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MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
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HOMOGENIZATION OF THE STOKES SYSTEM IN A THIN FILM FLOW WITH RAPIDLY VARYING THICKNESS (*)

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Abstract — *We study a problem with two small parameters, that models a fluid flow between two close rough surfaces We study the convergence by the energy method of the 3-dimensional Stokes system solution when the ratio* $\lambda = \eta/\varepsilon$ *is constant (*η *is linked to the fluid thickness and* ε *to the size of the roughness period) Then making* λ *tend to infinity (resp to zero) we show that the case in which the thickness is greater (resp smaller) than the period is an asymptotic limit of the intermediate case*

Résumé — *On considere un problème de passage à la limite à deux petits paramètres modélisant l'écoulement d'un fluide entre deux surfaces rapprochées supposées rugueuses Nous étudions d'abord la convergence par la méthode de l'énergie de la solution du système de Stokes tridimensionnel lorsque le rapport* $\lambda = \eta/\varepsilon$ *est constant (*η *est lié à l'epaisseur du domaine et* ε *à la période de la rugosité) Ensuite en faisant tendre* λ *vers l'infini (resp vers zéro) nous montrons que le cas ou l'epaisseur est plus grande (resp plus petite) que la periode est limite asymptotique du cas intermediaire*

I. INTRODUCTION

We study the asymptotic behavior of a viscous fluid flow in a narrow gap with mean thickness η whose surfaces are supposed to be rough, with a periodic roughness of wavelength ε, when the two small parameters ε and η tend to zero. This problem falls in the scope of the hydrodynamic lubrication.

In the mechanical literature most papers are based upon the Reynolds

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equation which is derived from the Stokes system by taking account of the small parameter η associated to the film thickness To evaluate the surfaces roughness effect, two different ways exist

First, lots of papers are concerned with the so called « Reynolds roughness » This approach which retains the validity of the Reynolds equation is commonly used if the characteristic wavelength of the roughness is much greater than the film thickness Proposed averaged equations appear in [9], [14], [16], some of them being devoted to the particular case of roughness pattern with small peak to valley height

The second way is associated to the « Stokes roughness », where the authors claim that the applicability of the Reynolds equation is not valid, especially when the roughness wavelength is small in front of the gap height The related studies retain the assumption of small roughness height, and use asymptotic expansions [15], [19]

All the results have given rise to many controversies both at view of their numerical results than for the heuristic assumptions on which they are based [8], [21] , intercomparisons are very difficult due to the various assumptions introduced

We are concerned with the mathematical aspect of this problem The most rigorous of the previously mentioned papers are based on formal asymptotic expansions , though this last method has already proved its effectiveness no real proof appears in the literature, the statistical surfaces descriptions rendering any mathematical proof very difficult We consider a deterministic way and assume a periodic roughness and the basic equations are the Stokes system Problems depending on two « small parameters » appear in various physical areas like electrical engineering [17], thermal effects in periodical structures [6] and mostly in the two dimensional approximation of the three dimensional plate models [7], [11], [12] This last problem is related to our study, especially when a rapidly varying thickness for the plate is considered

In most of the two small parameters problems, the way how the parameters tend to zero is primordial and the limiting equations are different whether ε tends to zero faster, slower or at the same rate as η In this paper, we show that this result is also true and it describes all the possibilities Mathematical tools are both asymptotic expansions [13] and the homogenization theory [5], [18]

The second section is concerned with the notations and a recall of the asymptotic equation that is proposed for the pressure in [2] by way of formal expansion when η/ε is a constant ratio λ

Section 3 is devoted to the validity of this equation in a rigorous way by the energy method [20] The result is obtained via a conjecture on the asymptotic behavior of the pressure we show the weak L^2 convergence and we have to suppose that the convergence is actually strong

In Section 4 we study the limit equation when the roughness is an actually periodic one and existence and uniqueness results are proved.

In Section 5, we study the limit of the previously mentionned equation when λ tends to infinity. The obtained equation can be associated to the Stokes roughness and seems to indicate that no flow occurs in the oscillating part of the gap. The last section is devoted to the limit equation obtained when λ tends to zero (Reynolds roughness). In that case, the study can be rigorously made by making first η tend to zero and then ε, which is nothing else than the homogenization of the classical Reynolds equation [4].

For a mechanical use, we summarize the conclusions so :

— all the three limiting equations are of the Reynolds type but different,

— the height of the roughness has no influence on the qualitative aspect of these equations,

— the second equation representative of the Stokes roughness is of very easy and cheap treatment, but a complete mathematical proof is missing,

— in the last case (Sect. 6) the results of Christensen [9] can be used with confidence for small roughness spacing.

A complete mathematical study is not yet available. If the assumptions of the full periodic roughness seems to be overcome by way of a space discretization, the conjecture of Section 3 is related to a finer difficulty. It is to be noted also that in the case of a thin plate with rapidly varying thickness, no complete proof of the different cases related to the ratio of the two small parameters exists at this time [12], contrary to the problem of a thin composite structure where only the elasticity coefficients are periodic and not the shape of the plate [7].

II. BASIC NOTATIONS AND ASYMPTOTIC PROPOSED EQUATION FOR CONSTANT $\lambda = \eta/\varepsilon$

II.1. Geometrical data and notations

We shall write $X = (x_1, x_2, x_3)$ for a current point in $\mathbb{R}^3$ and $x = (x_1, x_2)$ for its projection in $\mathbb{R}^2$.

ω is an open set in $\mathbb{R}^2$ with a Lipschitz boundary $\partial\omega$.

ε is a small parameter related to the roughness wavelength scale. h is a smooth function, defined for x in ω and y in $\mathbb{R}^2$, periodic with period Y_i in y_i $(i = 1, 2)$.

We set $Y = [0, Y_1] \times [0, Y_2]$, the periodic cell.

The real gap between the two surfaces is given by :

$$\eta h^{\varepsilon}(x) = \eta h(x, x/\varepsilon) \quad x \in \bar{\omega} .$$

The three dimensional domain occupied by the fluid is :

$$\Omega_{\varepsilon\eta} = \left\{X \in \mathbb{R}^3, x \in \omega, 0 < x_3 < \eta h^{\varepsilon}(x)\right\} .$$

II.2. The basic equations

We are concerned with the thin-film hydrodynamic lubrication of rough surfaces, that is the study of an incompressible viscous fluid flow between two surfaces in motion as the thickness of the gap is small. To make the model easier to study, we suppose that one of the surface is horizontal, smooth and moves with a constant velocity whereas the other one is rough and motionless.

The basic Stokes system is : (the viscosity is taken equal to 1)

$$(S_{\varepsilon\eta})\left|\begin{array}{ll} -\Delta \underline{u}^{\varepsilon\eta} + \underline{\nabla} \cdot p^{\varepsilon\eta} = 0 & (2.1) \\ & \text{in } \Omega_{\varepsilon\eta} \\ \operatorname{div}(\underline{u}^{\varepsilon\eta}) = 0 & (2.2) \end{array}\right.$$

Boundary conditions are added to solve equations (2.1) (2.2) ; classical operating conditions are Dirichlet ones :

$$\underline{u}^{\varepsilon\eta} = (\mathrm{k}^{\varepsilon\eta}, 0, 0) \quad \text{on} \quad \partial\Omega_{\varepsilon\eta} \tag{2.3}$$

with

$$k^{\varepsilon\eta} = \begin{bmatrix} 0 & \text{on} & \Sigma_{\varepsilon\eta} & \\ s & \text{on} & \omega & (s \in \mathbb{R}^+) \end{bmatrix} \tag{2.4}$$

where $\Sigma_{\varepsilon\eta}$ is the oscillating boundary of $\partial\Omega_{\varepsilon\eta}$ (*fig.* 1)

$$\Sigma_{\varepsilon\eta} = \left\{X \in \mathbb{R}^3, x \in \omega, x_3 = \eta h^{\varepsilon}(x)\right\}.$$

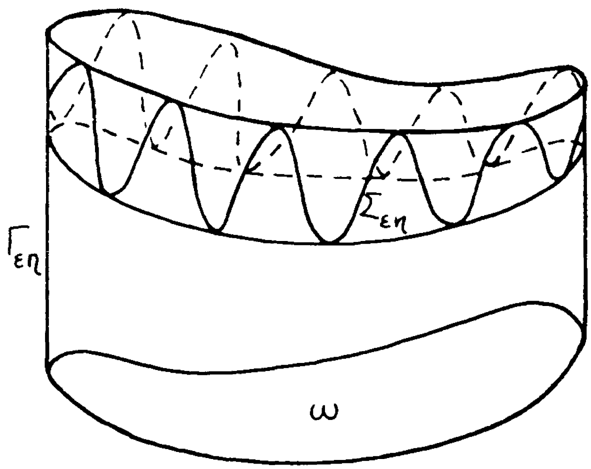


Figure 1. — Domain $\Omega_{\varepsilon\eta}$.

We denote by $\Gamma_{\varepsilon\eta}$ the lateral boundary $\partial\Omega_{\varepsilon\eta} - (\omega \cup \Sigma_{\varepsilon\eta})$; $k^{\varepsilon\eta}$ is not easy to evaluate experimentally on $\Gamma_{\varepsilon\eta}$ and we are led to make first the assumptions :

$$k^{\varepsilon\eta} \in H^{1/2}(\Gamma_{\varepsilon\eta}) \quad \text{and} \quad \int_{\partial\Omega_{\varepsilon\eta}} k^{\varepsilon\eta} \cos\,(\underline{n}, \underline{x}_1)\, d\sigma = 0 \tag{2.5}$$

(n is the outward unit normal vector on $\partial\Omega_{\varepsilon\eta}$ and $d\sigma$ denotes the surface measure).

