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A HOMOGENIZATION RESULT FOR PLANAR, POLYGONAL NETWORKS (*)

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Abstract — *We study the relation between planar electrical networks and continuous conductance models Our main theorem is a homogenization result for networks with infinitessimal edge length and rapidly varying resistor values We prove that any effective limit will always correspond to a second order equation of the form* $\nabla . (A \nabla u) = 0$ *In the last section we partially characterize the possible symmetric matrices, that can occur as effective limits of equilateral triangular networks*

Résume — *On étudie la relation entre des réseaux electriques plans et des modèles à conductance continue Le principal theorème est un résultat d'homogénéisation pour des réseaux à mailles infinitésimales et dont les valeurs des résistances varient très rapidement Nous montrons que toute limite effective correspond à une équation du second degré de la forme* $\nabla . (A \nabla u) = 0$ *Dans la dernière partie, nous caractérisons partiellement parmi les matrices symetriques, celles qui sont limites effectives de réseaux composes de triangles équilatéraux*

0. INTRODUCTION

In this paper we study the relationship between planar electrical networks and the boundary value problem

$$(1) \qquad \nabla . (A \nabla u) = 0 \text{ in } \Omega , \quad u = \phi \text{ on } \partial\Omega .$$

We prove that equations of the form (1) for an appropriate class of matrices A describe all possible weak limits of the voltage potentials corresponding to a sequence of networks with decreasing edge lengths and no interior sources or sinks. This result is a network analogue of homogenization convergence

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results by F. Murat and L. Tartar for second order elliptic boundary value problems (*cf.* [16, 17]). Networks may naturally be viewed as tools to calculate approximate solutions to second order elliptic boundary value problems with smoothly varying (or constant) coefficients (see for instance [7, 8, 10]). If the strength of the resistors in the network are permitted to vary rapidly then the network contains two natural length scales : the edge length (the discretization parameter) and the length scale of variation of the resistors. The result in this paper shows that asymptotically, as far as the *structure* of the limiting equations are concerned, the discreteness introduces no new phenomena. Subsequences of solutions to continuous conductance equations with rapidly varying coefficients converge weakly to solutions of « new », homogenized conductance equations, and so do subsequences of network voltages corresponding to rapidly varying resistor strength, as the edge length approaches zero. The situation is quite different when it comes to characterize the exact set of matrices A, that can arise in (1). The set of matrices that can appear in (1) when passing to the limit along sequences of solutions to the continuous conductance equations is frequently referred to as the G-closure of the admissible set of matrices for the original conductance equations. This G-closure has been determined in a number of interesting cases (see for example [14, 17]). In the last section of this paper we present one simple calculation related to the « G-closure » of locally equilateral, triangular networks with infinitessimal edge length and conductances lying between $\gamma_{\min} = 2\,\mu$ and $\gamma_{\max} = 2\,\gamma$. The « G-closure » appears to be different from any set one could predict based entirely on the continuous theory. For the mixture of two components of continuous media there is a natural notion of volume fraction and this leads to the much more informative (but still, geometry independent) Hashin-Shtrikman bounds [15, 17, 21]. For networks it is not in general clear how to define a limiting notion of volume fraction as the edge length approaches zero and so it is not obvious what corresponds to the Hashin-Shtrikman bounds. Several authors have derived selfconsistent effective medium theories for discrete networks with finite edge length (*cf.* [12, 20]). In cases (of simple geometry) where there is a natural notion of volume fraction, it has been noted that the discrete effective media theories may fall outside the corresponding Hashin-Shtrikman bounds [12]. This is not surprising, and a similar observation is in some sense confirmed (for networks with infinitessimal edge length) by the calculation in Section 4 of the present paper.

We now proceed with a description of the assumptions concerning the discrete network models. Let Ω be a bounded polygon in $\mathbb{R}^2$ and let $\{\omega^{(i)}\}$ be a convex polygonal tiling of Ω. By this we mean that

(2.*a*) each $\omega^{(i)}$ is a convex polygon,

(2.*b*) $\omega^{(i)} \cap \omega^{(j)} = \emptyset$ for $i \neq j$, and

(2.*c*) $\bar{\Omega} = \cup\, \overline{\omega^{(i)}}$.

The vertices and the edges of $\partial\omega^{(i)}$ form a planar, polygonal network spanning Ω ; it is exactly networks $\mathfrak{N}$ that arise in this way which we shall study in this paper. We refer to $\{\omega^{(i)}\}$ as the polygonal tiling associated with $\mathfrak{N}$. Note that all edges of our networks are linesegments and note that different edges of $\mathfrak{N}$ only meet at a vertices of $\mathfrak{N}$. A pattern, such as the Wheatstone bridge, schematically shown in figure 1, can therefore not be part of our networks.

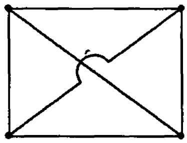

Figure 1.

We denote the vertices of $\mathfrak{N}$ by $\{x_k\}$. If the network $\mathfrak{N}$ contains an edge joining x_k to x_l then we denote this edge by $e_{k,l}$. The conductance (the reciprocal resistance) of the edge $e_{k,l}$ is denoted by $\gamma_{k,l}$. If there is no edge $e_{k,l}$ in $\mathfrak{N}$ then the corresponding conductance $\gamma_{k,l}$ vanishes. Let U_k denote the voltage potential at the vertex x_k. Kirchhoff's laws of electrical conductance in the network $\mathfrak{N}$ (given a boundary voltage potential ϕ) assert that $\{U_k\}$ minimize the energy expression

$$\frac{1}{2}\sum_{k,l}\gamma_{k,l}(U_k - U_l)^2 \tag{3}$$

subject to the boundary conditions

$$U_{k_0} = \phi_{k_0} \text{ at any vertex of } \mathfrak{N} \text{ lying on } \partial\Omega . \tag{4}$$

Associated to any network is a maximal edge length $h = \max\{|e_{k,l}| : e_{k,l} \in \mathfrak{N}\}$. In the remainder of this paper, whenever we write $\mathfrak{N}_h$, the subscript h refers to the maximal edge length of the network.

We call a sequence of planar, polygonal networks $\mathfrak{N}_h$ regular, provided

(5.*a*) $c_0 h < |e_{k,l}| < C_0 h$ for any $e_{k,l} \in \mathfrak{N}_h$,

(5.*b*) No vertex of a polygon associated with $\mathfrak{N}_h$ lies in the interior of an edge of another,

(5.*c*) all interior angles of the polygons associated with $\mathfrak{N}_h$ satisfy :

$$0 < d_0 < \theta < \pi - d_0 .$$

The constants C_0, $0 < c_0$ and d_0 are independent of h. We note that the upper bound on the angles in the condition (5.*c*) in itself implies that all the polygons $\omega_h^{(i)}$ must be convex ; the conditions (5.a) and (5.c) imply that the maximal number of edges in any of these polygons is bounded independently of h (and i).

We assume that the conductances $\gamma_{k,l}^h$ satisfy

$$0 < \gamma_{\min} \leq \gamma_{k,l}^h \leq \gamma_{\max}$$

with $\gamma_{\min}$ and $\gamma_{\max}$ independent of h, k and l. Given $\varepsilon > 0$, $H^{\frac{1}{2}+\varepsilon}(\partial\Omega)$ denotes the set of functions that are continuous on $\partial\Omega$ and furthermore are in the classical Sobolev space $H^{\frac{1}{2}+\varepsilon}(l)$ for any linesegment l contained in $\partial\Omega$. The boundary conditions ϕ_{k_0} are assumed to be of the form

$$\phi_{k_0} = \phi(x_{k_0}) , \tag{6}$$

for some function ϕ in $H^{\frac{1}{2}+\varepsilon}(\partial\Omega)$. Most importantly we assume that the maximal edge length of $\mathfrak{N}_h$ converges to 0, i.e.,

$$h = \max \{ |e_{k,l}| : e_{k,l} \in \mathfrak{N}_h \} \to 0 .$$

In order to formulate the main theorem of this paper it is necessary to discuss in some more detail possible interpolations of the discrete voltage potentials U_k. Let $\mathfrak{T}^{(i)}$ be a triangulation of $\omega^{(i)}$ in which each triangle has at least one entire edge in common with $\partial\omega^{(i)}$ and such that all vertices of $\mathfrak{T}^{(i)}$ are vertices of $\partial\omega^{(i)}$. Let $\mathfrak{T} = \cup \mathfrak{T}^{(i)}$ be the associated triangulation of the domain Ω. We shall denote a triangulation $\mathfrak{T}$, which arises in this way a minimal triangulation of the network $\mathfrak{N}$. To illustrate this concept consider the hexagon shown in figure 2. The triangulation in figure 2a is admissible according to our definition of minimal triangulation, whereas the triangulations shown in figure 2b and 2c are not (in 2b the center triangle has no edges in common with the hexagon, in 2c the center vertex is not a vertex of the hexagon).

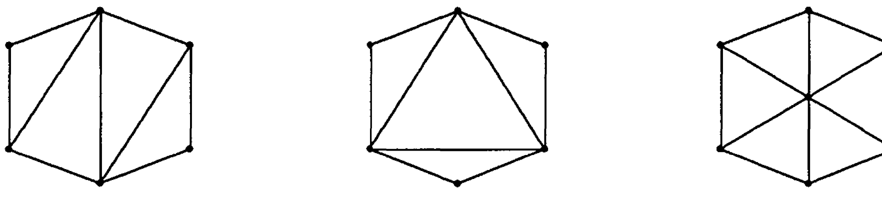

Figure 2*a*. **Figure 2*b*.** **Figure 2*c*.**

Given a regular sequence of planar, polygonal networks $\mathfrak{N}_h$ it is possible to find a quasiuniform sequence of minimal triangulations $\mathfrak{T}_h$ of $\mathfrak{N}_h$. By quasiuniform we mean that there exist constants c and C such that

(7) each triangle $\tau \in \mathfrak{T}_h$ has an inscribed circle of radius ch, and a circumscribed circle of radius Ch.

To verify this consider any polygon $\omega_h^{(i)} \in \mathfrak{N}_h$ and select a fixed vertex, x_0, on the boundary of $\omega_h^{(i)}$. Connect this fixed vertex by straight lines to all other vertices of $\omega_h^{(i)}$. This construction is illustrated in figure 3.

