (preliminaries).

As specified in § 17, we continue in this section, to take our field k to be $\mathbf{R}$ or $\mathbf{C}$ (though this assumption is not always necessary; e.g., for Propositions (20.1), (20.2), and Lemma (20.4)). We let $\lambda_i\in\mathfrak{h}_k^e(\widetilde{\mathrm{A}})^*$, the dual space of $\mathfrak{h}_k^e(\widetilde{\mathrm{A}})$, be defined by the conditions

$$\lambda_i(h_j)=\delta_{ij}, \quad i,j=1, \ldots, \ell+1.$$

The following proposition is an immediate Corollary of Lemma (15.7):

Proposition **(20.1)**. — *If* $\lambda\in\mathbf{D}$ *is any normal element, then there is a positive integer* m, *such that* $\Xi_{m\lambda_i}\subset\Xi_\lambda$, *for each* $i=1, \ldots, \ell+1$.

The next proposition is an immediate Corollary of Lemma (15.7) and Theorem (15.9):

Proposition **(20.2)**. — *Let* $\lambda, \mu\in\mathbf{D}$ *be two normal elements. Then there exists an integer* $m_0>0$, *such that if* m *is any positive multiple of* m_0, $m=jm_0$, $j>0$, *there is a well defined group homomorphism*

$$\pi(\lambda, m\mu) : \hat{\mathrm{G}}_k^\lambda\to\hat{\mathrm{G}}_k^{m\mu},$$

satisfying (and uniquely determined by) the conditions

(20.3) $\pi(\lambda, m\mu)(\chi_\alpha^\lambda(\sigma(t))) = \chi_\alpha^{m\mu}(\sigma(t)), \quad \alpha \in \Delta(A), \ \sigma(t) \in \mathscr{L}.$

The next result will be needed presently;

Lemma **(20.4)**. — *If, relative to a coherently ordered basis of* V_k^λ, *the element* $g \in \hat{G}_k^\lambda$ *is represented by a diagonal matrix, then g is in* H_k^λ *(and, of course, conversely).*

Proof. — Let $g \in \mathscr{I} w \mathscr{I}$, $w \in \mathbf{W}$ and assume that, relative to a coherently ordered basis, g is represented by a diagonal matrix. We first show that w is the identity. Assume this is not so. Recall that by Lemma (15.2), we have

(20.5) $\Xi_r \subset \Xi_\lambda,$

and that in § 17, we proved that there is an isomorphism $\Phi_0 : \mathbf{W} \to W$, of $\mathbf{W}$ onto W ($W \subset \mathrm{Aut}(\mathfrak{h}_k^e(\widetilde{A})^*)$ being the Weyl group of $\mathfrak{g}^e(\widetilde{A})$, defined in § 3—(also see § 17) uniquely defined by the conditions

(20.6) $\Phi_0(w_{a_i}(1)(\mathscr{I} \cap N)) = r_i, \quad i = 1, \ldots, \ell + 1.$

By Lemma (11.2), (i), and by (20.6), we see that if $n \in N$ represents w, then

(20.7) $n(V_{\mu,k}^\lambda) = V_{\Phi_0(w)(\mu),k}^\lambda$

for every weight μ of V^λ. Since $\Xi_r \subset \Xi_\lambda$ (see (20.5)) we therefore have that $\Phi_0(w)(\mu') \neq \mu'$ for some weight μ' of V^λ. The point here, is that since we have assumed w different from the identity, and since Φ_0 is an isomorphism, $\Phi_0(w)$ cannot be the identity. But then $\Phi_0(w)$, restricted to Ξ_r, is not the identity (see [8], § 2, the discussion preceeding Proposition (2.5)). Choose a weight μ of V^λ of minimal depth among all weights μ' of V^λ such that $\Phi_0(w)(\mu') \neq \mu'$ (for the definition of depth see the beginning of § 12).

Also, write $g \in \mathscr{I} w \mathscr{I}$ as

$$g = xny, \quad x, y \in \mathscr{I}.$$

If $v \in V_{\mu,k}^\lambda$, $v \neq 0$, then, thanks to our minimality condition on the depth of μ, and to Corollary (14.18), we have

$$g.v = cn.v \text{ (modulo summands of strictly smaller depth)},$$

where $c \in k^*$. But, by our choice of μ, and by (20.7), we have that v and $n.v$ are in different weight spaces. Hence the element g is not represented by a diagonal matrix, relative to a coherently ordered basis; i.e., our assumption that $w \neq e$ has led to a contradiction. Hence $g \in \mathscr{I}$. But then $g \in H_k^\lambda$, by Corollary (14.18).

Now Theorem (14.10) implies that $(\hat{G}_k^\lambda, \mathscr{I}, N, \mathbf{S})$ is a Tits system. Hence, we have the Bruhat decomposition

$$\hat{G}_k^\lambda = \bigcup_{w \in W} \mathscr{I} w \mathscr{I},$$

(see [3], Chapter IV, § 2, Theorem 1, p. 25), and every element g of $\hat{G}_k^\lambda$ is in $\mathscr{I}w\mathscr{I}$, for some $w \in \mathbf{W}$. We thus obtain the Lemma. ∎

Now let $\lambda \in \mathbf{D}$ be a normal element, and let $\hat{G}_k^{\lambda, e} \subset \operatorname{Aut} V_k^\lambda$ denote the subgroup generated by the elements of $\hat{G}_k^\lambda$ and by the elements e^{rD}, $r \in \mathbf{R}$ (recall from § 6 that D acts on $V_{\mathbf{Z}}^\lambda$ (we assumed $\lambda(D) \in \mathbf{Z}$) and hence on V_k^λ for *every* field k, and that e^{rD} was defined in § 17). We then have:

Lemma **(20.8)**. — *Let* $\lambda, \mu \in \mathbf{D}$ *be two normal elements and assume* $\Xi_\mu \subset \Xi_\lambda$. *Then the homomorphism*

$$\pi(\lambda, \mu) : \hat{G}_k^\lambda \to \hat{G}_k^\mu$$

of Theorem (15.9) *has a unique extension to a homomorphism*

$$\pi(\lambda, \mu) : \hat{G}_k^{\lambda, e} \to \hat{G}_k^{\mu, e},$$

such that $\pi(\lambda, \mu)(e^{rD}) = e^{rD}$ *for all* $r \in \mathbf{R}$.

Proof. — For each $r \in \mathbf{R}$, the automorphism e^{rD} of V_k^ν normalizes $\hat{G}_k^\nu$ ($\nu = \mu$ or λ) and one has

$$\pi(\lambda, \mu)(e^{rD} g e^{-rD}) = e^{rD} \pi(\lambda, \mu)(g) e^{-rD}, \quad g \in \hat{G}_k^\lambda.$$

Hence, in order to prove the Lemma, it suffices to show that

$$\textbf{(20.9)} \qquad \hat{G}_k^\lambda \cap \{e^{rD}\}_{r \in \mathbf{R}} = \{\text{identity}\}.$$

Indeed, by Lemma (20.4), this intersection is contained in H_k^λ. However, $(e^{rD})^{a_i} = 1$, for $i = 1, \ldots, \ell$. Then, if $e^{rD} \in H_k^\lambda$, we must have, as a consequence of these equalities, that $(e^{rD})^{a_{\ell+1}} = 1$. But

$$(e^{rD})^{a_{\ell+1}} = e^r,$$

and thus $r = 0$, and this proves (20.9), and hence proves Lemma (20.8). ∎

For $\sigma > 0$, we write $\mathfrak{S}_\sigma^\lambda$ (resp. H_σ^λ) for $\mathfrak{S}_\sigma$ (resp. for $H_\sigma \subset H_k^\lambda$) whenever we wish to keep track of the λ-dependence (see Definitions (19.1) and (19.2) for the definition of $\mathfrak{S}_\sigma$ and H_σ). We let

$$H_\sigma^{\lambda, \mathbf{R}} = H_\sigma^{\mathbf{R}}$$

consist of all $h' \in H_\sigma$ such that

$$h' = h e^{-rD}, \quad h \in H_{k,+}, \ r > 0.$$

We set

$$\mathfrak{S}_\sigma^{\lambda, \mathbf{R}} = \mathfrak{S}_\sigma^{\mathbf{R}} =_{\mathrm{df}} \hat{K} H_\sigma^{\mathbf{R}} \mathscr{I}_{U, \mathscr{D}}.$$

We write $\hat{\Gamma}_0^\lambda$ for $\hat{\Gamma}_0$ in $\hat{G}_k^\lambda$, and similarly, $\mathscr{I}_U^\lambda$ for $\mathscr{I}_U$, $H_{k,+}^\lambda$ for $H_{k,+}$, and $\hat{K}^\lambda$ for $\hat{K}$, whenever we wish to keep track of the λ-dependence of these groups. We have

Lemma **(20.10)**. — *Let* $\lambda, \mu \in \mathbf{D}$ *be normal elements such that* $\Xi_\mu \subset \Xi_\lambda$. *Let* $\pi(\lambda, \mu) : \hat{G}_k^{\lambda, e} \to \hat{G}_k^{\mu, e}$ *be the homomorphism given by Lemma* (20.8). *We then have*

(20.11) $$\pi(\lambda, \mu)(\hat{\Gamma}_0^\lambda) = \hat{\Gamma}_0^\mu$$
$$\pi(\lambda, \mu)(\hat{K}^\lambda) = \hat{K}^\mu$$
$$\pi(\lambda, \mu)(\mathfrak{S}_\sigma^{\lambda, \mathbf{R}}) = \mathfrak{S}_\sigma^{\mu, \mathbf{R}} \qquad (\sigma > 0).$$

Proof. — From Theorem (15.9) it follows that

(20.12) $$\pi(\lambda, \mu)(\chi_\alpha^\lambda(\sigma(t))) = \chi_\alpha^\mu(\sigma(t)), \qquad \alpha \in \Delta(A),\ \sigma(t) \in \mathscr{L}_k,$$

and then, as a special case of this equality, we have

(20.12′) $$\pi(\lambda, \mu)(\chi_a^\lambda(s)) = \chi_a^\mu(s), \qquad a \in \Delta_W(\tilde{A}),\ s \in k$$

(where we write $\chi_a^\nu(s)$ for $\chi_a(s)$ ($\nu = \lambda$ or μ), to keep track of the dependence on the highest weight). The first equality of (20.11) follows from (20.12) and from the definition of $\hat{\Gamma}_0^\nu$ ($\nu = \lambda$ or μ) in § 17 (after Lemma (17.15)).

In order to prove the second equality of (20.11) we note that if $\hat{K}_0^\nu \subset \hat{G}^\nu$ ($\nu = \lambda$ or μ) is the subgroup of $\hat{K}^\nu$ generated by the subgroups $\Psi_{a_i}(K)$, $i = 1, \ldots, \ell+1$ (see § 16 for the definition of Ψ_a, $a \in \Delta_W(\tilde{A})$, and of $K \subset SL_2(k)$), then the proof given in § 16 for Lemma (16.14) also shows that

$$\hat{G}^\nu = \hat{K}_0^\nu H_{k,+}^\nu \mathscr{I}_U^\nu, \qquad \nu = \lambda \text{ or } \mu,$$

and then, by the uniqueness assertion of Lemma (16.14), we have

(20.13) $$\hat{K}_0^\nu = \hat{K}^\nu, \qquad \nu = \lambda \text{ or } \mu.$$

On the other hand, if we write Ψ_a^ν for $\Psi_a : SL_2(k) \to \hat{G}_k^\nu$, $\nu = \lambda$ or μ, $a \in \Delta_W(\tilde{A})$, then from the definition of Ψ_a in § 16, and from (20.12′), we have

$$\Psi_a^\mu = \pi(\lambda, \mu) \circ \Psi_a^\lambda, \qquad a \in \Delta_W(\tilde{A}),$$

and hence, from (20.13),

$$\pi(\lambda, \mu)(\hat{K}^\lambda) = \hat{K}^\mu;$$

i.e., we have proved the second equality of (20.11).

From (20.12′) we also have

$$\pi(\lambda, \mu)(h_{a_i}^\lambda(s)) = h_{a_i}^\mu(s), \qquad s \in k^*,\ i = 1, \ldots, \ell+1,$$

where (as usual) we write $h_a^\nu(s)$ for $h_a(s)$, $a \in \Delta_W(\tilde{A})$, $s \in k^*$, to keep track of the dependence on the highest weight. The third equality of (20.11) then follows from this, from the second equality of (20.11), and from (20.12). This proves Lemma (20.10). ■

We can now prove the following extension of Theorem (19.3):

Theorem **(20.14).** — *Let* $\lambda \in \mathbf{D}$ *be normal. For any* $g \in \widehat{G}_k^\lambda$, *and* $r > 0$, *we can find* $\gamma \in \widehat{\Gamma}_0$ *such that*

$$g e^{-rD} \gamma \in \mathfrak{S}_{\sigma_0}^{\mathbf{R}}.$$

Proof. — Fix $\rho \in \mathbf{D}$, such that $\rho(h_i) = 1$, for $i = 1, \ldots, \ell + 1$. Then, by Theorem (19.3), we know that Theorem (20.14) is valid for $\lambda = \rho$. If $\lambda \in \mathbf{D}$ is any normal element, then by Lemma (15.7), Proposition (20.2) and Lemma (20.8), there is a positive integer m so that

$$\Xi_{m\rho} \subset \Xi_\rho$$
$$\Xi_{m\rho} \subset \Xi_\lambda,$$

and we have well defined homomorphisms

$$\pi(\rho, m\rho) : \widehat{G}_k^{\rho, e} \to \widehat{G}_k^{m\rho, e},$$
$$\pi(\lambda, m\rho) : \widehat{G}_k^{\lambda, e} \to \widehat{G}_k^{m\rho, e}.$$

Using Theorem (19.3), and Lemma (20.10) (with $\lambda = \rho$, $\mu = m\rho$) we see that Theorem (20.14) is true for $\lambda = m\rho$. Presently we shall prove:

(20.15) If $\lambda, \mu \in \mathbf{D}$ are normal, and if $\Xi_\mu \subset \Xi_\lambda$, then the kernel of the homomorphism

$$\pi(\lambda, \mu) ; \widehat{G}_k^{\lambda, e} \to \widehat{G}_k^{\mu, e}$$

is contained in $\widehat{K}^\lambda \cap H_k^\lambda \cap (\text{center } \widehat{G}_k^\lambda)$.

We note that (20.15), Lemma (20.10) and Theorem (20.14) for $m\rho$, which we have proved, then imply Theorem (20.14) for λ, thanks to the existence of $\pi(\lambda, m\rho)$. We now prove (20.15). We first note that it suffices to consider $\pi(\lambda, \mu) : \widehat{G}_k^\lambda \to \widehat{G}_k^\mu$. Then, using the Bruhat decomposition corresponding to the Tits system of Theorem (14.10) (for *both* $\widehat{G}_k^\lambda$ and $\widehat{G}_k^\mu$) we see that first $\pi(\lambda, \mu)$ respects these Bruhat decompositions, and then, as a consequence, that the kernel of $\pi(\lambda, \mu)$ is contained in $\mathscr{I} \subset \widehat{G}_k^\lambda$.

On the other hand, we have a commutative diagram

(20.16)
$$\begin{array}{ccc} \widehat{G}_k^\lambda & \xrightarrow{\pi(\lambda, \mu)} & \widehat{G}_k^\mu \\ & {\scriptstyle \Phi' \circ \mathrm{Ad}} \searrow \quad \swarrow {\scriptstyle \Phi' \circ \mathrm{Ad}} & \\ & G_{\mathrm{ad}, \mathscr{L}}(A) & \end{array}$$

where, of course, $\Phi' \circ \mathrm{Ad}$ on the right and left are two different homomorphisms. But (either) $\Phi' \circ \mathrm{Ad}$ is injective on $\mathscr{I}_U$ (by Lemma (18.1)). Since $\mathscr{I} = H_k \mathscr{I}_U$ (semi-direct product) by Lemma (14.12), by (14.13) and (14.16), we therefore have that kernel $\pi(\lambda, \mu)$ is contained in $H_k = H_k^\lambda$. Also, by Lemmas (8.14) and (9.1), we have: $\text{kernel}(\Phi' \circ \mathrm{Ad}) \subset \text{center } \widehat{G}_k^\nu$, $\nu = \lambda$ or μ (recall that for $k = \mathbf{C}$ or $\mathbf{R}$, the algebra $\mathfrak{g}_k^c(\widetilde{A})$ is perfect). Hence, by the commutativity of (20.16), we have:

$$\text{kernel } \pi(\lambda, \mu) \subset \text{center } \widehat{G}_k^\lambda.$$

Thus, to prove (20.15), we need only verify that $\text{kernel } \pi(\lambda, \mu) \subset \widehat{K}^\lambda$.