Following [2], we introduce a supplementary condition for $k^{\varepsilon\eta}$. According to the rescaling, we suppose that there exists a regular link between 0 and s which does not depend on ε and λ such that :

and
$$\begin{cases} k^{\varepsilon\eta}(X) = K(x, x_3/\lambda\varepsilon) \\ K(x, z) = 0 \quad \text{for} \quad h_{\min}(x) < z \end{cases} \tag{2.6}$$

where

$$h_{\min}(x) = \min_{y \in Y} h(x, y) \quad \forall x \in \bar{\omega} \quad \text{and} \quad h_{\min} > 0\,.$$

We define :

$$t(x) = Y_1\, Y_2 \int_{0 < z < h_{\min}(x)} K(x, z)\, dz \cos\,(n, x_1)\,.$$

Due to (2.5) there exists a unique solution

$$(\underline{u}^{\varepsilon\eta}, p^{\varepsilon\eta}) \quad \text{of} \quad (S_{\varepsilon\eta}) \quad \text{in} \quad H^1(\Omega_{\varepsilon\eta})^3 \times L^2(\Omega_{\varepsilon\eta})/\mathbb{R} \quad [10]\,.$$

II.3. The new variables and the local auxiliary problems

We define first the rescaled thickness $z = x_3/\lambda\varepsilon$ and we introduce the following operators :

$$\begin{aligned} \operatorname{div}_\lambda &= \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} + \frac{1}{\lambda}\frac{\partial}{\partial z} \\ \Delta_\lambda &= \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} + \frac{1}{\lambda^2}\frac{\partial^2}{\partial z^2} \qquad \lambda = \eta/\varepsilon \\ \nabla_\lambda &= \left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{1}{\lambda}\frac{\partial}{\partial z} \right). \end{aligned}$$

We introduce three auxiliary problems $(L^0)(L^1)(L^2)$ in the following weak formulation :

Let :

$$B_x = \{(y, z) \in \mathbb{R}^3, y \in Y, 0 < z < h(x, y)\}$$
$$H^Y = \{\underline{\phi} \in (H^1(B_x))^3, \underline{\phi} \text{ is } Y \text{ periodic in the } y_i \text{ variables}\}$$
$$H_0^Y = \{\underline{\phi} \in H^Y, \underline{\phi}(y, 0) = \underline{\phi}(y, h(x, y)) = 0\}$$
$$a(\underline{\phi}^1, \underline{\phi}^2) = \sum_{i=1,3} \sum_{j=1,2} \int_{B_x} \frac{\partial \phi_i^1}{\partial y_j} \frac{\partial \phi_i^2}{\partial y_j} + \frac{1}{\lambda^2} \int_{B_x} \frac{\partial \phi_i^1}{\partial z} \frac{\partial \phi_i^2}{\partial z}.$$

Problem (L^0) :

$$\left|\begin{array}{l} \text{Fin } \underline{\alpha}^0 \text{ in } H^Y \text{ and } q^0 \text{ in } L^2(B_x) \text{ such that :} \\ a(\underline{\alpha}^0, \underline{\phi}) = \displaystyle\int_{B_x} q^0 \operatorname{div}_\lambda(\underline{\phi}) \quad \forall \underline{\phi} \in H_0^Y \\ \displaystyle\int_{B_x} \xi \operatorname{div}_\lambda(\underline{\alpha}^0) = 0 \quad \forall \xi \in L^2(B_x) \\ \underline{\alpha}^0(y, h(x, y)) = 0 \quad \underline{\alpha}^0(y, 0) = (s, 0, 0). \end{array}\right.$$

Problem (L^i) $i = 1, 2$:

$$\left|\begin{array}{l} \text{Fin } \underline{\alpha}^i \text{ in } H_0^Y \text{ and } q^i \text{ in } L^2(B_x) \text{ such that :} \\ a(\underline{\alpha}^i, \underline{\phi}) = \displaystyle\int_{B_x} q^i \operatorname{div}_\lambda(\underline{\phi}) - \int_{B_x} \phi_i \quad \forall \underline{\phi} \in H_0^Y \\ \displaystyle\int_{B_x} \xi \operatorname{div}_\lambda(\underline{\alpha}^i) = 0 \quad \forall \xi \in L^2(B_x). \end{array}\right.$$

II.4. Asymptotic proposed equation

A formal study by asymptotic expansion (see [2]) for ε and η tending to zero with a constant ratio $\lambda = \eta/\varepsilon$ shows that $p^{\varepsilon\eta} \sim \dfrac{p^{-2}}{\varepsilon^2}$ such that p^{-2} is solution of :

$$\left\{\begin{array}{l} \text{Find } p^{-2} \text{ in } H^1(\omega) \text{ such that :} \\ \displaystyle\sum_{i,j=1,2} \int_\omega [\alpha_i^j] \frac{\partial p^{-2}}{\partial x_i} \frac{\partial \phi}{\partial x_j} dx + \sum_{i=1,2} \int_\omega [\alpha_i^0] \frac{\partial \phi}{\partial x_i} dx = \int_{\partial\omega} t\phi \, d\sigma \end{array}\right. \quad (2.7)$$

where $[\phi]$ denotes the integral of ϕ on B_x.

III. THEORETICAL STUDY OF THE ASYMPTOTIC BEHAVIOR OF THE FLOW

In this section we prove the result obtained first by formal asymptotic expansions in [2]. From now on we suppose :

$$h^{\varepsilon}(x) = h(x/\varepsilon)\,.$$

If we want to cancel this assumption, regularity results for solutions of Stokes problem with respect to the domain have to be obtained.

The behavior of the velocity is easy to study because it can be extended by zero to a fixed domain including Ω_ε. We use the technic introduced by Tartar ([18] appendix) to extend the pressure and then the standard energy method [20]. We suppose the strong convergence of the pressure to have a complete proof, but only the weak one is proved ; as the formal limit doesn't depend on the micro variable this seems to be a reasonable conjecture.

III.1. The rescaled weak formulation

We use a mixed weak formulation for the Stokes system (2.1)-(2.3) in the rescaled domain

$$\Omega_\varepsilon = \{(x, z) \in \mathbb{R}^3, x \in \omega, 0 < z < h^{\varepsilon}(x)\}\,. \tag{3.1}$$

We point out that in spite of the rescaling, this domain is not a fixed one and this will lead to further difficulties when letting ε tend to zero. Therefore we need to introduce a fixed Ω involving Ω_ε, in which convergences can be proved :

$$\begin{aligned}
\Omega &= \left\{(x, z) \in \mathbb{R}^3, x \in \omega, 0 < z < h_{\max}\right\}\\
\Sigma &= \left\{(x, z) \in \mathbb{R}^3, x \in \omega, z = h_{\max}\right\}\\
\Omega^- &= \left\{(x, z) \in \mathbb{R}^3, x \in \omega, 0 < z < h_{\min}\right\}\\
\Sigma^- &= \left\{(x, z) \in \mathbb{R}^3, x \in \omega, z = h_{\min}\right\}
\end{aligned}$$

where

$$h_{\max} = \max_{y \in Y} h(y)\,.$$

Set :

$$a^{\varepsilon}(\underline{u}, \phi) = \sum_{i=1,3} \int_{\Omega_\varepsilon} \left(\sum_{j=1,2} \frac{\partial u_i}{\partial x_j}\frac{\partial \phi_i}{\partial x_j} + \frac{1}{\lambda^2 \varepsilon^2}\frac{\partial u_i}{\partial z}\frac{\partial \phi_i}{\partial z} \right) dx\,dz \tag{3.2}$$

$$b^{\varepsilon}(q, \phi) = \int_{\Omega_\varepsilon} q \left(\frac{\partial \phi_1}{\partial x_1} + \frac{\partial \phi_2}{\partial x_2} + \frac{1}{\lambda\varepsilon}\frac{\partial \phi_3}{\partial z} \right) dx\,dz \tag{3.3}$$

$$L_0^2(\Omega_\varepsilon) = \left\{ q \in L^2(\Omega_\varepsilon), \int_{\Omega_\varepsilon} q\,dx\,dz = 0 \right\}.$$

We denote by $(\underline{u}^{\varepsilon}, p^{\varepsilon})$ the rescaled solution of $(S_{\varepsilon\eta})$ because the two small parameters are of the same size. So $(\underline{u}^{\varepsilon}, p^{\varepsilon})$ satisfies :

$$a^{\varepsilon}(\underline{u}^{\varepsilon}, \underline{\phi}) = b^{\varepsilon}(p^{\varepsilon}, \underline{\phi}) \quad \forall \underline{\phi} \in H_0^1(\Omega_{\varepsilon})^3 \tag{3.4}$$

$$b^{\varepsilon}(q, \underline{u}^{\varepsilon}) = 0 \quad \forall q \in L^2(\Omega_{\varepsilon}) \tag{3.5}$$

$$\underline{u}^{\varepsilon}/\partial\Omega_{\varepsilon} = (K, 0, 0) \quad \text{where } K \text{ is given by (2.6)}.$$

III.2. Behavior of the velocity

We introduce

$$V_z = \left\{ \underline{v} \in L^2(\Omega)^3, \frac{\partial \underline{v}}{\partial z} \in L^2(\Omega)^3 \right\}.$$

For any function v defined on Ω_{ε}, we denote by $\bar{v}$ the function equal to v on Ω_{ε} and extended by zero to Ω. Obviously $v \in H^1(\Omega_{\varepsilon})$ and $v = 0$ on Σ_{ε} imply $\bar{v} \in H^1(\Omega)$. We set for $\underline{\phi}$ in $H^1(\Omega_{\varepsilon})^3$:

$$\|\underline{\phi}\|_{\varepsilon} = \left(\sum_{i=1,3} \left(\sum_{j=1,2} \left\| \frac{\partial \phi_i}{\partial x_j} \right\|^2_{L^2(\Omega_{\varepsilon})} + \frac{1}{\lambda^2 \varepsilon^2} \left\| \frac{\partial \phi_i}{\partial z} \right\|^2_{L^2(\Omega_{\varepsilon})} \right) \right)^{1/2}. \tag{3.6}$$

THEOREM 3.1 : *There exists $\underline{u}^*$ in V_z such that :*

$$\bar{\underline{u}}^{\varepsilon} \rightharpoonup \underline{u}^* \quad L^2(\Omega)^3 \ \textit{weak},$$

$$\frac{\partial \bar{\underline{u}}^{\varepsilon}}{\partial z} \rightharpoonup \frac{\partial \underline{u}^*}{\partial z} \quad L^2(\Omega)^3 \ \textit{weak},$$

$$\varepsilon \frac{\partial \bar{\underline{u}}^{\varepsilon}}{\partial x_i} \rightharpoonup 0 \quad L^2(\Omega)^3 \ \textit{weak}, \quad i = 1, 2.$$

Moreover $u_3^* \equiv 0$, $\underline{u}^* = 0$ on Σ, $\underline{u}^* = (s, 0, 0)$ on ω.