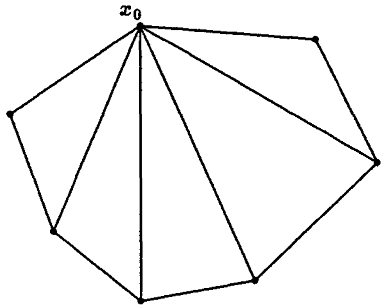

Figure 3.

Since $\omega_h^{(i)}$ is convex one obtains a triangulation $\mathfrak{T}_h^{(i)}$ of $w_h^{(i)}$. The union $\mathfrak{T}_h = \cup \mathfrak{T}_h^{(i)}$ is a minimal triangulation of $\mathfrak{N}_h$. Due to the conditions (5*a*)-(5*c*) (and the fact that the maximal number of edges in any of the polygons $\omega_h^{(i)}$ is bounded independently of h) it follows that this sequence of triangulations $\mathfrak{T}_h$ is quasiuniform. We want to emphasize that the particular construction method outlined above is not the only method to produce a minimal triangulation of $\mathfrak{N}_h$ (*cf. fig.* 2*a*). It is, however, very easy to see that any minimal triangulation of $\mathfrak{N}_h$ is quasiuniform due to the conditions (5*a*)-(5*c*).

THEOREM 1 : *Let $\mathfrak{N}_h$ be a regular sequence of planar, polygonal networks, spanning Ω and with maximal edge length approaching zero. Let $\{U_k^h\}$ denote the minimizer of* (3) *subject to the boundary conditions* (4), (6). *There exists a subsequence, for simplicity also denoted by $\mathfrak{N}_h$, and a measurable, matrix valued function A, $A(x)$ is symmetric, $0 < c\gamma_{\min} \leq A(x) \leq C\gamma_{\max}$, such that : for any $\phi \in H^{\frac{1}{2}+\varepsilon}(\partial\Omega)$ and any sequence of minimal triangulations $\mathfrak{T}_h$ of the subsequence $\mathfrak{N}_h$, the piecewise linear interpolants, U^h, of the discrete voltages $\{U_k^h\}$, satisfy*

$$U^h \text{ converges weakly in } H^1(\Omega) \text{ towards } U^0,$$

where U^0 denotes the solution to the elliptic boundary value problem

$$\nabla \cdot (A \nabla U^0) = 0 \text{ in } \Omega , \quad U^0 = \phi \text{ on } \partial\Omega .$$

Furthermore if ω is a smooth subdomain $\subset\subset \Omega$, then the local power dissipation in the network

$$\frac{1}{2} \sum_{x_k, x_l \in \omega} \gamma_{k,l}^h (U_k^h - U_l^h)^2 \quad (\dagger)$$

converges to

$$\frac{1}{2} \int_\omega (A \nabla U^0, \nabla U^0)\, dx ,$$

along the subsequence $\mathfrak{N}_h$. The constants c and C depend on c_0, C_0 and d_0, but are otherwise independent of the subsequence $\mathfrak{N}_h$.

1. PRELIMINARIES

Part of the difficulty in formulating Theorem 1 is the need to interpolate the discrete voltage potentials. If the original networks $\mathfrak{N}_h$ are themselves triangulations then it is not very difficult to rewrite the energy expression (3) as a integral over Ω. If the edge $e_{k,l}$ is common to the triangles τ and τ' then we assign half of the conductance $\gamma_{k,l}$ to τ the other half to τ'. For an edge on the boundary of Ω we assign the entire conductance to the single triangle to which this edge belongs. Consider now a single triangle, τ, as shown in figure 4, with vertices x_1, x_2 and x_3 and edges given by the vectors e_1, e_2 and e_3. The corresponding assigned conductances are denoted γ_1, γ_2 and γ_3. The expression

$$\frac{1}{2} (\gamma_1 (U_1 - U_2)^2 + \gamma_2 (U_2 - U_3)^2 + \gamma_3 (U_3 - U_1)^2)$$

may be written

$$\frac{1}{2} \int_\tau (A_\tau \nabla U, \nabla U)\, dx .$$

where U is the linear interpolant of $\{U_i\}$ and A_τ is the symmetric positive definite matrix

$$A_\tau = \sum_{j=1}^{3} e_j \gamma_j e_j^T / |\tau| . \tag{8}$$

($\dagger$) What we mean by this notation is a sum over those edges whose endpoints x_k and x_ℓ lie in ω Each edge is therefore represented only once in the sum

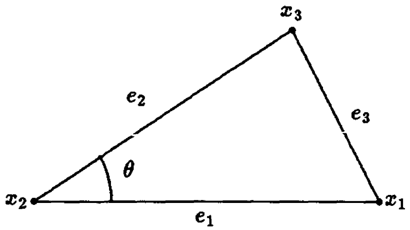

Figure 4.

With the above assignment strategy it is not difficult to see that

$$c\gamma_{\min} \leq A_\tau \leq C\gamma_{\max} ,$$

where the constants c and C depend on the constants from the conditions (5*a*) and (5*c*).

The entire minimization problem may now be formulated in terms of an integral over Ω. We seek the minimizer of

$$\frac{1}{2}\int_\Omega (A(\mathfrak{T}_h)\nabla U, \nabla U)\,dx \tag{9}$$

in the set of continuous piecewise linear functions relative to the triangulation $\mathfrak{T}_h(=\mathfrak{N}_h)$, subject to the following boundary condition

$$U(x_{k_0}) = \phi(x_{k_0}) \text{ at any vertex of } \mathfrak{T}_h \text{ lying on } \partial\Omega .$$

The matrix valued function $A(\mathfrak{T}_h)$ is defined by

$$A(\mathfrak{T}_h)(x) = A_\tau \; x \in \tau , \quad \tau \in \mathfrak{T}_h . \tag{10}$$

In the following $P^{[1],0}(\mathfrak{T}_h)$ denotes the set of continuous piecewise linear functions relative to $\mathfrak{T}_h$, and $P_0^{[1],0}(\mathfrak{T}_h)$ denote those functions of $P^{[1],0}(\mathfrak{T}_h)$ that vanish on $\partial\Omega$. Taking first variations of the expression (9) we seek the minimizer U^h as the solution to

$$\begin{aligned} & U^h \in P^{[1],0}(\mathfrak{T}_h) \\ & \int_\Omega (A(\mathfrak{T}_h)\nabla U^h, \nabla V^h)\,dx = 0 \quad \forall V^h \in P_0^{[1],0}(\mathfrak{T}_h) \\ & U^h(x_{k_0}) = \phi(x_{k_0}) \text{ at any vertex of } \mathfrak{T}_h \text{ lying on } \partial\Omega . \end{aligned} \tag{11}$$

If the original networks $\mathfrak{N}_h$ are not triangulations then there are edges in minimal triangulations $\mathfrak{T}_h$ that are not part of $\mathfrak{N}_h$. Let e_{kl} be an edge of $\mathfrak{T}_h$ which is not contained in $\mathfrak{N}_h$ and let this edge be common to the two triangles τ and τ'. We assign the energy contribution $-\gamma^*(U_k - U_l)^2$ to τ

and the contribution $\gamma^*(U_k - U_l)^2$ to τ' or vice versa, such as to make these contributions cancel when summation is performed over all triangles. It is possible to select the signs so that any triangle of the minimal triangulation $\mathfrak{T}_h$ has at most one of its edges contributing a negative term. To see this it suffices to consider a single polygon $\omega^{(i)}$ (not a triangle) associated with $\mathfrak{N}_h$. Let n denote the number of vertices of $\partial\omega^{(i)}$ and let m denote the number of triangles of $\mathfrak{T}^{(i)}$ that have 2 edges in common with $\partial\omega^{(i)}$. A simple counting argument gives that $\mathfrak{T}^{(i)}$ must have exactly $2n - \frac{3}{2}m$ edges, $n - m$ triangles and n vertices. According to Euler's formula we get $(n - m) + n - \left(2n - \frac{3}{2}m\right) = 1$, from which we deduce that m must be 2. Pick one of the two triangles that has two edges in common with $\partial\omega^{(i)}$ and assign it the number 1, it has exactly one neighboring triangle (one with which it shares an edge), assign that triangle the number 2. Either triangle number 2 coincides with the second triangle that shares two edges with $\partial\omega^{(i)}$ (and we terminate) or else it has exactly one neighboring triangle that has not yet been numbered, in the latter case we assign that triangle the number 3. We proceed this way till we reach the second triangle that shares two edges with $\partial\omega^{(i)}$. Upon termination we have obtained a numbering of all the triangles in $\mathfrak{T}^{(i)}$ with the property that triangle τ_k has one edge in common with triangle τ_{k+1}. This enumeration strategy is illustrated in figure 5, using the two triangulations from figure 2*a* and figure 3 respectively. If e is the common edge between triangle τ_k and triangle τ_{k+1}, then we assign the negative energy contribution to triangle τ_k the positive contribution to triangle τ_{k+1}. In this way no triangle has more than one side representing a negative conductance.

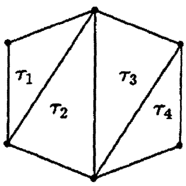

Figure 5*a*.

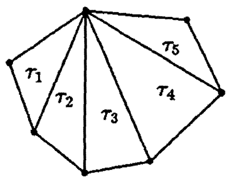

Figure 5*b*.

We now get the formula

$$\frac{1}{2} \sum_{e_{k,l} \in \mathfrak{N}_h} \gamma_{k,l}(U_k - U_l)^2$$
$$= \frac{1}{2} \sum_{e_{k,l} \in \mathfrak{T}_h} \gamma_{k,l}(U_k - U_l)^2$$
$$= \frac{1}{2} \int_\Omega (A(\mathfrak{T}_h)\nabla U, \nabla U)\, dx\,,$$

where the matrix valued function, $A(\mathfrak{T}_h)$, as before is given by the formulas (8) and (10). In contrast to before one of the γ_j's in (8) may now be negative. We observe that

LEMMA 1 : *If γ_1 and γ_2 are positive numbers and γ_3 is of arbitrary sign then the symmetric matrix $A_\tau = \sum_{j=1}^{3} e_j \gamma_j e_j^T/|\tau|$ is positive definite iff* $-\dfrac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2} < \gamma_3$.