But from (16.13), we have $H_k^\nu = H_{k,+}^\nu H_{k,\theta}^\nu$ ($\nu = \lambda$ or μ), where we introduce the superscript ν to keep track of the dependence on the highest weight. Also, we clearly have from the equality

$$\pi(\lambda, \mu)(h_{a_i}^\lambda(s)) = h_{a_i}^\mu(s), \qquad s \in k^*, \ i = 1, \ldots, \ell+1,$$

which we noted earlier, that

(20.17)
$$\pi(\lambda, \mu)(H_{k,+}^\lambda) = H_{k,+}^\mu$$
$$\pi(\lambda, \mu)(H_{k,\theta}^\lambda) = H_{k,\theta}^\mu.$$

Also, $H_{k,+}^\nu \cap H_{k,\theta}^\nu$ consists of diagonalizable automorphisms of V_k^ν ($\nu = \lambda$ or μ) each of whose eigenvalues is positive and has modulus one (see § 16). Hence

$$H_{k,+}^\nu \cap H_{k,\theta}^\nu = \{\text{identity}\}.$$

Thus, since $\text{kernel}\, \pi(\lambda, \mu) \subset H_k^\lambda$, and since $H_{k,\theta}^\lambda \subset \hat{K}^\lambda$, we see from (20.17) that if we show the injectivity of $\pi(\lambda, \mu)$ restricted to $H_{k,+}^\lambda$, we will have $\text{kernel}\, \pi(\lambda, \mu) \subset \hat{K}^\lambda$. But the fact that $\pi(\lambda, \mu)$ is injective on $H_{k,+}^\lambda$ follows easily from the fact that Ξ_μ spans $\mathfrak{h}^e(\tilde{A})^*$. Hence we have proved (20.15), and, as we already noted, we obtain Theorem (20.14) from (20.15). ■

21. The stucture of arithmetic quotients (conclusion).

As specified in § 17, we take our field k in this section, to be **R** or **C**. We fix $\rho \in \mathfrak{h}^e(\tilde{A})^*$ such that $\rho(h_i) = 1$, $i = 1, \ldots, \ell+1$. We set

$$\Phi_w = \Delta_+(\tilde{A}) \cap w(\Delta_-(\tilde{A})), \qquad w \in W,$$

and we let $\langle \Phi_w \rangle$ denote the sum of all the roots in Φ_w. Then from [8], Proposition (2.5), page 50, we have

(21.1) $$\langle \Phi_w \rangle = \rho - w.\rho, \qquad \text{for} \quad w \in W.$$

Thus, for $w \in W$, we have an expression for $\rho - w.\rho$ of the form

$$\rho - w.\rho = \sum_{j=1}^{\ell+1} k_j a_j,$$

where the k_j are non-negative integers.

Let $w = r_{i_1} \ldots r_{i_\tau}$ be an expression of minimal length of w in terms of the generators r_i. Set

(21.2) $$b_j = r_{i_1} \ldots r_{i_{j-1}}(a_{i_j}), \qquad j = 1, \ldots, \tau.$$

Then from Proposition (2.2), page 49, in [8], we have $\Phi_w = \{b_1, \ldots, b_\tau\}$, and Φ_w has exactly τ elements. Hence the b_j are mutually distinct.

For each $i = 1, \ldots, \ell+1$, we let $W_i \subset W$ denote the subgroup generated by the r_j, $j \neq i$. We then have:

Lemma **(21.3)**. — *Fix an integer* $i\in\{1, \ldots, \ell+1\}$. *If* $w\in W$ *is not in* W_i, *then*

$$w.\rho-\rho=-\sum_{j=1}^{\ell+1} k_j a_j, \qquad k_j\geq 0,\ k_j\in \mathbf{Z},$$

and $k_i>0$.

Proof. — Let

$$w=r_{i_1}\ldots r_{i_\tau}$$

be an expression of minimal length of w in terms of the generators r_i. Let $\varkappa$ be the smallest integer between one and τ, such that $i_\varkappa=i$ (w is not in W_i, so such a $\varkappa$ does exist). Then

$$\begin{aligned} b_\varkappa &= r_{i_1}\ldots r_{i_{\varkappa-1}}(a_{i_\varkappa}) \\ &= r_{i_1}\ldots r_{i_{\varkappa-1}}(a_i) \\ &= \sum_{j=1}^{\ell+1} q_j a_j, \qquad q_i\neq 0, \end{aligned}$$

and hence

$$\rho-w.\rho=\langle\Phi_w\rangle=b_1+\ldots+b_\tau=\sum_{j=1}^{\ell+1} k_j a_j,$$

with $k_i>0$. ■

Lemma **(21.4)**. — *Fix an* $i\in\{1, \ldots, \ell+1\}$. *Then there is a real number* $c>0$, *such that if* $b\in\Delta_+(\widetilde{A})$, *and* $b=\sum_{j=1}^{\ell+1} k_j a_j$, *with* $k_i>0$, *then*

$$k_i\geq c\Big(\sum_{j=1}^{\ell+1} k_j\Big). \tag{21.5}$$

Proof. — Let $\alpha_0\in\Delta(A)$ denote the highest root, and express α_0 as a linear combination of the simple roots of $\Delta(A)$:

$$\alpha_0=\sum_{j=1}^{\ell} m_j\alpha_j.$$

We let $m=\sum_{j=1}^{\ell} m_j$. Next, let ι denote the imaginary root

$$\iota=\sum_{j=1}^{\ell} m_j a_j+a_{\ell+1},$$

and recall the description of $\Delta_+(\widetilde{A})$ given in (4.3).

Thus, if $b\in\Delta_+(\widetilde{A})$, then b has an expression

$$b=\Big(\sum_{j=1}^{\ell} q_j a_j\Big)+n\Big(\sum_{j=1}^{\ell+1} m_j a_j\Big),$$

where we have set $m_{\ell+1}=1$, and where

$$0\leq q_j\leq m_j, \qquad j=1, \ldots, \ell$$

and $n \geq 0$. On the other hand, in the statement of Lemma (21.4), we have set $b = \sum_{j=1}^{\ell+1} k_j a_j$, with $k_i > 0$ for our fixed i. We set

$$c = \frac{1}{2(m+1)},$$

and we let $q_{\ell+1} = 0$. We have

(21.6) $$k_j = q_j + nm_j, \quad j = 1, \ldots, \ell+1,$$

where

(21.7) $$m_j \geq 1, \quad q_j \leq m_j, \quad j = 1, \ldots, \ell+1.$$

Then

$$\begin{aligned} \frac{k_i}{\sum_{j=1}^{\ell+1} k_j} &= \frac{q_i + nm_i}{(\sum_{j=1}^{\ell+1} q_j) + n(m+1)}, \quad \text{by (21.6)} \\ &\geq \frac{q_i + n}{m + n(m+1)}, \quad \text{by (21.7)} \\ &\geq \frac{q_i + n}{(m+1)(n+1)} \\ &\geq \begin{cases} \dfrac{1}{m+1}, & \text{if } n = 0 \quad (\text{then } q_i = k_i \geq 1) \\ \dfrac{n}{n+1} \dfrac{1}{m+1} \geq \dfrac{1}{2(m+1)}, & \text{if } n \geq 1, \end{cases} \end{aligned}$$

and in either case, we obtain (21.5). ■

Lemma **(21.8)**. — *Fix an* $i \in \{1, \ldots, \ell+1\}$. *There exists* $\varkappa > 0$ *such that for all* $w \in W - W_i$, *if we set*

$$\rho - w.\rho = \sum_{j=1}^{\ell+1} k_j a_j,$$

then

(21.9) $$k_i \geq \varkappa\left(\sum_{j=1}^{\ell+1} k_j\right).$$

Proof. — For a root

$$f = \sum_{j=1}^{\ell+1} n_j a_j$$

in $\Delta_+(\widetilde{A})$, we let $\mathrm{ht}(f)$ (= height of f) be defined by

$$\mathrm{ht}(f) = \sum_{j=1}^{\ell+1} n_j.$$

The set $F \subset \Delta_+(\widetilde{A})$ of all roots $f \in \Delta_+(\widetilde{A})$, such that

$$f = \sum_{j=1}^{\ell+1} n_j a_j, \qquad n_i = 0,$$

is a finite subset. We let $f_1, \ldots, f_q$ denote the elements of F, and we set

$$\mu = \sum_{\sigma=1}^{q} \mathrm{ht}(f_\sigma).$$

Recalling (21.2), and that if $w = r_{i_1} \ldots r_{i_\tau}$ is an expression of minimal length of w in terms of the generators r_i, then $\Phi_w = \{b_1, \ldots, b_\tau\}$, we have from (21.1) that

$$\rho - w.\rho = b_1 + \ldots + b_\tau.$$

We write $$b_u = \sum_{v=1}^{\ell+1} k_{vu} a_v, \qquad u = 1, \ldots, \tau,$$

and note that $k_j = \sum_{u=1}^{\tau} k_{ju}$.

We set $$T(F) = \{u \in \{1, \ldots, \tau\} \mid b_u \notin F\},$$

and $$M = \sum_{u \in T(F)} \sum_{j=1}^{\ell+1} k_{ju}.$$

Then, since $w \notin W_i$, it follows that $M \geq 1$. As a result:

$$\begin{aligned} k_i\Big(\sum_{j=1}^{\ell+1} k_j\Big)^{-1} &= \Big(\sum_{u=1}^{\tau} k_{iu}\Big)\Big(\sum_{j=1}^{\ell+1} \sum_{u=1}^{\tau} k_{ju}\Big)^{-1} \\ &\geq \Big(\sum_{u \in T(F)} k_{iu}\Big)(M+\mu)^{-1} \\ &\geq cM(M+\mu)^{-1}, \text{ by Lemma (21.4),} \\ &= c(1+\mu/M)^{-1} \geq c(1+\mu)^{-1}. \end{aligned}$$

We may thus take $\varkappa = c(1+\mu)^{-1}$, in (21.9). ∎

For each $\sigma > 0$, we defined $H_\sigma^{\mathbf{R}}$ in § 20, to be the set of all $h' \in H_\sigma$ such that $h' = he^{-rD}$, $h \in H_{k,+}$, $r > 0$. Thus, in particular

$$(h')^{a_i} < \sigma, \qquad i = 1, \ldots, \ell+1.$$

We now prove:

Lemma **(21.10)**. — *For all* $\varepsilon > 0$, *there exists* $M > 0$ *such that if* $r > M$ *and* $h' = he^{-rD} \in H_\sigma^{\mathbf{R}}$, $h \in H_{k,+}$, *then for some* $i = 1, \ldots, \ell+1$, *we have* $(h')^{a_i} < \varepsilon$. *Conversely, for all* $M > 0$, *there exists an* $\varepsilon > 0$, *such that if* $h' = he^{-rD} \in H_\sigma^{\mathbf{R}}$, $h \in H_{k,+}$, *and if* $(h')^{a_i} < \varepsilon$ *for some* $i = 1, \ldots, \ell+1$, *then* $r > M$.

Proof. — If $h' = he^{-rD}$, with $r > 0$ and $h \in H_{k,+}$, then

(21.11) $$(h')^{a_i} = h^{a_i}, \qquad i = 1, \ldots, \ell,$$
$$(h')^{a_{\ell+1}} = e^{-r} h^{a_{\ell+1}}.$$

Writing the highest root $\alpha_0 \in \Delta(A)$ as an integral linear combination of simple roots of $\Delta(A)$, we have

$$\alpha_0 = \sum_{j=1}^{\ell} m_j \alpha_j.$$

For $h \in H_{k,+}$, we then have

(21.12) $$h^{a_{\ell+1}} = \prod_{j=1}^{\ell} h^{-m_j a_j}.$$

We let $m = \sum_{j=1}^{\ell} m_j$. Assume we are given $\varepsilon > 0$, and let

$$\varepsilon' = \varepsilon^{-m}.$$

Then choose $M > 0$ so that

$$\varepsilon' e^{-M} < \varepsilon.$$

We then consider $h' = he^{-rD}$ in H_σ^R, with $h \in H_{k,+}$, and $r > M$. We claim that $(h')^{a_i} < \varepsilon$ for some $i = 1, \ldots, \ell+1$. If this is not true for some $i = 1, \ldots, \ell$, then $h^{a_i} \geq \varepsilon$, for $i = 1, \ldots, \ell$ (by (21.11)). Hence, by (21.12), we have

$$h^{a_{\ell+1}} \leq \varepsilon'.$$

But then
$$\begin{aligned}(h')^{a_{\ell+1}} &= (he^{-rD})^{a_{\ell+1}} \\ &= h^{a_{\ell+1}} e^{-r} \leq \varepsilon' e^{-r} \\ &< \varepsilon' e^{-M} < \varepsilon.\end{aligned}$$

Conversely, assume, for some $M > 0$, that $h' = he^{-rD}$ is in H_σ^R, where $h \in H_{k,+}$ and $0 < r < M$. Then

$$h'^{a_i} < \sigma, \quad i = 1, \ldots, \ell+1,$$

and by (21.11) and (21.12), we have

$$h^{a_{\ell+1}} > \sigma',$$

where $\sigma' = \sigma^{-m}$. On the other hand

$$h^{a_{\ell+1}} e^{-r} = (h')^{a_{\ell+1}} < \sigma,$$

by (21.11) and our assumption that h' is in H_σ^R. Thus

$$\sigma' < h^{a_{\ell+1}} < \sigma e^M.$$

For $i = 1, \ldots, \ell$, we let $m(i) = m - m_i$ and $b(i) = (\sum_{j=1}^{\ell} m_j a_j) - m_i a_i$. We then have

$$\begin{aligned}h^{m_i a_i} &= h^{-a_{\ell+1}} h^{-b(i)}, \quad \text{by (21.12)} \\ &> (\sigma e^M)^{-1} \sigma^{-m(i)}, \quad i = 1, \ldots, \ell,\end{aligned}$$

and thus the $(h')^{a_i} = h^{a_i}$, $i = 1, \ldots, \ell$ (see (21.11)), are bounded below. But

$$(h')^{a_{\ell+1}} = h^{a_\ell + 1} e^{-r} \quad \text{(by (21.11))}$$
$$> \sigma' e^{-M},$$

so all the $(h')^{a_i}$, $i = 1, \ldots, \ell + 1$, are bounded below, and this proves Lemma (21.10). ■

In the proof of Lemma (20.8), we showed that if we write $g \in \hat{G}_k^{\lambda, e}$ as

(21.13) $$g = g' e^{rD}, \quad g' \in \hat{G}_k^\lambda, \ r \in \mathbf{R},$$

then g' and r are uniquely determined by g (see (20.9)). It follows from this fact, and from the Iwasawa decomposition of Lemma (16.14), that in the decomposition

(21.14) $$\mathfrak{S}_\sigma^{\mathbf{R}} = \hat{K} H_\sigma^{\mathbf{R}} \mathscr{I}_{U, \mathscr{D}}, \quad \sigma > 0,$$

we have uniqueness of expression. For $x \in \mathfrak{S}_\sigma^{\mathbf{R}}$, we let

$$x = x_K x_H x_U, \quad x_K \in \hat{K}, \ x_H \in H_\sigma^{\mathbf{R}}, \ x_U \in \mathscr{I}_{U, \mathscr{D}},$$

denote the decomposition of x, relative to the decomposition (21.14).

The operator e^{rD}, $r \in \mathbf{R}$, normalizes $\hat{G}_k^\lambda$, and hence, from the uniqueness of the decomposition (21.13), we have:

(21.15) If $g_1, g_2, g_3 \in \hat{G}_k^\lambda$, if $r', r'' \in \mathbf{R}$, and if $g_1 e^{-r'D} g_2 = g_3 e^{-r''D}$, then $r' = r''$.