Proof: From (2.5) and (2.6) we are able to construct a fixed $\underline{J}$ in $H^1(\Omega^-)^3$ such that :

$$\underline{J} = (K, 0, 0) \quad \text{on} \quad \partial\Omega^- \quad \text{and} \quad \operatorname{div} \underline{J} = 0.$$

Setting $\underline{K}^{\varepsilon} = (J_1, J_2, \varepsilon\lambda J_3)$ in Ω^-, extended by zero in Ω,

$$\underline{K}^{\varepsilon} \in H^1(\Omega)^3, \quad \frac{\partial K_1^{\varepsilon}}{\partial x_1} + \frac{\partial K_2^{\varepsilon}}{\partial x_2} + \frac{1}{\lambda\varepsilon} \frac{\partial K_3^{\varepsilon}}{\partial z} = 0 \tag{3.7}$$

$$\|\underline{K}^{\varepsilon}\|_{\varepsilon}^2 = \sum_{i=1}^{3} \int \left\{ \left(\frac{\partial K_i^{\varepsilon}}{\partial x_1} \right)^2 + \left(\frac{\partial K_i^{\varepsilon}}{\partial x_2} \right)^2 + \frac{1}{\lambda^2 \varepsilon^2} \left(\frac{\partial K_i^{\varepsilon}}{\partial z} \right)^2 \right\} \leqslant C/\varepsilon^2.$$

We choose $\underline{\phi} = (\underline{u}^\varepsilon - \underline{K}^\varepsilon)$ as a test function in (3.4) and $q = p^\varepsilon$ in (3.5). So :

$$a^\varepsilon(\underline{u}^\varepsilon, \underline{u}^\varepsilon) = a^\varepsilon(\underline{u}^\varepsilon, \underline{K}^\varepsilon)$$

and
$$\|\underline{u}^\varepsilon\|_\varepsilon \leqslant \|\underline{K}^\varepsilon\|_\varepsilon \leqslant C/\varepsilon^2 \tag{3.8}$$

C being a constant with respect to ε, by taking the Poincaré inequality in the z-direction we obtain :

$$\|\bar{u}_i^\varepsilon\|_{L^2(\Omega)} \leqslant \left\|\frac{\partial \bar{u}_i^\varepsilon}{\partial z}\right\|_{L^2(\Omega)} \leqslant C$$

and
$$\left\|\frac{\partial \bar{u}_i^\varepsilon}{\partial x_j}\right\|_{L^2(\Omega)} \leqslant C/\varepsilon \quad i = 1, 3 \quad j = 1, 2$$

and we can extract subsequences such that $\underline{\bar{u}}^\varepsilon, \dfrac{\partial \underline{\bar{u}}^\varepsilon}{\partial z}, \varepsilon \dfrac{\partial \underline{\bar{u}}^\varepsilon}{\partial x_j}$ weakly converge in $L^2(\Omega)^3$. This implies that $\dfrac{\partial \underline{\bar{u}}^\varepsilon}{\partial x_j}$ also converges in $\mathcal{D}'(\Omega)$ and the last a priori estimate gives that the limit of $\varepsilon \dfrac{\partial \underline{\bar{u}}^\varepsilon}{\partial x_j}$ is zero. Moreover for each function in V_z, we can define a trace on $\partial\Omega$ such that the application $\underline{v} \to \underline{v} n_z$ is a linear continuous operator from V_z^3 in $H^{-1/2}(\partial\Omega)^3$. So the values of $\underline{\bar{u}}_\varepsilon$ on the boundary $\Sigma \cup \omega$ which are constant are preserved by letting ε tend to zero.

(3.5) is true for any ϕ in $L^2(\Omega)$. Taking the limits of all terms in $b^\varepsilon(\phi, \underline{u}^\varepsilon)$, it follows :

$$\int_\Omega \phi \frac{\partial u_3^*}{\partial z} \, dx \, dz = 0 \quad \forall \phi \in \mathcal{D}(\Omega)$$

and $u_3^* = 0$ because of its values on the boundary. □

III.3. A priori estimates for the pressure

PROPOSITION 3.1 :

$$\left\|\frac{\partial p^\varepsilon}{\partial x_j}\right\|_{H^{-1}(\Omega_\varepsilon)} \leqslant C/\varepsilon^2 \quad (j = 1, 2) ; \quad \left\|\frac{\partial p^\varepsilon}{\partial z}\right\|_{H^{-1}(\Omega_\varepsilon)} \leqslant C/\varepsilon .$$

Proof: Taking successively $(\phi_1, 0, 0)$, $(0, \phi_2, 0)$ and $(0, 0, \phi_3)$ with $\phi_i \in H_0^1(\Omega_\varepsilon)$ in (3.4) we get for $i = 1, 2$:

$$\int_{\Omega_\varepsilon} p^\varepsilon \frac{\partial \phi_i}{\partial x_i} \, dx \, dz = \int_{\Omega_\varepsilon} \left(\sum_{j=1,2} \frac{\partial u_i^\varepsilon}{\partial x_j} \frac{\partial \phi_i}{\partial x_j} + \frac{1}{\lambda^2 \varepsilon^2} \frac{\partial u_i^\varepsilon}{\partial z} \frac{\partial \phi_i}{\partial z} \right) dx \, dz$$

and the estimates on the velocity induce

$$\left| \left\langle \varepsilon^2 \frac{\partial p^\varepsilon}{\partial x_i}, \phi_i \right\rangle_{H^{-1}(\Omega_\varepsilon)\, H_0^1(\Omega_\varepsilon)} \right| \leq C \| \phi_i \|_{H_0^1(\Omega_\varepsilon)}$$

By the same way

$$\left| \left\langle \varepsilon \frac{\partial p^\varepsilon}{\partial z}, \phi_3 \right\rangle_{H^{-1}(\Omega_\varepsilon)\, H_0^1(\Omega_\varepsilon)} \right| \leq C \| \phi_3 \|_{H_0^1(\Omega_\varepsilon)}$$

which ends the proof □

Ω_ε being a bounded Lipschitz domain, we have [22]

$$\| q \|_{L_0^2(\Omega_\varepsilon)} \leq C(\Omega_\varepsilon) \| \underline{\nabla} q \|_{H^{-1}(\Omega_\varepsilon)^3}$$

where the constant depends on the domain, that is to say on ε and we don't know how it depends on ε So we have to define a continuation of the pressure to Ω to prove convergence

III.4. Continuation of the pressure to Ω

L Tartar has introduced a continuation of the pressure for a flow in porous media (see [18] appendix) This construction applies to periodic holes in a domain $\Omega_{\varepsilon\eta}$ when each hole is strictly contained into the period cell We cannot use directly the results in our case because the « holes » are along the boundary $\Sigma_{\varepsilon\eta}$ of $\Omega_{\varepsilon\eta}$, moreover the scales of the geometry of the actual domain $\Omega_{\varepsilon\eta}$ are different in the x-direction (the macro one) and in the x_3-direction (the micro one) This fact will induce several limitations in the results obtained by using the method, especially in view of the convergence for the pressure

Recalling that h is a function of y only, the basic cell B_x defined in § II 2 is now a fixed one

$$B = \{(y, z) \in \mathbb{R}^3, \quad y \in Y, \quad 0 < z < h(y)\}$$

We suppose from now on

H1 the surface roughness is made of detached smooth humps periodically given on the upper part of the gap (*) ,

H2 ω is covered by an exact finite number of period εY ,

We consider a smooth surface included in B and surrounding the hump such that B is split into two areas B_f and B_m such that (*fig* 2)

H3 ∂B_m is a C^1 manifold

(*) From the fluid point of view !

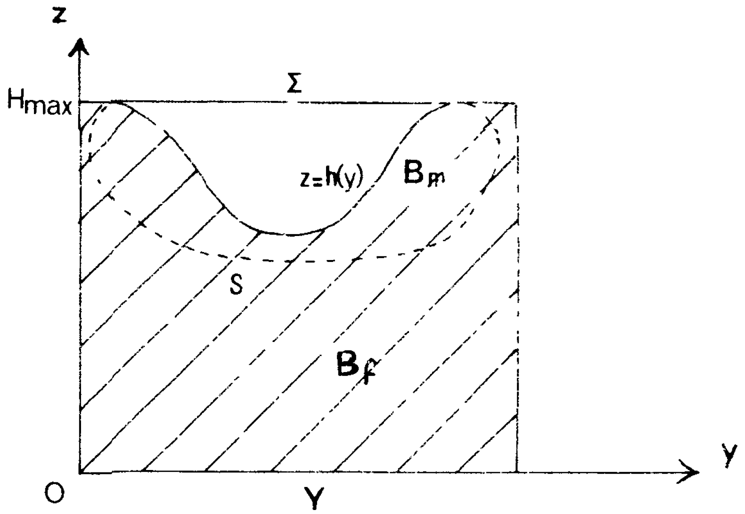


Figure 2. — The basic cell B.

We note

$$\Pi = Y \times \,]0, h_{\max}[$$
$$B_s = \Pi \backslash (B_m \cup B_f)$$
$$S = \partial B_m \cap \partial B_f .$$

We set

$$V = \{ \underline{v} \in H^1(\Pi)^3, \quad \underline{v} = 0 \text{ on } \Sigma \} .$$

In the following, we'll use the Poincaré norm both in $H^1(\Pi)$ and $H^1(B_m)$, all the function involved being zero on a part of the boundary. Moreover C denotes constant with respect to ε which can be function of λ.

LEMMA 3.1 : *For given $\underline{v}$ in V, there exists $\underline{w}$ in $H^1(B_m)^3$ such that :*

$$\underline{w}/S = \underline{v}/S \quad \text{and} \quad \underline{w}/\partial B_m \backslash S = 0 .$$

Moreover there exists a constant C which does not depend on v such that :

$$\left.\begin{aligned} &\|\underline{w}\|_{H^1(B_m)^3} \leqslant C \|\underline{v}\|_{H^1(\Pi)^3} \\ \textit{and} \qquad &\\ &\operatorname{div}_\lambda (v) = 0 \Rightarrow \operatorname{div}_\lambda (\underline{w}) = 0 \end{aligned}\right\} \tag{3.9}$$

Proof First we construct a lift of the boundary condition From H3 $\underline{v}/\partial B_m$ lies in $(H^{1/2}(\partial B_m))^3$ and we define $\underline{\beta}^1$ in $H^1(B_m)^3$ by

$$-\Delta_\lambda \underline{\beta}^1 = 0, \quad \underline{\beta}^1 = \underline{v} \text{ on } S, \quad \underline{\beta}^1 = \underline{0} \text{ on } \partial B_m \backslash S$$

By regularity theorem

$$\|\underline{\beta}^1\|_{H^1(B_m)^3} \leq C_1 \|\underline{\beta}^1\|_{H^{1/2}(\partial B_m)^3} = C_1 \|\underline{v}\|_{H^{1/2}(\partial B_m)^3} \leq C_2 \|\underline{v}\|_{H^1(\Pi)^3}$$

from H3 and classical trace theorem

Secondly we introduce

$$F = -\operatorname{div}_\lambda (\underline{\beta}^1) + \operatorname{div}_\lambda (\underline{v}) + \left(\int_{B_s} \operatorname{div}_\lambda (\underline{v})\, dy\, dz \right) / \text{mes } (B_m)$$

$$\int_{B_m} F = -\int_{\partial B_m} \underline{\beta}^1 \cdot \underline{\tilde{n}} + \int_{\partial B_m} \underline{v} \cdot \underline{\tilde{n}} + \int_{\partial B_s} \underline{v} \cdot \underline{\tilde{n}}$$

where $\underline{\tilde{n}} = (n_1, n_2, \lambda n_3)$, n outward normal to B_m

H1 and the definition of $\underline{\beta}^1$ imply

$$\int_{B_m} F = 0$$

So there exists $\underline{\beta}^2$ in $H_0^1(B_m)^3$ such that [22]

$$\operatorname{div}_\lambda (\underline{\beta}^2) = F\,, \quad \|\underline{\beta}^2\|_{H^1(B_m)^3} \leq C\, \|F\|_{L^2(B_m)} \leq C\, \|\underline{v}\|_{H^1(\Pi)^3} \quad \text{(by H3)}$$