Proof : We use the notation shown in figure 4 and pick a coordinate system which has origin at x_2, x-axis parallel to e_1, y-axis parallel to $e_1^\perp$, and such that τ lies in the first two quadrants. The angle θ denotes the angle between e_1 and e_2. A simple computation now gives that

$$\sum_{j=1}^{3} e_j \gamma_j e_j^T = |e_1|^2 \begin{pmatrix} (\gamma_1 + \gamma_3) + (\gamma_2 + \gamma_3) r^2 \cos^2 \theta - 2 \gamma_3 r \cos \theta & (\gamma_2 + \gamma_3) r^2 \cos \theta \sin \theta - \gamma_3 r \sin \theta \\ (\gamma_2 + \gamma_3) r^2 \cos \theta \sin \theta - \gamma_3 r \sin \theta & (\gamma_2 + \gamma_3) r^2 \sin^2 \theta \end{pmatrix}$$

with $r = |e_2|/|e_1|$. Computing the determinant of this matrix we get

$$\begin{aligned} &\det \left(\sum_{j=1}^{3} e_j \gamma_j e_j^T \right) \\ &= [(\gamma_1 + \gamma_3)(\gamma_2 + \gamma_3) - \gamma_3^2] |e_1|^2 \, |e_2|^2 \sin^2 \theta \\ &= 4[(\gamma_1 + \gamma_3)(\gamma_2 + \gamma_3) - \gamma_3^2] \, |\tau|^2 . \end{aligned}$$

The matrix $\sum_{j=1}^{3} e_j \gamma_j e_j^T/|\tau|$ is therefore positive definite iff $\gamma_2 + \gamma_3 > 0$ and $(\gamma_1 + \gamma_3)(\gamma_2 + \gamma_3) - \gamma_3^2 > 0$. Since $\gamma_1 \gamma_2/(\gamma_1 + \gamma_2) < \min(\gamma_1, \gamma_2)$ these latter conditions are equivalent to $\gamma_3 > -\gamma_1 \gamma_2/(\gamma_1 + \gamma_2)$. □

Consider a single polygon $\omega^{(i)}$ (not a triangle) associated with $\mathfrak{N}_h$. Let the triangles of $\mathfrak{T}^{(i)}$ be labeled through the strategy described earlier (and illustrated in *fig.* 5*a* and 5*b*). Corresponding to the common edge between τ_k and τ_{k+1} we assign conductivity $-\gamma_k^* = -\alpha_k \gamma_{\min}$ to τ_k and $\gamma_k^* = \alpha_k \gamma_{\min}$ to τ_{k+1}. From lemma 1 it now follows that the matrices A_{τ_k} are positive definite if

$$0 < \alpha_1 \gamma_{\min} < \frac{1}{2/\gamma_{\min} + 2/\gamma_{\min}}$$

$$0 < \alpha_k \gamma_{\min} < \frac{1}{1/\alpha_{k-1} \gamma_{\min} + 2/\gamma_{\min}}, \quad k = 2, \ldots, K^{(i)} - 1 ,$$

i.e., the matrices A_{τ_k} are positive definite if we select the constants $\alpha_k > 0$ to satisfy

$$\alpha_1 < 1/4 \tag{12}$$

$$\alpha_k < \frac{\alpha_{k-1}}{1 + 2\,\alpha_{k-1}} \quad k = 2, \ldots, K^{(i)} - 1 .$$

Here $K^{(i)}$ denotes the number of triangles from $\mathfrak{T}_h$ which lie in $\omega^{(i)}$. Since there is a uniform upper bound on the number of triangles of $\mathfrak{T}_h$ which lie in any one polygon from the tiling associated with $\mathfrak{N}_h$, it is furthermore possible to select the α_k so that the matrix valued function $A(\mathfrak{T}_h)$ satisfies

$$c\gamma_{\min} \leq A(\mathfrak{T}_h) \leq C\gamma_{\max} , \tag{13}$$

with constants c and C depending on the constants from (5a) and (5c). The problem of finding the discrete voltage potentials $\{U_k\}$ has thus again been reduced to the variational problem (11). In the next section we shall prove that there exists a subsequence of networks $\mathfrak{N}_h$ and a positive definite matrix A such that the functions U^h converge weakly in $H^1(\Omega)$ towards the solution to $\nabla \cdot (A\,\nabla U^0) = 0$ in Ω, $U^0 = \phi$ on $\partial\Omega$, for any ϕ.

2. PROOF OF THEOREM 1

The proof of Theorem 1 consists in an appropriate combination of ideas of F. Murat and L. Tartar, concerning homogenization convergence [16, 17], with well known results from approximation theory. A crucial role is played by the following localization lemma, which not surprisingly also plays a fundamental role in most proofs of interior estimates for the finite element method [18].

LEMMA 2 : *Let W^h be in $P^{[1],0}$ relative to the triangulation $\mathfrak{T}_h$ and let $\psi \in C_0^\infty(\Omega)$. Then there exist $V^h \in P_0^{[1],0}(\mathfrak{T}_h)$ such that*

$$\| \psi W^h - V^h \|_{H^1(\Omega)} \leq Ch \, \| W^h \|_{H^1(\Omega)} ,$$

where h denotes the meshsize of $\mathfrak{T}_h$. The constant C is independent of h and W^h but depends on ψ.

Proof : This result follows directly from the proof of property (A.2) in Section 2 of [18]. For completeness we provide a brief sketch of the argument : it is well known that there exist $V^h \in P_0^{[1],0}(\mathfrak{T}_h)$ such that

$$\| \psi W^h - V^h \|^2_{H^1(\Omega)} \leq Ch^2 \sum_i \, | \psi W^h |^2_{2, \tau^{(i)}} ,$$

where the sum is taken over all those triangles that intersect the support of ψ, and where $|\cdot|_{2,\tau^{(i)}}$ denotes the seminorm

$$|u|_{2,\tau^{(i)}} = \left(\sum_{|\alpha| = 2} \| D^{(\alpha)} u \|^2_{L^2(\tau^{(i)})} \right)^{1/2}$$

We shall not provide a proof of this estimate here, rather we refer the interested reader to [4]. Since W^h is linear we also have

$$\sum_i |\psi W^h|^2_{2,\tau^{(i)}} \leq C \sum_i \| W^h \|^2_{H^1(\tau^{(i)})} \leq C \| W^h \|^2_{H^1(\Omega)},$$

and the desired result follows. □

As stated in Theorem 1 the limiting second order differential operator and the subsequence along which the piecewise linear interpolants converge to U^0 *do not* depend on the particular choice of minimal triangulations of $\mathfrak{N}_h$. To establish this independence of the choice of triangulations we shall at various places make use of the following two simple lemmata.

LEMMA 3 : *Let $\mathfrak{T}_h$ and $\tilde{\mathfrak{T}}_h$ be minimal triangulations of the regular sequence of networks $\mathfrak{N}_h$. Let V^h and $\tilde{V}^h$ be continuous, piecewise linear polynomials relative to $\mathfrak{T}_h$ and $\tilde{\mathfrak{T}}_h$ respectively. Assume that $V^h(x) = \tilde{V}^h(x)$ at all vertices of $\mathfrak{N}_h$. Then there exists a constant C, independent of h, V^h and $\tilde{V}^h$ such that*

$$\| V^h - \tilde{V}^h \|_{L^2(\Omega)} \leq Ch \| V^h - \tilde{V}^h \|_{H^1(\Omega)} .$$

Proof : Consider a single polygon, ω_h, in the tiling corresponding to $\mathfrak{N}_h$. Let S_h denote a square with side length Ch containing ω_h. Let

$$W^h = \begin{cases} V^h - \tilde{V}^h, & \text{in } \omega_h \\ 0, & \text{in } S_h \backslash \omega_h . \end{cases}$$

This function W^h is in $H^1(S_h)$ and vanishes on the boundary of S_h. From the rescaled version of Poincaré's inequality it follows that

$$\| W^h \|_{L^2(S_h)} \leq Ch \| W^h \|_{H^1(S_h)} .$$

Consequently

$$\| V^h - \tilde{V}^h \|_{L^2(\omega_h)} \leq Ch \| V^h - \tilde{V}^h \|_{H^1(\omega_h)} . \tag{14}$$

Summation of the inequalities (14) over all polygons ω_h associated with $\mathfrak{N}_h$ leads to the desired estimate. □

LEMMA 4 : *Let V^h and $\tilde{V}^h$ be as in Lemma 3. There exist positive constants c, C such that*

$$c\| V^h\|_{H^1(\Omega)} \leq \| \tilde{V}^h \|_{H^1(\Omega)} \leq C \| V^h\|_{H^1(\Omega)} .$$

Proof : The proof of this follows immediately from the observation that $\| V^h\|_{H^1(\Omega)}$ and $\| \tilde{V}^h \|_{H^1(\Omega)}$ are both equivalent to the expression

$$\left(\sum_{e_{kl} \in \mathfrak{N}_h} (V_k^h - V_l^h)^2 \right)^{1/2},$$

where V_k^h denote the common values of V^h and $\tilde{V}^h$ at the vertices of $\mathfrak{N}_h$. □

Let $\mathfrak{T}_h$ be a sequence of minimal triangulations of the networks $\mathfrak{N}_h$. An important ingredient of the proof of Theorem 1 is the construction of auxiliary mappings $R_0 : H^{-1}(\Omega) \to H_0^1(\Omega)$ and $S_0 : H^{-1}(\Omega) \to (L^2(\Omega))^2$. Given any $F \in H^{-1}(\Omega)$, $R_0(F)$ is our candidate for the solution of the boundary value problem

$$- \nabla . (A \nabla W) = F, \text{in } \Omega , \quad W = 0 \text{ on } \partial\Omega ,$$

involving the effective conductivity A. The vector field $S_0(F)$ is our candidate for the flux associated with $R_0(F)$, and the effective conductivity A will therefore implicitly be defined by $S_0(F) = A \nabla R_0(F)$.