For each $i = 1, \ldots, \ell + 1$, we let $\mathbf{W}_i \subset \mathbf{W}$ denote the subgroup generated by the elements $w_{a_j}(1)(\mathscr{I} \cap N)$, with $j \neq i$. For $F \subset \{1, \ldots, \ell + 1\}$, we let $\mathbf{W}_F = \bigcap_{i \in F} \mathbf{W}_i$, and we denote by $P_F \subset \hat{G}_k^\lambda$ the parabolic subgroup

$$P_F = \mathscr{I} \mathbf{W}_F \mathscr{I}.$$

Indeed, the fact that P_F is a subgroup of $\hat{G}_k^\lambda$ follows from Theorem (14.10) and from standard properties of Tits systems (see [3], Chapter 4, Theorem 3, page 27).

Theorem **(21.16)**. — *Fix* $\sigma > 0$. *Then there exists* $\varepsilon > 0$ *such that if* $x, y \in \mathfrak{S}_\sigma^{\mathbf{R}}$, *if* $\gamma \in \hat{\Gamma}_0$, *if for some* $i_0 = 1, \ldots, \ell + 1$,

(21.17) $$x_H^{a_{i_0}} < \varepsilon,$$

and if

(21.18) $$x\gamma = y,$$

then $\gamma \in P_{i_0}$.

Proof. — We first note that as a consequence of Lemma (21.10) and of (21.15), we have that for all $\varepsilon' > 0$, there exists $\varepsilon > 0$ such that if $x, y \in \mathfrak{S}_\sigma^{\mathbf{R}}$, $\gamma \in \hat{\Gamma}_0$ satisfy (21.17) and (21.18), then $y_H^{a_j} < \varepsilon'$ for some $j = 1, \ldots, \ell+1$. Thus, in order to prove Theorem (21.16), it suffices to prove

(21.19) There exists $\varepsilon > 0$, such that if $x, y \in \mathfrak{S}_\sigma^{\mathbf{R}}$, if $\gamma \in \hat{\Gamma}_0$, if for some $i_0, j_0 \in \{1, \ldots, \ell+1\}$, we have

(21.17′) $$x_H^{a_{i_0}} < \varepsilon, \qquad y_H^{a_{j_0}} < \varepsilon,$$

and if $x\gamma = y$, then $\gamma \in P_{\{i_0, j_0\}}$.

Now if $x, y \in \mathfrak{S}_\sigma^{\mathbf{R}}$, $\gamma \in \hat{\Gamma}_0$ satisfy (21.18), we have

(21.20) $$x_K x_H x_U \gamma = y_K y_H y_U.$$

Moreover, $\gamma . v_0 = m_\gamma \in V_J^\lambda$ (see § 17 and recall we are taking $\hat{G}_k^\lambda \subset \mathrm{Aut}\, V_k^\lambda$, where $\lambda \in \mathbf{D}$ is normal). If μ is a weight of V^λ, then the weight component $m_\gamma(\mu)$ of m_γ in V_μ^λ is also in V_J^λ (see (6.3)). Thus, if μ_0 is a weight of V^λ of maximal depth such that $m_\gamma(\mu_0) \neq 0$, we have

$$\| x_K x_H x_U \gamma . v_0 \| = \| x_H x_U . m_\gamma \| \geq \| x_H . m_\gamma(\mu_0) \| \geq x_H^{\mu_0}.$$

On the other hand, from (21.20), we have

$$\| x_K x_H x_U \gamma . v_0 \| = \| y_K y_H y_U . v_0 \| = y_H^\lambda.$$

Thus, if

$$\mu_0 = \lambda - \sum_{i=1}^{\ell+1} q_i a_i, \qquad q_i \in \mathbf{Z},\ q_i \geq 0,$$

(see (6.7)), and if we set $\tilde{a} = \sum_{i=1}^{\ell+1} q_i a_i$, we then have

(21.21) $$y_H^\lambda \geq x_H^\lambda x_H^{-\tilde{a}}.$$

Now in § 17, we defined an isomorphism

$$\Phi_0 : \mathbf{W} \to W,$$

and hence we may identify $\mathbf{W}$ with W, by means of Φ_0. We observe that if $\gamma \in \mathscr{I} w \mathscr{I}$, $w \in \mathbf{W}$, we may take

$$\mu_0 = \lambda - \sum_{i=1}^{\ell+1} q_i a_i = w . \lambda,$$

thanks to Lemma (11.2), Corollary (14.18) and the definition of Φ_0.

On the other hand, interchanging the roles of x and y, we also obtain

(21.21′) $$x_H^\lambda \geq y_H^\lambda y_H^{-\tilde{b}}, \text{ where } w^{-1}(\lambda) = \lambda - \tilde{b}, \text{ with } \tilde{b} = \sum_{i=1}^{\ell+1} p_i a_i, \quad p_i \in \mathbf{Z},\ p_i \geq 0.$$

Combining (21.21) and (21.21′), we find

(21.22) $$1 \leq x_H^{\tilde{a}} y_H^{\tilde{b}}, \quad \tilde{a} = \sum_{i=1}^{\ell+1} q_i a_i, \quad \tilde{b} = \sum_{i=1}^{\ell+1} p_i a_i,$$
$$p_i, q_i \in \mathbf{Z}, \quad p_i, q_i \geq 0, \quad i = 1, \ldots, \ell+1,$$
$$w(\lambda) = \lambda - \tilde{a}, \quad w^{-1}(\lambda) = \lambda - \tilde{b}.$$

Now assume λ is a multiple of ρ; i.e., assume that for some positive integer n, we have

$$\lambda(h_i)=n, \quad i=1, \ldots, \ell+1.$$

To prove (21.19) for this choice of λ, we first note that from Lemma (21.3), Lemma (21.8) and (21.22), we can conclude that there exists $\varepsilon'>0$ so that if (21.17') holds for ε' in place of ε, and if $x\gamma=y$, then either γ is in P_{i_0} or γ is in P_{j_0}. Fixing such an ε', we may in particular, assume γ lies in one of finitely many double cosets $\mathscr{I}w\mathscr{I}$, $w\in\mathbf{W}$. We then obtain from Lemma (21.3) and from (21.22) that there exists $\varepsilon>0$ so that if (21.17') holds, and if $x\gamma=y$, then γ is in both P_{i_0} and P_{j_0}; i.e., we obtain (21.19), and hence Theorem (21.16), for $\lambda=n\rho$. The theorem for general λ then follows in a manner parallel to the proof of Theorem (20.14). ■

For $r>0$, we set $\mathfrak{S}_\sigma^{\mathbf{R}}(r)=\mathfrak{S}_\sigma^{\mathbf{R}}e^{r\mathrm{D}}\cap\hat{G}_k^\lambda$ and $\hat{\Gamma}_r=e^{-r\mathrm{D}}\hat{\Gamma}_0e^{r\mathrm{D}}\subset\hat{G}_k^\lambda$. Then

$$\hat{G}_k^\lambda=\mathfrak{S}_{\sigma_0}^{\mathbf{R}}(r)\hat{\Gamma}_r \quad (r>0),$$

and we also have

Corollary 1. — For r sufficiently large and $\gamma\in\hat{\Gamma}_r$ *such that*

$$\mathfrak{S}_\sigma^{\mathbf{R}}(r)\gamma\cap\mathfrak{S}_\sigma^{\mathbf{R}}(r)\neq\emptyset,$$

we have $\gamma\in P_i$ *for some* $i=1, \ldots, \ell+1$.

Proof. — This follows from Theorem (21.16), from Lemma (21.10) and from the fact that $e^{-r\mathrm{D}}P_ie^{r\mathrm{D}}=P_i$.

Corollary 2. — For r sufficiently large, $\hat{\Gamma}_r$ *acts on* $\hat{K}/\hat{G}_k^\lambda$ *with finite isotropy groups.*

Proof. — This follows from Corollary 1, and from the fact that $\hat{\Gamma}_r\cap P_i$ intersected with a conjugate of $\hat{K}$ is finite. Indeed, $\hat{G}=\hat{K}\mathscr{I}=\hat{K}P_i$ (see Theorem (16.8)), and hence our last assertion is equivalent to the assertion that the intersection of $\hat{K}\cap P_i$ with a conjugate $p(\hat{\Gamma}_r\cap P_i)p^{-1}$, $p\in P_i$, is finite. But if $M_i\subset P_i$ is the finite-dimensional subgroup generated by the $\chi_{\pm a_j}(s)$, $j\neq i$, $s\in k$, then $\hat{K}\cap P_i\subset M_i$ is compact, and $p(\hat{\Gamma}_r\cap P_i)p^{-1}\cap M_i$ is discrete. ■

Corollary 3. — For r sufficiently large, $\hat{\Gamma}_r$ *is not conjugate to* $\hat{\Gamma}_0$ *by an element of* $\hat{G}_k^\lambda$.

Proof. — The group $\hat{\Gamma}_0\cap\hat{K}$ is infinite, as it contains all $w_a(1)$, $a\in\Delta(\widetilde{A})$. Hence Corollary 3 follows from Corollary 2. ■

Appendix 1

We note that D, the $\ell+1$st degree derivation of $\mathfrak{g}(\widetilde{A})$, maps $\mathfrak{g}_{\mathbf{Z}}(\widetilde{A})$ into itself. Hence, as in § 6, we may adjoin D to $\mathfrak{g}_{\mathbf{Z}}(A)$, to obtain the extended integral algebra $\mathfrak{g}^e_{\mathbf{Z}}(\widetilde{A})=\mathfrak{g}^e_{\mathbf{Z}}=\mathfrak{g}_{\mathbf{Z}}(\widetilde{A})\oplus\mathbf{Z}D$. We then set $\mathfrak{h}^e_{\mathbf{Z}}(\widetilde{A})=\mathfrak{h}^e_{\mathbf{Z}}=\mathfrak{h}_{\mathbf{Z}}(\widetilde{A})\oplus\mathbf{Z}D$, and for a commutative ring (with unit) R, we let

$$\mathfrak{h}^e_R(\widetilde{A})=\mathfrak{h}^e_R=R\otimes_{\mathbf{Z}}\mathfrak{h}^e_{\mathbf{Z}},$$

$$\mathfrak{g}^e_R(\widetilde{A})=\mathfrak{g}^e_R=R\otimes_{\mathbf{Z}}\mathfrak{g}^e_{\mathbf{Z}}.$$

For $a\in\Delta_W(\widetilde{A})$, $s\in R$, we let $\mathscr{Y}_a(s)$ denote the automorphism of $\mathfrak{g}^e_R$ defined by

$$\mathscr{Y}_a(s)=\sum_{j\geq 0}s^j\,\mathrm{ad}(\xi_a^j/j!).$$

For $s\in R^*$, the group of units of R, we set

$$\widetilde{w}_a(s)=\mathscr{Y}_a(s)\mathscr{Y}_{-a}(-s^{-1})\mathscr{Y}_a(s),$$

$$\widetilde{h}_a(s)=\widetilde{w}_a(s)\widetilde{w}_a(1)^{-1}.$$

We then have:

Lemma **(A.1)**. — *For* $h\in\mathfrak{h}^e_R$, *and* $s\in R^*$, $a\in\Delta_W(\widetilde{A})$,

$$\widetilde{w}_a(s)(h)=h-a(h)h_a.$$

Proof. — The following equalities are obtained from a direct computation:

$$\mathscr{Y}_a(s)(h)=h-sa(h)\xi_a,$$

$$\mathscr{Y}_{-a}(-s^{-1})(h-sa(h)\xi_a)=h-a(h)h_a-sa(h)\xi_a,$$

$$\mathscr{Y}_a(s)(h-a(h)h_a-sa(h)\xi_a)=h-a(h)h_a.$$

The lemma follows. ■

We now proceed to prove Lemma (11.2). First we take s to be an indeterminate over $\mathbf{Z}$, and consider

$$V^\lambda_{\mathbf{Z}[s,s^{-1}]}=\mathbf{Z}[s,s^{-1}]\otimes_{\mathbf{Z}}V^\lambda_{\mathbf{Z}},$$

and $\mathfrak{g}_{\mathbf{Z}[s,s^{-1}]}(\widetilde{A})=\mathbf{Z}[s,s^{-1}]\otimes_{\mathbf{Z}}\mathfrak{g}_{\mathbf{Z}}(\widetilde{A})$. We fix an embedding of $\mathbf{Z}[s,s^{-1}]$ in $\mathbf{C}$, and thus obtain embeddings of $V^\lambda_{\mathbf{Z}[s,s^{-1}]}$, $\mathfrak{g}_{\mathbf{Z}[s,s^{-1}]}(A)$ in $V^\lambda_{\mathbf{C}}$, $\mathfrak{g}_{\mathbf{C}}(\widetilde{A})$, respectively. We may define $w_a(s)$, $h_a(s)\in\mathrm{Aut}(V^\lambda_{\mathbf{C}})$, as automorphisms of $V^\lambda_{\mathbf{C}}$ which leave $V^\lambda_{\mathbf{Z}[s,s^{-1}]}$ invariant. Thus, if we can prove Lemma (11.2) over $\mathbf{C}$, we may then specialize s and so obtain Lemma (11.2) over an arbitrary field k.

So now take $k=\mathbf{C}$. Our argument follows that in Steinberg [21], Lemma 19, page 27. For $s\in\mathbf{C}$, $v\in V_\mu^\lambda$, we set $v(s)=w_a(s).v$, and we note that we may write $v(s)$ as

$$\textbf{(A.2)}\qquad v(s)=\sum_{j\in\mathbf{Z}} s^j v_j \quad \text{(finite sum)}, \qquad v_j\in V^\lambda_{\mu+ja}.$$

On the other hand, noting that Lemma (A.1) implies $\widetilde{w}_a(s)(h)=\widetilde{w}_a(s)^{-1}(h)$, for $h\in\mathfrak{h}^e(\widetilde{A})$, we have

$$\pi^\lambda(h).(w_a(s).v)=r_a(\mu)(h)w_a(s).v,$$

where $$r_a(\mu)=\mu-\mu(h_a)a.$$

Hence, in (A.2), we have $v_j=0$ unless $r_a(\mu)=\mu+ja$; i.e., unless $j=-\mu(h_a)$. Thus, taking $v'=v_j$, for $j=-\mu(h_a)$, we obtain Lemma (11.2) (i).

To prove Lemma (11.2) (ii), we first note that, by a direct computation from the definition of $w_a(s)$ (see (7.23)), we have

$$w_a(s)^{-1}=w_a(-s),$$

and thus for, $v\in V_\mu^\lambda$,

$$\textbf{(A.3)}\qquad h_a(s).v=w_a(-s)^{-1}w_a(-1).v.$$

Now, by Lemma (11.2) (i),

$$w_a(-1).v=(-1)^{-\mu(h_a)}v'$$
$$w_a(-s).v=(-s)^{-\mu(h_a)}v',$$

and it then follows from (A.3) that

$$h_a(s).v=s^{\mu(h_a)}.v,$$

so we obtain (ii) of Lemma (11.2). ■

We now wish to prove an analogue of Lemma (11.2) for the adjoint representation:

Lemma **(A.4)**. — *Let k be any field, and let $a, b\in\Delta_W(\widetilde{A})$; then in $\mathfrak{g}_k(\widetilde{A})$ we have*

$$\widetilde{w}_b(s)(\xi_a)=\gamma s^{-a(h_b)}\xi_{r_b(a)},$$

where $s\in k$, $\gamma=\gamma(b,a)=\pm 1$ is independent of s and k, with $\gamma(b,a)=\gamma(b,-a)$, and where $r_b(a)=a-a(h_b)b$.

Proof. — For the most part, the proof of Lemma (A.4) is parallel to that of Lemma (11.2), so we only give a sketch of the proof. Thus, just as in the proof of Lemma (11.2), we may take $k=\mathbf{C}$ and then show that

$$\widetilde{w}_b(s)(\xi_a)=\gamma s^{-a(h_b)}\xi_{r_b(a)},$$

for some $\gamma\in\mathbf{C}$. But taking $s=1$, and noting that $\widetilde{w}_b(1)(\xi_a)$ is in $\mathfrak{g}_{\mathbf{Z}}(\widetilde{A})$, and is in fact a primitive vector, since $\widetilde{w}_b(1)$ is an automorphism of $\mathfrak{g}_{\mathbf{Z}}(\widetilde{A})$, we see that $\gamma=\pm 1$.