It remains to solve an homogeneous Stokes system There exists a pair $(\underline{\beta}^3, q)$ in $H_0^1(B_m)^3 \times L^2(B_m)$ such that

$$\begin{aligned} \Delta_\lambda \underline{\beta}^3 &= \Delta_\lambda (\underline{v} - \underline{\beta}^1 - \underline{\beta}^2) + \nabla_\lambda q \\ \operatorname{div}_\lambda \underline{\beta}^3 &= 0 \end{aligned}$$

For this problem the classical estimation gives

$$\|\underline{\beta}^3\|_{H^1(B_m)^3} \leq C\, \|\underline{v} - \underline{\beta}^1 - \underline{\beta}^2\|_{H^1(B_m)^3} \leq C\, \|\underline{v}\|_{H^1(\Pi)^3}$$

Then $\underline{w} = \underline{\beta}^1 + \underline{\beta}^2 + \underline{\beta}^3$ is solution in $H^1(B_m)^3$ of the following Stokes system

$$\left| \begin{aligned} &\Delta_\lambda \underline{w} = \Delta_\lambda \underline{v} + \nabla_\lambda q \\ &\operatorname{div}_\lambda \underline{w} = \operatorname{div}_\lambda \underline{v} + \left(\int_{B_s} \operatorname{div}_\lambda \underline{v} \right) / \text{mes } (B_m) \\ &\underline{w}/S = \underline{v}/S \quad \underline{w}/\partial B_m \backslash S = 0 \end{aligned} \right\} \tag{3 10}$$

and obviously we have :

$$\| \underline{w} \|_{H^1(B_m)^3} \leq C \| \underline{v} \|_{H^1(\Pi)^3} ,$$

which ends the proof. □

This lemma allows us to construct a restriction Ω^ε defined in the physical variables (x_1, x_2, x_3) and to deduce a continuation of the pressure.

LEMMA 3.2 : *There exists an operator :*

$$R^\varepsilon : H_0^1(\Omega_\eta)^3 \to H_0^1(\Omega_{\varepsilon_\eta})^3 \quad \textit{such that :}$$

$$\left.\begin{array}{l} \underline{v} \in H_0^1(\Omega_{\varepsilon_\eta})^3 \Rightarrow R^\varepsilon(\underline{v}) = \underline{v} \\ \| R^\varepsilon(\underline{v}) \| \leq C \| \underline{v} \| \\ \operatorname{div} \underline{v} = 0 \Rightarrow \operatorname{div} R_\varepsilon(v) = 0 \end{array}\right\} . \tag{3.11}$$

Proof: For any $\underline{v}$ en V, lemma 3.1 allows us to define $R(\underline{v})$ in $H^1(\Pi)^3$ by :

$$R(\underline{v}) = \begin{cases} \underline{v} & \text{if} \quad (y, z) \in B_f \\ \underline{w} & \text{if} \quad (y, z) \in B_m \\ 0 & \text{if} \quad (y, z) \in B_s \end{cases}$$

which satisfies

$$\| R(\underline{v}) \|_{H^1(\Pi)^3} \leq C \| \underline{v} \|_{H^1(\Pi)^3} . \tag{3.12}$$

Suppose that $C_0 =]0, \varepsilon Y_1[\times]0, \varepsilon Y_2[\times]0, \eta h_{\max}[$ is contained in Ω_η and define $C_{\underline{k}}$ where $\underline{k} = (k_1, k_2)$ by :

$$C_{\underline{k}} = \{ X \in \mathbb{R}^3, x \in \omega, (x_1 - k_1 \varepsilon Y_1, x_2 - k_2 \varepsilon Y_2, x_3) \in C_0 \}$$

$$\mathrm{H2} \Rightarrow \Omega = \bigcup_{\underline{k}} C_{\underline{k}} .$$

We define R^ε by applying R to each period. More precisely for any $\underline{v}$ in $H_0^1(\Omega_\eta)^3$, we call $\underline{v}_{\underline{k}}$ its restriction to C_k, and $\hat{\underline{v}}_{\underline{k}}(y_1, y_2, z) = \underline{v}_{\underline{k}}(\varepsilon y_1 + k_1 \varepsilon Y_1, \varepsilon y_2 + k_2' \varepsilon Y_2, z)$ is defined on Π. So R^ε is defined on each $C_{\underline{k}}$ by applying R to $\hat{\underline{v}}_{\underline{k}}$. Obviously $R^\varepsilon(\underline{v})$ lies in $H_0^1(\Omega_{\varepsilon\eta})^3$ and is equal to $\underline{v}$ if $\underline{v}$ is zero on $\Omega_\eta \backslash \Omega_{\varepsilon\eta}$. Now using (3.12) we obtain :

$$\| R^\varepsilon(\underline{v}) \|^2 = \varepsilon \sum_{\underline{k}} \left\| R(\hat{v}_{\underline{k}}) \right\|_{H^1(\Pi)^3}^2 \leq C \varepsilon \sum_{\underline{k}} \left\| \hat{v}_{\underline{k}} \right\|_{H^1(\Pi)^3}^2 = C \| \underline{v} \|^2$$

$(3.11)_4$ is obvious from $(3.9)_2$ and the definition of R^ε. □

THEOREM 3.2 : *There exists P^{ε} in $L^2(\Omega_\eta)$ such that $\underline{\nabla} P^{\varepsilon}$ is an extension of $\underline{\nabla} p^{\varepsilon}$. Moreover there exists P^* in $L^2(\Omega)/\mathbb{R}$ such that a subsequence verifies :*

$$\varepsilon^2 P^{\varepsilon} \rightharpoonup P^* \quad L^2(\Omega)/\mathbb{R} \text{ weak} \tag{3.13}$$

$$\frac{\partial P^*}{\partial z} = 0 . \tag{3.14}$$

Proof: For any ϕ in $H_0^1(\Omega)^3$, we define $\underline{F}^{\varepsilon}$ by :

$$\langle \underline{F}^{\varepsilon}, \phi \rangle = \left\langle \frac{\partial p^{\varepsilon}}{\partial x_1}, R_1^{\varepsilon}(\phi) \right\rangle + \left\langle \frac{\partial p^{\varepsilon}}{\partial x_2}, R_2^{\varepsilon}(\phi) \right\rangle + \left\langle \frac{\partial p^{\varepsilon}}{\partial x_3}, R_3^{\varepsilon}(\phi) \right\rangle$$

where $\langle\ ,\ \rangle$ is for the duality product between H^{-1} and H_0^1 either on Ω or on Ω_ε

$$\langle \underline{F}^{\varepsilon}, \phi \rangle = - \int_{\Omega_\varepsilon} p^{\varepsilon} \left(\frac{\partial R_1^{\varepsilon}(\phi)}{\partial x_1} + \frac{\partial R_2^{\varepsilon}(\phi)}{\partial x_2} + \frac{\partial R_3^{\varepsilon}(\phi)}{\partial x_3} \right) dX = - a(\underline{u}^{\varepsilon}, R^{\varepsilon}(\phi)) .$$

From (3.8) and (3.11) :

$$| \langle \underline{F}^{\varepsilon}, \phi \rangle | \leqslant \|\underline{u}^{\varepsilon\eta}\| \, \|R^{\varepsilon}(\phi)\| \leqslant C \sqrt{\varepsilon} \|\underline{u}^{\varepsilon}\|_{\varepsilon} \|\phi\| \leqslant C \varepsilon^{-3/2} \|\phi\| . \tag{3.15}$$

So $\underline{F}^{\varepsilon} \in H^{-1}(\Omega_\eta)^3$.

If div $(\phi) = 0$ then by (3.11) div $(R^{\varepsilon}(\phi)) = 0$ and $\langle \underline{F}^{\varepsilon}, \phi \rangle = 0$. So there exists P^{ε} in $L^2(\Omega_\eta)$ such that :

$$\underline{F}^{\varepsilon} = \underline{\nabla} P^{\varepsilon} . \tag{3.16}$$

We remark that if ϕ belongs to $H_0^1(\Omega_{\varepsilon\eta})^3$, $R^{\varepsilon}(\phi) = \phi$ and $\underline{F}^{\varepsilon}$ reduces to $\underline{\nabla} p^{\varepsilon}$. So we have constructed a continuation of the pressure gradient. We use (3.15) to give a priori estimates on the « new pressure » in the fixed domain Ω. (To simplify we keep the same notation for the pressure in Ω^+) :

$$\int_{\Omega} P^{\varepsilon} \left(\frac{\partial \phi_1}{\partial x_1} + \frac{\partial \phi_2}{\partial x_2} + \frac{1}{\lambda\varepsilon} \frac{\partial \phi_3}{\partial z} \right) dx_1 \, dx_2 \, dz \leqslant C \varepsilon^{-2} \|\phi\|_{\varepsilon} .$$

We get :

$$\left\| \varepsilon^2 \frac{\partial P^{\varepsilon}}{\partial x_i} \right\|_{H^{-1}(\Omega)} \leqslant C \quad (i = 1, 2) \quad \left\| \varepsilon \frac{\partial P^{\varepsilon}}{\partial z} \right\|_{H^{-1}(\Omega)} \leqslant C$$

and (3.13) follows from the inequality

$$\|P^{\varepsilon}\|_{L^2(\Omega)/\mathbb{R}} \leq C \|\nabla P^{\varepsilon}\|_{H^{-1}(\Omega)^3}$$

and (3.14) by noting that $\varepsilon \dfrac{\partial P^{\varepsilon}}{\partial z}$ converges also in $H^{-1}(\Omega)$ weak. □

III.3. The limit equation

THEOREM 3.3 : *With the conjecture that the pressure convergence is strong in $L^2(\Omega)$, P^* satisfies :*

$$\sum_{i=1,2} \frac{\partial}{\partial x_i} \left(\sum_{j=1,2} \frac{\partial}{\partial x_j} ([\alpha_j^i] P^*) \right) = 0$$

with the boundary condition :

$$\sum_{i=1,2} \left(\sum_{j=1,2} \frac{\partial}{\partial x_j} ([\alpha_j^i] P^*) + [\alpha_i^0] n_i \right) = Y_1 Y_2 t(x) .$$

Proof: New rescaled problems (L_ε^i) can be defined in (Ω_ε) from problem (L^i) defined in the basic cell, which doesn't depend on x, like function h. We extend the functions $\underline{\alpha}^i$ and q^i used in the proof of lemma 3.2 by periodicity for y in $\mathbb{R}^2$ and $0 < z < h(y)$.