Consider the problem

(16)
$$\begin{aligned} &\text{find } W^h \in P_0^{[1],0}(\mathfrak{T}_h) \text{ such that} \\ &\int_\Omega A(\mathfrak{T}_h) \nabla W^h \nabla V^h \, dx = \langle F, V^h \rangle \quad \forall V^h \in P_0^{[1],0}(\mathfrak{T}_h) . \end{aligned}$$

Define $R_h(F) := W^h$. The mapping $R_h : H^{-1}(\Omega) \to H_0^1(\Omega)$ satisfies

$$\| R_h(F) \|_{H^1(\Omega)} \leq C \| F \|_{H^{-1}(\Omega)} .$$

Passing to a subsequence (which we for simplicity also index by h), we may obtain that

(17)
$$\begin{aligned} &R_h(F) \to R_0(F) \text{ weakly in } H_0^1(\Omega) \\ &S_h(F) := A(\mathfrak{T}_h) \nabla R_h(F) \to S_0(F) \text{ weakly in } L_2(\Omega) , \end{aligned}$$

where the linear operators R_0 and S_0 satisfy

$$\| R_0(F) \|_{H^1(\Omega)} \leq C \| F \|_{H^{-1}(\Omega)} \text{ and}$$

(18)

$$\| S_0(F) \|_{L_2(\Omega)} \leq C \| F \|_{H^{-1}(\Omega)} ,$$

(in order to show that one may take the same subsequence for all $F \in H^{-1}(\Omega)$ we first verify, by a diagonalization procedure, that (17) holds for a dense, countable subset of $H^{-1}(\Omega)$ and then we apply a density argument). Let P_h denote the orthogonal projection $H_0^1(\Omega) \to P_0^{[1],0}(\mathfrak{T}_h) \subset H_0^1(\Omega)$, then it is clear that

$$\sup_{V^h \in P_0^{[1],0}} \frac{\langle F, V^h \rangle}{\| V^h \|_{H^1(\Omega)}} = \sup_{V \in H_0^1(\Omega)} \frac{\langle F, P_h V \rangle}{\| V \|_{H^1(\Omega)}} = \| P_h^* F \| , \tag{19}$$

where $P_h^* : H^{-1}(\Omega) \to H^{-1}(\Omega)$ is the dual of P_h. Due to the fact that the matrices $A(\mathfrak{T}_h)$ are uniformly bounded away from 0 it follows from (16) and (19) that

$$\| P_h^* F \|_{H^{-1}(\Omega)} \leq C \| R_h(F) \|_{H^1(\Omega)} . \tag{20}$$

By insertion of $V^h = R_h(F)$ into (16) and use of (20) we now get

$$\| P_h^* F \|^2_{H^{-1}(\Omega)} \leq C \langle F, R_h(F) \rangle . \tag{21}$$

Since the triangulations $\mathfrak{T}_h$ have meshsize approaching 0 it is well known that

$$P_h^* F \to F \text{ weakly in } H^{-1}(\Omega) \text{ for any } F \in H^{-1}(\Omega) ;$$

because of (17), (21) and the weak lower semicontinuity of the norm it now follows that

$$\| F \|^2_{H^{-1}(\Omega)} \leq C \langle F, R_0(F) \rangle , \tag{22}$$

or

$$\| F \|_{H^{-1}(\Omega)} \leq C \| R_0(F) \|_{H^1(\Omega)} . \tag{23}$$

R_0 is therefore an isomorphism $H^{-1}(\Omega) \to R_0(H^{-1}(\Omega))$, and the latter space is a closed subspace of $H_0^1(\Omega)$, due to (18) and (23). If $G \in H^{-1}(\Omega)$ is such that G anihilates $R_0(H^{-1}(\Omega))$, then $\langle G, R_0(G) \rangle = 0$ and using (22) we get that $G = 0$; it follows that $R_0(H^{-1}(\Omega)) = H_0^1(\Omega)$. We may therefore conclude that

(24) R_0 is an isomorphism from $H^{-1}(\Omega)$ onto $H_0^1(\Omega)$.

LEMMA 5 : *The limiting operators R_0 and S_0 depend on the particular subsequence of networks, $\mathfrak{N}_h$, which has been extracted, but they are independent of the choice of minimal triangulations.*

Proof: Let $\mathfrak{T}_h$ and $\tilde{\mathfrak{T}}_h$ denote two minimal triangulations of the network $\mathfrak{N}_h$. If V^h is an element in $P^{[1],0}(\mathfrak{T}_h)$, i.e. piecewise linear respective to the triangulation $\mathfrak{T}_h$, then we let $\tilde{V}^h$ denote the element in $P^{[1],0}(\tilde{\mathfrak{T}}_h)$ which agrees with V^h at all the vertices. From the very definition of the matrices $A(\mathfrak{T}_h)$ and $A(\tilde{\mathfrak{T}}_h)$ we get

$$\begin{aligned}
&\int_\Omega (A(\tilde{\mathfrak{T}}_h)\,\nabla \widetilde{R_h(F)}, \nabla \tilde{V}^h)\,dx \\
(25)\qquad &= \int_\Omega (A(\mathfrak{T}_h)\,\nabla R_h(F), \nabla V^h)\,dx \\
&= \langle F, V^h\rangle, \quad \forall V^h \in P^{[1],0}(\mathfrak{T}_h)\,.
\end{aligned}$$

Let $\tilde{R}_h$ denote the analog of the operator R_h, only corresponding to the triangulation $\tilde{\mathfrak{T}}_h$. $\tilde{R}_h(F)$ satisfies

$$\begin{aligned}
&\int_\Omega (A(\tilde{\mathfrak{T}}_h)\,\nabla \tilde{R}_h(F), \nabla \tilde{V}^h)\,dx \\
&= \left\langle F, \tilde{V}^h\right\rangle \quad \forall V^h \in P^{[1],0}(\mathfrak{T}_h)\,.
\end{aligned}$$

We thus have

$$\begin{aligned}
&\int_\Omega (A(\tilde{\mathfrak{T}}_h)\,\nabla(\widetilde{R_h(F)} - \tilde{R}_h(F)), \nabla \tilde{V}^h)\,dx \\
&= \left\langle F, V^h - \tilde{V}^h\right\rangle \\
(26)\qquad &\leq C\,\|F\|_{L^2(\Omega)}\,\left\|V^h - \tilde{V}^h\right\|_{L^2(\Omega)} \\
&\leq C'\,h\,\|F\|_{L^2(\Omega)}\,\left\|V^h - \tilde{V}^h\right\|_{H^1(\Omega)} \\
&\leq C''\,h\,\|F\|_{L^2(\Omega)}\,\left\|\tilde{V}^h\right\|_{H^1(\Omega)},
\end{aligned}$$

where we used Lemma 3 for the next to last estimate, and Lemma 4 for the last estimate. Based on (26) we conclude that

$$(27)\qquad \left\|\widetilde{R_h(F)} - \tilde{R}_h(F)\right\|_{H^1(\Omega)} \leq Ch\,\|F\|_{L^2(\Omega)}\,.$$

By Lemma 3 and Lemma 4

$$\begin{aligned}
&\left\|R_h(F) - \widetilde{R_h(F)}\right\|_{L^2(\Omega)} \\
(28)\qquad &\leq Ch\,\|R_h(F)\|_{H^1(\Omega)} \\
&\leq C'\,h\,\|F\|_{H^{-1}(\Omega)}\,.
\end{aligned}$$

A combination of (27) and (28) yields

$$\left\| R_h(F) - \tilde{R}_h(F) \right\|_{L^2(\Omega)} \leq Ch \|F\|_{L^2(\Omega)} . \tag{29}$$

We assume that $\mathfrak{T}_h$ form a sequence for which $R_h(F) \to R_0(F)$ weakly in $H_0^1(\Omega)$. Through extraction of a subsequence from $\tilde{\mathfrak{T}}_h$ we may assume that $\tilde{R}_h(F) \to \tilde{R}_0(F)$ weakly in $H_0^1(\Omega)$. From (29) it now follows that $\tilde{R}_0(F) = R_0(F)$ for any $F \in L^2(\Omega)$. By continuity the same identity holds for all $F \in H^{-1}(\Omega)$. Since the limit is independent of the particular subsequence of $\tilde{\mathfrak{T}}_h$ it furthermore follows that $\tilde{R}_h(F)$ converges to $R_0(F)$ along the entire sequence. This shows that R_0 is independent of the choice of minimal triangulations.

We now proceed to show that the limit S_0 is also independent of the choice of minimal triangulations. Our verification of this fact furthermore provides a construction of the desired limiting conductivity A. Let $\chi, \psi \in C_0^\infty(\Omega)$, with $\chi \equiv 1$ on supp (ψ), and let x_i, $i = 1, 2$ denote the two coordinate functions in $\mathbb{R}^2$. With $W_i^h = R_h(R_0^{-1}(\chi(x)\, x_i))$, we evaluate

$$\begin{aligned} &\int_\Omega (A(\mathfrak{T}_h)\, \nabla R_h(F), \nabla W_i^h)\, \psi(x)\, dx \\ &= \int_\Omega (A(\mathfrak{T}_h)\, \nabla R_h(F), \nabla (W_i^h \psi))\, dx \\ &- \int_\Omega (A(\mathfrak{T}_h)\, \nabla R_h(F), \nabla \psi)\, W_i^h\, dx . \end{aligned} \tag{30}$$

According to Lemma 2 there exist $V^h \in P_0^{[1],0}(\mathfrak{T}_h)$, such that

$$\begin{aligned} &\int_\Omega (A(\mathfrak{T}_h)\, \nabla R_h(F), \nabla (W_i^h \psi))\, dx \\ &= \int_\Omega (A(\mathfrak{T}_h)\, \nabla R_h(F), \nabla V^h)\, dx + O(h \|R_h(F)\|_{H^1(\Omega)} \, \|W_i^h\|_{H^1(\Omega)}) \\ &= \int_\Omega (A(\mathfrak{T}_h)\, \nabla R_h(F), \nabla V^h)\, dx + O(h \|F\|_{H^{-1}(\Omega)}), \text{ and} \end{aligned} \tag{31}$$

V^h converges towards $\psi(x)\, x_i$ weakly in $H_0^1(\Omega)$.