Finally, noting that $[\xi_a, \xi_{-a}] = h_a$, and then applying $\widetilde{w}_b(1)$ to both sides of this equality, we obtain (using Lemma (A.1));

$$\gamma(b, a)\gamma(b, -a)h_{r_b(a)} = h_a - b(h_a)h_b = h_{r_b}(a),$$

and so $\gamma(b, a) = \gamma(b, -a)$. The independence assertion is clear, and so we obtain the Lemma. ■

Appendix II

In this appendix, we take k to be the finite field with q elements. We wish to prove

Theorem **(B.1)**. — *Assume $\lambda \in \mathbf{D}$ is normal, and that $\lambda(h_i)$ is divisible by $q-1$, for each $i = 1, \ldots, \ell+1$. Then the homomorphism $\pi_1 : \hat{G}_k^\lambda \to G_{\mathscr{L}_k}^\lambda$ of § 12 is an isomorphism.*

Proof. — By definition of π_1, we know that the kernel of π_1 is the group C generated by the central elements

$$h_\alpha(\sigma_1)h_\alpha(\sigma_2)h_\alpha(\sigma_1\sigma_2)^{-1}, \quad \alpha \in \Delta(A), \ \sigma_1, \sigma_2 \in \mathscr{L}_k^*.$$

By Lemma (10.1), we have the homomorphism

$$\Psi^e : E(G_{\mathscr{L}_k}) \to \hat{G}_k^\lambda,$$

defined by

$$\Psi^e(\chi_\alpha^e(\sigma)) = \chi_\alpha(\sigma), \quad \alpha \in \Delta(A), \ \sigma \in \mathscr{L}_k.$$

We let $C^e \subset E(G_{\mathscr{L}_k})$ be the subgroup generated by the central elements

$$h_\alpha^e(\sigma_1)h_\alpha^e(\sigma_2)h_\alpha^e(\sigma_1\sigma_2)^{-1}, \quad \alpha \in \Delta(A), \ \sigma_1, \sigma_2 \in \mathscr{L}_k^*.$$

If β is a fixed long root in $\Delta(A)$, then by Moore [18], Lemma (8.2), $C = \Psi^e(C^e)$ is generated by the elements

$$b_\beta(\sigma_1\sigma_2) = h_\beta(\sigma_1)h_\beta(\sigma_2)h_\beta(\sigma_1\sigma_2)^{-1}, \quad \sigma_1, \sigma_2 \in \mathscr{L}_k^*.$$

By Theorem (12.24), we have

$$b_\beta(\sigma_1, \sigma_2) = c_T(\sigma_1, \sigma_2)^\omega \mathbf{I},$$

where $\mathbf{I}$ denotes the identity operator of V_k^λ, where $\omega = -2\lambda(h_i')(\beta, \beta)^{-1}$, and where $c_T(\ ,\ )$ denotes the tame symbol (see (12.20)). However, the only thing we have to note here is that $c_T(\sigma_1, \sigma_2) \in k^*$. Now $2h_i'/(\alpha, \alpha)$ is an integral linear combination of $h_1, \ldots, h_{\ell+1}$ (see the discussion preceeding Remark (4.7)). Hence, since $\lambda(h_i)$ is divisible by $q-1$, for each $i = 1, \ldots, \ell+1$, we must have

$$c_T(\sigma_1, \sigma_2)^\omega = 1,$$

and π_1 is injective. Since π_1 is in any case surjective, we obtain the Theorem. ■

The locally compact group $G_{\mathscr{L}_k}$ admits a certain irreducible, unitary representation, called the special representation (see [2]), which is an analogue of the Steinberg representation of a finite Chevalley group. On the other hand, when $\lambda \in \mathbf{D}$ satisfies the conditions of Theorem (B.1), we have a well-defined modular representation

$$\pi^{\lambda} = \pi_1^{-1} \circ \pi_2 : \ G_{\mathscr{L}_k} \to \hat{G}_k^{\lambda} \subset \mathrm{Aut}\, V_k^{\lambda},$$

where $\pi_2 : G_{\mathscr{L}_k} \to \hat{G}^{\lambda}_{\mathscr{L}_k}$ is defined in § 12. In analogy with known results concerning the Steinberg representation of a finite Chevalley group, we propose:

Conjecture. — Let $\rho \in \mathbf{D}$ satisfy $\rho(h_i) = 1$, for $i = 1, \ldots, \ell + 1$. The special representation of $G_{\mathscr{L}_k}$ admits an integral subrepresentation such that, when we tensor this integral representation with k, we obtain the modular representation $\pi^{(q-1)\rho}$ (1).

Appendix III

In this appendix we wish to discuss how one can extend some of the results of this paper to the non-split case. The idea is to use the universal properties of our central extensions in order to lift Galois automorphisms, and then use descent.

In this section we take k to be a field of characteristic zero. We let $\mathfrak{g}$ denote a split, simple Lie algebra over k, and we let $\tilde{\mathfrak{g}}^c = \tilde{\mathfrak{g}}_k$ denote the infinite-dimensional k-Lie algebra

$$\tilde{\mathfrak{g}}_k^c = \mathscr{L}_k \otimes_k \mathfrak{g}.$$

We define

$$\tau_0 : \ \mathscr{L}_k \times \mathscr{L}_k \to k$$

by

(C1) $$\tau_0(\sigma_1, \sigma_2) = \mathrm{residue}(\sigma_1 d\sigma_2), \qquad \sigma_1, \sigma_2 \in \mathscr{L}_k.$$

Then the τ_0 of § 2 is just the restriction to $k[t, t^{-1}] \times k[t, t^{-1}]$ of the τ_0 defined here. We then define

$$\tau : \ \tilde{\mathfrak{g}}_k^c \times \tilde{\mathfrak{g}}_k^c \to k$$

exactly as in § 2; i.e., we set

(C2) $$\tau(\sigma_1 \otimes x, \sigma_2 \otimes y) = -\tau_0(\sigma_1, \sigma_2)(x, y), \qquad \sigma_1, \sigma_2 \in \mathscr{L}_k, \ x, y \in \tilde{\mathfrak{g}}_k^c.$$

Then $\tau \in Z^2(\tilde{\mathfrak{g}}^c, k)$ (where we regard k as a trivial $\tilde{\mathfrak{g}}^c$ module), and corresponding to τ we have the central extension

(C3) $$0 \to k \to \hat{\mathfrak{g}}_k^c \underset{\smile}{\overset{\tilde{\omega}_k}{\to}} \tilde{\mathfrak{g}}_k^c \to 0,$$

(1) Recently, J. Annon has proved a modified, topological version of this conjecture. Roughly speaking, Annon introduces a topology on $V^{(q-1)\rho}$, and then proves that a k-form of the special representation is isomorphic to a dense subrepresentation of $\pi^{(q-1)\rho}$ (the density being with respect to Annon's topology).

where we take

(C4) $\qquad \hat{\mathfrak{g}}_k^c = \tilde{\mathfrak{g}}_k^c \oplus k$ (as a vector space over k),

with multiplication defined by

(C5) $\qquad [(\xi, s), (\xi', s')] = ([\xi, \xi'], \tau(\xi, \xi')), \qquad \xi, \xi' \in \tilde{\mathfrak{g}}_k^c, \ s, s' \in k;$

and where $\tilde{\omega}_k$ is the projection of $\hat{\mathfrak{g}}_k^c = \tilde{\mathfrak{g}}_k^c \oplus k$ onto the first factor.

We note that $\hat{\mathfrak{g}}_k^c$ is perfect. Indeed, it is obvious that $\tilde{\mathfrak{g}}_k^c$ is perfect, since $\mathfrak{g}$ is perfect. But then it suffices to show the commutator subalgebra of $\hat{\mathfrak{g}}_k^c$ contains the direct summand k of (C4). For this we need only check that $\tau(\xi, \xi')$ is not zero for some pair ξ, ξ'. But if $x \in \mathfrak{g}$ is any element such that $(x, x) \neq 0$, then

$$\tau(t \otimes x, t^{-1} \otimes x) \neq 0,$$

by (C2). Thus we have proved:

Lemma **(C6)**. — *The Lie algebra* $\hat{\mathfrak{g}}_k^c$ *is perfect.*

We may now recall the comments of Remarks (5.11) to the present context. Thus we let $\mathfrak{g}_{\mathcal{O}} \subset \tilde{\mathfrak{g}}_k^c$ denote the k-subalgebra

$$\mathfrak{g}_{\mathcal{O}} = \mathcal{O} \otimes_k \mathfrak{g},$$

where $\mathcal{O} = \mathcal{O}_k \subset \mathcal{L}_k$ is the subring of formal power series (see Remarks (5.11)). By a *congruence subalgebra* of level n in $\tilde{\mathfrak{g}}_k^c$ we mean the subalgebra of all $\xi \in \mathfrak{g}_{\mathcal{O}}$ such that $\xi \equiv 0 \bmod t^n$. We then note

Lemma **(C7)**. — *The central extension* (C3) *is universal in the category of all central extensions of* $\tilde{\mathfrak{g}}_k^c$ *which split over the congruence subalgebra of level* n, *for some* n.

The proof is as in § 2, with the one additional point discussed in Remarks (5.11).

Since $\tilde{\mathfrak{g}}_k^c = \mathcal{L}_k \otimes_k \mathfrak{g}$, we may topologize $\tilde{\mathfrak{g}}_k^c$ by means of the t-adic topology of $\mathcal{L}_k$. We call $\hat{\mathfrak{g}}_k^c$ the *universal topological covering* of $\tilde{\mathfrak{g}}_k^c$.

Lemma **(C8)**. — *For every automorphism*

$$\tilde{\varkappa} : \tilde{\mathfrak{g}}_k^c \to \tilde{\mathfrak{g}}_k^c,$$

with $\tilde{\varkappa}$ *and* $\tilde{\varkappa}^{-1}$ *continuous, there is a unique automorphism*

$$\hat{\varkappa} : \hat{\mathfrak{g}}_k^c \to \hat{\mathfrak{g}}_k^c,$$

such that the diagram

$$\begin{array}{ccc} \hat{\mathfrak{g}}_k^c & \xrightarrow{\hat{\varkappa}} & \hat{\mathfrak{g}}_k^c \\ \downarrow{\scriptstyle \tilde{\omega}_k} & & \downarrow{\scriptstyle \tilde{\omega}_k} \\ \tilde{\mathfrak{g}}_k^c & \xrightarrow{\tilde{\varkappa}} & \tilde{\mathfrak{g}}_k^c \end{array}$$

is commutative.

Proof. — The existence of $\hat{\varkappa}$ follows from the universality property of Lemma (C7). The uniqueness of $\hat{\varkappa}$ follows from Lemma (1.5) and Lemma (C6). ■

Now let $k' \supset k$ be a Galois extension. Then we have $\mathscr{L}_{k'} \supset \mathscr{L}_k$ and $\mathscr{L}_{k'}$ is a Galois extension of $\mathscr{L}_k$. Moreover, we have a natural inclusion

(C9) $$\mathrm{Gal}(k'/k) \hookrightarrow \mathrm{Gal}(\mathscr{L}_{k'}/\mathscr{L}_k)$$

of the Galois group of k' over k into the Galois group of $\mathscr{L}_{k'}$ over $\mathscr{L}_k$, and this inclusion is an isomorphism, as one sees immediately from Galois theory. We now let $\mathfrak{l}$ denote an absolutely simple Lie algebra which is defined over $\mathscr{L}_k$ and which splits over $\mathscr{L}_{k'}$. We let $\tilde{\mathfrak{g}}^c_{k'}$ denote the Lie algebra

(C10) $$\tilde{\mathfrak{g}}^c_{k'} = \mathscr{L}_{k'} \otimes_{\mathscr{L}_k} \mathfrak{l}.$$

By assumption, we have an isomorphism of Lie algebras over $\mathscr{L}_{k'}$ (and hence over k')

(C11) $$\tilde{\mathfrak{g}}^c_{k'} \cong \mathscr{L}_{k'} \otimes_k \mathfrak{g} \qquad (\cong \mathscr{L}_{k'} \otimes_{k'} (k' \otimes_k \mathfrak{g}))$$

where $\mathfrak{g}$ is a simple Lie algebra which is defined and split over k. We fix the isomorphism in (C11).

We now wish to show that we can lift Galois automorphisms of $\tilde{\mathfrak{g}}^c_{k'} = \mathscr{L}_{k'} \otimes_{\mathscr{L}_k} \mathfrak{l}$, to the corresponding central extension $\hat{\mathfrak{g}}^c_{k'}$ of (C3). Thus, let $\varkappa : \mathscr{L}_{k'} \to \mathscr{L}_{k'}$ be an element of $\mathrm{Gal}(\mathscr{L}_{k'}/\mathscr{L}_k)$, and let $\tilde{\varkappa} : \tilde{\mathfrak{g}}^c_{k'} \to \tilde{\mathfrak{g}}^c_{k'}$ be defined by

$$\tilde{\varkappa}(\sigma \otimes \xi) = \varkappa(\sigma) \otimes \xi, \qquad \sigma \in \mathscr{L}_{k'}, \ \xi \in \mathfrak{l},$$

(see (C10)). Then we have

Lemma **(C12)**. — *For every* $\varkappa \in \mathrm{Gal}(\mathscr{L}_{k'}/\mathscr{L}_k)$, *there exists a unique* $\hat{\varkappa} : \hat{\mathfrak{g}}^c_{k'} \to \hat{\mathfrak{g}}^c_{k'}$ *such that* $\hat{\varkappa}$ *preserves brackets and sums, such that*

(C13) $$\hat{\varkappa}(s\xi) = \varkappa(s)\hat{\varkappa}(\xi), \qquad s \in k', \ \xi \in \hat{\mathfrak{g}}^c_{k'},$$

and such that the diagram

$$\begin{array}{ccc} \hat{\mathfrak{g}}^c_{k'} & \xrightarrow{\hat{\varkappa}} & \hat{\mathfrak{g}}^c_{k'} \\ \downarrow{\scriptstyle \tilde{\omega}_{k'}} & & \downarrow{\scriptstyle \tilde{\omega}_{k'}} \\ \tilde{\mathfrak{g}}^c_{k'} & \xrightarrow{\tilde{\varkappa}} & \tilde{\mathfrak{g}}^c_{k'} \end{array}$$

is commutative.