Setting $\underline{\alpha}^{i\varepsilon} = \underline{\alpha}^i(x/\varepsilon, z)$, $q^{i\varepsilon} = q^i(x/\varepsilon, z)$ and $H^\# = \{\underline{\phi} \in H^1(\Omega_\varepsilon)^3,\ \underline{\phi}|_\omega = \underline{\phi}|_{\Sigma_\varepsilon} = \underline{0},\ \underline{\phi}$ is εY periodic in $x\}$.

Rescaled problems L_ε^i are : $(i = 1, 2)$

$$L_\varepsilon^i \left| \begin{array}{ll} \text{Find } (\underline{\alpha}^{i\varepsilon}, q^{i\varepsilon}) \text{ in } H^\# \times L_0^2(\Omega_\varepsilon) \text{ such that :} & \\ a^\varepsilon(\varepsilon^2 \underline{\alpha}^{i\varepsilon}, \underline{\phi}) = b^\varepsilon(\varepsilon q^{i\varepsilon}, \underline{\phi}) - \displaystyle\int_{\Omega_\varepsilon} \underline{f}^{i\varepsilon} \cdot \underline{\phi}\, dx\, dz \quad \forall \underline{\phi} \in H^\# & (3.17) \\ b^\varepsilon(q, \underline{\alpha}^{i\varepsilon}) = 0 \quad \forall q \in L^2(\Omega_\varepsilon) & (3.18) \end{array} \right.$$

$$\underline{f}^{1\varepsilon} = (1, 0, 0), \quad \underline{f}^{2\varepsilon} = (0, 1, 0)$$

and problem L_ε^0 is defined in the same way with $\underline{\alpha}^{0\varepsilon}|_\omega = (s, 0, 0)$ and $\underline{f}^{0\varepsilon} = 0$. Extending $\underline{\alpha}^{i\varepsilon}$ by zero to the whole Ω, we have for $i = 0, 1, 2$:

$$\|\underline{\alpha}^{i\varepsilon}\|_{L^2(\Omega)^3} \leq C, \quad \|q^{i\varepsilon}\|_{L^2(\Omega)} \leq C \tag{3.19}$$

and taking account of the rescaling in z :

$$\left\| \frac{\partial \underline{\alpha}^{i\varepsilon}}{\partial z} \right\|_{L^2(\Omega)^3} \leq C, \quad \varepsilon \left\| \frac{\partial \underline{\alpha}^{i\varepsilon}}{\partial x_j} \right\|_{L^2(\Omega)^3} \leq C \quad (j = 1, 2). \tag{3.20}$$

Now we use the energy method [20] to prove convergence. For any ϕ in $\mathscr{D}(\omega)$, we use $\phi(\underline{u}^\varepsilon - \underline{\alpha}^{0\varepsilon})$ as test function in (3.17) and $\varepsilon^2 \phi\underline{\alpha}^{i\varepsilon}$ in (3.4). We obtain : $(i = 1, 2)$

$$a^\varepsilon(\varepsilon^2 \underline{\alpha}^{i\varepsilon}, \phi\underline{u}^\varepsilon) = b^\varepsilon(\varepsilon q^{i\varepsilon}, \phi(\underline{u}^\varepsilon - \underline{\alpha}^{0\varepsilon})) + a^\varepsilon(\varepsilon^2 \underline{\alpha}^{i\varepsilon}, \phi\underline{\alpha}^{0\varepsilon}) - \int_{\Omega_\varepsilon} \phi(u_i^\varepsilon - \alpha_i^{0\varepsilon})\, dx\, dz$$

and

$$a^\varepsilon(\underline{u}^\varepsilon, \phi\varepsilon^2 \underline{\alpha}^{i\varepsilon}) = b^\varepsilon(p^\varepsilon, \varepsilon^2 \phi\underline{\alpha}^{i\varepsilon}) .$$

We calculate :

$$I_i^\varepsilon = a^\varepsilon(\varepsilon^2 \underline{\alpha}^{i\varepsilon}, \phi\underline{u}^\varepsilon) - a^\varepsilon(\varepsilon^2 \underline{u}^\varepsilon, \phi\underline{\alpha}^{i\varepsilon}) \tag{3.21}$$

$$I_i^\varepsilon = b^\varepsilon(\varepsilon q^{i\varepsilon}\phi, \underline{u}^\varepsilon - \underline{\alpha}^{0\varepsilon}) + \int_{\Omega_\varepsilon} \varepsilon q^{i\varepsilon} \sum_{j=1,2} \frac{\partial\phi}{\partial x_j}(u_j^\varepsilon - \alpha_j^{0\varepsilon}) + a^\varepsilon(\varepsilon^2 \underline{\alpha}^{i\varepsilon}, \phi\underline{\alpha}^{0\varepsilon})$$
$$- b^\varepsilon(\varepsilon^2 p^\varepsilon \phi, \underline{\alpha}^{i\varepsilon}) - \int_{\Omega_\varepsilon} \varepsilon^2 p^\varepsilon \sum_{j=1,2} \frac{\partial\phi}{\partial x_j}\alpha_j^{i\varepsilon} - \int_{\Omega_\varepsilon} \phi(u_i^\varepsilon - \alpha_i^{0\varepsilon})\, dx\, dz .$$

The bilinear form b^ε are zero because of (3.5), (3.18). Moreover :

$$\left| \int_{\Omega_\varepsilon} \varepsilon q^{i\varepsilon} \sum_{j=1,2} \frac{\partial\phi}{\partial x_j}(u_j^\varepsilon - \alpha_j^{0\varepsilon}) \right| \leqslant C\varepsilon \|q^{i\varepsilon}\|_{L^2(\Omega_\varepsilon)} \|\underline{u}^\varepsilon - \underline{\alpha}^{0\varepsilon}\| \leqslant K\varepsilon \underset{(\varepsilon\to 0)}{\to} 0$$

and using $\phi\underline{\alpha}^{i\varepsilon}$ as test function in problem L_ε^0 :

$$a^\varepsilon(\varepsilon^2 \underline{\alpha}^{i\varepsilon}, \phi\underline{\alpha}^{0\varepsilon}) = a^\varepsilon(\varepsilon^2 \underline{\alpha}^{0\varepsilon}, \phi\underline{\alpha}^{i\varepsilon}) + \sum_{k=1,2}\sum_{j=1,3} \varepsilon^2 \frac{\partial\phi}{\partial x_k}\left(\frac{\partial\alpha_j^i}{\partial x_k}\alpha_j^0 - \frac{\partial\alpha_j^0}{\partial x_k}\alpha_j^i \right)$$
$$= \int_{\Omega_\varepsilon} \varepsilon q^{0\varepsilon}\left(\sum_{j=1,2} \frac{\partial\phi}{\partial x_j}\alpha_j^{i\varepsilon} \right) \leqslant K\varepsilon .$$

So

$$\lim_{\varepsilon\to 0} I_i^\varepsilon = \lim_{\varepsilon\to 0}\left(-\int_\Omega \varepsilon^2 P^\varepsilon \sum_{j=1,2} \frac{\partial\phi}{\partial x_j}\alpha_j^{i\varepsilon}\, dx\, dz \right) - \int_\Omega \phi(u_i^* - \alpha_i^{0*})\, dx\, dz .$$

On the other hand, the right handside of (3.21) can be written :

$$I_i^\varepsilon = \int_\Omega \varepsilon^2 \sum_{j=1,2} \frac{\partial\phi}{\partial x_j} \sum_{k=1,3}\left(\frac{\partial\alpha_k^{i\varepsilon}}{\partial x_j}u_k^\varepsilon - \frac{\partial u_k^\varepsilon}{\partial x_j}\alpha_k^{i\varepsilon} \right) dx\, dz .$$

The estimates of theorem 3.1, (3.19) and (3.20) easily induce that the limit of this last writing is zero. So for $i = 1, 2$:

$$\int_{\Omega} \phi (u_i^* - \alpha_i^{0*})\, dx\, dz = - \lim_{\varepsilon \to 0} \sum_{j=1,2} \int_{\Omega} \varepsilon^2 P^{\varepsilon} \frac{\partial \phi}{\partial x_j} \bar{\alpha}_j^{i\varepsilon}\, dx\, dz\,. \quad (3.22)$$

As already mentioned, to take the limit in this product we have to suppose something better than the convergence in theorem 3.2, for instance the strong L^2 convergence of $\varepsilon^2 P^{\varepsilon}$. Then the limit equation for P^* is easy to obtain.

By classical lemma for Y-periodic functions (see for instance [7]), we get :

$$\int_0^{h(y)} \alpha_j^{i\varepsilon}(x, z)\, dx\, dz \to \frac{1}{\text{mes } [Y]} \int_B \alpha_j^i(y, z)\, dy\, dz =$$
$$= [\alpha_j^i]/Y_1\ Y_2 \text{ in } L^2(\omega) \text{ weak}\,.$$

$$(3.22) \Rightarrow \sum_{j=1,2} \frac{1}{Y_1 Y_2} \int_{\omega} \frac{\partial \phi}{\partial x_j} P^* [\alpha_j^i]\, dx =$$
$$= - \int_{\omega} \phi \left(\int_0^{h_{\max}} u_i^* - [\alpha_i^0]/Y_1\, Y_2 \right) dx$$

which is :

$$\sum_{j=1,2} \frac{\partial}{\partial x_j} ([\alpha_j^i]\, P^*) + [\alpha_i^0] = Y_1\, Y_2 \int_0^{h_{\max}} u_i^*(x, z)\, dz \quad \text{in} \quad \mathscr{D}'(\omega)\,. \quad (3.23)$$

Now, we use (3.5) with $\phi \in \mathscr{D}(\omega)$ as a test function.
We obtain :

$$\int_{\Omega} u_1^* \frac{\partial \phi}{\partial x_1} + u_2^* \frac{\partial \phi}{\partial x_2} = 0$$

which implies :

$$\operatorname{div}_x \left(\int_0^{h_{\max}} \underline{u}^*\, dz \right) = 0 \quad \text{in} \quad \mathscr{D}'(\omega)\,. \quad (3.24)$$

Combining (3.23) and (3.24), we obtain the limit equation for P^* :

$$\sum_{i=1,2} \frac{\partial}{\partial x_i} \left(\sum_{j=1,2} \frac{\partial}{\partial x_j} ([\alpha_j^i]\, P^*) \right) + \sum_{i=1,2} \frac{\partial}{\partial x_i} [\alpha_i^0] = 0\,. \quad (3.25)$$

The last term is zero because $\underline{\alpha}^0$ doesn't depend on x with the hypothesis made on h.