For the derivation of the last statement we use that $\chi \equiv 1$ on supp (ψ). We also have

$$\int_\Omega (A(\mathfrak{T}_h)\, \nabla R_h(F), \nabla V^h)\, dx = \langle F, V^h \rangle .$$

Use of this identity in connection with (31) now gives that

$$\int_\Omega (A(\mathfrak{T}_h)\, \nabla R_h(F), \nabla (W_i^h \psi))\, dx \text{ converges to } \langle F, \psi(x)\, x_i \rangle . \tag{32}$$

A the same time

$$\begin{aligned} &\int_\Omega (A(\mathfrak{T}_h)\, \nabla R_h(F), \nabla\psi)\, W_i^h\, dx \text{ converges to} \\ &\int_\Omega (S_0(F), \nabla\psi)\, \chi(x)\, x_i\, dx . \end{aligned} \tag{33}$$

Combining (30), (32) and (33) with the fact that $\nabla . S_0(F) = -F$ (and integration by parts) we conclude that

$$\begin{aligned} &\int_\Omega (A(\mathfrak{T}_h)\, \nabla R_h(F), \nabla W_i^h)\, \psi(x)\, dx \text{ converges to} \\ &\int_\Omega (S_0(F), \nabla (\chi(x)\, x_i))\, \psi\, dx = \int_\Omega (S_0(F))_i\, \psi\, dx . \end{aligned} \tag{34}$$

By interchanging the roles played by $R_h(F)$ and W_i^h in the preceeding argument we get

$$\begin{aligned} &\int_\Omega (A(\mathfrak{T}_h)\, \nabla R_h(F), \nabla W_i^h)\, \psi(x)\, dx \\ &= \int_\Omega (A(\mathfrak{T}_h)\, \nabla W_i^h, \nabla (R_h(F)\, \psi))\, dx \\ &- \int_\Omega (A(\mathfrak{T}_h)\, \nabla W_i^h, \nabla\psi)\, R_h(F)\, dx . \end{aligned} \tag{35}$$

According to Lemma 2 there exist $V^h \in P_0^{[1],0}(\mathfrak{T}_h)$, such that

$$\begin{aligned} &\int_\Omega (A(\mathfrak{T}_h)\, \nabla W_i^h, \nabla (R_h(F)\, \psi))\, dx \\ &= \int_\Omega (A(\mathfrak{T}_h)\, \nabla W_i^h, \nabla V^h)\, dx + O(h \| W_i^h \|_{H^1(\Omega)} \| R_h(F) \|_{H^1(\Omega)}) \\ &= \int_\Omega (A(\mathfrak{T}_h)\, \nabla W_i^h, \nabla V^h)\, dx + O(h \| F \|_{H^{-1}(\Omega)}), \text{ and} \\ &V^h \text{ converges towards } R_0(F)\, \psi \text{ weakly in } H_0^1(\Omega) . \end{aligned} \tag{36}$$

We also have

$$\int_\Omega (A(\mathfrak{X}_h)\, \nabla W_i^h, \nabla V^h)\, dx = \langle R_0^{-1}(\chi(x)\, x_i), V^h \rangle .$$

Use of this identity in connection with (36) now gives that

$$\int_\Omega (A(\mathfrak{X}_h)\, \nabla W_i^h, \nabla (R_h(F)\, \psi))\, dx \text{ converges to} \tag{37}$$
$$\langle R_0^{-1}(\chi(x)\, x_i), R_0(F)\, \psi \rangle .$$

At the same time

$$\int_\Omega (A(\mathfrak{X}_h)\, \nabla W_i^h, \nabla\psi)\, R_h(F)\, dx \text{ converges to} \tag{38}$$
$$\int_\Omega (S_0(R_0^{-1}(\chi(x)\, x_i)), \nabla\psi)\, R_0(F)\, dx .$$

Combining (35), (37) and (38) with the fact that $\nabla \cdot S_0(R_0^{-1}(\chi(x)\, x_i)) = - R_0^{-1}(\chi(x)\, x_i)$ (and integration by parts) we conclude that

$$\int_\Omega (A(\mathfrak{X}_h)\, \nabla R_h(F), \nabla W_i^h)\, \psi(x)\, dx \text{ converges to} \tag{39}$$
$$\int_\Omega (S_0(R_0^{-1}(\chi(x)\, x_i)), \nabla R_0(F))\, \psi\, dx .$$

Given $\omega \subset\subset \Omega$ select $\chi \in C_0^\infty(\Omega)$ so that $\chi \equiv 1$ in a neighborhood of $\bar{\omega}$ and define a matrix values function by

$$\text{row } i \text{ of } A = S_0(R_0^{-1}(\chi(x)\, x_i)) \text{ in } \omega . \tag{40}$$

It is not difficult to see that $A|_\omega$ is independent of the particular choice of χ, and therefore (40) provides a well defined A (on all of Ω), see [16, 17]. The matrix A can easily be shown to be symmetric, uniformly positive definite and uniformly bounded in Ω. Due to (34) and (39)

$$S_0(F) = A\, \nabla R_0(F) \text{ in } \omega ,$$

and since $\omega \subset\subset \Omega$ may be selected arbitrarily it follows that $S_0(F) = A\, \nabla R_0(F)$ in Ω. For any $F \in H^{-1}(\Omega)$ we therefore have

$$S_0(F) = A\, \nabla R_0(F) \text{ and} \tag{41}$$
$$\nabla \cdot (A\, \nabla R_0(F)) = -F \text{ in } \Omega , \quad R_0(F) = 0 \text{ on } \partial\Omega .$$

If we had made a different choice of minimal triangulations $\tilde{\mathfrak{T}}_h$, then after possibly extracting a subsequence we would obtain limits $\tilde{R}_0 = R_0$ and $\tilde{S}_0$ and a matrix valued function $\tilde{A}$ such that for any $F \in H^{-1}(\Omega)$

$$\tilde{S}_0(F) = \tilde{A}\,\nabla R_0(F) \text{ and}$$

(42)

$$\nabla\,.\,(\tilde{A}\,\nabla R_0(F)) = -F \text{ in } \Omega\,, \quad R_0(F) = 0 \text{ on } \partial\Omega\,.$$

Since R_0 maps onto all of $H_0^1(\Omega)$ it follows directly from (41) and (42) that

$$\text{(43)} \qquad \int_\Omega (A\,\nabla V, \nabla W)\,dx = \int_\Omega (\tilde{A}\,\nabla V, \nabla W)\,dx \quad \forall V, W \in H_0^1(\Omega)\,.$$

Select complex V of the form $\psi(x)\,e^{(\zeta + i\xi, x)}$ and complex W of the form $\psi(x)\,e^{(-\zeta + i\xi, x)}$, with $\psi \in C_0^\infty(\Omega)$ and ζ, $\xi \in \mathbb{R}^2$. Then

$$\nabla V = \psi(x)\,e^{(\zeta + i\xi, x)}(\zeta + i\xi) + e^{(\zeta + i\xi, x)}\,\nabla\psi(x)$$
$$\nabla W = \psi(x)\,e^{(-\zeta + i\xi, x)}(-\zeta + i\xi) + e^{(-\zeta + i\xi, x)}\,\nabla\psi(x)\,,$$

so that

$$\text{(44)} \quad (A\,\nabla V, \nabla W) = -\psi(x)^2\,e^{2i(\xi, x)}(A(x)\,\zeta, \zeta) + O(|\zeta|) \text{ as } |\zeta| \to \infty\,,$$

and

$$\text{(45)} \quad (\tilde{A}\,\nabla V, \nabla W) = -\psi(x)^2\,e^{2i(\xi, x)}(\tilde{A}(x)\,\zeta, \zeta) + O(|\zeta|) \text{ as } |\zeta| \to \infty\,.$$

By insertion of (44) and (45) into (43), division by $|\zeta|^2$ and passage to the limit in $|\zeta|$ (for a fixed direction $e = \zeta/|\zeta|$) we get

$$\int_{\mathbb{R}^2} (A(x)\,e, e)\,\psi(x)^2\,e^{2i(\xi, x)}\,dx = \int_{\mathbb{R}^2} (\tilde{A}(x)\,e, e)\,\psi(x)^2\,e^{2i(\xi, x)}\,dx\,.$$

It now follows, through inversion of the Fourier transform, that

$$(A(x)\,e, e)\,\psi(x)^2 = (\tilde{A}(x)\,e, e)\,\psi(x)^2 \quad \forall \psi \in C_0^\infty(\Omega)\,, \quad e \in \mathbb{R}^2\,.$$

Therefore $\tilde{A} = A$ and $\tilde{S}_0(F) = \tilde{A}\,\nabla R_0(F) = A\,\nabla R_0(F) = S_0(F)$. Since the limit $\tilde{S}_0$ is independent of the particular subsequence it furthermore follows that $A(\tilde{\mathfrak{T}}_h)\,\nabla \tilde{R}_h(F)$ converges towards $S_0(F)$ along the entire sequence $\tilde{\mathfrak{T}}_h$. This completes the proof of Lemma 5. □

2.1. Remark

There is a certain arbitrariness in the way the matrices $A(\mathfrak{T}_h)$ are constructed. By a slight extension of the proof of Lemma 5 it follows that R_0 and S_0 are also independent of the particular choice of (cancelling) conductances corresponding to edges which are in $\mathfrak{T}_h$ but not in $\mathfrak{N}_h$. It is furthermore of no consequence for the limiting operators R_0 and S_0, whether we distribute the conductances, corresponding to edges in $\mathfrak{N}_h$, equally on neighboring triangles or in some other ratio.

For later use it shall be convenient to have the following lemma :

LEMMA 6 : *Let $\mathfrak{N}_h$ be a subsequence of networks for which the limits in (17) are attained. Let $\mathfrak{T}_h$ be a minimal triangulation of $\mathfrak{N}_h$, and let ψ be in $C_0^\infty(\Omega)$. Then*

$$\int_\Omega (A(\mathfrak{T}_h)\, \nabla R_h(F), \nabla R_h(G))\, \psi\, dx \text{ converges to}$$

$$\int_\Omega (A\, \nabla R_0(F), \nabla R_0(G))\, \psi\, dx$$

for any $F, G \in H^{-1}(\Omega)$.

Proof : The proof follows the exact same lines as were used to derive (34) (or (39)) ; it shall for reasons of brevity not be given here. □

We are now in a position to give a proof of Theorem 1.