Proof. — Assume $\tilde{\varkappa}$ and $\tilde{\varkappa}^{-1}$ are continuous (relative to the isomorphism (C11) and to the t-adic topology introduced just before Lemma (C8)). We note that $\tilde{\varkappa}$ preserves brackets and that

(C14) $$\tilde{\varkappa}(s\eta) = \varkappa(s)\tilde{\varkappa}(\eta), \qquad s \in k', \ \eta \in \tilde{\mathfrak{g}}^c_{k'}.$$

We wish to define a second such mapping

$$\varkappa^{\#}: \tilde{\mathfrak{g}}^c_{k'} \to \tilde{\mathfrak{g}}^c_{k'}.$$

To do this we use the isomorphism (C11):

(C15) $$\varkappa^{\#}(\sigma \otimes x) = \varkappa(\sigma) \otimes x, \qquad \sigma \in \mathscr{L}_{k'},\ x \in \mathfrak{g}.$$

Then $\varkappa^{\#}$ preserves brackets and sums, and satisfies

(C16) $$\varkappa^{\#}(s\eta) = \varkappa(s)\varkappa^{\#}(\eta), \qquad s \in k,\ \eta \in \tilde{\mathfrak{g}}^c_{k'}.$$

But then $\tilde{\varkappa} \circ (\varkappa^{\#})^{-1}$ still preserves brackets and sums, and is k'-linear by (C14) and by (C16); i.e., $\tilde{\varkappa} \circ (\varkappa^{\#})^{-1}$ is a Lie algebra endomorphism (over k') of $\tilde{\mathfrak{g}}^c_{k'}$. In fact, $\tilde{\varkappa}$ and $\varkappa^{\#}$ are each clearly bijective, so $\tilde{\varkappa} \circ (\varkappa^{\#})^{-1}$ is a Lie algebra automorphism of $\tilde{\mathfrak{g}}^c_{k'}$ (as a Lie algebra over k'). But $\varkappa^{\#}$ and $(\varkappa^{\#})^{-1}$ are clearly continuous, and hence, so are $(\tilde{\varkappa} \circ (\varkappa^{\#})^{-1})^{\pm 1}$, by our assumption that $\tilde{\varkappa}$ and $\tilde{\varkappa}^{-1}$ are continuous. Hence, by Lemma (C8), there is an automorphism $\hat{\varkappa}^{*}; \hat{\mathfrak{g}}^c_{k'} \to \hat{\mathfrak{g}}^c_{k'}$ such that the diagram

$$\begin{array}{ccc} \hat{\mathfrak{g}}^c_{k'} & \xrightarrow{\hat{\varkappa}^{*}} & \hat{\mathfrak{g}}^c_{k'} \\ \Big\downarrow{\scriptstyle \tilde{\omega}_{k'}} & & \Big\downarrow{\scriptstyle \tilde{\omega}_{k'}} \\ \tilde{\mathfrak{g}}^c_{k'} & \xrightarrow{\tilde{\varkappa} \circ (\varkappa^{\#})^{-1}} & \tilde{\mathfrak{g}}^c_{k'} \end{array}$$

is commutative. On the other hand, if we let $\hat{\varkappa}^{\#}: \hat{\mathfrak{g}}^c_{k'} \to \hat{\mathfrak{g}}^c_{k'}$ be defined, relative to the decomposition (C4)

$$\hat{\mathfrak{g}}^c_{k'} = \tilde{\mathfrak{g}}^c_{k'} \oplus k',$$

by

$$\hat{\varkappa}^{\#}((\xi, s)) = (\varkappa^{\#}(\xi), \varkappa(s)), \qquad \xi \in \tilde{\mathfrak{g}}^c_{k'},\ s \in k',$$

then $\hat{\varkappa}^{\#}$ preserves brackets (as one checks from (C5)) and sums (as one checks directly), and thanks to (C16), one has

$$\hat{\varkappa}^{\#}(s\eta) = \varkappa(s)\varkappa^{\#}(\eta), \qquad s \in k',\ \eta \in \hat{\mathfrak{g}}^c_{k'}.$$

Moreover, from the definition of $\hat{\varkappa}^{\#}$, we have

$$\tilde{\omega}_{k'} \circ \hat{\varkappa}^{\#} = \varkappa^{\#} \circ \tilde{\omega}_{k'},$$

and hence, if we set

$$\hat{\varkappa} = \hat{\varkappa}^{*} \circ \hat{\varkappa}^{\#},$$

we obtain the Lemma.

The only remaining point then, is to show that $\tilde{\varkappa}$ and $\tilde{\varkappa}^{-1}$ are continuous. To see this, we consider the tensor product decomposition (C11) and we fix a basis $X_1, \ldots, X_n$ ($n=\dim \mathfrak{g}$) of $\mathfrak{g}$. We then have

$$\tilde{\varkappa}(X_i)=\sum_{j=1}^{n} \tilde{\varkappa}_{ji} X_j,$$

where $\tilde{\varkappa}_{ji} \in \mathscr{L}_{k'}$, $i,j=1, \ldots, n$. But then if

$$X=\sum_{i=1}^{n} \sigma_i X_i, \quad \sigma_i \in \mathscr{L}_{k'},$$

is a general element of $\tilde{\mathfrak{g}}_{k'}^c \cong \mathscr{L}_{k'} \otimes_k \mathfrak{g}$, we have from the definition of $\tilde{\varkappa}$ (preceeding Lemma (C12))

$$\tilde{\varkappa}(X)=\sum_{i=1}^{n} \sigma_i' X_i,$$

where

$$\sigma_i'=\sum_{j=1}^{n} \tilde{\varkappa}_{ij} \varkappa(\sigma_j), \quad i=1, \ldots, n.$$

Since σ_j and $\varkappa(\sigma_j)$ have the same t-adic absolute value ($\varkappa$ only acts on the coefficients of the Laurent series σ_j), we obtain the desired continuity of $\tilde{\varkappa}$. The same argument shows that $\tilde{\varkappa}^{-1}$ is continuous, and hence we obtain the lemma. ■

We let $\mathscr{G}=\operatorname{Gal}(\mathscr{L}_{k'}/\mathscr{L}_k) \cong \operatorname{Gal}(k'/k)$, and note that if we consider the $\mathscr{G}$-module k', then we have

(C17) $\quad H^1(\mathscr{G}, k')=0.$

This is a simple consequence of the facts that $\mathscr{G}$ is finite and k' has characteristic zero.

Now, consider the central extension

$$0 \rightarrow k' \rightarrow \hat{\mathfrak{g}}_{k'}^c \xrightarrow{\tilde{\omega}_{k'}} \tilde{\mathfrak{g}}_{k'}^c \rightarrow 0.$$

Then, by Lemma (C12), for every $\varkappa \in \mathscr{G}$, we have maps $\hat{\varkappa}$, $\tilde{\varkappa}$ of $\hat{\mathfrak{g}}_{k'}^c$, $\tilde{\mathfrak{g}}_{k'}^c$, respectively, which preserve sums and brackets, satisfy (C13), (C14), respectively, and satisfy $\tilde{\omega}_{k'} \circ \hat{\varkappa}=\tilde{\varkappa} \circ \tilde{\omega}_{k'}$. In particular, note that for $k' \subset \hat{\mathfrak{g}}_{k'}^c$, we have from (C13):

(C18) $\quad \hat{\varkappa}(s)=\varkappa(s)\hat{\varkappa}(1).$

Since $\hat{\varkappa}$ is unique (given $\varkappa \in \mathscr{G}$), we have

$$\widehat{\varkappa_1 \varkappa_2}=\hat{\varkappa}_1 \hat{\varkappa}_2, \quad \varkappa_1, \varkappa_2 \in \mathscr{G}.$$

Thus, we obtain the following from (C18):

$$\widehat{\varkappa_1 \varkappa_2}(1)=\varkappa_1(\hat{\varkappa}_2(1))\hat{\varkappa}_1(1);$$

i.e., $\varkappa \mapsto \hat{\varkappa}(1)$ is an element of $Z^1(\mathscr{G}, k'^*)$, the one-cocycles of $\mathscr{G}$ with coefficients in k'^*. By Hilbert's theorem 90, there exists $s_0 \in k'^*$, such that, for all $\varkappa \in \mathscr{G}$,

(C19) $\quad \hat{\varkappa}(1)=\varkappa(s_0)s_0^{-1}.$

But then

$$\begin{aligned}\hat{\varkappa}(s_0^{-1}) &= \varkappa(s_0^{-1})\hat{\varkappa}(1), \quad \text{by (C18)}\\ &= \varkappa(s_0)^{-1}(\varkappa(s_0)s_0^{-1}), \quad \text{by (C19)}\\ &= s_0^{-1},\end{aligned}$$

and hence

$$\begin{aligned}\hat{\varkappa}(ss_0^{-1}) &= \varkappa(s)\hat{\varkappa}(s_0^{-1}), \quad \text{by (C13)}\\ &= \varkappa(s)s_0^{-1},\end{aligned}$$

by the above computation; i.e., we have

(C20) $$\hat{\varkappa}(ss_0^{-1}) = \varkappa(s)s_0^{-1}, \qquad s \in k'.$$

We let $\hat{\mathfrak{l}}$ denote the k-subalgebra of $\hat{\mathfrak{g}}_{k'}^c$

(C21) $$\hat{\mathfrak{l}} = \{\xi \in \hat{\mathfrak{g}}_{k'}^c \mid \hat{\varkappa}(\xi) = \xi, \quad \text{for all} \quad \varkappa \in \mathscr{G}\}.$$

We also have, of course

$$\mathfrak{l} = \{\xi \in \tilde{\mathfrak{g}}_{k'}^c \mid \tilde{\varkappa}(\xi) = \xi, \quad \text{for all} \quad \varkappa \in \mathscr{G}\}.$$

Hence $\tilde{\omega}_{k'}$ induces a homomorphism of k-Lie algebras

$$\pi : \hat{\mathfrak{l}} \to \mathfrak{l}.$$

We now wish to show that π is surjective, and has kernel isomorphic to k. Indeed, if $\xi \in \mathfrak{l}$, there exists $\xi' \in \hat{\mathfrak{g}}_{k'}^c$ such that

$$\tilde{\omega}_{k'}(\xi') = \xi.$$

But then, since ξ is $\mathscr{G}$-invariant,

$$\tilde{\omega}_{k'}(\hat{\varkappa}(\xi')) = \tilde{\omega}_{k'}(\xi'), \qquad \text{for all} \quad \varkappa \in \mathscr{G}.$$

Utilizing the decomposition $\hat{\mathfrak{g}}_{k'}^c = \tilde{\mathfrak{g}}_{k'}^c \oplus k'$, of (C4), we may take $\xi' = (\xi, 0)$, $\xi \in \tilde{\mathfrak{g}}_{k'}^c$, and we see that for $\varkappa \in \mathscr{G}$, there exists $s(\varkappa) \in k'$ such that

$$\hat{\varkappa}(\xi') = (\xi, s(\varkappa)s_0^{-1}).$$

But then for $\varkappa_1, \varkappa_2 \in \mathscr{G}$

$$\widehat{\varkappa_1\varkappa_2}(\xi') = (\xi, s(\varkappa_1\varkappa_2)s_0^{-1}),$$

and on the other hand

$$\begin{aligned}\widehat{\varkappa_1\varkappa_2}(\xi') &= \hat{\varkappa}_1\hat{\varkappa}_2(\xi')\\ &= \hat{\varkappa}_1(\xi, s(\varkappa_2)s_0^{-1}) = (\xi, s(\varkappa_1)s_0^{-1} + \hat{\varkappa}_1(s(\varkappa_2)s_0^{-1}))\\ &= (\xi, (s(\varkappa_1) + \varkappa_1(s(\varkappa_2)))s_0^{-1}), \quad \text{by (C20)}.\end{aligned}$$

Thus, $\varkappa \mapsto s(\varkappa)$ satisfies the cocycle identity

$$s(\varkappa_1\varkappa_2) = s(\varkappa_1) + \varkappa_1(s(\varkappa_2)), \qquad \varkappa_1, \varkappa_2 \in \mathscr{G},$$

and hence by (C17), there exists $s_1 \in k'$ such that

(C22) $$s(\varkappa) = s_1 - \varkappa(s_1), \qquad \varkappa \in \mathscr{G}.$$

But then for $\varkappa \in \mathscr{G}$,

$$\begin{aligned}\varkappa((\xi, s_1 s_0^{-1})) &= (\xi, s(\varkappa)s_0^{-1} + \hat{\varkappa}(s_1 s_0^{-1})) \\ &= (\xi, (s_1 - \varkappa(s_1))s_0^{-1} + \hat{\varkappa}(s_1)s_0^{-1}), \quad \text{by (C20) and (C22)} \\ &= (\xi, s_1 s_0^{-1}),\end{aligned}$$

and so

$$(\xi, s_1 s_0^{-1}) \in \hat{\mathfrak{l}}.$$

Moreover we have

$$\pi((\xi, s_1 s_0^{-1})) = \tilde{\omega}_{k'}((\xi, s_1 s_0^{-1})) = \xi,$$

and thus we have shown π is surjective. Moreover, thanks to (C20), we see that kernel $\pi = \{(0, s s_0^{-1}) \mid s \in k\}$ is isomorphic to k. Hence we obtain the central extension

(C23) $$0 \to k \to \hat{\mathfrak{l}} \xrightarrow{\pi} \mathfrak{l} \to 0.$$

We summarize what we have just proved in:

Theorem **(C24)**. — *Let* $\mathfrak{l}$ *denote an absolutely simple Lie algebra which is defined over* $\mathscr{L}_k$ *and splits over the unramified Galois extension* $\mathscr{L}_{k'}$, *with* $\mathscr{L}_{k'} \otimes_{\mathscr{L}_k} \mathfrak{l} \cong \tilde{\mathfrak{g}}^c_{k'}$. *We let* $\hat{\mathfrak{l}} \subset \hat{\mathfrak{g}}^c_{k'}$ *be the* k*-subalgebra defined in* (C21), *and we let* $\pi : \hat{\mathfrak{l}} \to \mathfrak{l}$ *denote the restriction of* $\tilde{\omega}_{k'} : \hat{\mathfrak{g}}^c_{k'} \to \tilde{\mathfrak{g}}^c_{k'}$. *Then* π *is surjective, and* kernel $\pi \cong k$. *We thus obtain the central extension* (C23) *of* $\mathfrak{l}$.

Remark. — We can generalize Theorem (C24) to the case when $\mathfrak{l}$ is semi-simple, and we will now sketch the necessary argument. In this more general case, we have $\mathscr{L}_{k'} \otimes_{\mathscr{L}_k} \mathfrak{l} \cong \tilde{\mathfrak{g}}^c_{k'}$, where $\tilde{\mathfrak{g}}^c_{k'}$ is a direct sum of simple components. We then let $\hat{\mathfrak{g}}^c_{k'}$ be the corresponding direct sum of universal topological coverings of these components, and note that, in an obvious sense, $\hat{\mathfrak{g}}^c_{k'}$ is the universal topological covering of $\tilde{\mathfrak{g}}^c_{k'}$. Then, as in Lemma (C12), the action of $\mathscr{G}$ on $\tilde{\mathfrak{g}}^c_{k'}$ lifts to $\hat{\mathfrak{g}}^c_{k'}$ (we let $\hat{\varkappa}$ denote the lift of $\varkappa \in \mathscr{G}$), and we again let $\hat{\mathfrak{l}}$ denote the fixed points in $\hat{\mathfrak{g}}^c_{k'}$, of $\mathscr{G}$. As before, the projection $\tilde{\omega}_{k'} : \hat{\mathfrak{g}}^c_{k'} \to \tilde{\mathfrak{g}}^c_{k'}$ induces a Lie algebra homomorphism $\pi : \hat{\mathfrak{l}} \to \mathfrak{l}$. Since $H^1(\mathscr{G}, V) = 0$ whenever V is a finite-dimensional vector space over k with $\mathscr{G}$-action (recall char. k is now assumed equal to zero), we obtain that π is surjective. Moreover, if $n = \dim_{k'}(\text{kernel } \tilde{\omega}_{k'})$, then $\dim_k(\text{kernel } \pi) = n$. This last assertion follows from

(*) Let V′ be a finite-dimensional vector space over k', and assume we are given a $\mathscr{G}$-action on V′, where each element of $\mathscr{G}$ acts as a semi-linear automorphism; i.e., for $\sigma \in \mathscr{G}$, we have

$$\sigma(\alpha v) = \sigma(\alpha)\sigma(v), \quad \alpha \in k', \ v \in V',$$

where $\sigma(\alpha)$ denotes the Galois action of σ on α. Let $V \subset V'$ be the space of fixed points of this action; then $V' \cong k' \otimes_k V$.

The following proof of $(*)$ was communicated to us by T. Tamagawa: Let $\omega_1, \ldots, \omega_r$ be a basis of k' over k, let $x \in \mathrm{V}'$, and for each $i = 1, \ldots, r$, let

$$y_i = \sum_{\sigma \in \mathscr{G}} \sigma(\omega_i x).$$

Since $(\sigma(\omega_i))_{i,\sigma}$ is a non-singular $r \times r$ matrix, it follows that x is a linear combination of the y_i. Since x was arbitrary, and since the y_i are in V, we obtain $(*)$.

We will continue to use the notation suggested by the last Remark. Thus if $\mathfrak{g}$ is a semi-simple Lie algebra defined and split over k, and if $\widetilde{\mathfrak{g}}^c_{k'} = \mathscr{L}_{k'} \otimes_k \mathfrak{g}$, then we let $\widehat{\mathfrak{g}}^c_{k'}$ be the direct sum of the universal topological coverings of the direct summands of $\widetilde{\mathfrak{g}}^c_{k'}$, and we let $\widetilde{\omega}_{k'} : \widehat{\mathfrak{g}}^c_{k'} \to \widetilde{\mathfrak{g}}^c_{k'}$ denote the projection map.