Now (3.23) induces that ∇P^* lies in $L^2(\omega)$ while (3.24) induces that, as $\int_0^{h_{\max}} \underline{\bar{u}}^\varepsilon \, dz$, $\int_0^{h_{\max}} \underline{\bar{u}}^* \, dz$ lies also in :

$$H(\mathrm{div}, \omega) = \left\{ \underline{v} \in L^2(\omega)^2, \mathrm{div}_x (\underline{v}) \in L^2(\omega) \right\} .$$

From theorem 3.1 the convergence takes place in $H(\mathrm{div}, \omega)$ and

$$\int_0^{h_{\max}} \underline{u}^* \cdot \underline{n} \, dz = \lim_{\varepsilon \to 0} \int_0^{h_{\max}} \underline{u}^\varepsilon \cdot \underline{n} \, dz = \int_0^{h_{\min}} K(x, z) \cos (n, x_1) \, dz = t(x) .$$

By (3.23) $\nabla_x P^*$ lies also in $H(\mathrm{div}, \omega)$ so that $\nabla_x P^* \cdot \underline{n}$ makes sense and

$$\sum_{i=1,2} \left(\sum_{j=1,2} \frac{\partial}{\partial x_j} ([\alpha_j^i] P^*) + [\alpha_i^0] \right) n_i = Y_1 Y_2 t(x) \tag{3.26}$$

(3.25) and (3.26) are nothing else than the strong formulation of equation (2.7).

IV. STUDY OF THE HOMOGENIZED THIN FILM EQUATION

IV.1. Existence and uniqueness

From now on, to recall that the limit equation (3.25) is obtained when ε and η tend to zero with a constant ratio $\lambda = \eta/\varepsilon$, we write p^λ instead of p^{-2} (in Sect. 2) or P^* (in Sect. 3).

So we consider the problem :

$$\mathscr{P}^\lambda \left| \begin{array}{l} \text{Find } p^\lambda \text{ in } H^1(\omega) \text{ such that :} \\ \displaystyle\sum_{i,j=1,2} \int_\omega [\alpha_i^j] \frac{\partial p^\lambda}{\partial x_i} \frac{\partial \phi}{\partial x_j} \, dx + \sum_{i=1,2} \int_\omega [\alpha_i^0] \frac{\partial \phi}{\partial x_i} \, dx = \\ \displaystyle\qquad = \int_{\partial\omega} t\phi \, d\sigma \quad \forall \phi \in H^1(\omega) \end{array} \right. \tag{4.1}$$

where $[\phi] = \int_B \phi(x, y, z) \, dy \, dz$ and $t = Y_1 Y_2 \left(\int_0^{h_{\min}} K(x, z) \, dz \right) (n \cdot x_1)$.

By taking $\underline{\alpha}^i$ as a test function for the local problem L^i, we remark that : $[\alpha_i^j] = -a(\underline{\alpha}^i, \underline{\alpha}^j)$ $i, j = (1, 2)$ (the bilinear form a is defined in § II.3).

Setting

$$A = (a_{ij}(x)) \quad \text{with} \quad a_{ij}(x) = a(\underline{\alpha}^i, \underline{\alpha}^j) . \tag{4.2}$$

A is a symmetric matrix, moreover :

LEMMA 4.1 : *A is a positive definite matrix.*

Proof: Set $\underline{\sigma} = \sum_{i=1,2} \xi_i \underline{\alpha}^i$ for any $\underline{\xi}$ in $\mathbb{R}^2$

$$I = \sum_{i,j=1,2} a_{ij} \xi_i \xi_j = \sum_{i,j=1,2} \xi_i \xi_j a(\underline{\alpha}^i, \underline{\alpha}^j) = a(\underline{\sigma}, \underline{\sigma}) \geqslant 0 .$$

For any $\underline{\phi}$ in H_0^Y with $\operatorname{div}_y \underline{\phi} = 0$, one has

$$a(\underline{\sigma}, \underline{\phi}) = \sum_{i=1,2} \xi_i a(\underline{\alpha}^i, \underline{\phi}) = - \sum_{i=1,2} \xi_i \int_B \phi_i \, dy \, dz \tag{4.3}$$

by definition of problem L^i.

We choose

$$\underline{\beta} = (6 \xi_1 z(z - h_{\min})/Y_1 Y_2 h_{\min}^3 , \quad 6 \xi_2 z(z - h_{\min})/Y_1 Y_2 h_{\min}^3, 0)$$

for $z < h_{\min}$ and we continue it by zero to define it on B. $\underline{\beta}$ lies obviously in H_0^Y with

$$\operatorname{div}_y \underline{\beta} = 0 , \quad [\beta_i] = - \xi_i , \quad \text{and} \quad a(\underline{\sigma}, \underline{\beta}) = - \int_B \sum_{i=1,2} \xi_i \beta_i \, dy \, dz = \Sigma \xi_i^2 .$$

If we suppose $I = 0$ then $a(\sigma, \sigma) = 0 \Rightarrow \sigma = 0 \Rightarrow a(\sigma, \beta) = 0$. Then $\xi_i = 0 \ \forall i$, which ends the proof. □

We can now show the existence of the homogenized pressure :

THEOREM 4.1 : *There exists a unique p^λ in $H^1(\omega)/\mathbb{R}$ solution of $\mathcal{P}^\lambda$.*

Proof: Lemma 4.1 induces that the bilinear form :

$$\gamma(\phi, \psi) = \sum_{i,j=1,2} \int_\omega a_{ij} \frac{\partial \phi}{\partial x_i} \frac{\partial \psi}{\partial x_j} dx$$

is definite positive but it is not coercive on $H^1(\omega)$. As the solutions of

$$\gamma(\phi, \gamma) = 0$$

are the constant functions, a necessary and sufficient condition for the existence of p^λ in $H^1(\omega)$ is that the second member is orthogonal to the constants but from (2.5)

$$\int_{\partial\omega} t \, d\sigma = Y_1 Y_2 \int_{\partial\Omega} - K(x, z)(n \cdot x_1) \, dx \, dz = 0 .$$

Then all the solution of (4.1) differ from one another of a constant and there is one and only one solution in $H^1(\omega)/\mathbb{R}$.

IV.2. A priori estimates

In this section C will denote any constant with respect to λ.

LEMMA 4.2 : *We have the following estimates :*

$$\|\alpha_j^i\|_{L^2(B)} \leqslant C\lambda^2 , \quad \left\|\frac{\partial \alpha_j^i}{\partial z}\right\|_{L^2(B)} \leqslant C\lambda^2 , \quad \left\|\frac{\partial \alpha_j^i}{\partial y_k}\right\|_{L^2(B)} \leqslant C\lambda$$
$$i, k = 1, 2 \quad j = 1, 2, 3 .$$

Proof: We take $\phi = \underline{\alpha}^i$ as a test function in problem (L^i) ; so :

$$a(\underline{\alpha}^i, \underline{\alpha}^i) = -\int_B \alpha_i^i \, dy \, dz \leqslant |B|^{1/2} \|\alpha_i^i\|_{L^2(B)} \leqslant |B|^{1/2} h_{\max} \left\|\frac{\partial \alpha_i^i}{\partial z}\right\|_{L^2(B)}$$

by Poincaré inequality. So :

$$\frac{1}{\lambda^2}\left\|\frac{\partial \alpha_i^i}{\partial z}\right\|_{L^2(B)} \leqslant C \quad \text{and} \quad \left\|\frac{\partial \alpha_i^i}{\partial y_k}\right\|_{L^2(B)}^2 \leqslant \left\|\frac{\partial \alpha_i^i}{\partial z}\right\|_{L^2(B)} \leqslant C\lambda^2 . \quad \square$$

LEMMA 4.3 : *We have the following estimates for problem* L^0 :

$$\|\alpha_j^0\|_{L^2(B)} \leqslant C , \quad \left\|\frac{\partial \alpha_j^0}{\partial z}\right\|_{L^2(B)} \leqslant C , \quad \left\|\frac{\partial \alpha_j^0}{\partial y_k}\right\|_{L^2(B)} \leqslant C/\lambda$$
$$k = 1, 2 \quad j = 1, 2, 3 .$$

Proof: We have to find a function of $H^1(B)^3$ periodic in y which satisfies the same boundary conditions as $\underline{\alpha}^0$.

Let :

$$d(z) = \begin{cases} (h_{\min} - z)\, s/h_{\min} & \text{for} \quad 0 < z < h_{\min} \\ 0 & \text{for} \quad h_{\min} < z \end{cases}$$

$(\alpha_1^0 - d, \alpha_2^0, \alpha_3^0)$ is then a test function for L^0 so that :

$$a(\underline{\alpha}^0, \underline{\alpha}^0) = \frac{1}{\lambda^2}\int_B \frac{\partial \alpha_1^0}{\partial z}\frac{\partial d}{\partial z}\, dy \, dz \leqslant \frac{C}{\lambda^2}\left\|\frac{\partial \alpha_1^0}{\partial z}\right\|_{L^2(B)} .$$

So :

$$\left\|\frac{\partial \alpha_1^0}{\partial z}\right\|_{L^2(B)} \leqslant C .$$

The other estimates follow as in lemma 4.2. $\square$

Remark : The previous estimates induce the rewritting of the weak formulation (4.1), so :

$$\sum_{i,j=1,2} \int_\omega a_{ij}^\lambda \frac{\partial}{\partial x_i}(\lambda^2 p^\lambda) \frac{\partial \phi}{\partial x_j} dx = \sum_{i=1,2} \int_\omega [\alpha_i^0] \frac{\partial \phi}{\partial x_i} dx - \int_{\partial\omega} t\phi \, d\sigma \tag{4.4}$$

with $a_{ij}^\lambda = a_{ij}/\lambda^2$.

V. THE STOKES ROUGHNESS ($\lambda \to +\infty$)

We are dealing now with the limit of p^λ when λ tends to infinity ; this describes the situation when the roughness wavelength is very small both in front of the gap and the roughness height.