Proof : Since the boundary value ϕ is in $H^{1/2+\varepsilon}(\partial\Omega)$ there exists $W \in H^{1+\varepsilon}(\Omega)$ such that $W|_{\partial\Omega} = \phi$ and $\|W\|_{H^1+_\varepsilon} \leq C\,\|\phi\|_{H^{1/2+\varepsilon}}$ ([1, 11]). Let $W^h \in P^{[1],0}(\mathfrak{T}_h)$ denote the piecewise linear interpolant of W. Then

$$\begin{aligned} &\|W^h\|_{H^1(\Omega)} \leq C\,\|\phi\|_{H^{1/2+\varepsilon}(\partial\Omega)} \\ &W^h(x_{k_0}) = \phi(x_{k_0}) \text{ at all vertices of } \mathfrak{T}_h \text{ on } \partial\Omega \\ &W^h|_{\partial\Omega} \to \phi \text{ in } L^2(\partial\Omega)\,. \end{aligned} \tag{46}$$

The function $U_*^h = U^h - W^h$ satisfies

$$\begin{aligned} &U_*^h \in P_0^{[1],0}(\mathfrak{T}_h), \text{ and} \\ &\int_\Omega A(\mathfrak{T}_h)\, \nabla U_*^h\, \nabla V^h\, dx = -\int_\Omega A(\mathfrak{T}_h)\, \nabla W^h\, \nabla V^h\, dx \\ &\forall V^h \in P_0^{[1],0}(\mathfrak{T}_h)\,. \end{aligned}$$

Therefore $\| U_*^h \|_{H^1} \leq C \| W^h \|_{H^1} \leq C' \| \phi \|_{H^{1/2+\varepsilon}}$, and consequently

$$\| U^h \|_{H^1(\Omega)} \leq C \| \phi \|_{H^{1/2+\varepsilon}(\partial\Omega)}$$
$$\| A(\mathfrak{T}_h) \nabla U^h \|_{L^2(\Omega)} \leq \| \phi \|_{H^{1/2+\varepsilon}(\partial\Omega)} .$$

By extraction of a subsequence we may obtain that

$$U^h \to U^0 \text{ weakly in } H^1(\Omega), \text{ and}$$
$$A(\mathfrak{T}_h) \nabla U^h \to \xi^0 \text{ weakly in } L^2(\Omega)$$
$$\text{for any } \phi \in H^{1/2+\varepsilon}(\partial\Omega) .$$

Here we are relying on a similar diagonalization procedure, and density argument as that which was used to derive (17). One may also show that U^0 and ξ^0 are independent of the choice of minimal triangulations and the particular conductance assignment strategy. We shall not give the proof of that here (since it is nearly identical to the proof of Lemma 5); we only briefly outline the steps that lead to an equation for U^0. Let χ, $\psi \in C_0^\infty(\Omega)$, with $\chi \equiv 1$ on supp (ψ). Let x_i, $i = 1, 2$ denote the two coordinate functions in $\mathbb{R}^2$ and define $W_i^h = R_h(R_0^{-1}(\chi(x)\, x_i))$. Performing the same calculations as in (30)-(34) we get

$$\text{(47)} \quad \int_\Omega (A(\mathfrak{T}_h) \nabla U^h, \nabla R_h(R_0^{-1}(\chi(x)\, x_i)))\, \psi \, dx \text{ converges to } \int_\Omega (\xi^0)_i \, \psi \, dx.$$

Interchanging the roles played by U^h and $R_h(R_0^{-1}(\chi(x)\, x_i))$ and performing the same calculations as in (35)-(39) we get

$$\text{(48)} \quad \begin{aligned} &\int_\Omega (A(\mathfrak{T}_h) \nabla U^h, \nabla R_h(R_0^{-1}(\chi(x)\, x_i)))\, \psi \, dx \text{ converges to} \\ &\int_\Omega (S_0(R_0^{-1}(\chi(x)\, x_i)), \nabla U^0)\, \psi \, dx . \end{aligned}$$

Let A denote the matrix valued function defined by (40). The statements (47) and (48) show that $\xi^0 = A \nabla U^0$ in Ω, and therefore

$$\nabla \cdot (A \nabla U^0) = \nabla \cdot \xi^0 = 0 \text{ in } \Omega .$$

Since $U^h|_{\partial\Omega} = W^h|_{\partial\Omega} \to \phi$ in $L^2(\partial\Omega)$ it follows that

$$U^0 = \phi \text{ on } \partial\Omega .$$

This verifies the first part of Theorem 1, except for the inequalities $c\gamma_{\min} \leq A \leq C\gamma_{\max}$. These inequalities follow immediately from the inequalities in Lemma 7 in combination with (13). Instead of providing a proof of these inequalities here, we refer the reader to the proof of Lemma 7 in the following section We now turn to the proof of the second statement about convergence of the local power dissipation

Given any positive ε, we can find smooth subdomains ω' and ω'' with $\omega \subset\subset \omega \subset\subset \omega$ for which

$$\int_{\omega} (A \nabla U^0, \nabla U^0)\, dx - \varepsilon/2 < \int_{\omega} (A \nabla U^0, \nabla U^0)\, dx \leqslant \int_{\omega} (A \nabla U^0, \nabla U^0)\, dx < \int_{\omega} (A \nabla U^0, \nabla U^0)\, dx + \varepsilon/2 \tag{49}$$

Select $0 \leqslant \psi \leqslant 1$ and $0 \leqslant \psi' \leqslant 1$ in $C_0^\infty(\Omega)$, such that

$$\begin{gathered} \omega' \subset\subset \{x . \psi' \equiv 1\} \subset \operatorname{supp}(\psi') \subset\subset \omega \text{ and }, \\ \omega \subset\subset \{x \quad \psi \equiv 1\} \subset \operatorname{supp}(\psi) \subset\subset \omega'' \end{gathered} \tag{50}$$

From (49) and (50) it follows immediately that

$$\int_{\omega} (A \nabla U^0, \nabla U^0)\, dx - \varepsilon/2 < \int_{\Omega} (A \nabla U^0, \nabla U^0)\, \psi'\, dx \leqslant \int_{\Omega} (A \nabla U^0, \nabla U^0)\, \psi\, dx < \int_{\omega} (A \nabla U^0, \nabla U^0)\, dx + \varepsilon/2 \tag{51}$$

Let ω_h be the union of all those polygons, associated with $\mathfrak{N}_h$, whose closure lie strictly inside ω and let $\tilde{\omega}_h$ denote the union of all those polygons, associated with $\mathfrak{N}_h$, whose closure intersect $\bar{\omega}$ Using (50) we get

$$\begin{aligned} \int_{\Omega} (A \nabla U^h, \nabla U^h)\, \psi'\, dx & \leqslant \int_{\omega_h} (A \nabla U^h, \nabla U^h)\, dx \\ & \leqslant \int_{\tilde{\omega}_h} (A \nabla U^h, \nabla U^h)\, dx \\ & \leqslant \int_{\Omega} (A \nabla U^h, \nabla U^h)\, \psi\, dx , \end{aligned} \tag{52}$$

for h sufficiently small. Performing calculations similar to those in (30)-(34) (or (35)-(39)) it is not difficult to see that

$$(53) \qquad \begin{aligned} &\int_\Omega (A \nabla U^h, \nabla U^h)\, \psi'\, dx \to \int_\Omega (A \nabla U^0, \nabla U^0)\, \psi'\, dx \text{ and} \\ &\int_\Omega (A \nabla U^h, \nabla U^h)\, \psi\, dx \to \int_\Omega (A \nabla U^0, \nabla U^0)\, \psi\, dx\,. \end{aligned}$$

Combining (53) with (52) and (51) we conclude that

$$(54) \qquad \begin{aligned} \int_\omega (A \nabla U^0, \nabla U^0)\, dx - \varepsilon/2 &< \int_{\omega_h} (A \nabla U^h, \nabla U^h)\, dx \\ &\leq \int_{\tilde{\omega}_h} (A \nabla U^h, \nabla U^h)\, dx \\ &< \int_\omega (A \nabla U^0, \nabla U^0)\, dx + \varepsilon/2\,, \end{aligned}$$

for h sufficiently small. Since ε is arbitrary and since

$$\int_{\omega_h} (A \nabla U^h, \nabla U^h)\, dx \leq \sum_{x_k, x_l \in \omega} \gamma_{k,l}^h (U_k^h - U_l^h)^2 \leq \int_{\tilde{\omega}_h} (A \nabla U^h, \nabla U^h)\, dx\,,$$

it follows immediately from (54) that the local power dissipation

$$\frac{1}{2} \sum_{x_k, x_l \in \omega} \gamma_{k,l}^h (U_k^h - U_l^h)^2$$

converges to

$$\frac{1}{2} \int_\omega (A \nabla U^0, \nabla U^0)\, dx$$

along the subsequence $\mathfrak{N}_h$. This completes the proof of Theorem 1. □

2.2. Remark

As mentioned earlier, network geometries containing Wheatstone bridges are not covered by our convergence analysis. Consider the single Wheatstone bridge shown in figure 6*a*, with associated edge conductances γ_{ij}. The condition

$$(55) \qquad \min \{\gamma_{12}\gamma_{34}, \gamma_{23}\gamma_{14}\} \geqslant \gamma_{13}\gamma_{24}$$

is necessary and sufficient in order that it be possible to replace the Wheatstone bridge by a configuration, as shown in figure 6*b*, without changing the voltages at any of the nodes in the original network (including nodes x_1 through x_4). New conductances for the edges of the graph shown in figure 6*b* are found by inversion of the so-called star-mesh transformation [7]. These conductances are not unique. The condition (55) expresses the fact, that the presence of diagonal edges has not changed the direction of current flow along any of the edges on the boundary of the Wheatstone bridge [9]. The configuration shown in figure 6*b* is included in our analysis, and consequently so are Wheatstone bridges for which the edge conductances satisfy (55).

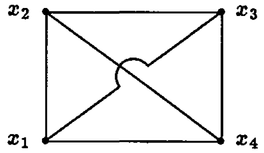

Figure 6a.

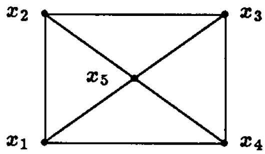

Figure 6b.

Consider a rectangular array of Wheatstone bridges. If all the Wheatstone bridges have horizontal and vertical edges of zero conductance (meaning these edges are really not part of the network), then we have in effect two disconnected networks, as shown in figure 7.