Our next topic is to investigate the possibility of obtaining an analogue of Lemma (C12), and of Theorem (C24), for the groups $\widehat{\mathrm{G}}^\lambda_k$. In view of Theorem (12.24), one expects such an analogue to be intimately connected with some universal property of the tame symbol. However, here we take a different approach, and continue to utilize the Lie algebra point of view.

We begin with some representation theory. We let $\mathfrak{g}$ denote a semi-simple Lie algebra which is defined and split over k. We fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and let Δ denote the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. We fix an order on Δ and let $\Delta_\pm$ denote the corresponding set of $\pm$ roots. We let $\mathfrak{b} \subset \mathfrak{g}$ denote the Borel subalgebra spanned by $\mathfrak{h}$ and the positive root vectors. We set $\mathfrak{g}' = k' \otimes_k \mathfrak{g}$, $\mathfrak{b}' = k' \otimes_k \mathfrak{b}$ and $\mathfrak{h}' = k' \otimes_k \mathfrak{h}$. We let $\widetilde{\mathfrak{i}} \subset \mathcal{O}_{k'} \otimes_k \mathfrak{g}$ (resp. $\widetilde{\mathfrak{i}}_{\mathrm{U}} \subset \mathcal{O}_{k'} \otimes_k \mathfrak{g}$) denote the k'-subalgebra of elements whose reduction mod t is in $\mathfrak{b}'$ (resp. is in $[\mathfrak{b}', \mathfrak{b}']$). We then set

(C25)
$$\widehat{\mathfrak{h}}' = \widetilde{\omega}_{k'}^{-1}(\mathfrak{h}'),$$
$$\widehat{\mathfrak{i}} = \widetilde{\omega}_{k'}^{-1}(\widetilde{\mathfrak{i}}),$$
$$\widehat{\mathfrak{i}}_{\mathrm{U}} = \widetilde{\omega}_{k'}^{-1}(\widetilde{\mathfrak{i}}_{\mathrm{U}}).$$

We remark that if $\mathfrak{g}'$ is simple, with corresponding classical Cartan matrix A, then $\widehat{\mathfrak{g}}^c_{k'}$ is just the algebra $\mathfrak{g}^c_{k'}(\widetilde{\mathrm{A}})$ of § 5. In general, $\mathfrak{g}'$ decomposes into a direct sum of simple algebras $\mathfrak{g}'_i$, $i \in \mathrm{I}_0$, each with corresponding classical Cartan matrix A_i. We write A for the direct sum matrix of the A_i (so A is the Cartan matrix corresponding to $\mathfrak{g}'$), and we let $\widetilde{\mathrm{A}}$ denote the direct sum matrix of the affine matrices $\widetilde{\mathrm{A}}_i$. We then write $\mathfrak{g}^c_{k'}(\widetilde{\mathrm{A}})$ for the direct sum $\coprod_{i \in \mathrm{I}_0} \mathfrak{g}^c_{k'}(\widetilde{\mathrm{A}}_i)$. Then the algebra $\widehat{\mathfrak{g}}^c_{k'}$ is isomorphic to $\mathfrak{g}^c_{k'}(\widetilde{\mathrm{A}})$, and we let $\rho : \mathfrak{g}^c_{k'}(\widetilde{\mathrm{A}}) \to \widehat{\mathfrak{g}}^c_{k'}$ denote the isomorphism. We set $\mathfrak{h}'(\widetilde{\mathrm{A}}) = \rho^{-1}(\widehat{\mathfrak{h}}')$, $\mathfrak{i} = \rho^{-1}(\widehat{\mathfrak{i}})$, $\mathfrak{i}_{\mathrm{U}} = \rho^{-1}(\widehat{\mathfrak{i}}_{\mathrm{U}})$. We may regard $\mathfrak{g}^c_{k'}(\widetilde{\mathrm{A}})$ as the completion of the Kac-Moody algebra (see § 3) $\mathfrak{g}_{k'}(\widetilde{\mathrm{A}})$. Then $\mathfrak{h}'(\widetilde{\mathrm{A}})$ is contained in $\mathfrak{g}_{k'}(\widetilde{\mathrm{A}})$, and in fact may be taken to be the Cartan subalgebra defined in § 3. We say that $\lambda \in (\widehat{\mathfrak{h}}')^*$ (the dual space of $\widehat{\mathfrak{h}}'$) is *dominant integral*, in case $\lambda \circ \rho \in \mathfrak{h}'(\widetilde{\mathrm{A}})^*$ is dominant integral, and that λ is *normal* if $\lambda \circ \rho \in \mathfrak{h}'(\widetilde{\mathrm{A}})^*$ is normal (see § 15) in the sense that for each $i \in \mathrm{I}$, $\lambda \circ \rho|_{\mathfrak{h}'(\widetilde{\mathrm{A}}_i)}$

is normal. We let $\hat{\mathbf{D}}$ denote the set of dominant integral elements of $(\hat{\mathfrak{h}}')^*$. Then given $\lambda \in \hat{\mathbf{D}}$, we let $M(\lambda)$ denote the one dimensional $\hat{\mathfrak{i}}$-module, with generator v_0, and defined by

$$\xi . v_0 = \begin{cases} \lambda(\xi) v_0, & \xi \in \hat{\mathfrak{h}}', \\ 0, & \xi \in \hat{\mathfrak{i}}_U. \end{cases}$$

We let $V_k^{M(\lambda)} = \mathscr{U}(\hat{\mathfrak{g}}^c_{k'}) \otimes_{\hat{\mathfrak{i}}} M(\lambda)$, and note that left multiplication gives $V_k^{M(\lambda)}$ a $\mathscr{U}(\hat{\mathfrak{g}}^c_{k'})$- (and hence a $\hat{\mathfrak{g}}^c_{k'}$-) module structure. We let v_0 also denote $1 \otimes v_0$ in $V_{k'}^{M(\lambda)}$. We then let $V_{k'}^\lambda$, be the quotient of $V_{k'}^{M(\lambda)}$ by the (unique) maximal $\hat{\mathfrak{g}}^c_{k'}$-submodule, not intersecting $k' v_0$, and we let v_0 also denote the image of $v_0 \in V_{k'}^{M(\lambda)}$ in $V_{k'}^\lambda$. We may regard $V_{k'}^\lambda$, as either a $\mathfrak{g}^c_{k'}$- or $\hat{\mathfrak{g}}^c_{k'}(\tilde{A})$- module, and hence also as a $\mathfrak{g}_{k'}(\tilde{A})$-module. The same holds for $V_{k'}^{M(\lambda)}$.

Now each A_i, $i \in I_0$, is an $\ell_i \times \ell_i$ matrix for some positive integer ℓ_i. We set $\ell' = \sum_{i \in I_0} (\ell_i + 1)$, so $\tilde{A}$ is an $\ell' \times \ell'$ matrix. We let $e_1, \ldots, e_{\ell'}, f_1, \ldots, f_{\ell'}, h_1, \ldots, h_{\ell'}$ be the canonical generators of the algebra $\mathfrak{g}_{k'}(\tilde{A})$, given by the Kac-Moody construction (see § 3). When we regard $V_{k'}^\lambda$ as a $\mathfrak{g}_{k'}(\tilde{A})$-module, as above, then $V_{k'}^\lambda$ is *quasisimple* in the sense that there exists a positive integer n such that $f_i^n . v_0 = 0$ for all $i = 1, \ldots, \ell'$ (see [8], § 6). Indeed, since $\lambda(h_i) \geq 0$, and $\lambda(h_i) \in \mathbf{Z}$, for each $i = 1, \ldots, \ell'$ (i.e., since λ is dominant integral), we may take $n = \max_{i=1,\ldots,\ell'} (\lambda(h_i) + 1)$. Also since λ is dominant integral, it follows from a theorem of Kac (see [8], Corollary (9.8)) that $V_{k'}^\lambda$ is the *unique* quasisimple $\mathfrak{g}_{k'}(\tilde{A})$-module with highest weight λ. We can then use $V_{k'}^\lambda$ to construct Chevalley groups $\hat{G}_{k'}^\lambda$, exactly as in § 7 (of course, we now allow that A correspond to any semisimple algebra).

We now return to the context of Theorem (C24), and of Lemma (C12). Our goal is, for each

$$\varkappa \in \mathscr{G} = \mathrm{Gal}(\mathscr{L}_{k'} / \mathscr{L}_k),$$

to define a bijection $\hat{\varkappa} : V_{k'}^\lambda \to V_{k'}^\lambda$ (actually—see (C34) below—one must replace $V_{k'}^\lambda$ by a direct sum of $V_{k'}^\mu$'s, in general) such that $\hat{\varkappa}^\lambda$ preserves sums, and satisfies

(C26)
$$\hat{\varkappa}^\lambda(sv) = \varkappa(s) \hat{\varkappa}^\lambda(v), \quad s \in k', \ v \in V_{k'}^\lambda,$$
$$\hat{\varkappa}^\lambda(\xi v) = \hat{\varkappa}(\xi) \hat{\varkappa}^\lambda(v), \quad \xi \in \mathfrak{g}^c_{k'}, \ v \in V_{k'}^\lambda.$$

If A is a classical Cartan matrix (and so, corresponds to a simple Lie algebra), we have defined, in § 10, the simply connected Chevalley group $G_{\mathscr{L}_{k'}}(A)$. More generally, if A is the finite direct sum of the classical Cartan matrices A_i, $i \in I_0$, we let $G_{\mathscr{L}_{k'}}(A)$ be the direct product

$$G_{\mathscr{L}_{k'}}(A) = \prod_{i \in I_0} G_{\mathscr{L}_{k'}}(A_i).$$

We now consider $\tilde{\varkappa}(\tilde{\mathfrak{i}})$, the image of $\tilde{\mathfrak{i}}$ under $\tilde{\varkappa}$. From the results of Bruhat and Tits (see [5]) we know that $\tilde{\varkappa}(\tilde{\mathfrak{i}}) = \mathrm{Ad}\, \chi(\tilde{\mathfrak{i}})$, for some $\chi \in G_{\mathscr{L}_{k'}}(A)$.

We let $\widetilde{\omega}^\lambda : \widehat{\mathfrak{g}}^c_{k'} \to \mathrm{End}\, V^\lambda_{k'}$ denote the representation corresponding to the $\widehat{\mathfrak{g}}^c_{k'}$-module structure of $V^\lambda_{k'}$. We set $\widehat{\mathfrak{g}}^{\lambda, c}_{k'} = \widetilde{\omega}^\lambda(\widehat{\mathfrak{g}}^c_{k'})$, and we let $\widetilde{\omega}_\lambda : \widehat{\mathfrak{g}}^{\lambda, c}_{k'} \to \widetilde{\mathfrak{g}}^c_{k'}$ denote the Lie algebra homomorphism defined by the condition that $\widetilde{\omega}_\lambda \circ \widetilde{\omega}^\lambda = \widetilde{\omega}_{k'}$. We choose $\chi^\lambda \in \widehat{G}^\lambda_{k'}$ and (by universality) a Lie algebra automorphism $\mathscr{Y}$ of $\widehat{\mathfrak{g}}^c_{k'}$, such that

(C27) $$\widetilde{\omega}^\lambda(\mathscr{Y}(\xi)) = \chi^\lambda \widetilde{\omega}^\lambda(\xi)(\chi^\lambda)^{-1}, \qquad \xi \in \widehat{\mathfrak{g}}^c_{k'},$$
$$\widetilde{\omega}_\lambda(\chi^\lambda \eta (\chi^\lambda)^{-1}) = \mathrm{Ad}\, \chi(\widetilde{\omega}_{k'}(\eta)), \qquad \eta \in \mathfrak{g}^{\lambda, c}_{k'}.$$

We then have, from Lemma (C12), from (C25), and from the above, that

(C28) $$\widehat{\mu}(\widehat{\mathfrak{i}}) = \widehat{\mathfrak{i}},$$

where $\widehat{\mu} = \mathscr{Y}^{-1} \circ \widehat{\chi}$. We define $\mathscr{I} \subset \widehat{G}^\lambda_{k'}$, as in § 7 (but now for our more general A).

If A is a Classical Cartan matrix (so corresponding to a simple Lie algebra), we have defined the adjoint group $G_{\mathrm{ad}, \mathscr{L}_{k'}}(A)$ in § 8. More generally, if A is the finite direct sum of the classical Cartan matrices A_i, $i \in I_0$, we let $G_{\mathrm{ad}, \mathscr{L}_{k'}}$ be the direct product group

$$G_{\mathrm{ad}, \mathscr{L}_{k'}}(A) = \prod_{i \in I_0} G_{\mathrm{ad}, \mathscr{L}_{k'}}(A_i).$$

Also, as in § 8 (but for our more general A) we construct the group $G_{\mathrm{ad}}(\widetilde{A})$ and the homomorphism $\Phi' : G_{\mathrm{ad}}(\widetilde{A}) \to G_{\mathrm{ad}, \mathscr{L}_{k'}}(A)$. We let $\widetilde{\mathscr{I}} = \Phi' \circ \mathrm{Ad}(\mathscr{I})$, where

$$\mathrm{Ad} : \widehat{G}^\lambda_{k'} \to G_{\mathrm{ad}}(\widetilde{A})$$

is the adjoint representation. We let $\widetilde{\mathscr{I}}_U \subset \widetilde{\mathscr{I}}$ be the subgroup of all elements whose reduction mod t, is in the unipotent radical of the Borel subgroup corresponding to $\mathfrak{b}'$. We let $\widetilde{\mathscr{I}}^{(0)}_U = \widetilde{\mathscr{I}}_U$, and for $j \geq 1$, we let $\widetilde{\mathscr{I}}^{(j)}_U$ be the subgroup of all elements in $\widetilde{\mathscr{I}}_U$ (and in $\widetilde{\mathscr{I}}$) whose reduction mod t^j, is the identity. As in § 18 (but again, for our more general A) we can define the Lie algebras $\mathfrak{g}_{\mathcal{O}_j}(A)$ over the truncated power series ring $\mathcal{O}_j$, $j \geq 0$.