We denote by B^+ the upper part of B and B^- the lower part of B :

$$B^+ = \{(y, z) \in B, h_{\min} < z < h(y)\}$$
$$B^- = B \backslash B^+ .$$

LEMMA 5.1 : *When λ tends to infinity, for $i = 1, 2$, $j = 1, 2, 3$*

$$\alpha_j^i/\lambda^2 \rightharpoonup \alpha_j^{i\,*} \quad H^1(B)\ \textit{weak}$$

where

$$\underline{\alpha}^{i\,*} = 0 \quad on \quad B^+$$
$$\alpha_i^{i\,*} = z(z - h_{\min})/2\,, \quad \alpha_j^{i\,*} = 0 \quad on \quad B^- \quad i \neq j = 1, 2$$
$$\alpha_3^{i\,*} = 0 \quad on \quad B^- .$$

Proof: From the estimates of lemma 4.2, it is clear that $\underline{\alpha}^i/\lambda^2$, $\frac{\partial \underline{\alpha}^i}{\partial z}/\lambda^2$ and $\frac{\partial \underline{\alpha}^i}{\partial y_k}/\lambda$ have a limit in $L^2(B)^3$ weak after extraction of a subsequence. Moreover $\frac{\partial \underline{\alpha}^i}{\partial y_k}/\lambda^2$ tends to zero in $L^2(B)^3$ which induces that $\underline{\alpha}^i/\lambda^2$ converges in fact in $H^1(B)^3$ to $\underline{\alpha}^{i\,*}$ with $\frac{\partial \underline{\alpha}^{i\,*}}{\partial y_k} = 0$. A direct consequence of the condition $\underline{\alpha}^i = 0$ on the boundary $z = h(y)$ which is preserved when $\lambda \to +\infty$ because the convergence takes place in $H^1(B)^3$, is :

$$\underline{\alpha}^i = 0 \quad \text{on} \quad B^+ .$$

Let $\theta(z)$ be a function of $\mathcal{D}(]0, h_{\min}[)$ extended by zero to B. We use successively $(\theta(z), 0, 0)$ and $(0, \theta(z), 0)$ as test functions in L^i. Letting λ tend to infinity, we find the limit equations :

$$\int_{B^-} \frac{\partial \alpha_j^{i*}}{\partial z} \frac{\partial \theta}{\partial z} dy\, dz = -\delta_j^i \int_{B^-} \theta(z)\, dy\, dz \quad (i, j = 1, 2)$$

$$\frac{\partial^2 \alpha_j^{i*}}{\partial z^2} = \delta_j^i \quad \text{in} \quad \mathscr{D}'(]0, h_{\min}[)\,. \tag{5.1}$$

Moreover :

$$\int_B q \operatorname{div}_y (\underline{\alpha}^i) = 0 \quad \forall q \in L^2(B)\,.$$

Taking $q = 1$ and the limit of each term we find $\frac{\partial \alpha_3^{i*}}{\partial z} = 0$ and the proof is ended by integrating (5.1). □

LEMMA 5.2 : *When λ tends to infinity, for $j = 1, 2, 3$*

$$\alpha_j^0 \rightharpoonup \alpha_j^{0*} \quad H^1(B) \text{ weak}$$

where

$$\begin{aligned} &\underline{\alpha}^{0*} = 0 \quad on \quad B^+ \\ &\alpha_1^{0*} = s(1 - z/h_{\min}), \quad \alpha_2^{0*} = 0 \quad on \quad B^- \\ &\alpha_3^{0*} = 0 \quad on \quad B^-\,. \end{aligned}$$

Proof: The existence of the limit is obvious via the estimates of lemma 4.3 and we have again $\frac{\partial \alpha_j^{0*}}{\partial y_k} = 0$. The proof is the same as that of previous lemma but the boundary conditions $(s, 0, 0)$ on Y and $(0, 0, 0)$ on $z = h_{\min}$ give different values for α^{0*}. □

THEOREM 5.1 : *When λ tends to infinity, the sequence $\lambda^2 p^\lambda$ converges in $H^1(\omega)$ weak to the unique solution in $H^1(\omega) \cap L_0^2(\Omega)$ of:*

$$\int_\omega h_{\min}^3 \sum_{i=1,2} \frac{\partial p^{*\infty}}{\partial x_i} \frac{\partial \phi}{\partial x_i} dx = \int_\omega 6\, s h_{\min} \frac{\partial \phi}{\partial x_1} dx - 12 \int_{\partial\omega} t\phi\, d\sigma\,. \tag{5.2}$$

Proof: We use the formulation (4.4) of $\mathscr{P}^\lambda$. $A^\lambda = (a_{ij}^\lambda)$ is a symmetric definite positive matrix for each fixed λ. As $a_{ij}^\lambda = -[\alpha_j^i]/\lambda^2$, lemma 5.1 induces that A^λ converges to $A^{*\infty}$ with :

$$A^{*\infty} = \begin{bmatrix} h_{\min}^3/12 & 0 \\ 0 & h_{\min}^3/12 \end{bmatrix}$$

with non zero diagonal entries. So for sufficiently large λ, there exists $C_0 > 0$ such that :

$$\sum_{i,j=1,2} a_{ij}^{\lambda} \xi_i \xi_j \geq C_0 \sum_{i=1,2} \xi_i^2 .$$

Taking $\phi = \lambda^2 p^{\lambda}$ in (4.4)

$$C_0 \sum_{i=1,2} \left\| \frac{\partial}{\partial x_i} (\lambda^2 p^{\lambda}) \right\|_{L^2(\omega)}^2 \leq \sum_{i,j=1,2} \int_{\omega} a_{ij}^{\lambda} \frac{\partial}{\partial x_i} (\lambda^2 p^{\lambda}) \frac{\partial}{\partial x_j} (\lambda^2 p^{\lambda}) \, dx$$
$$\leq C \|\lambda^2 p^{\lambda}\|_{H^1(\omega)} .$$

If we choose the element of $H^1(\omega)/\mathbb{R}$ which belongs to $L_0^2(\omega)$, we may take the Poincaré norm [22]. This last inequality implies :

$$\lambda^2 \|p^{\lambda}\|_{H^1(\omega)/\mathbb{R}} \leq C$$

and we can extract a subsequence that weakly converges in $H^1(\omega)$. □

Now we can find the limit equation.

We make λ tend to infinity in (4.4). We know the limit of all terms and t is independent of λ. So (5.2) is obvious and it has a unique solution in $H^1(\omega)/\mathbb{R}$ by same arguments as in theorem 4.1. □

Conclusion : If we come back to the basic equations (2.1)-(2.3) which describe the flow of a fluid between two surfaces in relative motion, with a roughness length ε and a gap between the surfaces of size η, it has been shown that the way how the two small parameters tend to zero leads to different results. If ε tends first to zero and then η tends to zero, the limit pressure is solution of a Reynolds equation with an effective height $h_{\min}$ [3] :

$$\operatorname{div} (h_{\min}^3 \nabla p) = 6 \, s \, \frac{\partial h_{\min}}{\partial x_1} \tag{5.3}$$

$$h_{\min}^3 \frac{\partial p}{\partial n} = \left(6 \, s h_{\min} - 12 \lim_{\eta \to 0} \int_0^{h_{\min}} K(x, z) \, dz \right) (n \, . \, x_1) \tag{5.4}$$

$$\int_{\omega} h_{\min} \, p \, dx = 0 \tag{5.5}$$

where $h_{\min}$ defines the non oscillating part of the rescaled domain Ω^- and K the velocity of the fluid given on :

$$\Gamma^- = \{(x, z), x \in \partial\omega, 0 < z < h_{\min}\} .$$

p is the $H^1(\omega)/\mathbb{R}$ solution of the Reynolds problem (5.3)-(5.4), the constant being fixed by (5.5). As h_{min} is independent of x, the right handside of (5.3) is zero.

We find exactly the same result in these two cases :

— ε and $\eta \to 0$ with constant $\lambda = \eta/\varepsilon$, then $\lambda \to +\infty$;

— $\varepsilon \to 0$ first, then $\eta \to 0$.

VI. THE REYNOLDS ROUGHNESS ($\lambda \to 0$)

It is less straightforward because we cannot compute the exact values of the limits of the solution of auxiliary problems as in lemma 5.1 and 5.2. The way to overcome this difficulty is to point out that making λ tend to zero is nothing else that making the height of the gap tend to zero. So the asymptotic behavior of $\underline{\alpha}^i$ and q^i are studied by the same way as in passing from Stokes to Reynolds [1], the difference being the periodic boundary conditions partly substituted to Dirichlet conditions.

LEMMA 6.1 : *When λ tends to zero for $i, k = 1, 2$; $j = 1, 2, 3$*

$$\begin{aligned}
\alpha_j^i/\lambda^2 &\rightharpoonup \alpha_j^{i*} && \textit{weakly in } L^2(B)\\
\frac{\partial \alpha_i^j}{\partial z}/\lambda^2 &\rightharpoonup \frac{\partial \alpha_j^{i*}}{\partial z} && \textit{weakly in } L^2(B)\\
\frac{\partial \alpha_j^i}{\partial y_k}/\lambda &\rightharpoonup 0 && \textit{weakly in } L^2(B)\\
q^i &\rightharpoonup q^{i*} && \textit{weakly in } L_0^2(B)\,.
\end{aligned}$$

Moreover $\dfrac{\partial q^{i*}}{\partial z} = 0.$

Proof: The first three limits are direct consequences of the estimates of lemma 4.2. Because $\dfrac{\partial \alpha_j^i}{\partial y_k}/\lambda^2$ converges to $\dfrac{\partial \alpha_j^{i*}}{\partial y_k}$ in $\mathscr{D}'(B)$, the third limit is equal to zero.

Now we choose any ϕ in $H_0^1(B)$ and we take successively $\underline{\phi}^1 = (\phi, 0, 0)$, $\underline{\phi}^2 = (0, \phi, 0)$ and $\underline{\phi}^3 = (0, 0, \phi)$ as test functions in the local problems L^i. Estimates of lemma 4.2 implies :

$$\left\| \frac{\partial q^i}{\partial y_k} \right\|_{H^{-1}(B)} \leqslant C \quad \text{and} \quad \left\| \frac{1}{\lambda}\frac{\partial q^i}{\partial z} \right\|_{H^{-1}(B)} \leqslant C \tag{6.1}$$

$L_0^2(B)$ being weakly closed in $H^{-1}(B)$, then :

$$\|q^i\|_{L_0^2(B)} \leqslant C$$

which insures the last limit. Now, by (6.1) $\frac{1}{\lambda}\frac{\partial q^i}{\delta z}$ also converges in $H^{-1}(B)$ weak when λ tends to zero and this induces that $\frac{\partial q^{i*}}{\partial z} = 0$. □

LEMMA 6.2 : *When λ tends to zero, for $j = 1, 2, 3$:*

$$\begin{array}{lll} \alpha_j^0 \rightharpoonup \alpha_j^{0*} & & \textit{weakly in } L^2(B) \\ \dfrac{\partial \alpha_j^0}{\partial z} \rightharpoonup \dfrac{\partial \alpha_j^{0*}}{\partial z} & & \textit{weakly in } L^2(B) \\ \lambda \dfrac{\partial \alpha_j^0}{\partial y_k} \rightharpoonup 0 & & \textit{weakly in } L^2(B) \\ \lambda^2 q^0 \rightharpoonup q^{0*} & & \textit{weakly in } L_0^2(B)\,. \end{array}$$

Moreover : $\frac{\partial q^{0*}}{\partial z} = 0.$

Proof: The same as lemma 6.1. □

We give now the equation of these limits :

LEMMA 6.3 : *q^{i*} is the unique solution in $L_0^2(B) \cap H_p^1(Y)$ of:*

$$\int_Y h^3 \nabla q^{i*}\, \nabla \phi = \int_Y \frac{\partial h^3}{\partial y_i}\, \phi\, dy\,, \quad \forall \phi \in H_p^1(Y) \tag{6.2}$$

with $H_p^1(Y) = \{\phi \in H^1(Y), \phi \text{ is } Y\text{-periodic}\}$.