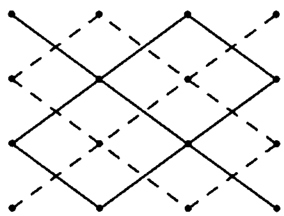

Figure 7.

If all the edges indicated by solid lines are assigned one conductance and all the edges indicated by dashed lines are assigned another conductance, then it is fairly obvious that a convergence result, such as that in Theorem 1 (with an appropriate interpolation) is no longer valid. It is not clear what happens if all edges in the Wheatstone bridges are required to have positive conductance, but one might speculate that if the horizontal and vertical conductances are sufficiently small (so that (55) is violated) then there will in general not be a limit, that can be characterized *via* a continuous conductance equation.

2.3. Remark

We chose here to perform the analysis within the framework of so-called homogenization convergence. Very related techniques are associated with the names of G- (and Γ-) convergence [5, 6]. Once the problem has been cast in a variational form, we could also have performed our analysis using these techniques. The method of Γ-convergence has recently been used to find effective limits of periodically reinforced structures [2, 3]. These structures are not discrete, rather they are modelled by continuous equations, including thin, web-like areas of extreme strength (conductance). The analysis is therefore entirely one of continuous problems. However, as far as effective limits are concerned, one would intuitively expect that these structures are quite related to geometrically periodic (discrete) networks with edges of constant conductance per unit length.

3. TWO SIMPLE INEQUALITIES FOR A

To provide some characterization of the possible effective matrices A, that can be attained, we prove two inequalities. The proof is extremely simple, and no different from that found in [17].

LEMMA 7: *Let $\mathfrak{T}_h$ be minimal triangulations corresponding to the subsequence $\mathfrak{N}_h$ from Theorem 1, and let A denote the limit whose existence is guaranteed by Theorem 1. If A^+ is a weak* limit of a subsequence of $A(\mathfrak{T}_h)$ in $L^\infty(\Omega)$, and if $(A^-)^{-1}$ is a weak* limit of a subsequence of $(A(\mathfrak{T}_h))^{-1}$ in $L^\infty(\Omega)$, then*

$$A^- \leqslant A \leqslant A^+ \text{ a.e. in } \Omega .$$

Proof: Take χ, $\psi \in C_0^\infty(\Omega)$ with $\chi \equiv 1$ on supp (ψ), and consider the expression

$$\int_\Omega (A(\mathfrak{T}_h)\nabla[R_h(F) - R_0(F)], \nabla[R_h(F) - R_0(F)])\,\psi\,dx \tag{56}$$

with $F \in H^{-1}(\Omega)$. Due to (17), (41) and Lemma 6 this expression converges to

$$\int_\Omega (A^+\nabla R_0(F), \nabla R_0(F))\,\psi\,dx - \int_\Omega (A\nabla R_0(F), \nabla R_0(F))\,\psi\,dx ,$$

along the subsequence, which corresponds to the weak* limit A^+. Since the expression (56) is non-negative it follows that

$$\int_\Omega (A\nabla R_0(F), \nabla R_0(F))\,\psi\,dx \leqslant \int_\Omega (A^+\nabla R_0(F), \nabla R_0(F))\,\psi\,dx . \tag{57}$$

Insertion of $F = \sum \eta_i R_0^{-1}(\chi(x) x_i)$ into (57) gives that

$$\int_\Omega (A\eta, \eta) \psi\, dx \leq \int_\Omega (A^+ \eta, \eta) \psi\, dx,$$

for any $\psi \in C_0^\infty(\Omega)$. This immediately proves the second inequality of this lemma.

The first inequality of this lemma may be derived in a similar fashion, based on analysis of the limit of the expression

$$\int_\Omega (A(\mathfrak{T}_h)^{-1} [A(\mathfrak{T}_h) \nabla R_h(F) - A^- \nabla R_0(F)], \times$$
$$\times [A(\mathfrak{T}_h) \nabla R_h(F) - A^+ \nabla R_0(F)]) \psi\, dx. \quad \square$$

In the following section we shall use Lemma 7 to partially characterize the set of all possible effective matrices, A, which correspond to locally equilateral, triangular networks with (interior) conductances ranging between $2\,\mu$ and $2\,\gamma$.

4. A CALCULATION RELATED TO EQUILATERAL, TRIANGULAR NETWORKS

Consider a single triangle as shown in figure 4. The edge e_j has conductance γ_j. Pick a coordinate system which has origin at x_2, x-axis parallel to e_1, y-axis parallel to $e_1^\perp$, and such that the triangle lies in the first two quadrants. A simple computation gives that the matrix

$$A_\tau = \sum_{j=1}^{3} e_j \gamma_j e_j^T / |\tau|,$$

has the formula

$$A_\tau = \frac{|e_1|^2}{|\tau|} \begin{pmatrix} (\gamma_1 + \gamma_3) + (\gamma_2 + \gamma_3) r^2 \cos^2\theta - 2\gamma_3 r \cos\theta & (\gamma_2 + \gamma_3) r^2 \cos\theta \sin\theta - \gamma_3 r \sin\theta \\ (\gamma_2 + \gamma_3) r^2 \cos\theta \sin\theta - \gamma_3 r \sin\theta & (\gamma_2 + \gamma_3) r^2 \sin^2\theta \end{pmatrix}$$

with $r = |e_2|/|e_1|$ and $|\tau| = \frac{1}{2}|e_1||e_2| \sin\theta$. Let $\lambda_1 \geq \lambda_2$ denote the eigenvalues of A_τ; they satisfy

$$\lambda_1 + \lambda_2 = \frac{2}{\sin\theta}\left((\gamma_1 + \gamma_3)\frac{1}{r} + (\gamma_2 + \gamma_3) r - 2\gamma_3 \cos\theta\right) \text{ and },$$
$$\lambda_1 . \lambda_2 = 4(\gamma_1\gamma_2 + \gamma_1\gamma_3 + \gamma_2\gamma_3).$$

We now specialize to the case of an equilateral triangle : $|e_1| = |e_2| = h$, and $\theta = \pi/3$, and proceed to characterize the set of all possible matrices A_τ. The equations above simplify to

$$\lambda_1 + \lambda_2 = 4(\gamma_1 + \gamma_2 + \gamma_3)/\sqrt{3} \text{ and },$$

(58)

$$\lambda_1 \cdot \lambda_2 = 4(\gamma_1 \gamma_2 + \gamma_1 \gamma_3 + \gamma_2 \gamma_3) .$$

We assume that the edge conductances for a single triangle, γ_j, $j = 1, 2, 3$, satisfy

$$0 < \mu \leqslant \gamma_j \leqslant \gamma .$$

In terms of an entire network this situation corresponds to equipartition of the edge conductances in the network onto neighboring triangles, with the original network conductances of interior edges lying between 2μ and 2γ (and conductances of boundary edges lying between μ and γ). If two of the conductances, γ_j, are in the set $\{\mu, \gamma\}$ then there are, as far as eigenvalues of A_τ are concerned, three possible cases :

(i) $\gamma_1 = \mu, \gamma_2 = \mu, \mu \leqslant \gamma_3 \leqslant \gamma,$

(ii) $\gamma_1 = \mu, \gamma_2 = \gamma, \mu \leqslant \gamma_3 \leqslant \gamma,$ or

(iii) $\gamma_1 = \gamma, \gamma_2 = \gamma, \mu \leqslant \gamma_3 \leqslant \gamma.$

In case (i) the solution to (58) is $\lambda_1 = 2(\mu + 2\gamma_3)/\sqrt{3}$, $\lambda_2 = 2\sqrt{3}\,\mu$. This is the dashed horizontal linesegment shown in figure 8.

In case (iii) the solution to (58) is $\lambda_1 = 2\sqrt{3}\,\gamma$, $\lambda_2 = 2(\gamma + 2\gamma_3)/\sqrt{3}$. This corresponds to the dashed vertical linesegment shown in figure 8.

Consider now case (ii). Through multiplication of the first equation in (58) by $\sqrt{3}(\gamma_1 + \gamma_2)$ and subtraction from the second equation we get

$$(\lambda_1 - \sqrt{3}(\gamma_1 + \gamma_2))(\lambda_2 - \sqrt{3}(\gamma_1 + \gamma_2)) = -(\gamma_1 - \gamma_2)^2 .$$

By inserting $\gamma_1 = \mu$, $\gamma_2 = \gamma$ and rearranging we get that λ_1 and λ_2 are related by

$$\lambda_2 - \sqrt{3}(\mu + \gamma) = -\frac{(\mu - \gamma)^2}{\lambda_1 - \sqrt{3}(\mu + \gamma)} .$$

This curve, which is a piece of a hyperbola, connects the point $P = (2(\mu + 2\gamma)/\sqrt{3}, 2\sqrt{3}\,\mu)$ to $Q = (2\sqrt{3}\,\gamma,\ 2(\gamma + 2\mu)/\sqrt{3})$. As γ_3 decreases from γ to μ the point (λ_1, λ_2) runs from Q to P. This curve is also shown (dashed) in figure 8.