We let $\widetilde{\mathfrak{i}}^{(j)}_U \subset \widetilde{\mathfrak{i}}_U$ be the subalgebra of all elements whose reduction mod t^j is zero, for $j \geq 1$, and we let $\widetilde{\mathfrak{i}}^{(0)}_U = \widetilde{\mathfrak{i}}_U$. For subgroups H_1, H_2 of a group G, we let $[H_1, H_2] \subset G$ denote the subgroup generated by the commutators $h_1 h_2 h_1^{-1} h_2^{-1}$, $h_1 \in H_1$, $h_2 \in H_2$. By considering the adjoint action of $\widetilde{\mathscr{I}}/\widetilde{\mathscr{I}}^{(j+1)}_U$ on $\mathfrak{g}_{\mathcal{O}_j}(A)$, we see that $\widetilde{\mathscr{I}}/\widetilde{\mathscr{I}}^{(j+1)}_U$ is a Lie group, and a linear algebraic group, with Lie algebra $\widetilde{\mathfrak{i}}/\widetilde{\mathfrak{i}}^{(j+1)}_U$. By considering Lie algebras, it then follows that there are integer $p > 0$, $q > 0$, such that

$$[\widetilde{\mathscr{I}}, \widetilde{\mathscr{I}}] = \widetilde{\mathscr{I}}_U,$$
$$\overbrace{[\widetilde{\mathscr{I}}_U, [\ldots, [\widetilde{\mathscr{I}}_U, [\widetilde{\mathscr{I}}_U}^{p\text{-times}}, \widetilde{\mathscr{I}}^{(j)}_U]] \ldots]] = \widetilde{\mathscr{I}}^{(j+1)}_U \qquad j \geq 1.$$
$$\overbrace{[\widetilde{\mathscr{I}}_U, [\ldots, [\widetilde{\mathscr{I}}_U, [\widetilde{\mathscr{I}}_U}^{q\text{-times}}, \mathscr{I}]] \ldots]] = \mathscr{I}^{(1)}_U.$$

It follows that for each $j \geq 0$, the group $\widetilde{\mathscr{I}}_{U}^{(j)}$ is a characteristic subgroup of $\widetilde{\mathscr{I}}$. Hence, $\mathrm{Ad}\,\chi^{-1} \circ \widetilde{\varkappa}$ induces an automorphism of $\widetilde{\mathscr{I}}/\widetilde{\mathscr{I}}_{U}^{(j+1)}$, for each $j \geq 0$. Since $\widetilde{\mathscr{I}}/\widetilde{\mathscr{I}}_{U}^{(j+1)}$ is algebraic, it follows that, regarding $\mathfrak{h}'$ (the Cartan subalgebra of $\mathfrak{g}'$) as a subalgebra of $\widetilde{\mathfrak{i}}/\widetilde{\mathfrak{i}}_{U}^{(j+1)}$, we have that $\mathrm{Ad}\,\chi^{-1} \circ \widetilde{\varkappa}(\mathfrak{h}')$ is conjugate to $\mathfrak{h}'$ by an element of $\widetilde{\mathscr{I}}/\widetilde{\mathscr{I}}_{U}^{(j+1)}$. By a simple argument one can then pass to the limit as $j \to \infty$, and show that $\mathfrak{h}'$ and $\mathrm{Ad}\,\chi^{-1} \circ \widetilde{\varkappa}(\mathfrak{h}')$ are conjugate in $\widetilde{\mathfrak{i}}$ by an element of $\widetilde{\mathscr{I}}$ (to pass from $\widetilde{\mathscr{I}}/\widetilde{\mathscr{I}}_{U}^{(j+1)}$ to $\widetilde{\mathscr{I}}/\widetilde{\mathscr{I}}_{U}^{(j+2)}$, it suffices to conjugate by an element of $\widetilde{\mathscr{I}}_{U}^{(j+1)}/\widetilde{\mathscr{I}}_{U}^{(j+2)}$). Thus, multiplying χ by a suitable element of $\widetilde{\mathscr{I}}$, we may assume (setting $\widetilde{\mu} = \mathrm{Ad}\,\chi^{-1} \circ \widetilde{\varkappa}$),

(C29) $$\widetilde{\mu}(\widetilde{\mathfrak{i}}) = \widetilde{\mathfrak{i}}$$
$$\widetilde{\mu}(\mathfrak{h}') = \mathfrak{h}',$$

and hence we may assume in (C28) that we also have

(C30) $$\widehat{\mu}(\widehat{\mathfrak{h}}') = \widehat{\mathfrak{h}}'.$$

From (C29), we have the map $\widetilde{\mu} : \mathfrak{h}' \to \mathfrak{h}'$, which induces a dual map (still denoted by $\widetilde{\mu}$) on $(\mathfrak{h}')^*$, the k'-dual space of $\mathfrak{h}'$. From (C29), and from our earlier observation that $\widetilde{\mathfrak{i}}_{U}^{(j)}$ is a characteristic subalgebra of $\widetilde{\mathfrak{i}}$, it follows that $\widetilde{\mu}$ (on $(\mathfrak{h}')^*$) leaves the set of roots Δ (and the set of simple roots, relative to our fixed order on Δ) invariant. Hence $\widetilde{\mu}$ induces an isomorphism of the root system Δ, and this isomorphism extends to an automorphism ρ of $\mathfrak{g}$. Of course, by $\mathscr{L}_{k'}$-linearity, ρ extends to an automorphism of $\widetilde{\mathfrak{g}}_{k'}^{c}$ (which we also denote by ρ). Then $\rho^{-1} \circ \widetilde{\mu}$ still satisfies (C29), and induces the identity on $\mathfrak{h}$. We let $H_1, \ldots, H_\ell$ ($\ell = \dim \mathfrak{h}$) denote the simple coroots in $\mathfrak{h}$ (we had fixed an order on the roots Δ). Also, let

$$\{H_i, E_\alpha\}_{i=1,\ldots,\ell;\ \alpha \in \Delta},$$

be a Chevalley basis of $\mathfrak{g}$. Thus, in particular, for each $\alpha \in \Delta$, E_α is a non-zero vector in the α root space and $[E_\alpha, E_{-\alpha}] = H_\alpha$, the coroot corresponding to α. In particular, for each simple root α_i, $i = 1, \ldots, \ell$, we let $E_i = E_{\alpha_i}$, $F_i = E_{-\alpha_i}$, and we then have $[E_i, F_i] = H_i$, $i = 1, \ldots, \ell$. We also have, as we just noted

(C31) $$\rho^{-1} \circ \widetilde{\mu}(H_i) = H_i, \quad i = 1, \ldots, \ell.$$

Also, since $\rho^{-1} \circ \widetilde{\mu}$ (in place of $\widetilde{\mu}$) satisfies (C29), it induces a map on $\widetilde{\mathfrak{i}}_U/\widetilde{\mathfrak{i}}_U^{(1)}$, and (C31) implies that this induced map leaves each root space invariant. Hence for each $\alpha \in \Delta_+$, we have

$$\rho^{-1} \circ \widetilde{\mu}(E_\alpha) = \sigma_\alpha E_\alpha, \quad \sigma_\alpha \in \mathcal{O}^*,$$

$\mathcal{O}^*(\subset \mathcal{O} \subset \mathscr{L}_{k'})$ denoting the group of units in the ring $\mathcal{O}$ of formal power series with coefficients in k'. But then (C31) implies for $\alpha \in \Delta_-$

$$\rho^{-1} \circ \widetilde{\mu}(E_\alpha) = \sigma_\alpha E_\alpha, \quad \sigma_\alpha \in \mathscr{L}_{k'}.$$

But for $\alpha\in\Delta_+$

$$[\sigma_\alpha E_\alpha, \sigma_{-\alpha}E_{-\alpha}]=\rho^{-1}\circ\widetilde{\mu}(H_\alpha)=H_\alpha,$$

and so $\sigma_{-\alpha}=\sigma_\alpha^{-1}$; in particular, $\sigma_{-\alpha}\in\mathcal{O}^*$. We also have $(\alpha\in\Delta_+)$

$$\rho^{-1}\circ\widetilde{\mu}(tE_{-\alpha})=t\tau_\alpha^{-1}E_{-\alpha},$$
$$\rho^{-1}\circ\widetilde{\mu}(t^{-1}E_\alpha)=t^{-1}\tau_\alpha E_\alpha, \qquad \tau_\alpha\in\mathcal{O}^*.$$

To summarize:

Lemma **(C32)**. — *The map* $\rho^{-1}\circ\widetilde{\mu}$ *on* $\widetilde{\mathfrak{g}}^c_{k'}$ *satisfies the following:*

(i) $\rho^{-1}\circ\widetilde{\mu}(\mathfrak{h}')=\mathfrak{h}'$, *and* $\rho^{-1}\circ\widetilde{\mu}$ *restricted to* $\mathfrak{h}'$ *is the identity.*

(ii) *For each* $\alpha\in\Delta_+$, *we have*

$$\rho^{-1}\circ\widetilde{\mu}(E_\alpha)=\sigma_\alpha E_\alpha,$$
$$\rho^{-1}\circ\widetilde{\mu}(E_{-\alpha})=\sigma_\alpha^{-1}E_{-\alpha},$$
$$\rho^{-1}\circ\widetilde{\mu}(tE_{-\alpha})=t\tau_\alpha^{-1}E_{-\alpha},$$
$$\rho^{-1}\circ\widetilde{\mu}(t^{-1}E_\alpha)=t^{-1}\tau_\alpha E_\alpha,$$

where $\sigma_\alpha, \tau_\alpha\in\mathcal{O}^*$.

We may lift ρ to an automorphism $\widehat{\rho}$ of $\widehat{\mathfrak{g}}^c_{k'}$ and we will now show that $\widehat{\rho}^{-1}\circ\widehat{\mu}(h)=h$, for all $h\in\widehat{\mathfrak{h}}$ (where $\widehat{\mathfrak{h}}=\widetilde{\omega}_k^{-1}(\mathfrak{h})$, with $\widetilde{\omega}_k:\widehat{\mathfrak{g}}^c_k\to\widetilde{\mathfrak{g}}^c_k$ being the natural projection). To do this, it suffices to assume $\mathfrak{g}$ simple (and hence absolutely simple, since $\mathfrak{g}$ is assumed to be split). But then we have $\widehat{\mathfrak{g}}^c_{k'}=\widetilde{\mathfrak{g}}^c_{k'}\oplus k'$, and $\widehat{\rho}$ is defined by setting $\widehat{\rho}(\xi, s)=(\rho(\xi), s)$, $\xi\in\widetilde{\mathfrak{g}}^c_{k'}$, $s\in k'$. Moreover, for $\alpha\in\Delta_+$

$$\widehat{\rho}^{-1}\circ\widehat{\mu}(H_\alpha, 0)=\widehat{\rho}^{-1}\circ\widehat{\mu}([(E_\alpha, 0), (E_{-\alpha}, 0)])$$
$$=[(\sigma_\alpha E_\alpha, 0), (\sigma_\alpha^{-1}E_{-\alpha}, 0)]=(H_\alpha, 0),$$

since σ_α is a unit. Also, for $\alpha\in\Delta_+$

$$\widehat{\rho}^{-1}\circ\widehat{\mu}\left(H_{-\alpha}, \frac{2}{(\alpha,\alpha)}\right)=\widehat{\rho}^{-1}\circ\widehat{\mu}([(tE_{-\alpha}, 0), (t^{-1}E_\alpha, 0)])$$
$$=[(t\tau_\alpha^{-1}E_{-\alpha}, 0), (t^{-1}\tau_\alpha E_\alpha, 0)]$$
$$=\left(H_{-\alpha}, -\operatorname{Res}(t\tau_\alpha^{-1}\,d(t^{-1}\tau_\alpha))\frac{2}{(\alpha,\alpha)}\right), \quad \text{where } \operatorname{Res}=\text{residue},$$
$$=\left(H_{-\alpha}, \frac{2}{(\alpha,\alpha)}\right),$$

and thus we have proved (also see (C30))

(C33) $\qquad \widehat{\rho}^{-1}\circ\widehat{\mu}(h)=h, \qquad$ for all $\qquad h\in\widehat{\mathfrak{h}}$.

But then it follows that if $\lambda\in(\widehat{\mathfrak{h}})^*$ is dominant integral, $\widehat{\mu}(\lambda)$ is dominant integral, where, by definition, we set

$$\widehat{\mu}(\lambda)(h)=\lambda(\widehat{\mu}(h)), \qquad h\in\widehat{\mathfrak{h}}.$$

Also, since ρ on $\mathfrak{h}$ induces an automorphism of the root system Δ, it follows that $\hat{\rho}$ on $\hat{\mathfrak{h}}'$, and hence (by C33), $\hat{\mu}$ on $\hat{\mathfrak{h}}'$ has finite order. As $\varkappa$ varies over $\mathscr{G}$, $\hat{\mu}$ (which depends on $\varkappa$) varies over a finite group. We let $\{\lambda_0, \lambda_1, \ldots, \lambda_q\}$ denote the set of distinct elements in the orbit of λ under this group and we set

$$\text{(C34)} \qquad \tilde{V} = \coprod_{0 \leq j \leq q} V_{k'}^{\lambda_j}.$$

We note that the group $\hat{G}_{k'}^{\lambda}$ and the Lie algebra $\hat{\mathfrak{g}}_{k'}^{e}$ admit diagonal actions on $\tilde{V}$, and also that if

$$\hat{\varkappa} = (\mathrm{Ad}\ \chi^{\lambda}) \circ \hat{\mu}$$

(see (C28), and the subsequent definition of $\hat{\mu}$) then $\hat{\varkappa}$ induces a bijection $\hat{\varkappa}^{\lambda}$ of $\tilde{V}$, and that this $\hat{\varkappa}^{\lambda}$ satisfies (C26) (but with $V_{k'}^{\lambda}$ replaced by $\tilde{V}$—we note that if the Cartan matrix A has no non-trivial automorphisms, then $\tilde{V} = V_{k'}^{\lambda}$).

Now, as earlier in this appendix, we let $\mathfrak{l}$ denote a semi-simple Lie algebra which is defined over $\mathscr{L}_k$ and which splits over $\mathscr{L}_{k'}$. Let L denote the group of $\mathscr{L}_k$-rational points of the simply connected, linear algebraic group with Lie algebra $\mathfrak{l}$ (more precisely, we should here replace $\mathfrak{l}$ by its tensor product with an algebraic closure of $\mathscr{L}_k$). We let $G_{\mathscr{L}_{k'}}$ denote the group of $\mathscr{L}_{k'}$-rational points of this algebraic group, and for $\varkappa \in \mathrm{Gal}(\mathscr{L}_{k'}/\mathscr{L}_k)$, we also let $\varkappa$ denote the corresponding automorphism of $G_{\mathscr{L}_{k'}}$. We then construct the fibered product

$$E^{\lambda}(G_{\mathscr{L}_{k'}}) \subset \hat{G}_{k'}^{\lambda} \times G_{\mathscr{L}_{k'}}, \qquad \lambda \text{ dominant integral},$$

as in § 12 (see (12.14)), but now for our more general A. Then $\hat{\varkappa}^{\lambda}$ induces an automorphism $\hat{\varkappa}$ of $\hat{G}_{k'}^{\lambda}$, by conjugation. We let

$$\varkappa'(g, h) = (\hat{\varkappa}(g), \varkappa(h)), \ (g, h) \in E^{\lambda}(G_{\mathscr{L}_{k'}}),$$

and thus obtain a lift of our Galois action on $G_{\mathscr{L}_{k'}}$ to $E^{\lambda}(G_{\mathscr{L}_k})$. We let $\hat{L} \subset E^{\lambda}(G_{\mathscr{L}_k})$ denote the Galois fixed points. Then the projection onto the second factor

$$E^{\lambda}(G_{\mathscr{L}_k}) \to G_{\mathscr{L}_k}$$

induces (by restriction) a group homomorphism

$$\hat{L} \to L,$$

and by Hilbert's theorem 90, this latter homomorphism is surjective, and hence yields a central extension of L.

BIBLIOGRAPHY

[1] A. Borel, *Introduction aux groupes arithmétiques*, Paris, Hermann, 1969.

[2] A. Borel, Admissible representations of a semi-simple group over a local field with vectors fixed under an Iwahori subgroup, *Inventiones math.*, **35** (1976), 233-259.

[3] N. Bourbaki, *Groupes et algèbres de Lie*, chap. 4, 5 et 6, Paris, Hermann, 1968.

[4] F. Bruhat et J. Tits, Groupes réductifs sur un corps local, *Publ. Math. I.H.E.S.*, **41** (1972), 1-251.

[5] F. Bruhat et J. Tits, Groupes algébriques simples sur un corps local, in the *Proceedings of a Conference on Local Fields*, held at Driebergen (The Netherlands), Edited by T. A. Springer, New York, Springer-Verlag, 1967, pp. 23-36.

[6] H. Garland, Dedekind's η-function and the cohomology of infinite dimensional Lie algebras, *Proc. Nat. Acad. Sci.* (U.S.A.), **72** (1975), 2493-2495.

[7] H. Garland, The arithmetic theory of loop algebras, *J. Algebra*, **53** (1978), 480-551.

[8] H. Garland and J. Lepowsky, Lie algebra homology and the Macdonald-Kac formulas, *Inventiones math.*, **34** (1976), 37-76.

[9] N. Iwahori and H. Matsumoto, On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups, *Publ. Math. I.H.E.S.*, **25** (1965), 237-280.

[10] V. G. Kac, Simple irreducible graded Lie algebras of finite growth (in Russian), *Izv. Akad. Nauk. SSSR*, **32** (1968), 1323-1367; English translation: *Math. USSR-Izvestija*, **2** (1968), 1271-1311.

[11] V. G. Kac, Infinite-dimensional Lie algebras and Dedekind's η-function (in Russian), *Funkt. Anal. i Ego Prilozheniya*, **8** (1974), 77-78; English translation: *Functional Analysis and its Applications*, **8** (1974), 68-70.

[12] R. Marcuson, Tits systems in generalized nonadjoint Chevalley groups, *J. Algebra*, **34** (1975), 84-96.

[13] H. Matsumoto, Sur les sous-groupes arithmétiques des groupes semi-simples déployés, *Ann. Scient. Éc. Norm. Sup.*, **2** (4th series) (1969), 1-62.

[14] J. Milnor, *Introduction to algebraic K-theory*, Princeton, Princeton University Press, 1971.

[15] R. V. Moody, A new class of Lie algebras, *J. Algebra*, **10** (1968), 211-230.