Proof: We still have $\alpha_3^{i*} = 0$ on B by the divergence equation. Taking $\underline{\phi} = (\phi, 0, 0)$, and $(0, \phi, 0)$ in H_0^Y, we obtain for the limit :

$$\frac{\partial^2 \alpha_j^{i*}}{\partial z^2} = \left(\frac{\partial q^{i*}}{\partial y_j} + \delta_i^j\right) \quad (i, j = 1, 2)\,. \tag{6.3}$$

The second member doesn't depend on z and we can integrate these equations. The convergence of α_j^i takes place in $L^2(Y\,; H_0^1(]0, h(y)[))$ in which there is a trace defined on the boundary ∂B such that $\phi \to \phi(n\,.\,z)$ is a linear and continuous application in $H^{-1/2}(\partial B)$. $\alpha_j^i/\lambda^2 = 0$ on the boundaries $z = 0$ and $z = h(y)$ and this value is kept also by the limit.

We just have to suppose that there is no part of the boundary $\{z = h(y)\}$ which is vertical and of non zero measure. But this is actually not a restriction for the shape of the roughness.

The integration of (6.3) with respect to z gives now :

$$\alpha_j^{i*} = \left(\frac{\partial q^{i*}}{\partial y_j} + \delta_i^j\right) z(z - h)/2 \quad (i, j = 1, 2)\,. \tag{6.4}$$

Taking $\xi = \xi(y)$ in the divergence equation of problem L^i, with ξ in $\mathfrak{D}(Y)$:

$$0 = \int_B \xi \operatorname{div}_y \underline{\alpha}^i \, dy \, dz = \int_B \xi \left(\sum_{j=1,2} \frac{\partial \alpha_j^i}{\partial y_j} \right) dy \, dz$$

$$= \int_Y \xi \left(\sum_{j=1,2} \frac{\partial}{\partial y_j} (\tilde{\alpha}_j^i) \right) dy \qquad \left(\tilde{v} = \int_0^{h(y)} v(y, z) \, dz \right).$$

When λ tends to zero, this gives a conservation law in $\mathscr{D}'(Y)$:

$$\sum_{j=1,2} \frac{\partial}{\partial y_i} (\tilde{\alpha}_j^i *) = 0 . \tag{6.5}$$

From (6.4) :

$$\tilde{\alpha}_j^i * = - \frac{h^3}{12} \left(\frac{\partial q^i *}{\partial y_j} + \delta_i^j \right) . \tag{6.6}$$

Putting it in (6.5) :

$$\sum_{j=1,2} \frac{\partial}{\partial y_j} \left(h^3 \frac{\partial q^i *}{\partial y_j} \right) = - \frac{\partial h^3}{\partial y_i} \quad (i = 1, 2) . \tag{6.7}$$

Using the same arguments as in ([1] theorem 8) by mean of (6.6) it is possible to find the boundary conditions associated to (6.7). $\tilde{\underline{\alpha}}^i *$ belongs to $H(\operatorname{div}, Y)$ and $\frac{\partial q^i *}{\partial n} / \partial Y$ which exists in $H^{-1/2}(\partial Y)$ is Y-periodic and given by :

$$h^3 \frac{\partial q^i *}{\partial n} = - h^3 n_i + 12 \, \tilde{\underline{\alpha}}^i * . n . \tag{6.8}$$

By regularity, we can show also that the traces of $q^i *$ are equal on each opposite side of Y. Now for any $\underline{\phi} = (\phi, 0, 0)$ with ϕ in H_Y^0, we have :

$$\int_B \frac{\partial \alpha_1^i *}{\partial z} \frac{\partial \phi}{\partial z} \, dy \, dz = \int_B q^i * \frac{\partial \phi}{\partial y_1} \, dy \, dz - \int_B \phi \, dy \, dz .$$

(6.3) $\Rightarrow \dfrac{\partial^2 \alpha_1^i *}{\partial z^2}$ belongs in fact to $L^2(B)$ and by Green formula :

$$0 = \int_{\partial B} \frac{\partial \alpha_1^i *}{\partial z} \phi (n . z) \, d\sigma = \int_{\partial B} q^i * \phi (n . y_1) \, d\sigma$$

and this gives the Y-periodicity of q^{i*}, so $q^{i*} \in H_p^1(Y)$. The weak formulation in $H_p^1(Y)$ of (6.7)-(6.8) is then exactly (6.2) which has a unique solution in $H_p^1(Y)/\mathbb{R}$ because $\int_Y \frac{\partial h^3}{\partial y_i} = 0$. □

LEMMA 6.4 : *q^{0*} is the unique solution in $L_0^2(B) \cap H_p^1(Y)$ of:*

$$\int_Y h^3 \nabla q^{0*} \nabla \phi = -6s \int_Y \frac{\partial h}{\partial y_1} \phi \, dy \quad \forall \phi \in H_p^1(Y) .$$

Proof: Same as lemma 6.3. We find the corresponding of (6.6) :

$$\tilde{\alpha}_j^{0*} = -\frac{h^3}{12} \frac{\partial q^{0*}}{\partial y_j} + \frac{h}{2} s \delta_j^1 . \quad \square \tag{6.9}$$

The local problems L^i being completely known when λ tends to zero, the limit of p^λ can be given :

THEOREM 6.1 : *When λ tends to zero, the sequence $\lambda^2 p^\lambda$ converges in $H^1(\omega)$ weak to the unique solution in $H^1(\omega) \cap L_0^2(\Omega)$ of:*

$$\int_\omega \sum_{i,j=1,2} a_{ij}^{*0} \frac{\partial p^{*0}}{\partial x_i} \frac{\partial \phi}{\partial x_j} dx = \int_\omega \sum_{i=1,2} \beta_i^{*0} \frac{\partial \phi}{\partial x_i} - \int_{\partial \omega} t \phi \, d\sigma \tag{6.10}$$

where a_{ij}^{*0} is given by (6.11) *and :*

$$\beta_i^{*0} = -\int_Y \frac{h^3}{12} \frac{\partial q^{0*}}{\partial y_i} dy + \int_Y h s \delta_i^1 \, dy .$$

Proof: We consider the weak formulation (4.4) for $\lambda^2 p^\lambda$.
A^λ converges to $A^{*0} = a_{ij}^{*0}$ given by (see (6.6)) :

$$a_{ij}^{*0} = -\int_Y \tilde{\alpha}_j^{i*} \, dy = +\int_Y \frac{h^3}{12} \left(\frac{\partial q^{i*}}{\partial y_j} + \delta_i^j \right) dy . \tag{6.11}$$

We prove that A^{*0} is symmetric definite positive : We choose $\phi = q^{j*}$ in (6.2). Setting $\Pi_j(y) = y_j$:

$$a_{ij}^{*0} = \int_Y \frac{h^3}{12} \frac{\partial}{\partial y_j} (q^{i*} + \Pi_i(y)) \, dy$$

$$= \int_Y \frac{h^3}{12} \sum_k \frac{\partial (q^{i*} + \Pi_i(y))}{\partial y_k} \frac{\partial (q^{j*} + \Pi_j(y))}{\partial y_k} dy$$

which proves that the matrix is symmetric and positive. Now for any ξ in $\mathbb{R}^2$, we set :

$$\sigma = \sum_{i=1,2} \xi_i (q^{i*} + \Pi_i(y))$$

$$I = \sum_{i,j=1,2} a_{ij}^{*0} \xi_i \xi_j = \int_Y \frac{h^3}{12} \nabla\sigma^2 \, dy \geq \frac{h_{\min}^3}{12} \int_Y \nabla\sigma^2 \, dy .$$

$I = 0$ induces :

$$\int_Y \sum_{i=1,2} \left(\frac{\partial \sigma}{\partial y_i} \right)^2 dy = \int_Y \left(\sum_{i=1,2} \xi_i^2 + 2 \sum_{i,j=1,2} \xi_i \xi_j \frac{\partial q^{i*}}{\partial y_j} + \sum_{i=1,2} \left(\sum_{j=1,2} \xi_j \frac{\partial q^{i*}}{\partial y_j} \right)^2 \right) = 0 .$$

Due to the periodicity of q^{i*}, the second term is zero.
So $I = 0$ implies $\sum_{i=1,2} \xi_i^2 = 0$ and then $\xi = 0$.

This proves that the matrix is definite positive. The estimates for p^λ and the limits are obtained as in the proof theorem 5.1. □

Remark : It can be shown that the matrix A^{*0} is definite positive even if we don't suppose that h is uniform in x [4].

VII. CONCLUDING REMARKS

If we want to relax the assumption « $h(x, y)$ doesn't depend on x » we can first consider the case where

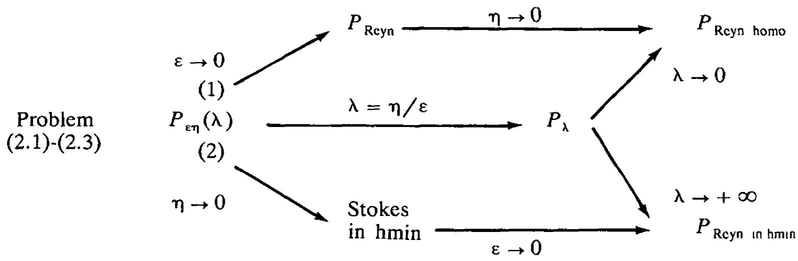
$$h(x, y) = f(x) \, g(y) .$$

For a mechanical aspect such a gap is well suited to take account of the elastic deformation of the surface coupled with the roughness appearing for instance in the gears.

In this case with minor changes all the coefficient estimates are still valid but the limit behavior of the pressure cannot be proved in a rigorous way.

If we want to consider a general form for h, regularity results for Stokes problem with coefficients are needed and this doesn't fall into the topic of this work.

The overall results can be summarized in a diagram with two small parameters ε and η and $\lambda = \eta/\varepsilon$.



There are three ways to make ε and η tend to zero : ways (1) (2) and (λ). Each gives a different result. But if way (λ) is driven on ($\lambda \to 0$ or $\lambda \to +\infty$), then the diagram is commutative. But ways (1) and (2) have strictly different issues.

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