If none, or at most one, of the γ_j are in the set $\{\mu, \gamma\}$, then we may without loss of generality assume that $\mu = \gamma_1 < \gamma_2 \leq \gamma_3 < \gamma$, $\mu < \gamma_1 \leq \gamma_2 < \gamma_3 = \gamma$ or $\mu < \gamma_1 \leq \gamma_2 \leq \gamma_3 < \gamma$. Consider the first possibility. Define a parametrized family of conductivities $\gamma_j(s)$ by

$$\gamma_1(s) = \gamma_1(= \mu), \quad \gamma_2(s) = \gamma_2 - s, \quad \gamma_3(s) = \gamma_3 + s,$$

for $0 \leq s \leq \min\{\gamma_2 - \mu, \gamma - \gamma_3\}$ (such that $\mu \leq \gamma_j(s) \leq \gamma$). Let $\lambda_i(s)$ denote the corresponding eigenvalues. Then

$$\text{(59)} \qquad \begin{aligned} \lambda_1(s) + \lambda_2(s) &= 4(\gamma_1 + \gamma_2 + \gamma_3)/\sqrt{3} \text{ and}, \\ \lambda_1(s) \,.\, \lambda_2(s) &= 4(\gamma_1 \gamma_2 + \gamma_1 \gamma_3 + (\gamma_2 - s)(\gamma_3 + s)) \,. \end{aligned}$$

By differentiation of (59) with respect to s we get

$$\text{(60)} \qquad \begin{aligned} &(\lambda_1)'(s) + (\lambda_2)'(s) = 0 \text{ and}, \\ &(\lambda_1)'(s)\,\lambda_2(s) + \lambda_1(s)(\lambda_2)'(s) = \gamma_2 - \gamma_3 - 2s < 0\,, \\ &\text{for } 0 < s < \min\{\gamma_2 - \mu, \gamma - \gamma_3\}\,, \end{aligned}$$

and therefore

$$\text{(61)} \quad [\lambda_1(s) - \lambda_2(s)](\lambda_2)'(s) < 0 \text{ for } 0 < s < \min\{\gamma_2 - \mu, \gamma - \gamma_3\}\,.$$

Since $\lambda_1(s) > \lambda_2(s)$ it follows immediately from (60) and (61) that

$$\text{(62)} \quad (\lambda_2)'(s) < 0, \text{ and } (\lambda_1)'(s) = -(\lambda_2)'(s) > 0 \qquad \text{for } 0 < s < \min\{\gamma_2 - \mu, \gamma - \gamma_3\}\,.$$

When s reaches the value $s_{max} = \min\{\gamma_2 - \mu, \gamma - \gamma_3\}$ then the corresponding point $(\lambda_1(s_{max}), \lambda_2(s_{max}))$ lies on one of the dashed curves in figure 8. From (62) it follows immediately that the points $(\lambda_1(s), \lambda_2(s))$, $0 \leq s < s_{max}$ lie above and to the left of the point $(\lambda_1(s_{max}), \lambda_2(s_{max}))$. We therefore conclude that the point $(\lambda_1, \lambda_2) = (\lambda_1(0), \lambda_2(0))$, which corresponds to $\mu = \gamma_1 < \gamma_2 \leq \gamma_3 < \gamma$ lies in the shaded area of figure 8 (or on the linesegment $2\sqrt{3}\,\mu < \lambda_1 = \lambda_2 < 2\sqrt{3}\,\gamma$). A similar argument shows that the points (λ_1, λ_2), which corresponds to $\mu < \gamma_1 \leq \gamma_2 < \gamma_3 = \gamma$ or $\mu < \gamma_1 \leq \gamma_2 \leq \gamma_3 < \gamma$ lie in the shaded area of figure 8 (or on the linesegment $2\sqrt{3}\,\mu < \lambda_1 = \lambda_2 < 2\sqrt{3}\,\gamma$). All sets of eigenvalues corresponding to the linesegment $\lambda_1 = \lambda_2$ between the points $(2\sqrt{3}\,\mu, 2\sqrt{3}\,\mu)$ and $(2\sqrt{3}\,\gamma, 2\sqrt{3}\,\gamma)$ are most easily attained by choosing $\gamma_1 = \gamma_2 = \gamma_3$ and letting this common value range between μ and γ. It is also fairly easy to show that, given any point in the shaded area of figure 8, it is possible to pick conductance values $\mu \leq \gamma_j \leq \gamma$, so that this point corresponds to the eigenvalues of the matrix A_τ. Through rotation of the triangle τ we may orient the orthogonal set of eigenvectors in any direction we desire. In summary, the set of all possible matrices A_τ consists exactly of those symmetric matrices whose eigenvalues $\lambda_1 \leq \lambda_2$ are in the set indicated in figure 8.

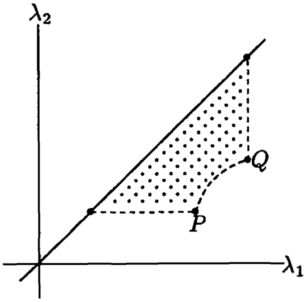

Figure 8.

For a given set of conductances $\mu \leq \gamma_j \leq \gamma$, consider the cell $\mathfrak{T}$, shown in figure 9*a*. It is not difficult to see that (with equipartition of the conductances onto neighboring triangles) $A(\mathfrak{T})$ is constantly equal to A_{τ_0}, where A_{τ_0} is the matrix corresponding to the single triangle shown in figure 9*b* (see (10) for the definition of $A(\mathfrak{T})$). By building a network from copies of the cell shown in figure 9*a* (doubling the conductances of shared edges), we see that any symmetric matrix whose eigenvalues lie in the set indicated in figure 8 corresponds to an equilateral, triangular network with internal conductances between $2\,\mu$ and $2\,\gamma$ (and boundary conductances between μ and γ).

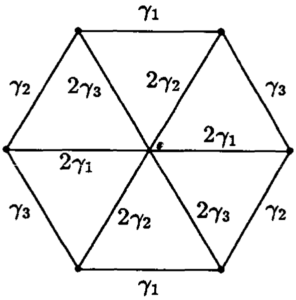

Figure 9*a*.

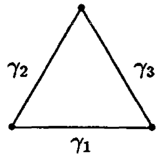

Figure 9*b*.

Let A denote the effective matrix (valued function) constructed in the proof of Theorem 1. We shall derive a partial characterization of the *constant* effective conductances A arising from (globally) equilateral, triangular networks (treating Ω as if it has no boundary !). If we only require that our triangulations be locally equilateral, than it is intuitively clear that this partial characterization leads to an almost everywhere, partial characterization of variable effective conductances. Let $A(\mathfrak{T}_h)$ be a sequence of matrices for which U^h converges to U^0 in the sense of Theorem 1. By extraction of a subsequence, if necessary, we may assume that

$$A^+ = \lim A(\mathfrak{T}_h), \quad (A^-)^{-1} = \lim A(\mathfrak{T}_h)^{-1}$$

exist as weak* limits in L^∞. According to Lemma 7 we now have

$$A^- \leqslant A \leqslant A^+ \text{ a.e..} \tag{63}$$

Let $\lambda_1^h \geqslant \lambda_2^h$ denote the eigenvalues of $A(\mathfrak{T}_h)$. From (63) we immediately get the bounds

$$\lambda^- \leqslant A \leqslant \lambda^+ \text{ a.e.,} \tag{64}$$

where λ^+ is the weak* limit of λ_1^h and $(\lambda^-)^{-1}$ is the weak* limit of $(\lambda_2^h)^{-1}$. The hyperbola

$$\lambda_1 + \frac{4\,\mu(2\,\mu + \gamma)}{\lambda_2} = 2\,\sqrt{3}(\gamma + \mu) \tag{65}$$

goes through the points P and Q, only it is convex, not concave as the hyperbola in figure 8. That piece of the hyperbola from (65), which lies between P and Q is shown in figure 10.

The eigenvalues $\lambda_1^h \geqslant \lambda_2^h$ of $A(\mathfrak{T}_h)$ (with equipartition) satisfy the inequality constraints corresponding to (the closure of) the shaded set in figure 8. Consequently they also satisfy the weaker constraints

$$\lambda_2^h \geqslant 2\,\sqrt{3}\,\mu\,, \quad \lambda_1^h \leqslant 2\,\sqrt{3}\,\gamma \text{ and ,}$$

$$\lambda_1^h + \frac{4\,\mu(2\,\mu + \gamma)}{\lambda_2^h} \leqslant 2\,\sqrt{3}(\gamma + \mu)\,.$$

Since the expression on the left hand side of the last inequality is linear in λ_1^h and $(\lambda_2^h)^{-1}$ it follows that the weak* limits λ^+ and λ^- continue to satisfy

$$\lambda^- \geqslant 2\,\sqrt{3}\,\mu\,, \quad \lambda^+ \leqslant 2\,\sqrt{3}\,\gamma \text{ and ,}$$
$$\lambda^+ + \frac{4\,\mu(2\,\mu + \gamma)}{\lambda^-} \leqslant 2\,\sqrt{3}(\gamma + \mu)$$

almost everywhere .

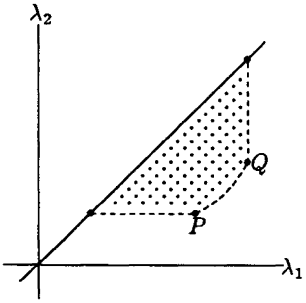

Figure 10.

From the inequality (64) it now follows that

$$\begin{aligned}&\text{The eigenvalues } \lambda_1 \geq \lambda_2 \text{ of } A \text{ satisfy}\\&\lambda_2 \geq 2\sqrt{3}\,\mu\,, \quad \lambda_1 \leq 2\sqrt{3}\,\gamma \text{ and}\,,\\&\lambda_1 + \frac{4\,\mu(2\,\mu+\gamma)}{\lambda_2} \leq 2\sqrt{3}(\gamma+\mu)\,,\end{aligned}$$

i.e., A has eigenvalues lying in (the closure of) the shaded set indicated in figure 10. In summary : the *constant* effective conductances associated with equilateral, triangular networks with infinitessimal edge length and with (interior) edge conductances between $2\,\mu$ and $2\,\gamma$ are among those symmetric matrices whose eigenvalues lie in the set indicated in figure 10, and any symmetric matrix whose eigenvalues lie in the set indicated in figure 8 may be attained as the effective conductance of such an infinitessimal, equilateral, triangular network.

Eigenvalues on the hyperbola in figure 10 may be attained by means of layering of networks which attain P (*cf. fig.* 9*a*) with 90 degree rotates of networks which attain Q (*cf. fig.* 9*a*). Unfortunately such layering does not preserve the equilateral, triangular structure of the network. It is unclear whether all the points in the set indicated in figure 10 may be attained as eigenvalues for effective conductances of equilateral, triangular networks.

4.1. Remark

It is possible to perform similar calculations for networks consisting of identical rectangles, or for networks that are made up of identical rectangles all with the same diagonal edge added, but these shall not be presented here. As seen from the example above the G-closure is in general a fairly large set. It should be interesting to examine examples in which there is a very restricted set of attainable limits. Such examples would necessarily come about through restrictions on the microstructure of the network (e.g.

through assumptions about periodicity or in the random case, stationarity). The simplest of such cases, corresponding to a network of squares with two possible edge conductances was analysed in [13], based on techniques developed in [19]. For certain applications it should also be interesting to understand exactly what restrictions lead to isotropic limits.

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