[16] R. V. Moody, Euclidean Lie algebras, *Can. J. Math.*, **21** (1969), 1432-1454.

[17] R. V. Moody and K. L. Teo, Tits' systems with crystallographic Weyl groups, *J. Algebra*, **21** (1972), 178-190.

[18] C. C. Moore, Group extensions of p-adic and adelic linear groups, *Publ. Math. I.H.E.S.*, **35** (1968), 157-222.

[19] J.-P. Serre, *Algèbres de Lie semi-simples complexes*, New York, Benjamin, 1966.

[20] R. Steinberg, *Générateurs, relations et revêtements de groupes algébriques, in* Colloque sur la Théorie des Groupes Algébriques, held in Brussels, 1962, Paris, Gauthier-Villars, 1962, pp. 113-127.

[21] R. Steinberg, *Lectures on Chevalley groups*, Yale University mimeographed notes, 1967.

Department of Mathematics
Yale University
Box 2155, Yale Station,
New Haven, Connecticut 06520

Manuscrit reçu le 27 juillet 1977,
révisé le 26 février 1979.

LIST OF NOTATIONS

0. R, $\mathscr{L}_R$, R^* (also, see the discussion at the end of § 0, concerning the use of "k")
1. **C, R, Q, Z**
   after (1.8): $B^2(\mathfrak{a}, V)$, $Z^2(\mathfrak{a}, V)$, $H^2(\mathfrak{a}, V)$ (where $\mathfrak{a}$ is a Lie algebra over a field k); $\mathfrak{a}_f$ (for $f \in Z^2(\mathfrak{a}, V)$)
2. $\mathfrak{g}$, $k[t, t^{-1}]$, $\tilde{\mathfrak{g}}$
   after (2.1): $\hat{\mathfrak{g}}$, $\mathfrak{c}$
   after (2.2): τ, (,) (on $\mathfrak{g}$), $\tilde{\omega}$
   after (2.19): $\mathfrak{h}$, Δ, $\mathfrak{g}^\alpha$ (for $\alpha \in \Delta$), H_α, E_α
   after (2.25): (,) (on $\mathfrak{h}^*$)
   after (2.35): h_i'
3. B, $\mathfrak{g}_1(B)$, e_i, f_i, h_i
   after (3.2): $\mathfrak{g}(B)$, $\mathfrak{d}$, $\mathfrak{g}^e(B)$, $\mathfrak{h}^e(B)$, $\mathfrak{h}^e(B)^*$, $a_1, \ldots, a_\ell$, $\Delta(B)$, $\Delta_\pm(B)$, R, σ, $D_1, \ldots, D_\ell$
   after (3.3): r_i (in $\mathrm{Aut}(\mathfrak{h}^e(B)^*)$), $W = W(B)$
   after Proposition (3.3): A, $\tilde{A}$, Δ, $\Delta_\pm$, $\mathfrak{h}$, E_i, F_i, H_i, $\alpha_1, \ldots, \alpha_\ell$, α_0, D, $\mathfrak{d}$ (as the span of D)
   in (3.8): π
   in (3.9): ψ
4. in (4.1): r
   in (4.2): Ψ
   after (4.2): ι
   in (4.3): $\Delta_+(\tilde{A})$
   after (4.3): $\Delta_W(\tilde{A})$, $\Delta_I(\tilde{A})$, $\Delta_{W,\pm}(\tilde{A})$, $\Delta_{I,\pm}(\tilde{A})$
   after (4.5): D_0, ξ_a, $\xi_i(h)$, h_i
   in (4.6): Φ
   after (4.8): $\mathfrak{u}^\pm(\tilde{A})$, $\tilde{\omega}^\pm$, $\mathfrak{u}^\pm$, $\mathfrak{g}_{\mathbf{Z}}(\tilde{A})$, $\mathfrak{g}_{\mathbf{Z}}(A)$, $\mathfrak{g}_R(\tilde{A})$, $\mathfrak{g}_R(A)$, $\tilde{\mathfrak{g}}_R$, $\tilde{\omega}_{\mathbf{Z}}$, $\mathfrak{u}^\pm_{\mathbf{Z}}(\tilde{A})$, $\tilde{\mathfrak{u}}^\pm_{\mathbf{Z}}$, $\mathfrak{u}^\pm_R(\tilde{A})$, $\tilde{\mathfrak{u}}^\pm_R$, $\mathfrak{h}_R(\tilde{A})$, $\mathfrak{h}_R(A)$, $\tilde{\omega}_R$, $\mathfrak{u}^\pm_R$, $\mathfrak{u}_R$, $\tilde{\mathfrak{u}}_R$
5. $\mathfrak{g}_R(\tilde{A})_i$, $\tilde{\mathfrak{g}}_{R,i}$
   in (5.2): | |
   after (5.6): $\mathfrak{g}^c_R(\tilde{A})$, $\tilde{\mathfrak{g}}^c_R$
   after (5.7): $\mathfrak{u}^c_R$, $\tilde{\mathfrak{u}}^c_R$, $\tilde{\omega}^+_R$
   in Remark (5.8): $(\mathscr{L}_R =) R[[t, t^{-1}]]$
   after (5.9): $\mathfrak{c}_R$
   after (5.12): $\mathscr{O}$
6. $\mathfrak{u}^\pm(B)$, $\mathfrak{p}^e$, $\mathscr{U}(\mathfrak{a})$ (for a Lie algebra $\mathfrak{a}$ over a field k), **D**, $M(\lambda)$, $\mathscr{P}^e$, $V^{M(\lambda)}$, V^λ, v_0, $\mathscr{U}_{\mathbf{Z}}(\tilde{A})$, $V^\lambda_{\mathbf{Z}}$, V^λ_μ, Ω, Ω_μ, $V^\lambda_{\mu,\mathbf{Z}}$
   after (6.1): V^λ_R
   in (6.2): $V^\lambda_{\mu,R}$
   after (6.3): $\mathfrak{h}_{\mathbf{Z}}(\tilde{A})$, $\mathfrak{g}^a_{\mathbf{Z}}$, $\mathfrak{h}_R(\tilde{A})$, $\mathfrak{g}^a_R$
   after (6.7): $\mathfrak{g}^e_{\mathbf{Z}}(\tilde{A})$, $\mathfrak{g}^e_R(\tilde{A})$, $\mathfrak{h}^e_{\mathbf{Z}}(\tilde{A})$, $\mathfrak{h}^e_R(\tilde{A})$, V^λ_j, $V^\lambda_{j,\mathbf{Z}}$, $V^\lambda_{j,R}$
   after (6.10): π^λ_R, π^λ
   after (6.12): $\tilde{r}_b$
   after (6.13): h_b, r_i (in $\mathrm{Aut}(\mathfrak{h}^e(\tilde{A}))$)

7. before (7.1): k
   after (7.1): $\mathscr{U}_{\mathrm{R}}(\widetilde{\mathrm{A}})$, $\pi^{\lambda}_{\mathrm{R}}$, $\mathrm{ad}_{\mathrm{R}}(\ )$, $\chi_{\pm a_i}(s)$
   after (7.3): $\mathscr{Y}_b(s)$
   after (7.6): r_i^{λ}
   in (7.9′): w^{λ}
   in (7.14): $\chi_a(s)$
   after (7.17): $\mathscr{P}$, $\mathscr{O}^*$
   in (7.19): $\chi^{\lambda}_{\alpha}(\sigma(t)) = \chi_{\alpha}(\sigma(t))$
   Definition (7.21): $\widehat{\mathrm{G}} = \widehat{\mathrm{G}}^{\lambda}_k = \widehat{\mathrm{G}}^{\lambda}_k(\widetilde{\mathrm{A}})$
   in (7.22): $w^{\lambda}_{\alpha}(\sigma(t)) = w_{\alpha}(\sigma(t))$, $h^{\lambda}_{\alpha}(\sigma(t)) = h_{\alpha}(\sigma(t))$
   in (7.23): $w^{\lambda}_a(s) = w_a(s)$, $h^{\lambda}_a(s) = h_a(s)$
   Definition (7.24): $\mathscr{I}$
8. in (8.2): $\mathscr{Y}_{\alpha}(\sigma(t))$
   after (8.7): $\mathscr{L}$, $\mathscr{X}_{\alpha}(\sigma(t))$
   after (8.12): $\mathrm{G}_{\mathrm{ad}.\mathscr{L}} = \mathrm{G}_{\mathrm{ad}.\mathscr{L}}(\mathrm{A})$, $\mathrm{G}_{\mathrm{ad}}(\widetilde{\mathrm{A}})$
   after (8.13): Φ'
9. after Lemma (9.1): $\{\ ,\ \}$, $*$, x^* (for $x \in \mathfrak{g}(\widetilde{\mathrm{A}})$)
   after (9.3): u^* (for $u \in \mathscr{U}(\mathfrak{g}(\widetilde{\mathrm{A}}))$)
10. $\mathrm{G}_{\mathscr{L}_k} = \mathrm{G}_{\mathscr{L}_k}(\mathrm{A})$, $\mathscr{X}'_{\alpha}(\sigma)$, $w'_{\alpha}(\sigma)$, $h'_{\alpha}(\sigma)$, G, $\mathrm{E}(\mathrm{G}_{\mathscr{L}_k})$, $\chi^e_{\alpha}(\sigma)$, φ^e
   in Lemma (10.1): Ψ^e
11. U^e_{α}, $w^e_{\alpha}(\sigma)$, $h^e_{\alpha}(\sigma)$, T^e, M^e_{α}
   in Proposition (11.1): U^e_+, U^e_-
   after Remarks (11.2): U_{α}, U_+, U_-, T, r_a, h_a
   after (11.3): T^{λ}, $\mathrm{U}^{\lambda}_{\alpha}$, U^{λ}_+, U^{λ}_-
   after (11.4): B^e, N^e
   after (11.5): $\mathrm{M}^{\lambda}_{\alpha}$
   after (11.7): ν, ν^e_{α}, ν^{λ}_{α}, ν_{α}, M_{α}, ν^*_{α}, U^*_{α}, M^*_{α}, T^*
   in Lemma (11.8): $\mathrm{U}^*_{\alpha, i}$
12. $\mathrm{dp}(\mu)$, coherently ordered basis, $\mathscr{I}_{\mathrm{U}}$
   after (12.4): g^h
   in (12.11): ω
   after (12.13): π_1, π_2, $\mathrm{E}^{\lambda}(\mathrm{G}_{\mathscr{L}_k})$, $\mathrm{G}^{\lambda}_{\mathscr{L}_k}$
   after (12.14): $\varphi_i = \varphi_i^{\lambda}$, $i = 1, 2, \pi$
   in (12.15): $b^{\lambda}_{\alpha}(\sigma, \tau)$
   after (12.15): $h''_{\alpha}(\sigma)$
   in (12.17): $b_{\alpha}(\sigma, \tau)$
   after (12.17): M, $\mathrm{S}(k^*, \mathrm{M})$
   after (12.18): $\mathrm{S}^0(k^*, \mathrm{M})$
   in (12.20): $c_{\mathrm{T}}(\ ,\ )$
   after Theorem (12.26): ρ^{λ}
13. in (13.1′): $s\mu^* + v$
   after (13.1′): definition of "equipollentes"
   after (13.1′): $\mathbf{N}^*$, ${}^v p$
   in (13.3): r_{α}
   in (13.4): $n.\mu^*$
   after (13.5): $\mathbf{A}$, $a_{\alpha, m}$ ${}^v a_{\alpha, m}$, Σ, $\mathscr{E}$, $\partial a_{\alpha, m}$, $r_{\alpha, m}$
   in Proposition (13.6): $p(n)$, U^*_a
   after Proposition (13.6): $\mathrm{T}^*_{\mathscr{O}}$
   after (13.13): Λ
   after (13.16′): f_{Ω}, $\mathrm{U}_{\Omega} = \mathrm{U}^{\lambda}_{\Omega}$, $\mathrm{U}_{\pm, \Omega}$

in Remark (13.17): C
after (13.19): $\mathbf{W}$, $\tilde{\mathbf{N}}^{\lambda}$, $\mathbf{S}$
14. in Proposition (14.8): N
after Theorem (14.10): $\mathrm{H}_k^{\lambda} = \mathrm{H}_k$
15. in (15.1): λ_i
after (15.1): Ξ, Ξ_r, Ξ_{λ}
in Theorem (15.9): $\pi(\lambda_1, \lambda_2)$
in (15.11): $c_{\alpha}(\sigma_1, \sigma_2)$
after (15.12): $\Psi^{e, \lambda}$
after (15.18): H_k'
16. after (16.1): $\{\ ,\ \}$
Lemma (16.3): Ψ_a
Definition (16.7): $\hat{\mathrm{K}} = \hat{\mathrm{K}}^{\lambda}$
after (16.12): $\mathrm{H}_{k,+}$, $\mathrm{H}_{k,\theta}$
17. before (17.1): $\mathfrak{h}_k^e(\tilde{\mathrm{A}})^*$
after (17.1): Φ_0
after (17.2′): $\tilde{\Phi}$
after (17.9): e^h
after Lemma (17.14): J, $\mathscr{L}_{\mathrm{J}}$, $\hat{\Gamma} = \hat{\Gamma}_k^{\lambda}$
18. after Lemma (18.2): $\mathscr{I}_{\mathrm{U}}^{(j)}$, $\Gamma_{\mathrm{U}}^{(j)}$, Γ_{U}
after (18.3): $\Psi^{(j)}$, $\pi^{(j)}$
after (18.5): $\mathscr{O}_j$
after (18.7): π^0, G_k, U_k, $\mathscr{A}_{\mathrm{U}}$, $\mathscr{D}$, $g_{\mathscr{D}}(\mathrm{A})$, $\mathrm{U}_{\mathscr{D}}$
after (18.8): $\mathrm{U}_{\mathscr{O}}$, $\mathrm{U}_{\mathscr{D}}^-$
in Definition (18.15): $\mathscr{O}_{\mathscr{D}}$, $\mathscr{O}_{\mathscr{D}}^*$, $\mathscr{I}_{\mathrm{U},\mathscr{D}}$
in Remark (18.17): $\Gamma_{0,\mathrm{U}}$, $\mathrm{J}[[t]]$
19. before Definition (19.1): σ_0
in Definition (19.1): H_{σ}
in Definition (19.2): $\mathfrak{S}_{\sigma}$
after Definition (19.2): $\hat{\Gamma}_0$
before (19.5): $g_{\mathscr{D}}$
20. before Proposition (20.1): λ_i
before Lemma (20.8): $\hat{\mathrm{G}}_k^{\lambda, e}$
before Lemma (20.10): $\mathfrak{S}_{\sigma}^{\lambda}$, $\mathrm{H}_{\sigma}^{\lambda}$, $\mathrm{H}_{\sigma}^{\mathbf{R}} = \mathrm{H}_{\sigma}^{\lambda,\mathbf{R}}$, $\mathfrak{S}_{\sigma}^{\mathbf{R}} = \mathfrak{S}_{\sigma}^{\lambda,\mathbf{R}}$, $\hat{\Gamma}_0^{\lambda} = \hat{\Gamma}_0$, $\mathscr{I}_{\mathrm{U}}^{\lambda} = \mathscr{I}_{\mathrm{U}}$, $\mathrm{H}_{k,+}^{\lambda} = \mathrm{H}_{k,+}$, $\hat{\mathrm{K}}^{\lambda} = \hat{\mathrm{K}}$
21. before (21.1): Φ_w
in (21.1): $\langle \Phi_w \rangle$
in (21.2): b_j
before Lemma (21.3): W_i
after (21.5): α_0, m_j, m
after (21.15): $\mathbf{W}_i$, $\mathbf{W}_{\mathrm{F}}$, $\mathbf{P}_{\mathrm{F}}$
after (21.22): $\hat{\Gamma}_r$, $\mathfrak{S}_{\sigma}^{\mathbf{R}}(r)$
Appendix I. before Lemma (A.1): $\mathfrak{h}_{\mathrm{R}}^e(\tilde{\mathrm{A}})$, $\mathfrak{g}_{\mathrm{R}}^e(\tilde{\mathrm{A}})$, $\mathscr{Y}_a(s)$, $\tilde{w}_a(s)$, $\tilde{h}_a(s)$
in Lemma (A.4): $\gamma = \gamma(b, a)$